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Integral Representation of the Solutions for Neutral Linear Fractional System with Distributed Delays

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Abstract: In the present paper, first we obtain sufficient conditions for the existence and uniqueness of the solution of the Cauchy problem for an inhomogeneous neutral linear fractional differential system with distributed delays (even in the neutral part) and Caputo type derivatives, in the case of initial functions with first kind discontinuities. This result allows to prove that the corresponding homogeneous system possesses a fundamental matrix $C(t,s)$ continuous in t, $t \in [a,\infty)$, $a \in \mathbb{R}$. As an application, integral representations of the solutions of the Cauchy problem for the considered inhomogeneous systems are obtained.

Keywords: fractional derivatives; neutral fractional systems; distributed delay; integral representation

MSC: 34A08; 34A12

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1. Introduction

It is well known that fractional calculus and fractional differential equations are an efficient tool for investigations in various fields of science. A good overview of the different areas of the fractional calculus theory and fractional differential equations can be found in the monographs of Kilbas et al. [1], and Kiryakova [2], Podlubny [3]. The book of Diethelm [4] is devoted to an application-oriented exposition, and some generalizations concerning distributed order fractional differential equations are considered in Jiao et al. [5]. The fundamental theory of the impulsive fractional differential equations (with and without delay) is presented in the monograph of Stamova and Stamov [6]; their applications are considered as well.

In comparison with the integer-order case, the main advantage of the delayed fractional differential equations is the possibility to describe the impact of history on the evolution of the processes, taking information from two sources (the memory source of the fractional derivative and the memory impact caused by the delay). It is well known that the problem to obtain an integral representation of the solution for linear fractional differential equations and/or systems (ordinary or with delay) plays a central role in the qualitative analysis of the studied objects. A mainstay tool for solving this problem is the existence of a fundamental matrix. That is why the road to the integral representations of the solutions crosses the problem of the existence of a fundamental matrix, an important evergreen theme for research. We think that this is an explanation for why a lot of papers are devoted to these problems. For the case of linear fractional ordinary differential equations and systems concerning this theme, we refer to [7–11] and the references therein. Some results in the autonomous case using the Laplace transform method are obtained in [12–14]. From the works devoted to the problem of establishing an integral representation for fractional differential equations and/or systems with delay, we point out [15–17] for the case of singular systems. Both problems (the existence of a fundamental matrix and integral representation of the solutions) for fractional systems with a single-order Caputo-type derivative of retarded and/or neutral type with distributed delays are studied in [18–26].

It is known that in the general case, the problem of the solvability of an initial problem for this system with discontinuous initial function is the basic result from which, as a corollary, we can prove the existence of a fundamental matrix for a homogeneous delayed (or neutral) fractional differential system. As far we know, in the last five years, published works related to the theme of the integral representation of the solutions for delayed and neutral fractional systems with Caputo-type derivatives are not so much. For articles considering delayed fractional systems with Caputo-type derivatives related to the theme, we refer to [18,22,25,27]. Concerning works devoted to this theme for autonomous neutral fractional systems case, we refer to [28] for the case of one constant delay; the case of a system with several constant delays is studied in [20]; and the general case of a system with distributed delays is considered in [26]. For nonautonomous neutral fractional systems, the case of one constant delay is considered in [17], and the case with an autonomous neutral part in [21].

In the present work, we consider as the most general case a neutral linear delayed system of the incommensurate type with distributed delay (even in the neutral part) with derivatives in the Caputo sense.

The paper is organized as follows: In Section 2, we recall some needed definitions of Riemann–Liouville and Caputo fractional derivatives, as well as the needed part of their properties. In the same section, the problem statement can be found, and some notations are introduced. Section 3 is devoted to the existence and the uniqueness of the solutions of the Cauchy problem for the considered systems with a picewise continuous initial function having bounded variation in the initial interval. As a corollary of the proved results, we obtain that the corresponding homogeneous system has a continuous, in t, $t \in [a, \infty)$, $a \in \mathbb{R}$, fundamental matrix, whose result extends the corresponding ones, even in the particular case of a fractional system with one constant delay and lower terminal at zero of the fractional derivatives. In Section 4 on the base of the obtained results in Section 3, we present integral representations of the solutions of the considered Cauchy problem. Some conclusions and comments are given in Section 5.

2. Preliminaries and Problem Statement

For the readers' convenience and to avoid possible misunderstandings, below we recall some definitions of Riemann–Liouville and Caputo fractional derivatives as well as some of their needed properties. For details and other properties, we refer to [1,3].

Let $\alpha \in (0,1)$ be an arbitrary number, and let us denote by $L_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$. Then, for each $a \in \mathbb{R}$ and $f \in L_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operator of order α and the corresponding left-side Riemann–Liouville derivative are defined for $t > a$ by the following:

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \ (D_{a+}^{0} f)(t) = f(t)$$
$$\text{and } {}_{RL}D_{a+}^{\alpha} f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(D_{a+}^{-(1-\alpha)} f(t)\right),$$

respectively. The Caputo fractional left-side derivative is defined by ${}_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t)$. We use the following relations (see Kilbas et al. [1]; see Lemmas 2.21 and 2.22):

(a) $D_{a+}^{-\alpha}\, {}_{C}D_{a+}^{\alpha} f(t) = f(t) - f(a)$;
(b) ${}_{C}D_{a+}^{\alpha} D_{a+}^{-\alpha} f(t) = f(t)$;
(c) ${}_{C}D_{a+}^{\alpha} f(t) = {}_{RL} D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}$.

Consider for $t > a$ the inhomogeneous and homogeneous linear neutral systems with incommensurate-type differential orders and distributed delays in the following general form:

$$D^{\alpha}_{a+}\Big(X(t) - \sum_{l=1}^{r}\int_{-\tau_l}^{0}\Big[\mathrm{d}_\theta V^l(t,\theta)\Big]X(t+\theta)\Big) = \sum_{i=0}^{m}\int_{-\sigma_i}^{0}\Big[\mathrm{d}_\theta U^i(t,\theta)\Big]X(t+\theta) + F(t), \quad (1)$$

$$D^{\alpha}_{a+}\Big(X(t) - \sum_{l=1}^{r}\int_{-\tau_l}^{0}\Big[\mathrm{d}_\theta V^l(t,\theta)\Big]X(t+\theta)\Big) = \sum_{i=0}^{m}\int_{-\sigma_i}^{0}\Big[\mathrm{d}_\theta U^i(t,\theta)\Big]X(t+\theta), \quad (2)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_k \in (0,1)$, $k \in \langle n \rangle = \{1,2,\ldots,n\}$, $l \in \langle r \rangle$, $r \in \mathbb{N}$, $\tau_l \in (0,\tau]$, $\sigma_i \in (0,\sigma]$, $h = \max(\tau,\sigma)$, $J_a = [a,\infty)$, $a \in \mathbb{R}$, $X, F : J_a \to \mathbb{R}^n$, $U^i, V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $V^l(t,\theta) = \Big\{v^l_{kj}(t,\theta)\Big\}^n_{k,j=1}$, $U^i(t,\theta) = \Big\{u^i_{kj}(t,\theta)\Big\}^n_{k,j=1}$, $V(t,\theta) = \sum_{l\in\langle r\rangle} V^l(t,\theta)$, $U(t,\theta) = \sum_{i=0}^{m} U^i(t,\theta)$, $X(t) = (x_1(t),\ldots,x_n(t))^T$, $F(t) = (f_1(t),\ldots,f_n(t))^T$, $D^{\alpha}_{a+}X(t) = \mathrm{diag}(D^{\alpha_1}_{a+}x_1(t),\ldots,D^{\alpha_n}_{a+}x_n(t))$, where $D^{\alpha_k}_{a+}$ denotes the left side Caputo fractional derivative ${}_CD^{\alpha_k}_{a+}$. The system (1) described with more details has the form

$$D^{\alpha_k}_{a+}(x_k(t) - \sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-\tau_l}^{0} x_j(t+\theta)\mathrm{d}_\theta v^l_{kj}(t,\theta))) = \sum_{i=0}^{m}(\sum_{j=1}^{n}\int_{-\sigma_i}^{0} x_j(t+\theta)\mathrm{d}_\theta u^i_{kj}(t,\theta)) + f_k(t)$$

In addition, we use also the following notations:

Let $Y : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $Y(t,\theta) = \{y_{k,j}(t,\theta)\}^n_{k,j=1}$. Then, $|Y(t,\theta)| = \sum_{k,j=1}^{n}|y_{k,j}(t,\theta)|$, $BV[-h,0]$ denotes the linear space of matrix valued functions $Y(t,\theta)$ with bounded variation in θ on $[-h,0]$ for every $t \in J_a$ and $Var_{[-h,0]}Y(t,\cdot) = \sum_{k,j=1}^{n} Var_{[-h,0]}y_{k,j}(t,\cdot)$. Everywhere below $\mathbb{R}_+ = (0,\infty)$, $\overline{\mathbb{R}}_+ = [0,\infty)$, $\mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $J_a = [a-h,\infty)$, $J_{a+M} = [a-h,a+M]$, $M \in \mathbb{R}_+$, and for $Y(t) = (y_1(t),...,y_n(t))^T : J_a \to \mathbb{R}^n$, $\beta = (\beta_1,...,\beta_n), \beta_k) \in [-1,1]$, $k \in \langle n \rangle$, we will use the notation $I_\beta(Y(t)) = \mathrm{diag}(y_1^{\beta_1}(t),...,y_n^{\beta_n}(t))$, where $\Theta, I \in \mathbb{R}^{n\times n}$ are the zero and identity matrices, respectively.

With $\mathbf{C}^*_a = PC([a-h,a],\mathbb{R}^n)$, $a \in \mathbb{R}$, we denote the Banach space of all right continuous piecewise vector functions $\Phi = (\phi_1,...,\phi_n)^T : [a-h,a] \to \mathbb{R}^n$ with norm $||\Phi|| = \sup_{t\in[a-h,a]}|\Phi(t)| = \sum_{k=1}^{n}\sup_{t\in[a-h,a]}|\phi_k(t)| < \infty$. For the set S^Φ of all jump points of each function $\Phi \in \mathbf{C}^*_a$, we assume that the number of jump points is finite. By $\mathbf{C}_a \subset \mathbf{C}^*_a$, we assume the subspace of all continuous functions in $\mathbf{C}^*_a$.

Consider the following standard initial conditions for the system (1):

$$X(t) = \Phi(t)(x_k(t) = \phi_k(t),\ k \in \langle n \rangle),\ t \in [a-h,a],\ \Phi \in \mathbf{C}^*_a. \quad (3)$$

We say that for the kernels U^i, $V^l : \overline{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $i \in \langle m \rangle_0$, $l \in \langle r \rangle$ the conditions (**S**) are fulfilled if the following conditions hold:

(**S1**) The functions $(t,\theta) \to U^i(t,\theta)$, $(t,\theta) \to V^l(t,\theta)$ are measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that $U^i(t,\theta) = 0$, $V^l(t,\theta) = 0$ for $\theta \geq 0$, $U^i(t,\theta) = U^i(t,-\sigma_i)$ for $\theta \leq -\sigma_i$, $V^l(t,\theta) = V^l(t,-\tau_l)$ for $\theta \leq -\tau_l$, $t \in J_a$. For each $t \in J_a$, $l \in \langle r \rangle$, $i \in \langle m \rangle_0 = \langle m \rangle \cup \{0\}$ the kernels $U^i(t,\theta)$ and $V^l(t,\theta)$ are continuous from the left in θ on $(-\sigma_i,0)$ and $(-\tau_l,0)$, respectively, and $U^i(t,\cdot)$, $V^l(t,\cdot) \in BV[-h,0]$.

