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New Results on Robust Synchronization for Memristive Neural Networks with Fractional Derivatives via Linear Matrix Inequality

Chao Song [1], Jinde Cao [2,3,*] and Mahmoud Abdel-Aty [4]

[1] School of Mathematics and Physics, Nanjing Institute of Technology, Nanjing 211167, China
[2] School of Mathematics, Southeast University, Nanjing 210096, China
[3] Yonsei Frontier Lab, Yonsei University, Seoul 03722, Korea
[4] Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt
* Correspondence: jdcao@seu.edu.cn

Abstract: This article mainly concentrates on the synchronization problem for a more general kind of the master–slave memristor-based neural networks with fractional derivative. By applying a continuous-frequency-distributed equivalent model tool, some new outcomes and sufficient conditions on the robust synchronization of the master–slave neural networks with uncertainty are proposed via linear matrix inequality (LMI). Finally, two memristive neural networks model with fractional derivatives are presented to validate the efficiency of the theoretical results.

Keywords: fractional derivative; memristive neural networks; uncertainty; synchronization

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1. Introduction

The mathematics theory and real application of fractional calculus have been greatly developed in the past two decades, which have renewed its vitality and re-attracted tremendous interest of experts and scholars from different fields, such as mathematics, physics, engineering, and materials science [1–12]. Compared with the mathematical model expressed by classical calculus, a fractional-order system (FOS) has great advantages when the system has memory and hereditary properties. For example, the theory and simulation results show that fractional-order recurrent neural networks are more beneficial in system estimation by Lundstrom et al. in [13].

Lyapunov stability theory is a powerful tool with which to research the qualitative theory of dynamical systems. Ten years ago, Trigeassou et al. in [14] proposed an indirect Lyapunov method to investigate the stability issue for linear and nonlinear fractional-order systems by utilizing a continuous-frequency-distributed equivalent model method (CFDEMM). Subsequently, the CFDEMM was used as a vital tool to analyze the stability of nonlinear FOSs in [15–21]. Boroujeni and Momeni considered a kind of nonlinear, Lipschitz, continuous FOS with bounded perturbations. The non-fragile fractional-order state observer (FOSO) was designed to achieve stability based on the CFDEMM in [15]. In [16,17], two kinds of nonlinear FOS and fractional-order complex networks were considered. By using of the technique of non-fragile observer, new sufficient conditions to realize robust asymptotic stability were established. A special class of 3-dimensional chaotic FOSs with parameter uncertainty and external disturbance was investigated in [18] where the adaptive sliding mode control (SMC) scheme was designed to realize the global robust stability via using the CFDEMM. A new incommensurate nonlinear FOS of lithium-ion batteries model was examined in [19]. A Luenberger-type observer was presented to evaluate lithium-ion battery state by means of the CFDEMM and the Lyapunov method. In [20], they studied the tracking control topic of positive switched FOS. A new sufficient condition to guarantee the exponential stability was derived in terms of LMI by designing an ideal observer and a CFDEMM. In [21], Tan et al. discussed the robust control of uncertain FOS with input saturation and measurement quantization. Aiding the Luenberger-type nonlinear

FOSO and the CFDEMM, new sufficient criteria were derived to get the robust stability for closed-loop systems.