(**S2**) For $t \in J_a$ and $\theta \in [-h,0]$, the Lebesgue decompositions of the kernels have the following form:
$U^i(t,\theta) = U^i_d(t,\theta) + U^i_{ac}(t,\theta) + U^i_s(t,\theta)$ and $V^l(t,\theta) = V^l_d(t,\theta) + V^l_{ac}(t,\theta) + V^l_s(t,\theta)$,

respectively, $U^i(t,\theta) = \{a^i_{kj}(t)H(\theta+\sigma_i(t))\}^n_{k,j=1}$, $\overline{V}^l(t,\theta) = \{\overline{a}^l_{kj}(t)H(\theta+\tau_l(t))\}^n_{k,j=1}$, $H(t)$ is the Heaviside function, $A^i(t) = \{a^i_{kj}(t)\}^n_{k,j=1} \in L^{loc}_1(J_a \times \mathbb{R},\ \mathbb{R}^{n\times n})$ are locally bounded, $\overline{A}^l(t) = \{\overline{a}^l_{kj}(t)\}^n_{k,j=1} \in C(J_a, \mathbb{R}^{n\times n})$, $\sigma_i(t)$, $\tau_l(t) \in C(J_a, \overline{\mathbb{R}}_+)$, $\sigma_0(t) \equiv 0$, for every $t \in J_a$, $\sup_{t\in J_a}\tau_l(t) \le \tau_l$, $\sup_{t\in J_a}\sigma_i(t) \le \sigma_i$, $l \in \langle r\rangle$, $i \in \langle m\rangle$ and $V^l_c(t,\theta) = \{\overline{v}^l_{k,j}\}_{k,j\in\langle n\rangle}$, $V^l_c(\cdot,\theta) = V^l_{ac}(\cdot,\theta) + V^l_s(\cdot,\theta) \in C([a,a+\tau],\ \mathbb{R}^{n\times n})$ for $\theta \in [-h,0]$.

(**S3**) The functions $Var_{[-\sigma,0]}U^i(t,\cdot)$ and $Var_{[-\tau,0]}V^l(t,\cdot)$ are locally bounded in J_a, and the kernels $V^l(t,\theta)$ are uniformly nonatomic at zero [29,30], i.e., for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $Var_{[-\delta,0]}V^l(t,\cdot) < \varepsilon$.

(**S4**) For each $t^* \in J_a$, the following relations hold:

$$\lim_{t\to t^*}\int_{-\sigma}^{0} |U^i(t,\theta) - U^i(t^*,\theta)|\mathrm{d}\theta = \lim_{t\to t^*}\int_{-\tau}^{0} |V^l(t,\theta) - V^l(t^*,\theta)|\mathrm{d}\theta = 0$$

and the sets $S^l_\Phi = \{t \in J_a|\ t-\tau_l(t) \in S^\Phi\}$, $S^i_\Phi = \{t \in J_a|\ t-\sigma_i(t) \in S^\Phi\}$ for every $l \in \langle r\rangle$ and $i \in \langle m\rangle$ do not have limit points [31].

Remark 1. *Note that from Condition* (**S4**)*, it follows that for all $l \in \langle r\rangle$, $i \in \langle m\rangle$ and $M \in \mathbb{R}_+$ the sets $S^i_\Phi(M) = \{t \in [a,a+M]|\ t-\sigma_i(t) \in S^\Phi\}$ and $S^l_\Phi(M) = \{t \in [a,a+M]|\ t-\tau_l(t) \in S^\Phi\}$ are finite.*

Definition 1. *The vector function $X(t) = (x_1(t),...,x_n(t))^T$ is a solution of the initial problem (IP)* (1), (3) *in $J_{a+M}(J_a)$ if $X|_{[a,a+M]} \in C([a,a+M],\ \mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$ satisfies system* (1) *for all $t \in (a,M](t \in (a,\infty))$ and initial condition* (3) *for $t \in [a-h,a]$.*

Let us assume that the conditions (**S**) hold. Consider the following auxiliary system:

$$\begin{aligned} X(t) &= \Phi(a) - \sum_{l=1}^{r}\int_{-\tau_l}^{0}[\mathrm{d}_\theta V^l(a,\theta)]X(a+\theta) + \sum_{l=1}^{r}\int_{-\tau_l}^{0}[\mathrm{d}_\theta V^l(t,\theta)]X(t+\theta) \\ &+ I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\eta)\sum_{i=0}^{m}\int_{-\sigma_i}^{0}[\mathrm{d}_\theta U^i(\eta,\theta)]X(\eta+\theta) \\ &+ I_{-1}(\Gamma(\alpha))\int_a^t I_{\alpha-1}(t-\eta)F(\eta)\mathrm{d}\eta, \end{aligned} \tag{4}$$

or in more detailed form for $k \in \langle n\rangle$

$$\begin{aligned} x_k(t) &= \phi_k(a) - \sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-h}^{0}\phi_j(a+\theta)\mathrm{d}_\theta v^l_{kj}(a,\theta)) + \sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-h}^{0}x_j(t+\theta)\mathrm{d}_\theta v^l_{kj}(t,\theta)), \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_a^t (t-\eta)^{\alpha_k-1}f_k(\eta)\mathrm{d}\eta \\ &+ \frac{1}{\Gamma(\alpha_k)}\int_a^t (t-\eta)^{\alpha_k-1}[\sum_{i=0}^{m}(\sum_{j=1}^{n}\int_{-h}^{0}x_j(\eta+\theta)\mathrm{d}_\theta u^i_{kj}(\eta,\theta))]\mathrm{d}\eta. \end{aligned} \tag{5}$$

Definition 2. *The vector function $X(t) = (x_1(t),...,x_n(t))^T$ is a solution of the IP* (4), (3) *in $J_{a+M}(J_a)$ if $X|_{[a,a+M]} \in C([a,a+M],\ \mathbb{R}^n)$, $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$ satisfies system* (4) *for all $t \in (a,M](t \in (a,\infty))$ and initial condition* (3) *for $t \in [a-h,a]$.*

Let $M_0 \in (0, h]$, $a + M_0$ be an arbitrary initial point; $\Phi \in \mathbf{C}_a^*$, $\Psi(t) \in C([a, a + M_0], \mathbb{R}^n)$ are arbitrary given functions with $\Psi(a) = \Phi(a)$. For the initial interval $[a - h, a + M_0]$, define the following initial function $\overline{\Phi}(t) : [a - h, a + M_0] \to \mathbb{R}^n$ as $\overline{\Phi}(t) = \Phi(t)$, $t \in [a - h, a]$, $\overline{\Phi}(t) = \Psi(t)$, $t \in [a, a + M_0]$ and introduce the following initial condition:

$$X(t) = \overline{\Phi}(t) \text{ for } t \in [a - h, a + M_0]. \tag{6}$$

Definition 3. *The vector function $X(t) = (x_1(t), ..., x_n(t))^T$ is a solution of IP (4) and (6) or of IP (1) and (6) in $(a + M_0, a + M_0 + M]$ ($(a + M_0, \infty)$), $M \in \mathbb{R}_+$ if $X|_{[a,a+M_0+M]} \in C([a, a + M_0 + M], \mathbb{R}^n)$, $(X|_{(a,\infty)} \in C((a, \infty), \mathbb{R}^n))$ satisfies system (4) or system (1), respectively, for all $t \in (a + M_0, a + M_0 + M]((a + M_0, \infty))$ and initial condition (6) for $t \in [a - h, a + M_0]$.*

Lemma 1 ([23])**.** *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ and is locally bounded.*

Then, every solution $X(t)$ of IP (1), (3) is a solution of IP (4) and (3) and vice versa.

The next lemma treats the same problem for IP (4) and (6) and IP (1) and (6) in the case when the initial point does not coincide with the lower terminal of the fractional derivative.

Lemma 2. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ and is locally bounded.*
3. *Let $\Phi \in \mathbf{C}_a^*$, $M^0 \in (0, h]$ be arbitrary and $\Psi(t) \in C([a, a + M^0], \mathbb{R}^n)$ with $\Psi(a) = \Phi(a)$ be an arbitrary given function.*

   Then, every solution $X(t)$ of IP (1) and (6) is a solution of IP (4) and (6) and vice versa.

Proof. The proof is standard and analogical of the proof of Lemma 4 in [23] and, therefore, will be omitted. □

Let $\Phi \in \mathbf{C}_a^*$ be arbitrary and introduce the sets $S_\Phi^l(M) = \{t \in [a, a + M] | \, t - \tau_l(t) \in S^\Phi\}$ for every $M \in [0, h]$ and define $S_\Phi^l(M) = \varnothing$ when $t - \tau_l(t) \notin S^\Phi$ for $t \in [a, a + M]$. Obviously, from condition (**S**4) it follows that the sets $\bigcup_{l \in r} S_\Phi^l(M)$, $M \in [0, h]$ are finite. Furthermore, it is not difficult to see that either $\bigcup_{l \in r} S_\Phi^l(0) \bigcap S^\Phi = \varnothing$, or $\bigcup_{l \in r} S_\Phi^l(0) \bigcap S^\Phi = \{a\}$. Note that it is possible that $a \notin \bigcup_{l \in r} S_\Phi^l(a)$, no matter whether $a \in S^\Phi$, or not. This case is considered in the next lemma.

3. Main Results

In our exposition below, we need the next two technical lemmas.

Lemma 3. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. $\prod_{l \in \langle r \rangle} \tau_l(a) > 0.$

Then, for every initial function $\Phi \in \mathbf{C}_a^*$, there exists a constant $M^* \in (0, h]$ (eventually depending from Φ) such that $\Phi(t - \tau_l(t))$ is continuous in $[a, a + M^*]$ for all $l \in \langle r \rangle$.

Proof. (a) Let $\bigcup_{l \in \langle r \rangle} S_\Phi^l(0) = \varnothing$. Then, the functions $t - \tau_l(t)$ are continuous at a for $l \in \langle r \rangle$ and since S^Φ and $\bigcup_{l \in \langle r \rangle} S_\Phi^l(0) = \varnothing$ are finite. Then, there exists $M_a \in (0, h]$ such that the set $\bigcup\limits_{l\in r} S^l_\Phi(M_a) = \varnothing$ and therefore $t - \tau_l(t) < a$ for $t \in (a, a+M_a]$. Thus, the functions $\Phi(t-\tau_l(t))$ are continuous in $[a, a+M_a]$ (right continuous at a), $l \in \langle r\rangle$.

(b) Let $\bigcup\limits_{l\in\langle r\rangle} S^l_\Phi(0) \neq \varnothing$. Let us assume the contrary, that for every $M_b \in (0,h]$, the set $\bigcup\limits_{l\in r} S^l_\Phi(M_b) \neq \varnothing$. Then, there exists a monotone decreasing sequence $\{M_q\}_{q\in\mathbb{N}} \subset (0,h]$, $M_{q+1} \subset M_q$ such that for each M_q and some $l \in \langle r\rangle$, the sets $S^l_\Phi(M_q) \neq \varnothing$. Since the numbers $l \in \langle r\rangle$ are finitely many, then for at least one number $l_* \in \langle r\rangle$, there exists an infinite monotone decreasing subsequence $\{M_{q_k}\}_{k\in\mathbb{N}} \subset \{M_q\}_{q\in\mathbb{N}}$ such that the sets $S^{l_*}_\Phi(M_{q_k}) \neq \varnothing$ for each $k \in \mathbb{N}$. Thus, the set $S^{l_*}_\Phi(h)$ has at least one limit point, which contradicts condition $(S6)$. Then, there exists $M_b \in (0,h]$ such that $t - \tau_l(t) < a$ for $t \in [a, a+M_b]$ and $\Phi(t-\tau_l(t))$ is continuous in $[a, a+M_b]$. □

Definition 4. *For an initial function $\Phi \in \mathbf{C}^*_a$, the point a is called a critical jump point concerning the kernel $V^l(t,\theta)$ if $a \in S^\Phi$ for this $l \in \langle r\rangle$ we have that $\tau_l(a) = 0$ and there exists a constant $\varepsilon \in (0,h]$ (eventually depending from τ_l), such that $t - \tau_l(t) \geq a$ for $t \in [a, a+\varepsilon]$.*

Remark 2. *It is simple to see that for an initial function $\Phi \in \mathbf{C}^*_a$, the point $a \in S^\Phi$ can be a critical jump point concerning more than one kernels $V^l(t,\theta)$ in partial for all of them.*

The next lemma considers the important case in view of the applications (existence of fundamental matrix, integral representations, etc.) when $\bigcup\limits_{l\in r} S^l_\Phi(a) \bigcap S^\Phi \neq \varnothing$, i.e., $a \in S^\Phi$ is a critical jump point concerning at least one kernel.

Lemma 4. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. $\prod\limits_{l\in\langle r\rangle} \tau_l(a) = 0.$

Then for every initial function $\Phi \in \mathbf{C}^*_a$ with $a \in S^\Phi$, one of the following statements holds:

(i) *There exists a constant $\varepsilon \in (0,h]$ (eventually depending from τ_l) such that $\Phi(t-\tau_l(t))$ is continuous for $t \in [a, a+\varepsilon]$.*

(ii) *The point $a \in S^\Phi$ is a critical jump point for Φ concerning a kernel $V^l(t,\theta)$, for some $l \in \langle r\rangle$.*

Proof. Condition 2 implies that $a \in \bigcup\limits_{l\in\langle r\rangle} S^l_\Phi(a)$ and let $l^* \in \langle r\rangle$ be an arbitrary number for which $\tau_{l^*}(a) = 0$. From condition (S4), it follows that there exists a constant $\varepsilon \in (0,h]$ (eventually depending from τ_l) such that either $t - \tau_{l^*}(t) < a$ and hence $\Phi(t - \tau^*_l(t))$ is continuous for $t \in [a, a+\varepsilon]$, or $t - \tau_{l^*}(t) \geq a$ for $t \in (a, a+\varepsilon]$. Hence, $a \in S^\Phi$ is a critical jump point and statement (ii) holds. For the case when $l \in \langle r\rangle$ and $\tau_l(a) > 0$ as in case (b) of Lemma 3, it can be proved that (i) holds. □

Let us, for every $\Phi \in \mathbf{C}^*_a$, consider the following set:

$$\begin{aligned} E^\Phi = &\{\widetilde{G} : [a-h,\infty) \to \mathbb{R}^n |\ \widetilde{G}|_{J_a} \in C(J_a,\mathbb{R}^n) \bigcap BV([a,a+\tau],\mathbb{R}^n), \\ &\widetilde{G}(t) = \Phi(t),\ t \in [a-h,a]\} \end{aligned}$$

and for $M \in (0,h]$ define the set $E^\Phi_M = \{G : [a-h, a+M] \to \mathbb{R}^n|\ \ G = \widetilde{G}|_{[a-h,a+M]},\ \widetilde{G} \in E^\Phi\}$ and a metric function $d^\Phi_M : E^\Phi_M \times E^\Phi_M \to \overline{\mathbb{R}}_+$, $d^\Phi_M(G,G^*) = \sum\limits_{k=1}^{n} \sup\limits_{t\in J_{a+M}} |g_k(t) - g^*_k(t)|$. It is clear that the set E^Φ_M endowed with the metric $d^\Phi_M : E^\Phi_M \times E^\Phi_M \to \overline{\mathbb{R}}_+$ is a complete metric space concerning this metric.

Remark 3 (Lemma 1, [25]). *Note that when $M \in (0, \tau]$ the defined above metric d_M^{Φ} is equivalent in E_M^{Φ} with the metric $\overline{d}_M^{\Phi}(G, G^*) = Var_{t\in[a,a+M]}(G(t) - G^*(t)) = \sum_{k=1}^{n} \underset{t\in J_{a+M}}{Var} |g_k(t) - g_k^*(t)|$.*

Following the approach introduced in [21] and supposing that the conditions (S) hold, for arbitrary $M \in (0, \tau]$, $k \in \langle n\rangle$ and for each $G_M = (g_1, ..., g_n)^T \in E_M^{\Phi}$ define the operator $\Re G(t) = (\Re_1 g_1(t), ..., \Re_n g_n(t))^T$ by the following:

$$\begin{aligned}\Re G(t) &= \Phi(a) - \sum_{l=1}^{r} \int_{-\tau_l}^{0} [\mathrm{d}_\theta V^l(a,\theta)]\Phi(a+\theta)) + \sum_{l=1}^{r} \int_{-\tau_l}^{0} [\mathrm{d}_\theta V^l(t,\theta)]G(t+\theta)) \\ &+ I_{-1}(\Gamma(\alpha)) \int_{a}^{t} I_{\alpha-1}(t-\eta)(\sum_{i=0}^{m} \int_{-\sigma_i}^{0} [\mathrm{d}_\theta U^i(\eta,\theta)]G(\eta+\theta) \\ &+ I_{-1}(\Gamma(\alpha)) \int_{a}^{t} I_{\alpha-1}(t-\eta)F(\eta)\mathrm{d}\eta,\end{aligned} \tag{7}$$

for $t \in (a, a+M)$ and the additional relations

$$\Re G(t) = \overline{\Phi}(t),\ t \in [a-h, a];\ \Re G(a+M) = \lim_{t\to(a+M)-0} \Re G(t). \tag{8}$$

Theorem 1. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold and the function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*
2. *The initial function $\Phi \in \mathbf{C}_a^*$ is without a critical jump point.*
3. *$\Phi \in BV([a-\tau, a], \mathbb{R}^n)$.*

Then there exists a constant $M_0 \in (0, h]$ such that the operator $(\Re G_{M_0})(t) = (\Re_1 g_1(t), \ldots, \Re_n g_n(t))^T$ has a unique fixed point in the complete metric space $E_{M_0}^{\Phi}$, i.e., the IP (4), (3) has a unique local solution with interval of existence J_{a+M_0}.

Proof. Let $\Phi \in \mathbf{C}_a^*, \Phi \in BV([a-\tau, a], \mathbb{R}^n)$ be arbitrary and $a \in S^{\Phi}$ be not a critical jump point. Then, according to Lemmas 3 and 4, we can choose $M^* = \min(M_a, M_b, \varepsilon, \tau)$ such that for every $l \in \langle r\rangle$, $t - \tau_l(t) < a$ for $t \in [a, a+M^*]$ and $\Phi(t - \tau_l(t))$ are continuous in $[a, a+M^*]$. For $t \in (a, a+M^*]$ and arbitrary $G = (g_1, \ldots, g_n)^T \in E_M^{\Phi}$ we have the following:

$$\begin{aligned}&\sum_{l=1}^{r}(\sum_{j=1}^{n} \int_{-h}^{0} (g_j(t+\theta))\mathrm{d}_\theta v_{kj}^l(t,\theta)) = \sum_{l=1}^{r}\sum_{j=1}^{n} \overline{a}_{kj}^l(t) g_j(t-\tau_l(t)) \\ &+ \sum_{j=1}^{n} \int_{-h}^{0} g_j(t+\theta)\mathrm{d}_\theta \overline{v}_{kj}^l(t,\theta) = \sum_{l=1}^{r}\sum_{j=1}^{n} \overline{a}_{kj}^l(t)\phi_j(t-\tau_l(t)) + \sum_{j=1}^{n} \int_{-h}^{0} g_j(t+\theta)\mathrm{d}_\theta \overline{v}_{kj}^l(t,\theta)\end{aligned} \tag{9}$$

Let $j \in \langle n\rangle$ be arbitrary and since $t + \theta \le t$ for each $t \in [a, a+\tau]$, we have that $Var_{\theta\in[-\tau,0]} g_j(t+\theta) < \infty$ and $\lim_{t\to\tilde{t}} \int_{-\tau}^{0} |g_j(\tilde{t}+\theta) - g_j(t+\theta)|\mathrm{d}\theta = 0$.

Then for every $k, j \in \langle n\rangle$ and $l \in \langle r\rangle$ integrating by parts, we obtain the following:

$$\begin{aligned}\int_{-h}^{0} g_j(t+\theta)\mathrm{d}_\theta \overline{v}_{kj}^l(t,\theta) &= \int_{-\tau_l}^{0} g_j(t+\theta)\mathrm{d}_\theta \overline{v}_{kj}^l(t,\theta) \\ &= g_j(t-\tau_l)\overline{v}_{kj}^l(t,-\tau_l) - \int_{-\tau_l}^{0} \overline{v}_{kj}^l(t,\theta)\mathrm{d}_\theta g_j(t+\theta)\,.\end{aligned} \tag{10}$$

Since $\overline{v}^l_{kj}(t,0) = 0$ and $\widetilde{v}^l_{kj}(t,\theta)$ are continuous functions for $t \in J_a$, $\theta \in [-\tau, 0]$, then $\overline{v}^l_{kj}(t,-\tau)$ is a continuous function for $t \in [a, a+M^*]$ too. Furthermore, according to Lemma 1 in [31], the integral in the right side of (10) exists and is a continuous function for $t \in [a, a+M^*]$. Thus, the integral in the left side of (10) also exists and is a continuous function for $t \in [a, a+M^*]$; since the first addend in the right side of (9) is continuous in $(a, a+M^*]$, then $\Re_k g_k(t)$ for each $k \in \langle n \rangle$ is a continuous function for $t \in (a, a+M^*)$ too. Since for $k \in \langle n \rangle$, the right side of (10) is right continuous at a and a continuous function for $t \in (a, a+M)$, $M \in (0, M^*]$, then conditions (S5) and (S6) imply that for $k, j \in \langle n \rangle$ and $l \in \langle r \rangle$ the following relation holds:

$$\lim_{t \to a+0} \int_{-h}^{0} g_j(t+\theta) \mathrm{d}_\theta \overline{v}^l_{kj}(t,\theta) = \int_{-h}^{0} g_j(a+\theta) \mathrm{d}_\theta \overline{v}^l_{kj}(a,\theta)$$

and hence $\Re_k g_k(t)$ is right continuous at a. Taking into account that the right side of (10) is left continuous at $a+M^*$ and a continuous function for $t \in (a, a+M^*)$, as stated above, we conclude that conditions (S3) and (S4) imply that for $k, j \in \langle n \rangle$ and $l \in \langle r \rangle$ the following relation holds:

$$\lim_{t \to a+M-0} \int_{-h}^{0} g_j(t+\theta) \mathrm{d}_\theta v^l_{kj}(t,\theta) = \int_{-h}^{0} g_j(a+M+\theta) \mathrm{d}_\theta v^l_{kj}(a+M,\theta) .$$

Then the right side of (7) is left continuous at $a+M^*$ and hence $\Re_k g_k(t)$ is a continuous function for $t \in [a, a+M^*]$. Thus, we proved that $\Re G \in E^\Phi_M$ and hence, $\Re E^\Phi_M \subseteq E^\Phi_M$ for $M \in (0, M^*]$.