In recently years, the dynamical behavior of fractional-order neural networks (FONN) was widely studied in [22–30], especially fractional-order memristive neural networks (FOMNNs) [31–41]. Chen et al. investigated Mittag–Leffler synchronization of a FOMNN by using an M-matrix method and set-valued theory in [31]. FOMNN with time-delay was discussed in [32]; some new results on the asymptotic stability and synchronization were achieved based on a comparison theorem of delayed FOSs. FOMNNs with uncertainty were addressed in [33]; new conditions on the synchronization were obtained by employing linear delay feedback control and the adaptive control schemes. In [34], novel criteria to identify the synchronization of FOMNNs with parameter uncertainty and time delay were proposed by considering different feedback control strategies and a comparison theorem. State estimation of delayed FOMNNs with uncertainty were studied in [35–37], and some new conclusions were obtained when the activation functions satisfy the different continuous and bound conditions. In [38], a new fractional-order memristive Wilson neuron model with the fractal-fractional derivative was intensively developed. The rich dynamic behaviors with different fractional-orders were proposed, such as complete synchronization, lag synchronization, phase synchronization, and sine-like synchronization, when the neurons are locally and diffusively coupled in a ring topology. In [39–41], finite time stability, finite time synchronization, and finite time Mittag–Leffler synchronization were extensively investigated. Some new sufficient criteria in the frame of linear matrix inequalities to assure the finite-time stability of FOMNNs were presented subject to the quantisation phenomenon, actuator failures, and time-varying delay by sampled-data control in [39]. Some results on asymptotic stability and finite-time stability were constructed by using the theory of fractional calculus for a general class of delayed inertial FOMNN [40]. In [41], finite-time Mittag–Leffler synchronization of complex-valued FOMNNs with time delay was achieved. Simultaneously, the upper bound of settling time was estimated by utilizing fractional differential inequality.

Taking into account the above factors, this paper discusses robust synchronization of FOMNNs by means of LMI and a CFDEMM. The major contributions are summarized as follows:

(1) In the literature, there are usually two kinds of uncertainties. One is norm-bounded uncertainty, and the other is bounded real uncertainty; see [42]. Differently from reference [26,27,36,37,39], a FOMNN with norm-bounded uncertainty is studied for the first time in this article.

(2) A novel uncertain memristive neural network model with fractional derivative is constructed. In addition, the new model is more practical and has potential application value in Engineering.

(3) Sufficient conditions to assure the robust synchronization of FOMNN are achieved by utilizing a continuous-frequency-distributed equivalent model method and an indirect Lyapunov method. The main results are expressed by LMI which can be easily realized by the Matlab toolbox.

(4) The theoretical contributions are confirmed and validated by two examples.

The article is made of five sections. Some necessary theoretical results of fractional calculus and differential inclusions are presented in Section 2. In addition, the problem description is also contained in this part. In Section 3, detailed results on robust synchronization for FOMNN are processed. In Section 4, 2-dimensional and 3-dimensional memristive neural network models with fractional derivatives are given to illustrate the validity of the proposed results. In the last section, main conclusions and possible interesting future work are appended.

2. Preliminaries and Problem Formulation

Some basic theoretical results of fractional calculus and differential inclusions are collected in this section which are useful in the following sections.

Definition 1 ([43])**.** *For a given function $g(t)$, the Riemann integral with order $\alpha > 0$ is defined by*

$$ {}_aI_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} g(u) du, $$

where $\Gamma(x) = \int_0^\infty e^{-\tau}\tau^{x-1}d\tau$ is the common gamma function.

Remark 1. *According to the definition of convolution operator, one has ${}_aI_t^\alpha g(t) = k(t) * g(t)$, where $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$.*

Definition 2 ([43])**.** *For a given function $g(t)$, the Riemann–Liouville derivative with order $\alpha > 0$ is defined by*

$$ {}_a^{RL}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-u)^{n-\alpha-1} g(u) du, \quad n-1 < \alpha < n, n \in \mathbb{N}^+. $$

Definition 3 ([43])**.** *For a given function $g(t)$, the Caputo derivative with order $\alpha > 0$ is defined by*

$$ {}_a^{C}D_t^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} \frac{d^n g(u)}{du^n} du, \quad n-1 < \alpha < n, n \in \mathbb{N}^+. $$

The notation D^α replaces the fractional-order Riemann–Liouville derivative ${}_0^{RL}D_t^\alpha$ for the convenience in the following sections. For more details on fractional calculus, one can refer to [43].