Let $G, \overline{G} \in E^\Phi_M$ be arbitrary and since for $t \in [a, a+M^*]$ and $l \in \langle r \rangle$ we have $t - \tau_l(t) < a$, then

$$G(t-\tau(t)) = \overline{G}(t-\tau(t)) = \Phi(t-\tau(t)) \tag{11}$$

for $t \in [a, a+M^*]$, where $G(t-\tau(t)) = (g_1(t-\tau_1(t)), ..., g_n(t-\tau_n(t)))^T$.

Let $\varepsilon \in (0, \frac{1}{2n})$ be arbitrary. From condition (S3) it follows that there exists $\delta(\varepsilon) \in (0,1)$ such that $Var_{[-\delta,0]} V^l(t,\cdot) < \varepsilon$ for every $l \in \langle r \rangle$ and $t \in J_a$. For every $t \in [a, a+\delta]$ and $\theta \leq -\delta$ we have $t+\theta \leq a$ and then for arbitrary $G, \overline{G} \in E^\Phi_M$ we have that the following equality holds:

$$G(t+\theta) = \overline{G}(t+\theta) = \Phi(t+\theta) \tag{12}$$

Let $\overline{M} = \min(M^*, \delta, 1)$; then, we obtain that $\sup\limits_{t \in [a, a+\delta]} \sum\limits_{l \in \langle r \rangle} Var_{\theta \in [-\delta, 0]} V^l_c(t,\theta) = \mathbf{V}_c < \frac{1}{2}$ for $t \in [a, a+\overline{M}]$. Then from (9) and (12) for $t \in [a, a+\overline{M}]$ it follows that

$$
\begin{aligned}
&\left|\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l(t,\theta)]G(t+\theta)-\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l(t,\theta)]\overline{G}(t+\theta)\right| \\
&\leq |\sum_{l=1}^{r}A^l(t)[G(t-\tau_l(t))-\overline{G}(t-\tau_l(t))] \\
&+\sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta))| \\
&\leq \sum_{l=1}^{r}|A^l(t)(\Phi(t-\tau_l(t))-\Phi(t-\tau_l(t)))| \\
&+\sum_{i=1}^{r}\left|\int_{-\tau}^{-\delta}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta))\right| \\
&+\sum_{i=1}^{r}\left|\int_{-\delta}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta)\right| \\
&\leq \sum_{i=1}^{r}\left|\int_{-\tau}^{-\delta}[\mathrm{d}_\theta V_c^l(t,\theta)](\Phi(t+\theta)-\Phi(t+\theta))\right| \\
&+\sum_{i=1}^{r}\left|\int_{-\delta}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta))\right| \leq \mathbf{V}_c d^{\Phi}_{\overline{M}}(G,\overline{G}) < \frac{1}{2}d^{\Phi}_{\overline{M}}(G,\overline{G}).
\end{aligned} \tag{13}
$$

Then for every for $t \in [a, a+\overline{M}]$ and $k \in \langle n\rangle$ from (7) and (12) we obtain the following estimation:

$$
\begin{aligned}
&|\Re G(t)-\Re\overline{G}(t)| \leq \frac{1}{2}d^{\Phi}_{\overline{M}}(G,\overline{G}) \\
&+\left|I_{-1}(\Gamma(\alpha))\int_{a}^{t}I_{\alpha-1}(t-\eta)(\sum_{i=0}^{m}\int_{-\sigma_i}^{0}[\mathrm{d}_\theta U^i(\eta,\theta)](G(\eta+\theta)-\overline{G}(\eta+\theta)))\mathrm{d}\eta\right| \\
&\leq \frac{1}{2}d^{\Phi}_{\overline{M}}(G,\overline{G})+(C^U\int_{a}^{t}|I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|\mathrm{d}\eta)d_{\overline{M}}(G,\overline{G}) \\
&\leq (\frac{1}{2}+C^U\int_{a}^{t}|I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|\mathrm{d}\eta)d_{\overline{M}}(G,\overline{G}),
\end{aligned} \tag{14}
$$

where $C^U = \sum\limits_{i\in\langle m\rangle_0}\sup\limits_{s\in J_{a+\overline{M}}} Var_{\theta\in[-\sigma,0]}U^i(s,\theta)$. For the integral in the right side of (14), we have the following:

$$
\begin{aligned}
&\int_{a}^{t}|I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|\mathrm{d}\eta) = \sum_{k\in\langle n\rangle}\Gamma^{-1}(\alpha_k)\int_{a}^{t}(t-\eta)^{\alpha_k-1}\mathrm{d}\eta \\
&= \sum_{k\in\langle n\rangle}\Gamma^{-1}(\alpha_k)\alpha_k^{-1}(t-a)^{\alpha_k} = \sum_{k\in\langle n\rangle}\frac{(t-a)^{\alpha_k}}{\Gamma(1+\alpha_k)} \\
&\leq (t-a)^{\alpha_{min}}\sum_{k\in\langle n\rangle}\frac{1}{\Gamma(1+\alpha_k)} = \Gamma^*(t-a)^{\alpha_{min}} \leq \Gamma^*\overline{M}^{\alpha_{min}},
\end{aligned} \tag{15}
$$

where $\Gamma = \max\limits_{k\in\langle n\rangle}(\frac{1}{\Gamma(1+\alpha_k)})$ and $a_{\min} = \min\limits_{k\in\langle n\rangle}(\alpha_k)$.

Let $q^* \in (\frac{1}{2}, 1)$ and choice $M_0 = \min((\frac{2q^*-1}{2C^U\Gamma^*})^{\frac{1}{\alpha_{\min}}}, \overline{M})$. Then from (14) and (15) it follows that $d_{M_0}(\Re G, \Re \overline{G}) \leq q^* d_{M_0}(\overline{G}, G)$ and hence, the operator $\Re$ is contractive in $\mathbf{E}^{\Phi}_{M_0}$. $\square$

Remark 4. *Note that without loss of generality, we can renumber all kernels so that those for which $a \in S^{\Phi}$ is a critical jump point have the numbers $V^1(t,\theta), \ldots, V^q(t,\theta)$, $0 \leq q \leq r$.*

In the next theorem, for convenience we assume that this renumbering is made.

Remark 5. *It is simple to see from the proof of Theorem 1 that the condition 3 of the Theorem 1 is necessary only for the proof that the Lebesgue–Stieltjes integral in the left side of (10) is a continuous function for $t \in [a-\tau, a]$. Obviously if the initial function $\Phi \in \mathbf{C}$, then condition 2 of Theorem 1 is ultimately fulfilled. According to Lemma 1 in [31], the mentioned integral is a continuous function for $t \in [a-\tau, a]$. Thus, in the case when $\Phi \in \mathbf{C}$ the condition $\Phi \in BV([a-\tau, a], \mathbb{R}^n)$ is unnecessary.*

Theorem 2. *Let the following conditions be fulfilled:*

1. *Condition 1 of Theorem 1 holds.*
2. *The point $a \in S^{\Phi}$ is a critical jump point for the initial function $\Phi \in \mathbf{C}^*_a$ concerning at least one kernel $V^l(t,\theta)$.*
3. $\sum_{l \in \langle r \rangle} \left| \overline{A}^l(a) \right| < 1.$

Then, there exists a constant $M_0 \in (0, h]$ such that the operator $(\Re G_{M_0})(t) = (\Re_1 g_1(t), \ldots, \Re_n g_n(t))^T$ has a unique fixed point in the complete metric space $E^{\Phi}_{M_0}$, i.e., the IP (4) and (3) have a unique local solution with an interval of existence J_{a+M_0}.

Proof. Let $\Phi \in \mathbf{C}^*_a$, $\Phi \in BV([a-\tau, a], \mathbb{R}^n)$ be arbitrary and $a \in S^{\Phi}$ be a critical jump point for Φ concerning the kernels $V^1(t,\theta), \ldots, V^q(t,\theta)$, for $1 \leq q \leq r$. Then, there exists a constant $\varepsilon(l_1) \in (0, h]$ (eventually depending from Φ), such that $t - \tau_1(t) \geq a$ for $t \in (a, a+\varepsilon(l_1)]$. Since the kernels $V^l(t,\theta)$, $l \in \langle q \rangle$ are finitely many, the constant $\varepsilon \in (0, h]$ can be chosen to be the same for all kernels. Obviously, for all other kernels $V^{q+1}(t,\theta), \ldots, V^r(t,\theta)$, according to Lemmas 3 and 4, there exists a constant $M^* \in (0, \varepsilon]$ (eventually depending from τ_l) such that $t - \tau_l(t) < a$ and $\Phi(t - \tau_l(t))$ are continuous for $t \in [a, a+M^*]$.

Let $M \in (0, M^*]$, $G \in E^{\Phi}_M$ be arbitrary and consider for $t \in (a, a+M)$ the operator $(\Re G)(t) = (\Re_1 g_1(t), ..., \Re_n g_n(t))^T$ defined with (7) and (8) in the proof of Theorem 1. As in Theorem 1, we can prove that for arbitrary $G \in E^{\Phi}_M$, we have $\Re G \in E^{\Phi}_M$ and hence, $\Re E^{\Phi}_M \subseteq E^{\Phi}_M$.

Let $G, \overline{G} \in E^{\Phi}_M$ be arbitrary. As shown above for $t \in [a, a+M^*]$ and $q+1 \leq l \leq r$ we have $t - \tau_l(t) < a$, and then (11) holds for $t \in [a, a+M^*]$ but only for $q+1 \leq l \leq r$. From condition 2, it follows that there exists $\widetilde{M} \in (0, \min(M^*, 1))$ such that $\sup_{t \in (a, a+\widetilde{M}]} \sum_{l \in \langle r \rangle} \left| \overline{A}^l(a) \right| = A^* < 1$. Let $\varepsilon \in (0, \frac{1}{2n})$ be arbitrary. From condition (S3) it follows that there exists $\delta(\varepsilon) \in (0,1)$ such that $Var_{[-\delta,0]} V^l(t,\cdot) < \varepsilon$ for every $l \in \langle r \rangle$ and $t \in J_a$. For every $t \in [a, a+\delta]$ and $\theta \leq -\delta$ we have that $t + \theta \leq a$ and then we have that the equality (12) holds for arbitrary $G, \overline{G}$ and $\sup_{t \in [a, a+\delta]} \sum_{l \in \langle r \rangle} Var_{\theta \in [-\delta, 0]} V^l_c(t,\theta) = \mathbf{V}_c < \frac{1-A^*}{2}$.
Then for every $t \in [a, a+\overline{M}]$, $\overline{M} = \min(\widetilde{M}, \delta)$, taking into account (12), we obtain the following:

$$
\begin{aligned}
&\left|\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l(t,\theta)]G(t+\theta)-\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l(t,\theta)]\overline{G}(t+\theta)\right| \\
&\leq |\sum_{l=1}^{q}\overline{A}^l(t)[G(t-\tau_l(t))-\overline{G}(t-\tau_l(t))] \\
&+\sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta)| \\
&\leq \sup_{t\in(a,a+\overline{M}]}\sum_{l\in\langle r\rangle}\left|\overline{A}^l(t)\right|\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G})+\sum_{i=1}^{r}\left|\int_{-\tau}^{-\delta}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta)\right| \\
&+\sum_{i=1}^{r}\left|\int_{-\delta}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta)\right| \\
&\leq A^*\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G})+\sum_{i=1}^{r}\left|\int_{-\delta}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(t+\theta))\right| \\
&< A^*\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G})+\frac{1-A^*}{2}\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G})=\frac{1+A^*}{2}\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G}).
\end{aligned}
\tag{16}
$$

Then for every $t \in [a, a+\overline{M}]$ as in (14) taking into account (15) and (16) we obtain the following:

$$
\begin{aligned}
|\Re G(t)-\Re\overline{G}(t)| &\leq \frac{1+A^*}{2}\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G})+C^U\Gamma^*\overline{M}^{\alpha_{\min}}\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G}) \\
&=(\frac{1+A^*}{2}+C^U\Gamma^*\overline{M}^{\alpha_{\min}})\mathrm{d}_{\overline{M}}^{\Phi}(G,\overline{G}).
\end{aligned}
\tag{17}
$$

Let $q^* \in (\frac{1+A^*}{2}, 1)$ and choose $M_0 = \left(\frac{2q^*-1-A^*}{2C^U\Gamma^*}\right)^{\frac{1}{\alpha_{\min}}}$. Then from (17) it follows that $d_{M_0}(\Re G, \Re\overline{G}) \leq q^* d_{M_0}(\overline{G}, G)$ and hence, the operator $\Re$ is contractive in $\mathbf{E}_{M_0}^{\Phi}$. □

Remark 6. *It is not difficult to see that the statement of Theorem 2 remains true if instead of condition 3, the following weakened condition $\sum_{l=1}^{q}\left|\overline{A}^l(a)\right| < 1$ holds, wherein in the left side, the matrices $\overline{A}^l$ in V_d^l take part only of these kernels V^l for which $a \in S^\Phi$ is a critical jump point concerning them.*

Remark 7. *Let $\Phi \in \mathbf{C}$. Then, since the point $a \notin S^\Phi$, it cannot be a critical jump point for the initial function $\Phi \in \mathbf{C}$. Thus, conditions 2 and 3 of Theorem 2 are unnecessary, i.e., in this case, Theorems 1 and 2 coincide.*

Let us assume that the initial function $\Phi \in \mathbf{C}_a^*$ is arbitrary and either the conditions of Theorem 1 or Theorem 2 hold. Then, according to Theorem 1 or Theorem 2, there exists a constant $M_0 \in (0, h]$ and a function $X_0(t) : J_{a+M_0} \to \mathbb{R}^n$ such that $X_0(t)$ is a local solution of IP (4) and (3). Since $X_0(t) \in C([a, a+M_0], \mathbb{R}^n)$ and $X_0(a) = \Phi(a)$, then we can define the initial function $\overline{\Phi}(t) : [a-h, a+M_0] \to \mathbb{R}^n$ for initial condition (6) as follows:

$$
\overline{\Phi}(t) = \Phi(t),\ t \in [a-h, a] \text{ and } \overline{\Phi}(t) = X^0(t),\ t \in [a, a+M_0]. \tag{18}
$$

Let us consider IP (4) and (6) for $t > a + M_0$ with the initial function $\overline{\Phi}(t)$ defined with (18). Then, for both cases considered in Theorem 1 and 2, the following statement holds.

Theorem 3. *Let the following conditions be fulfilled:*

1. *Condition 1 of Theorem 1 holds.*
2. *Either condition 2 of Theorem 1, or conditions 2 and 3 of Theorem 2 are fulfilled.*

Then there exists a constant $\widetilde{M}_0 \in (0, h]$ such that IP (4) and (6) have a unique local solution with interval of existence $J_{a+M_0+\widetilde{M}_0}$.

Proof. Let us, for definiteness, assume that the conditions of Theorem 2 hold for an initial function $\Phi \in \mathbf{C}_a^*$, and $X^0(t) \in C([a, a+M_0], \mathbb{R}^n)$ is the corresponding local solution for IP (4) and (3), with $M_0 \in (0, h]$ existing according to Theorem 2. Define the initial function $\overline{\Phi}(t) : [a-h, a+M_0] \to \mathbb{R}^n$ by (18) and consider IP (4) and (6) for $t > a + M_0$ with the initial function $\overline{\Phi}(t)$ (i.e., when (6) $\Psi(t) = X^0(t)$,
$t \in [a, a+M_0]$) and consider the IP.

Let us, for the defined $\overline{\Phi}(t)$ as above, introduce the following set:

$$E^{\overline{\Phi}} = \{\widetilde{G} : [a-h, \infty) \to \mathbb{R}^n |\ \widetilde{G}|_{[a+M_0,\infty)} \in C([a+M_0, \infty), \mathbb{R}^n) \bigcap BV([a, a+\tau], \mathbb{R}^n),$$
$$\widetilde{G}(t) = \overline{\Phi}(t),\ t \in [a-h, a+M_0]\}.$$

For $M \in \mathbb{R}_+$ define the set $E_M^{\overline{\Phi}} = \{G : [a-h, a+M_0+M] \to \mathbb{R}^n |\ G = \widetilde{G}|_{[a-h,a+M_0+M]}$, $\widetilde{G} \in E^{\overline{\Phi}}\}$ with a metric function $\mathrm{d}_M^{\overline{\Phi}}(G, \overline{G}) = \sum\limits_{k=1}^{n} \sup\limits_{t \in J_{a+M_0+M}} |g_k(t) - \overline{g}_k(t)|$, $G, \overline{G} \in E_M^{\overline{\Phi}}$ are arbitrary and then the set $E_M^{\overline{\Phi}}$ is a complete metric space concerning this metric. For each $G = (g_1, ..., g_n)^T \in E_M^{\overline{\Phi}}$ using (4), we define the point-wise operators $\Re G(t) = (\Re_1 g_1(t), ..., \Re_n g_n(t))^T$ by the following:

$$\begin{aligned} \Re G(t) = \Phi(a) - \sum_{l=1}^{r} \int_{-\tau_l}^{0} [\mathrm{d}_\theta V^l(a,\theta)]\Phi(a+\theta)) + \sum_{l=1}^{r} \int_{-\tau_l}^{0} [\mathrm{d}_\theta V^l(t,\theta)]G(t+\theta) \\ + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta)(\sum_{i=0}^{m} \int_{-\sigma_i}^{0} [\mathrm{d}_\theta U^i(\eta,\theta)]G(\eta+\theta) \\ + I_{-1}(\Gamma(\alpha)) \int_a^t I_{\alpha-1}(t-\eta)F(\eta)\mathrm{d}\eta \end{aligned} \tag{19}$$

for $t \in (a+M_0, a+M_0+M)$ with the following conditions:

$$\Re G(t) = \overline{\Phi}(t),\ t \in [a-h, a+M_0];\ \Re G(a+M^0+M) = \lim_{t \to (a+M_0+M)-0} \Re G(t). \tag{20}$$

It is clear that if $M_0 \geq \tau$ then the third addend is a continuous function for $t \in [a+M_0, a+M_0+M]$ for every $G \in E_M^{\overline{\Phi}}$. Then, $\Re G(t)$ is a continuous function in $[a+M_0, a+M_0+M]$ too, and the relations (18) hold. Thus $\Re E_M^{\Phi} \subseteq E_M^{\Phi}$ for every $M \in (0, M_0]$.

Consider the case when $M \in (0, \tau - M_0]$ and let $M \in (0, \tau - M_0]$ be arbitrary. Since the point $t = a + M_0$ is not a jump point for $\overline{\Phi}(t)$, then according to Lemma 3 point (b), there exists a constant $M_1 \in (0, \tau - M_0]$ such that $\overline{\Phi}(t - \tau_l(t))$ is continuous for $t \in [a+M_0, a+M_0+M_1]$. In addition, as was mentioned above, we can renumber all kernels so that for $l \in \langle q \rangle$ we have that $a + M_0 - \tau_l(a+M_0) \geq a$ and for $q+1 \leq l \leq r$ the inequality $a + M_0 - \tau_l(a+M_0) < a$ holds. Then, there exists a constant $\widetilde{M} \in (0, M_1]$ such that for every $t \in [a+M_0, a+M_0+\widetilde{M}]$, we have $t - \tau_l(t) \geq a$ when $l \in \langle q \rangle$, $q \in \langle r \rangle_0$ and $t - \tau_l(t) < a$ for $q+1 \leq l \leq r$.