Definition 4 ([14])**.** *Suppose $\xi(\omega)$ is the diffusive representation (TDR) of $w(t)$, which is the impulse response of a linear system; then, the following equality relation is satisfied:*

$$ w(t) = \int_0^\infty \xi(\omega) e^{-\omega t} d\omega $$

Let $k(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and the TDR of $k(t)$ be $\mu(\omega)$; then one has: $\mu(\omega) = \frac{sin(\alpha\pi)}{\pi}\omega^{-\alpha}$.

Lemma 1 ([14])**.** *Consider a nonlinear FOS*

$$ D^\alpha x(t) = f(x(t)) $$

Employing the CFDEMM of the fractional-order integrator, the above FOS has the equivalent form:

$$ \begin{cases} \frac{\partial u(\omega,t)}{\partial t} = -\omega u(\omega,t) + f(x(t)) \\ x(t) = \int_0^\infty \mu(\omega) u(\omega,t) d\omega \end{cases} $$

where $\mu(\omega)$ is the same as in Definition 4.

Lemma 2. *Let $\gamma > 0$, X and Y be real matrices with appropriate dimensions. Then, the following result holds:*

$$ 2X^TY \leq \gamma X^TX + \gamma^{-1}Y^TY $$

Lemma 3 (Boyd et al. [44])**.** *Given real symmetric matrix $X = X^T < 0$, the following assertions are equivalent:*

(i) $\quad X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} < 0;$

(ii) $X_{11} < 0, X_{22} - X_{12}^T X_{11}^{-1} X_{12} < 0;$

(iii) $X_{22} < 0, X_{11} - X_{12} X_{22}^{-1} X_{12}^T < 0.$

Consider the following uncertain FOMNN with a neuron n which is described by

$$D^\alpha m_i(t) = -(c_i + \Delta c_i(t))m_i(t) + \sum_{j=1}^{n} b_{ij}(m_j(t))f_j(m_j(t)) + I_i, \qquad t \geq 0, i = 1, 2, \cdots, n. \quad (1)$$

where $m_i(t)$ stands for the ith neuron state, $c_i > 0$ is a constant, $\Delta c_i(t)$ represents the uncertainty, I_i denotes the external input constant vector, $f_i(\cdot)$ denotes the neuron input activation functions and $b_{ij}(m_j(t))$ is a connection weight element, which can be given as

$$b_{ij}(m_j(t)) = \begin{cases} \hat{b}_{ij}, \ |m_j(t)| < T_j, \\ \check{b}_{ij}, \ |m_j(t)| > T_j, \end{cases}$$

where $b_{ij}(\pm T_j) = \{\hat{b}_{ij} \text{ or } \check{b}_{ij}\}$; switching times $T_j > 0, \hat{b}_{ij}, \check{b}_{ij}$ are constants.

Remark 2. *If all $b_{ij}(m_j(t)) \equiv b_{ij}, \Delta c_i(t) = 0$, where b_{ij} are constants and $\alpha = 1$, then System (1) will degenerate the classical Hopfield neural network.*

Since $b_{ij}(m_j(t))$ are discontinuous, the solution of FOMNN in this paper is understood in the differential inclusions sense. One can refer the books of [45,46] for more theories on set-valued maps and differential inclusions.

Definition 5 (Aubin and Cellina [45])**.** *Let $E \subset R^n$. If for each point $x \in E$, there is a corresponding nonempty set $\kappa(x) \subset R^n$, then $x \mapsto \kappa(x)$ is called a set-valued map defined on E. A set-valued map κ is convex (closed)-valued if $\kappa(x)$ is convex (closed) for all $x \in E$. A set-valued map κ is called upper semi-continuous at $x_0 \in E$, if the set $\kappa(x_0)$ is a nonempty closed subset of E, and for each open set N of E containing $\kappa(x_0)$, there exists an open neighborhood N_0 of x_0 such that $\kappa(N_0) \subset N$.*