Then for each $t \in [a + M_0, a + M_0 + \widetilde{M}]$ and every $G \in E^{\overline{\Phi}}_{\widetilde{M}}$ we have the following:

$$\begin{aligned} &\sum_{l=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l(t,\theta)]G(t+\theta) = \sum_{l=1}^{r} A^l(t)G(t-\tau(t)) + \sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l_c(t,\theta)]G(t+\theta) \\ &= \sum_{l=1}^{q}\overline{A}^l(t)X^0(t-\tau(t)) + \sum_{l=q+1}^{r}\overline{A}^l(t)\Phi(t-\tau(t)) + \sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l_c(t,\theta)]G(t+\theta). \end{aligned} \tag{21}$$

Note that the cases $q = 0$ or $q = r$ are not excluded; it depends only on the values at $t = a + M_0$ of the delays $\tau_l(t), l \in \langle r \rangle$). Thus, as shown above, we conclude that the third addend is a continuous function for $t \in [a + M_0, a + M_0 + \widetilde{M}]$ and hence, $\Re G(t)$ is a continuous function in $t \in [a + M_0, a + M_0 + \widetilde{M}]$. Hence, $\Re G(a + M_0 + \widetilde{M}) = \lim\limits_{t\to(a+M_0+\widetilde{M})-0} \Re G(t)$. Since $X^0(t) \in C([a, a + M_0], \mathbb{R}^n)$ and satisfy (4) for $t = a + M_0$, we obtain the following:

$$\begin{aligned} \lim_{t\to a+M_0+0}\Re G(t) &= \lim_{t\to a+M_0+0}[\Phi(a) - \sum_{l=1}^{r}\int_{-\tau_l}^{0}[\mathrm{d}_\theta V^l(a,\theta)\,]\Phi(a+\theta) \\ &+ \sum_{l=1}^{r}\int_{-\tau_l}^{0}[\mathrm{d}_\theta V^l(a+M_0,\theta)\,]G(a+M_0+\theta) \\ &+ I_{-1}(\Gamma(\alpha))\int_{a}^{a+M_0} I_{\alpha-1}(a+M_0-\eta)\sum_{i=0}^{m}\int_{-\sigma_i}^{0}[\mathrm{d}_\theta U^i(\eta,\theta)\,]G(\eta+\theta) \\ &+ I_{-1}(\Gamma(\alpha))\int_{a}^{a+M_0} I_{\alpha-1}(a+M_0-\eta)F(\eta)\mathrm{d}\eta] \\ &= \lim_{t\to a+M_0+0} X^0(a+M_0) = \overline{\Phi}(a+M_0) \end{aligned}$$

and therefore $\Re E^{\overline{\Phi}}_M \subseteq E^{\overline{\Phi}}_M$ for each $M \in (0, \widetilde{M}]$.

Let $G, \overline{G} \in E^{\overline{\Phi}}_{\widetilde{M}}$ be arbitrary. Taking into account (21), we have the following:

$$\begin{aligned} \left|\Re G(t) - \Re\overline{G}(t)\right| &\le \sum_{l=1}^{r}\left|\int_{-\tau_l}^{0}[\mathrm{d}_\theta V^l(t,\theta)\,](G(t+\theta)) - \overline{G}(t+\theta)))\right| \\ &+ \left|I_{-1}(\Gamma(\alpha))\int_{a}^{t} I_{\alpha-1}(t-\eta)(\sum_{i=0}^{m}\int_{-\sigma_i}^{0}[\mathrm{d}_\theta U^i(\eta,\theta)\,](G(\eta+\theta) - \overline{G}(\eta+\theta))\right| \\ &\le \left|\sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V^l_c(t,\theta)](G(t+\theta) - \overline{G}(\eta+\theta))\right| \\ &+ (C^U\int_{a}^{t}|I_{-1}(\Gamma(\alpha))I_{\alpha-1}(t-\eta)|\mathrm{d}\eta)\mathrm{d}^{\overline{\Phi}}_{M}(G,\overline{G}). \end{aligned} \tag{22}$$

Let $\varepsilon \in (0, \frac{1}{2n})$ be arbitrary. From condition (S3) it follows that there exists $\delta(\varepsilon) \in (0, 1)$ such that $Var_{[-\delta,0]}V^l(t, \cdot) < \varepsilon$ for every $l \in \langle r \rangle$ and $t \in J_a$. For every $t \in [a + M_0, a + M_0 + \delta]$ and $\theta \le -\delta$ we have that $t + \theta \le a$. Then for arbitrary $G, \overline{G} \in E^{\overline{\Phi}}_{\widetilde{M}}$ and $t \in [a + M_0, a + M_0 + \delta]$ we have that the following equality holds:

$$G(t+\theta) = \overline{G}(t+\theta) = \overline{\Phi}(t+\theta) \tag{23}$$

Let $\overline{M} = \min(\widetilde{M}, \delta)$. Then, we obtain that $\sup\limits_{t\in[a+M_0,a+M_0+\overline{M}]} \sum\limits_{l\in\langle r\rangle} Var_{\theta\in[-\delta,0]} V_c^l(t,\theta)$ $= \mathbf{V}_c < \frac{1}{2}$ for $t \in [a+M_0, a+M_0+\overline{M}]$.

For every $t \in [a+M_0, a+M_0+\overline{M}]$, from (23) it follows that

$$
\begin{aligned}
&\left|\sum_{i=1}^{r}\int_{-h}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(\eta+\theta))\right| \\
&\le \left|\sum_{i=1}^{r}\int_{-\tau}^{-\delta}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(\eta+\theta))\right| \\
&+ \left|\sum_{i=1}^{r}\int_{-\delta}^{0}[\mathrm{d}_\theta V_c^l(t,\theta)](G(t+\theta)-\overline{G}(\eta+\theta))\right| \\
&\le \left|\sum_{i=1}^{r}\int_{-\tau}^{-\delta}[\mathrm{d}_\theta V_c^l(t,\theta)](\overline{\Phi}(t+\theta)-\overline{\Phi}(\eta+\theta))\right| \\
&+ \frac{1}{2}\mathrm{d}^{\overline{\Phi}}_{\overline{M}}(G,\overline{G}) = \frac{1}{2}\mathrm{d}^{\overline{\Phi}}_{\overline{M}}(G,\overline{G}).
\end{aligned}
\tag{24}
$$

Thus from (15), (22) and (24) for $t \in [a+M_0, a+M_0+\overline{M}]$ we obtain the following:

$$
\left|\Re G(t) - \Re\overline{G}(t)\right| \le (\frac{1}{2} + C^U\Gamma^*\overline{M}^{\alpha_{min}})\mathrm{d}^{\overline{\Phi}}_{\overline{M}}(G,\overline{G}). \tag{25}
$$

Let, as shown above, $q^* \in (\frac{1}{2}, 1)$ and choose $\overline{M}_0 = \min(\left(\frac{2q^*-1}{2C^U\Gamma^*}\right)^{\frac{1}{\alpha_{\min}}}, \overline{M})$. Then from (25), it follows that $d_{M_0}(\Re G, \Re\overline{G}) \le q^* d_{M_0}(\overline{G}, G)$ and hence, the operator $\Re$ is contractive in $\mathbf{E}^{\overline{\Phi}}_{M_0}$. □

Remark 8. *We emphasize that the case when the right endpoint of the initial interval for some IP is a critical jump point concerning some kernel is possible only when the right endpoint coincides with the lower terminal of the fractional derivative. Thus, we can conclude that condition 3 of Theorem 2 is needed only for the existence of the local solution of the IP when the right endpoint is a jump point for the initial function and coincides with the lower terminal of the fractional derivative.*

Let us assume that there exist two solutions $X_1(t)$ and $X_2(t)$ of the IP (4), (3), with intervals of existence J_{a+M_1} and J_{a+M_2} where $M_1 < M_2$. Then obviously we have that $X_1(t) = X_2(t)$ for $t \in J_{a+M_1}$ and hence, $X_2(t)$ is a continuation of $X_1(t)$. A solution $X^*(t)$ with intervals of existence J_{a+M^*} is a maximal solution of the IP (4) and (3) if $X^*(t)$ is a continuation of each solution of IP (4) and (3).

Theorem 4. *Let the conditions of Theorem 3 hold.*

Then IP (4) and (3) have a unique global solution with the interval of existence J_a.

Proof. Let IP (4) and (3) have a maximal solution $X^*(t)$ and assume that the interval of existence J_{a+M^*} is closed from right. Then, using $X^*(t)$ instead $X^0(t)$ in (18) and applying Theorem 3, we obtain a continuation of $X^*(t)$ which contradicts our assumption. Thus the interval of existence of the maximal solution $X^*(t)$ has the form $[a, a+M^*)$. Let us assume that $M^* < \infty$. Then since the right side of Equation (4) is a continuous function in $(a, a+M^*)$ and the limit when $t \to a+M^*-0$ exists, then we obtain that $X^*(t)$ satisfies Equation (4) at $t = a+M^*$ too, i.e., we obtain a continuation of $X^*(t)$ which contradicts our assumption that $X^*(t)$ is a maximal solution. Thus $M^* = \infty$. □

Remark 9. *As shown above, we can establish that if $\Phi \in \mathbf{C}$, then condition 2 of Theorem 3 is unnecessary, and the statements of Theorems 3 and 4 are still true.*

4. Applications

Introduce the matrix functions $\Phi^*(t,s) = \{\varphi^*_{i,j}(t,s)\}_{i,j\in\langle n\rangle}, \overline{\Phi}^*(t,s) = \{\overline{\varphi}^*_{i,j}(t,s)\}_{i,j\in\langle n\rangle} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ as follows: for each $s \in J_a$, $\Phi^*(t,s) = \Theta,\ t < s; \Phi^*(t,s) = I,\ t = s;\ \Phi^*(t,s) = \Theta,\ s < a$. For every $s \in [a-h,a]$ we define the following: $\overline{\Phi}^*(t,s) = \Theta,\ t < s$; $\overline{\Phi}^*(t,s) = I,\ a-h \le s \le t \le a$; $\overline{\Phi}^*(t,s) = \Theta,\ s < a-h$.

Let $s \in J_*$ be an arbitrary fixed number, and consider the following matrix system:

$$D^\alpha_{a+}\Big(Y(t,s) - \int_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)]Y(t+\theta,s)\Big) = \int_{-h}^{0} [\mathrm{d}_\theta U(t,\theta)]Y(t+\theta,s),\ t \in (\max(a,s),\infty) \tag{26}$$

and the following two initial conditions

$$Y(t,s) = \Phi^*(t,s) \text{for } t \in [s-h,s],\ \text{when } s \ge a \tag{27}$$

and

$$Y(t,s) = \overline{\Phi}^*(t,s) \text{for } t \in [s,a], \text{when } s \in [a-h,a]. \tag{28}$$

For each fixed $s \ge a$, the matrix valued function $t \to C(t,s) = \{\gamma_{kj}(t,s)\}^n_{k,j=1}$, $C(\cdot,s) : J_s \to \mathbb{R}^{n\times n}$ is called a solution of matrix IP (26) and (27) if $C(\cdot,s) : J_s \to \mathbb{R}^{n\times n}$ is continuous in t on J_s and satisfies the matrix Equation (26) for $t \in (s,\infty)$, as well as the initial condition (27) too. As in the integer case, we call the matrix $C(t,s)$ a fundamental matrix of the system (2).

Analogically for each fixed $s \in [a-h,a]$, the matrix valued function $t \to Q(t,s) = \{q_{kj}(t,s)\}^n_{k,j=1}$, $Q(\cdot,s) : J_a \to \mathbb{R}^{n\times n}$ is called a solution of the IP (26), (28), if $Q(\cdot,s) : J_a \to \mathbb{R}^{n\times n}$ is continuous in t on J_a and satisfies the matrix Equation (26) for $t \in (a,\infty)$ as well as the initial condition (28).