According to Definition 5 and the theories of differential inclusions, the set-valued maps $m_i(t) \mapsto -(c_i + \Delta c_i(t))m_i(t) + \sum_{j=1}^n b_{ij}(m_j(t))f_j(m_j(t)) + I_i$ are nonempty convex compact values. Thus, the FOMNN with uncertainty can be rewritten as follows:

$$D^\alpha m_i(t) \in -(c_i + \Delta c_i(t))m_i(t) + \sum_{j=1}^{n} co[b_{ij}(m_j(t))]f_j(m_j(t)) + I_i, \qquad t \geq 0 \quad (2)$$

where

$$co[b_{ij}(m_j(t))] = \begin{cases} \hat{b}_{ij}, & |m_j(t)| < T_j, \\ co\{\hat{b}_{ij}, \check{b}_{ij}\}, & |m_j(t)| = T_j, \\ \check{b}_{ij}, & |m_j(t)| > T_j. \end{cases}$$

That is to say, there exist $\tau_{ij}(m_j(t)) \in co[b_{ij}(m_j(t))]$ such that

$$D^\alpha m_i(t) = -(c_i + \Delta c_i(t))m_i(t) + \sum_{j=1}^{n} \tau_{ij}(m_j(t))f_j(m_j(t)) + I_i, \quad a.e. \ \ t \geq 0 \quad (3)$$

To discuss the synchronization problem of FOMNN, we regard FOS (1) as the master system and the slave system as follows:

$$D^\alpha s_i(t) = -(c_i + \Delta c_i(t))s_i(t) + \sum_{j=1}^{n} b_{ij}(s_j(t))f_j(s_j(t)) + I_i + u_i(t), \quad i = 1, 2, \cdots, n \quad (4)$$

where $u_i(t)$ denotes the control input.

The slave system (4) can be described in similar forms owing to the theories of differential inclusions:

$$D^\alpha s_i(t) \in -(c_i + \Delta c_i(t))s_i(t) + \sum_{j=1}^{n} co[b_{ij}(s_j(t))]f_j(s_j(t)) + I_i + u_i(t), \quad t \geq 0 \tag{5}$$

where

$$co[b_{ij}(s_j(t))] = \begin{cases} \hat{b}_{ij}, & |s_j(t)| < T_j, \\ co\{\hat{b}_{ij}, \check{b}_{ij}\}, & |s_j(t)| = T_j, \\ \check{b}_{ij}, & |s_j(t)| > T_j. \end{cases}$$

That is, one has $\eta_{ij}(s_j(t)) \in co[b_{ij}(s_j(t))]$ such that

$$D^\alpha s_i(t) = -(c_i + \Delta c_i(t))s_i(t) + \sum_{j=1}^{n} \eta_{ij}(s_j(t))f_j(s_j(t)) + I_i + u_i(t), \ \ a.e. \ \ t \geq 0 \tag{6}$$

The robust synchronization error of the master–slave system is given as $e_i(t) = s_i(t) - m_i(t)$. Thus, we have the following synchronization error system:

$$D^\alpha e_i(t) = -(c_i + \Delta c_i(t))e_i(t) + \sum_{j=1}^{n}[b_{ij}(s_j(t))f_j(s_i(t)) - b_{ij}(m_j(t))f_j(m_j(t))] + u_i(t) \tag{7}$$

Therefore, one has the following equivalent equation:

$$D^\alpha e_i(t) = -(c_i + \Delta c_i(t))e_i(t) + \sum_{j=1}^{n}[b_{ij}(e_j(t) + m_j(t))f_j(e_j(t) + m_j(t)) - b_{ij}(m_j(t))f_j(m_j(t))] + u_i(t) \tag{8}$$

In order to guarantee the uniqueness and existence of the solutions of FOMNN (1) and obtain the primary conclusions, the following basic assumptions are listed as follows.

$A1$. Every neuron activation function $f_i(i = 1, \cdots, n)$ is a Lipschitz function and bounded. That is, for any $u, v \in R$, there exist $l_i \geq 0, n_i \geq 0$ such that

$$|f_i(u) - f_i(v)| \leq l_i|u - v|, |f_i(u))| \leq n_i$$

$A2$. The uncertainty term $\Delta c_i(t)(i = 1, \cdots, n)$ satisfies the norm-bounded uncertainty. That is, for all $\Delta c_i(t)$, there exist $\rho_i \geq 0$ such that $|\Delta c_i(t)| \leq \rho_i$.

Remark 3. *According to* [34,40]*, the uniqueness and existence of the solutions for FOMNN (1) can be obtained. Assumptions 1 and 2 are general assumptions for neural network activation functions; see* [40].

Remark 4. *Norm-bounded uncertainty and bounded real uncertainty exist in the control system* [42]. *Compared with the effect of bounded real uncertainty to dynamical behavior for FOS or FONN in* [36,37,39]*, norm-bounded uncertainty is addressed for the first time in this article.*

3. Main Results

A suitable control scheme and the main results are achieved for the robust synchronization of the FOMNN in this section.

Theorem 1. *Under $A1 - A2$, FOS (8) is asymptotically stable based on the controller*

$$u_i(t) = -\sum_{j=1}^{n} b_{ij}(e_j(t) + m_j(t))f_j(m_j(t)) + \sum_{j=1}^{n} b_{ij}(m_j(t))f_j(m_j(t)) + k_i e_i(t) \tag{9}$$

if there exist $\epsilon_1 > 0, \epsilon_2 > 0$*, such that*

$$\begin{bmatrix} D + \frac{\epsilon_1}{2}I + \frac{\epsilon_2}{2}\hat{A}\hat{A}^T & \rho I & LI \\ \bullet & -2\epsilon_1 I & 0 \\ \bullet & 0 & -2\epsilon_2 I \end{bmatrix} < 0, \tag{10}$$

where $D = diag(k_1 - c_1, \cdots, k_n - c_n), \hat{A} = (b_{ij}^u) = \max\{|\hat{b}_{ij}|, |\check{b}_{ij}|\}, \rho^2 = \max\{\rho_1^2, \cdots, \rho_n^2\}$, $L^2 = \max\{l_1^2, \cdots, l_n^2\}$.

Proof. According to error system (8) and controller (9), one can rewrite the following error system:

$$D^\alpha e_i(t) = (k_i - c_i - \Delta c_i(t))e_i(t) + \sum_{j=1}^{n} b_{ij}(e_j(t) + m_j(t))(f_j(e_j(t) + m_j(t)) - f_j(m_j(t))) \tag{11}$$

where $i = 1, \cdots, n$.

Therefore, there exist $\theta_{ij}(e_j(t) + m_j(t)) \in co[b_{ij}(e_j(t) + m_j(t))]$ such that

$$D^\alpha e_i(t) = (k_i - c_i - \Delta c_i(t))e_i(t) + \sum_{j=1}^{n} \theta_{ij}(e_j(t) + m_j(t))(f_j(e_j(t) + m_j(t)) - f_j(m_j(t))) \tag{12}$$

Let $e(t) = (e_1(t), \cdots, e_n(t))^T, D = diag(k_1 - c_1, \cdots, k_n - c_n), \Delta C = diag(\Delta c_1(t), \cdots, \Delta c_n(t))$, $A(e(t) + m(t))) = (\theta_{ij}(e_j(t) + m_j(t)))_{n\times n}$, $F(e(t) + m(t)) = (f_1(e_1(t) + m_1(t)) - f_1(m_1(t)), \cdots, f_n(e_n(t) + m_n(t)) - f_n(m_n(t)))^T$; then, System (12) can be represented in the following form:

$$D^\alpha e(t) = (D - \Delta C)e(t) + A(e(t) + m(t))F(e(t) + m(t)) \tag{13}$$

By virtue of the *Lemma 1*, we can reformulate the new error equation (Equation (13)) as follows:

$$\begin{cases} \frac{\partial u(\omega,t)}{\partial t} = -\omega u(\omega, t) + (D - \Delta C)e(t) + A(e(t) + m(t))F(e(t) + m(t)) \\ e(t) = \int_0^\infty \mu(\omega)u(\omega, t)d\omega \end{cases} \tag{14}$$

Define the Lyapunov function as:

$$V(t) = \int_0^\infty \mu(\omega)\nu(\omega, t)d\omega$$

where $\nu(\omega, t) = u(\omega, t)^T u(\omega, t)$ is a function on the frequency ω. It is obvious that $V(t)$ is a positive definite lyapunov function, and it is the sum of the $\nu(\omega, t)$ with the weighted coefficient $\mu(\omega)$.