Lemma 5. *Let the conditions of Theorem 3 hold.*

Then the following statements hold:

1. *For every fixed $s \in J_a$ the IP (26), (27) has a unique solution $C(\cdot,s) : J_s \to \mathbb{R}^{n\times n}$ in J_s.*
2. *For every fixed $s \in [a-h,a]$ the IP (26), (28) has a unique solution $Q(\cdot,s) : J_a \to \mathbb{R}^{n\times n}$ in J_a.*

Proof. The proof is evident but for completeness, we will sketch them. It is simple to see that for each $j \in \langle n\rangle$ the column vector function $\overline{\Phi}^*_j(t,s) = (\overline{\varphi}^*_{i,j}(t,s),\ldots,\overline{\varphi}^*_{n,j}(t,s))^T \in \mathbf{C}_a{}^*$for every $s \in [a-h,a]$ and $\Phi^*_j(t,s) = (\varphi^*_{i,j}(t,s),\ldots,\varphi^*_{n,j}(t,s))^T \in \mathbf{C}_s{}^*$ for each fixed $s \ge a$.

Let $s \in J_a$ be an arbitrary fixed number and for $j \in \langle n\rangle$ consider the IP (2), (3) in J_s with initial point $s \in J_a$ and initial function $\Phi^*_j(t,s)$ and hence, all conditions of Theorem 3 are fulfilled. Then it follows that for each $j \in \langle n\rangle$, IP (2) and (3) possess a unique solution $X^j(t,s) = (\gamma_{1j}(t,s),\ldots,\gamma_{nj}(t,s))^T$ in J_s and thus, the matrix $C(t,s) = (X^1(t,s),\ldots,X^n(t,s))$, $C(\cdot,s) : [s,\infty) \to \mathbb{R}^{n\times m}$ is a unique solution of IP (26) and (7) in J_s. The proof of point 2 is fully analogical. □

Remark 10. *Note that for every $t \in J_a$, the functions $C(t,\cdot) : J_a \to \mathbb{R}^{n\times n}$ and $Q(t,\cdot) : [a-h,a] \to \mathbb{R}^{n\times n}$ are locally bounded and locally Lebesgue integrable in s (it follows from Theorem 3 in [23]). Furthermore, the relation $C(t,s) = Q(t,s)$ for all $t \in J_a$ holds, only when $s = a$, i.e., $G(t,a) = Q(t,a)$.*

For arbitrary initial function $\Phi \in \mathbf{C}^*_a$ with $\Phi \in BV([a-h,a],\mathbb{R}^n)$, we introduce the following vector function:

$$X_\Phi(t) = \int_{a-h}^{a} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s),\ t \in J_a, \tag{29}$$

where $\widetilde{\Phi}(t) = \Phi(t)$ *for* $t \in (a-h,a]$, $\widetilde{\Phi}(a-h) = \mathbf{0}$ *and since* $Q(t,\cdot) : [a-h,a] \to \mathbb{R}^{n\times n}$ *are locally bounded and locally Lebesgue integrable in s. Then, the Lebesgue–Stieltjes integral in* (29) *exists. Furthermore, virtue of from Lemma 1 in* [31], *we can conclude that the vector function* X_Φ *defined by equality* (29) *and the integral* $\int\limits_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)]Q(t+\theta,s)$ *are continuous functions for* $t \in J_a$.

Theorem 5. *Let the conditions of Theorem 3 hold.*

Then the vector function $X_\Phi(t)$ *defined with* (29) *is a unique solution of IP* (2) *and* (3) *for arbitrary* $\Phi \in \mathbf{C}_a^*$ *with* $\Phi \in BV([a-h,a],\mathbb{R}^n)$.

Proof. It is not difficult to check that for $t \in [a-h,a]$,

$$\begin{aligned} X_\Phi(t) &= \int\limits_{a-h}^{a} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s) = \int\limits_{a-h}^{t} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s) + \int\limits_{t}^{a} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s) \\ &= \int\limits_{a-h}^{t} I\mathrm{d}_s\widetilde{\Phi}(s) = \widetilde{\Phi}(t) - \widetilde{\Phi}(a-h) = \Phi(t), \end{aligned}$$

i.e., $X_\Phi(t)$ satisfies the initial condition (3).

Substituting in the first addend in the left side of (2) the vector function $X_\Phi(t)$ we obtain the following:

$$ {}_C D_{a+}^{\alpha} X_\Phi(t) = {}_C D_{a+}^{\alpha} \int\limits_{a-h}^{a} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s) = {}_{RL} D_{a+}^{\alpha} \int\limits_{a-h}^{a} [Q(t,s) - Q(a,s)]\mathrm{d}_s\widetilde{\Phi}(s). \tag{30}$$

Let $t \in J_a$ be an arbitrary fixed number, $\Pi^* = [-h,0] \times [a-h,a]$ and μ_θ and μ_s be the Lebesgue-Stiltjes measures constructed via $V(t,\theta)$and $\widetilde{\Phi}(s)$. Then for the product measure $\mu_\theta \times \mu_s$ we have that $\mu_\theta \times \mu_s(\Pi^*) = \mu_\theta([-h,0])\mu_s([a-h,a])$ and since $Q(t+\theta,s)$ is locally bounded and locally Lebesgue integrable in t and s; hence $\iint\limits_{\Pi^*} |Q(t+\theta,s)|\mu_\theta \times \mu_s < \infty$ for every $t \in J_a$. Then substituting $X_\Phi(t)$ in the second addend of the left side of (2) and applying the Fubini theorem (see Proposition 5.4 in [32]) we have the following:

$$\begin{aligned} \int\limits_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] X_\Phi(t+\theta) &= \int\limits_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] \left(\int\limits_{a-h}^{a} Q(t+\theta,s)\mathrm{d}_s\widetilde{\Phi}(s) \right) \\ &= \int\limits_{a-h}^{a} [\mathrm{d}_s\widetilde{\Phi}(s)] \left(\int\limits_{-h}^{0} [\mathrm{d}_\theta V(t,\theta)] Q(t+\theta,s) \right). \end{aligned} \tag{31}$$

Analogically for arbitrary fixed $t \in J_a$, $\widetilde{\Pi} = [a,t] \times [a-h,a]$ for the product measure $\mu_l \times \mu_s$, we have that $\mu_l \times \mu_s(\widetilde{\Pi}) = \mu_l([a,t])\mu_s([a-h,a])$, where μ_l is the Lebesgue measure and since the expression $\left(\int\limits_{-h}^{0} [\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s) - \int\limits_{-h}^{0} [\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s) \right)$, $\eta \in [a,t]$ is locally bounded and locally Lebesgue integrable in η and s, therefore $\iint\limits_{\widetilde{\Pi}} \left| \int\limits_{-h}^{0} [\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s) - \int\limits_{-h}^{0} [\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s) \right| \mu_\theta \times \mu_s < \infty$. Then using (31) and applying again the Fubini theorem for the second addend in the left side of (2), we have the following:

$$\begin{aligned}
&{}_CD^{\alpha}_{a+}\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X_\Phi(t+\theta) = {}_CD^{\alpha}_{a+}\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)]\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]Q(t+\theta,s)\right)\\
&= {}_{RL}D^{\alpha}_{a+}\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)]\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]Q(t+\theta,s)-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s)\right)\\
&= (\Gamma^{-1}(1-\alpha)\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t} t-\eta)^{-\alpha}(\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)](\int_{-h}^{0}[\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s)\\
&-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s)))\mathrm{d}\eta\\
&= (\Gamma^{-1}(1-\alpha)\frac{\mathrm{d}}{\mathrm{d}t}(\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)](\int_{a}^{t}(t-\eta)^{-\alpha}(\int_{-h}^{0}[\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s)\\
&-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s))\mathrm{d}\eta)).
\end{aligned} \tag{32}$$

Note that we can differentiate under the integral sign in the right side of (32) because the expression $\left(\int_{a}^{t}(t-\eta)^{-\alpha}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s)-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s)\right)\mathrm{d}\eta\right)$ is an absolutely continuous function in $t\in J_a$. Then, from (32) we have the following:

$$\begin{aligned}
&{}_CD^{\alpha}_{a+}\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X_\Phi(t+\theta)\\
&=\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)](\Gamma^{-1}(1-\alpha)\frac{\mathrm{d}}{\mathrm{d}t}(\int_{a}^{t}(t-\eta)^{-\alpha}(\int_{-h}^{0}[\mathrm{d}_\theta V(\eta,\theta)]Q(\eta+\theta,s)\\
&-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s))\mathrm{d}\eta))\\
&=\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)]_{RL}D^{\alpha}_{a+}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]Q(t+\theta,s)-\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]Q(a+\theta,s)\right)\\
&=\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)]_CD^{\alpha}_{a+}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]Q(t+\theta,s)\right).
\end{aligned} \tag{33}$$

Let $t\in J_a$ be an arbitrary fixed number and substitute $X_\Phi(t)$ in the right side of (2). Then applying the Fubini theorem using the same reasons such as those for (31), we obtain the following:

$$\begin{aligned}
&\sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]X(t+\theta)=\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X_\Phi(t+\theta)\\
&=\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]\left(\int_{a-h}^{a}Q(t+\theta,s)\mathrm{d}_s\widetilde{\Phi}(s)\right)\\
&=\int_{a-h}^{a}[\mathrm{d}_s\widetilde{\Phi}(s)]\left(\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]Q(t+\theta,s)\right).
\end{aligned}\tag{34}$$

Thus from (31), (33) and (34) it follows that $X_\Phi(t)$ satisfies (2) for $t\in(a,\infty)$. □

For arbitrary locally bounded function $F\in L_1^{loc}(J_a,\mathbb{R}^n)$, since $C(t,\cdot):J_s\to\mathbb{R}^{n\times n}, s\geq a$ are locally bounded and locally Lebesgue integrable in s, and hence we can introduce the following vector function:

$$X^F(t)=\int_a^t C(t,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s.\tag{35}$$

Theorem 6. *Let the conditions of Theorem 3 hold.*

Then, the vector function $X^F(t)$ defined by equality 35 is a unique solution of IP (1) and (3) with initial function $\Phi(t)=\mathbf{0}$ and interval of existence J_a.

Proof. It is not difficult to conclude that the vector function $X^F(t)$ defined by equality (35) is a continuous function for $t\in J_a$. Furthermore, for $t\in J_a$, the following equalities hold:

$$\begin{aligned}
&{}_CD_{a+}^{\alpha}X^F(t)={}_{RL}D_{a+}^{\alpha}[X^F(t)-X^F(a)](t)={}_{RL}D_{a+}^{\alpha}X^F(t);\\
&{}_CD_{a+}^{\alpha}\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X^F(t+\theta)={}_{RL}D_{a+}^{\alpha}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X^F(t+\theta)\right)\\
&-{}_{RL}D_{a+}^{\alpha}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(a,\theta)]X^F(a+\theta)\right)={}_{RL}D_{a+}^{\alpha}\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X^F(t+\theta).
\end{aligned}\tag{36}$$

Then denoting $K(t,s)=C(t,s)T(s)$, where $T(s)={}_{RL}D^{1-\alpha}F(s)$ from the first equality in (36) it follows that

$$\begin{aligned}
&{}_CD_{a+}^{\alpha}X^F(t)={}_{RL}D_{a+}^{\alpha}X^F(t)={}_{RL}D_{a+}^{\alpha}[\int_a^t K(t,s)\mathrm{d}s](t)\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(t-\eta)^{-\alpha}(\int_a^\eta K(\eta,s))\mathrm{d}s)\mathrm{d}\eta\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t(\int_s^t(t-\eta)^{-\alpha}K(\eta,s)\mathrm{d}\eta)\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t\widetilde{K}(t,s)\mathrm{d}s,
\end{aligned}\tag{37}$$

where $\widetilde{K}(t,s)=\frac{1}{\Gamma(1-\alpha)}\int_s^t(t-\eta)^{-\alpha}K(\eta,s)\mathrm{d}\eta$. Since $C(a,s)=\Theta$ for $t<s$ we obtain the following:

$$\widetilde{K}(t,s)=\frac{1}{\Gamma(1-\alpha)}\int_s^t(t-\eta)^{-\alpha}K(\eta,s)\mathrm{d}\eta=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-\eta)^{-\alpha}K(\eta,s)\mathrm{d}\eta$$

and hence from (37) it follows that

$$
\begin{aligned}
{}_C D^{\alpha}_{a+} X^F(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \widetilde{K}(t,s)\mathrm{d}s = \int_a^t \frac{\partial}{\partial t}\widetilde{K}(t,s)\mathrm{d}s + \lim_{s\to t-0}\widetilde{K}(t,s) \\
&= \int_a^t {}_{RL}D^{\alpha}_{a+}K(t,s)\mathrm{d}s + \lim_{s\to t-0} D^{\alpha-1}_{a+}K(t,s).
\end{aligned} \tag{38}
$$

For the first addend in the right side of (38) since $C(a,s) = \Theta$, when $s > a$, we have the following:

$$
\begin{aligned}
&\int_a^t {}_{RL}D^{\alpha}_{a+}K(t,s)\mathrm{d}s = \int_a^t {}_C D^{\alpha}_{a+}K(t,s)\mathrm{d}s + \int_a^t {}_{RL}D^{\alpha}_{a+}K(a,s)\mathrm{d}s \\
&= \int_a^t {}_C D^{\alpha}_{a+}K(t,s)\mathrm{d}s + \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\eta)^{-\alpha}\Big(\int_a^{\eta} C(a,s)\mathrm{d}\eta\Big)\mathrm{d}s = \int_a^t {}_C D^{\alpha}_{a+}K(t,s)\mathrm{d}s
\end{aligned}
$$

and for the second addend of (38) in virtue of formula (2.1.40) and Lemma 3.2 in [1], we have the following:

$$
\begin{aligned}
\lim_{s\to t-0} D^{\alpha-1}_{a+}K(t,s) &= \frac{1}{\Gamma(1-\alpha)}\int_{a+}^t (t-\eta)^{-\alpha}\lim_{s\to\eta-0}K(\eta,s)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-\alpha)}\int_{a+}^t (t-\eta)^{-\alpha}C(\eta,\eta)T(\eta)\mathrm{d}\eta \\
&= D^{\alpha-1}T(t) = D^{\alpha-1}_{a+}\,{}_{RL}D^{1-\alpha}_{a+}F(t) \\
&= F(t) - \frac{(t-a)^{-\alpha}{}_{RL}D^{-\alpha}_{a+}F(a+)}{\Gamma(\alpha)} \\
&= F(t) - \frac{(t-a)^{-\alpha}}{\Gamma(\alpha)}\Gamma(\alpha)\lim_{t\to a+}(t-a)^{\alpha}F(t) = F(t)
\end{aligned}
$$

and hence we obtain the following:

$$
{}_C D^{\alpha}_{a+}X^F(t) = \int_{a+}^t {}_C D^{\alpha}_{a+}K(t,s)\mathrm{d}s + F(t) = \int_{a+}^t D^{1-\alpha}F(s)\,{}_C D^{\alpha}_{a+}C(t,s)\mathrm{d}s + F(t). \tag{39}
$$

Let for shortness denote $\widetilde{K}(t,s) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-\eta)^{-\alpha}\overline{K}(\eta,s)\mathrm{d}\eta\right)$, where $\overline{K}(t,s) = D^{1-\alpha}_{a+}F(s)\int_{-h}^{0}[\mathrm{d}_{\theta}V(t,\theta)]\,C(t+\theta,s)$. Taking into account that for $\theta \in [-h,0)$ and $s \in [t+\theta,t]$ we have that $C(t+\theta,s) = 0$. Then from the second relation in (36) we obtain the following:

$$\begin{aligned}
&\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]\left(\int_{a}^{t+\theta}C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\right)\\
&=\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]\left(\int_{a}^{t}C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&-\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]\left(\int_{t+\theta}^{t}C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&=\int_{a}^{t}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]C(t+\theta,s)\right)D_{a+}^{1-\alpha}F(s)\mathrm{d}s=\int_{a}^{t}\overline{K}(t,s)\mathrm{d}s.
\end{aligned} \tag{40}$$

Furthermore for each fixed $t \in J_a$ we have the following:

$$\lim_{s\to t-0}\widetilde{K}(t,s)=\frac{1}{\Gamma(1-\alpha)}\lim_{s\to t-0}\left(\int_{a}^{t}(t-\eta)^{-\alpha}\overline{K}(\eta,s)\mathrm{d}\eta\right) \quad . \tag{41}$$

Indeed for arbitrary $\eta < t+\theta$ and $s \in [\eta, t)$ we have the following: $\overline{K}(\eta,s) = D_{a+}^{1-\alpha}F(s)\int_{-h}^{0}[\mathrm{d}_\theta V(\eta,\theta)]\,C(\eta+\theta,s) = 0$ and for $\eta = t$ (thus $\theta = 0$) we have that $\overline{K}(t,t) = D_{a+}^{1-\alpha}F(t)\int_{-h}^{0}[\mathrm{d}_\theta V(t,0)]\,C(t,t) = 0$ too. Taking into account (41) we obtain the following:

$$\begin{aligned}
&{}_{C}D_{a+}^{\alpha}\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)]X^{F}(t+\theta)={}_{C}\,D_{a+}^{\alpha}\int_{a}^{t}\overline{K}(t,s)\mathrm{d}s\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a}^{t}(t-\eta)^{-\alpha}\left(\int_{a}^{\eta}\overline{K}(\eta,s)\mathrm{d}s\right)\mathrm{d}\eta\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a}^{t}\frac{1}{\Gamma(1-\alpha)}\left(\int_{s}^{t}(t-\eta)^{-\alpha}\overline{K}(\eta,s)\mathrm{d}\eta\right)\mathrm{d}s\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a}^{t}\frac{1}{\Gamma(1-\alpha)}\left(\int_{a}^{t}(t-\eta)^{-\alpha}\overline{K}(\eta,s)\mathrm{d}\eta\right)\mathrm{d}s\right)\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a}^{t}\widetilde{K}(t,s)\mathrm{d}s\right)=\int_{a}^{t}\frac{\partial}{\partial t}\widetilde{K}(t,s)\mathrm{d}s+\lim_{s\to t-0}\widetilde{K}(t,s)\\
&=\int_{a}^{t}{}_{RL}D_{a+}^{\alpha}\overline{K}(t,s)\mathrm{d}s+\lim_{s\to t-0}{}_{C}\frac{1}{\Gamma(1-\alpha)}\left(\int_{a}^{t}(t-\eta)^{-\alpha}\overline{K}(\eta,s)\mathrm{d}\eta\right)\\
&=\int_{a}^{t}D_{a+}^{1-\alpha}F(s){}_{C}D_{a+}^{\alpha}\left(\int_{-h}^{0}[\mathrm{d}_\theta V(t,\theta)\,]C(t+\theta,s)\right)\mathrm{d}s.
\end{aligned} \tag{42}$$

On the other hand, for the first addend in the right side (1) after applying the Fubini theorem (under the same reasons as above), we have the following:

$$
\begin{aligned}
\sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]X^F(t+\theta) &= \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]\left(\int_{a}^{t+\theta} C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&= \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]\left(\int_{a}^{t} C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&- \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]\left(\int_{t+\theta}^{t} C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&= \sum_{i=0}^{m}\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]\left(\int_{a}^{t} C(t+\theta,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s\right)\\
&= \sum_{i=0}^{m}\int_{a}^{t} D_{a+}^{1-\alpha}F(s)\left(\int_{-h}^{0}[\mathrm{d}_\theta U^i(t,\theta)]C(t+\theta,s)\right)\mathrm{d}s.
\end{aligned}
\tag{43}
$$

Then taking into account that $C(t,s)$ is a fundamental matrix, from (1), (39), (42) and (43) it follows that the vector function defined by equality (35) is a solution of the system (1). □

Corollary 1. *Let the conditions of Theorem 3 hold.*

Then for arbitrary initial function $\Phi \in \mathbf{C}_a^*$ with $\Phi \in BV([a-h,a],\mathbb{R}^n)$, the unique solution of the IP (1) and (3) for $t \in J_a$ has the following representation:

$$
X_\Phi^F(t) = \int_{a}^{t} C(t,s)D_{a+}^{1-\alpha}F(s)\mathrm{d}s + \int_{a-h}^{a} Q(t,s)\mathrm{d}_s\widetilde{\Phi}(s). \tag{44}
$$

Proof. Using the superposition principle, we define $X_\Phi^F(t) = X^F(t) + X_\Phi(t)$, where $X^F(t)$ and $X_\Phi(t)$ are defined by (35) and (29), respectively. Then the statement of Corollary 1 follows from Theorems 5 and 6. □

5. Comments and Conclusions

As the first result in the present work, it is proved the existence and uniqueness of the solutions of an initial problem (IP) for the most general case of neutral linear differential systems with Caputo fractional derivatives of incommensurate order, for each piecewise continuous (PC) initial function with a bounded variation on the interval $[a-\tau,a]$. Then, as a consequence, it is proved the existence and uniqueness of a fundamental matrix $C(t,s)$, which is continuous in t, $t \in [a,\infty)$ and locally Lebesgue integrable concerning s, $s \in [a-h,\infty)$. Using the existence of the fundamental matrix, integral representations for a particular solution of the inhomogeneous system with zero initial conditions as well as for the general solution of the homogeneous system are obtained. As a corollary, from these representations it is obtained an integral representation for the general solution of the inhomogeneous system with nonzero initial conditions too.

Our results improve the corresponding results given in [16,20,21,26]. Note that our results are proved under weaker conditions, even in the case of one constant delay as considered in Theorem 5.3 in [16]. First, for their result, the authors used the definition of the Caputo derivative that is applicable only for absolute continuous functions, an assumption which is essentially used in their proofs. Second, we point out that in [16], as the space of initial functions, the space $C^1([-\tau,0])$ is considered for which obviously the inclusion $C^1([-\tau,0],\mathbb{R}^n) \subset C_0^* = PC([-\tau,0],\mathbb{R}^n)\bigcap BV([-\tau,0],\mathbb{R}^n)$ holds.

We think that as in the case of delayed fractional systems, our obtained integral representation of the solutions for neutral linear fractional systems will play a central role in the qualitative analysis of such systems with nonlinear perturbation, especially in the case when the perturbation is unbounded. The same representation also can be an useful tool for establishing finite time stability results for neutral linear fractional systems.

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