The derivative of lyapunov function $V(t)$, which takes into account the trajectories of (14), is given as follows:

$$\begin{aligned} \frac{dV(t)}{dt} &= \int_0^\infty \mu(\omega)\frac{\partial \nu(\omega, t)}{\partial t}d\omega \\ &= \frac{1}{2}\int_0^\infty \mu(\omega)[-\omega u^T(\omega, t) + e(t)^T(D - \Delta C)^T + F^T(e(t) + m(t))A^T(e(t) + m(t))]u(\omega, t)d\omega \\ &\quad + \frac{1}{2}\int_0^\infty \mu(\omega)u^T(\omega, t)[-\omega u(\omega, t) + (D - \Delta C)e(t) + A(e(t) + m(t))F(e(t) + m(t))]d\omega \\ &= -\int_0^\infty \omega\mu(\omega)u^T(\omega, t)u(\omega, t)d\omega + e(t)^T(D - \Delta C)e(t) + e(t)^T A(e(t) + m(t))F(e(t) + m(t)) \end{aligned}$$

By employing the Lyapunov stability theorem, System (8) is asymptotically stable if

$$e(t)^T(D-\Delta C)e(t)+e(t)^TA(e(t)+m(t))F(e(t)+m(t))<0$$

Using $A2$ and *Lemma 2* on the first term, one can derive:

$$\begin{aligned} e(t)^T(D-\Delta C)e(t) &\leq e(t)^TDe(t)+\frac{\epsilon_1}{2}e(t)^Te(t)+\frac{1}{2\epsilon_1}e(t)^T\Delta C^T\Delta Ce(t) \\ &= e(t)^T(D+\frac{\epsilon_1}{2}I+\frac{1}{2\epsilon_1}\rho^2 I)e(t) \end{aligned}$$

where $\epsilon_1>0, \rho^2=\max\{\rho_1^2,\cdots,\rho_n^2\}$.

According to $A1$ and *Lemma 2*, one has the following estimation expression on the second term:

$$\begin{aligned} e(t)^TA(e(t)+m(t))F(e(t)+m(t)) &\leq \frac{\epsilon_2}{2}e(t)^TA(e(t)+m(t))A^T(e(t)+m(t))e(t)+\frac{1}{2\epsilon_2}F^T(e(t)+m(t))F(e(t)+m(t)) \\ &\leq \frac{\epsilon_2}{2}e(t)^T\hat{A}\hat{A}^Te(t)+\frac{1}{2\epsilon_2}L^2e(t)^Te(t) \\ &= e(t)^T(\frac{\epsilon_2}{2}\hat{A}\hat{A}^T+\frac{1}{2\epsilon_2}L^2I)e(t) \end{aligned}$$

where $\epsilon_2>0, \hat{A}=(b_{ij}^u)=\max\{|\hat{b}_{ij}|,|\check{b}_{ij}|\}, L^2=\max\{l_1^2,\cdots,l_n^2\}$.

Therefore, System (8) is asymptotically stable if

$$D+\frac{\epsilon_1}{2}I+\frac{\epsilon_2}{2}\hat{A}\hat{A}^T+\frac{\rho^2}{2\epsilon_1}I+\frac{L^2}{2\epsilon_2}I<0 \tag{15}$$

By means of the *Lemma 3*, $V'(t)<0$ is satisfied if LMI (10) holds. □

Remark 5. *The above method is also suitable for investigating other styles of synchronization of the FOMNN, such as anti-synchronization and projective synchronization.*

Remark 6. *Compared to* [33–36]*, we used a new method which is called the CFDEMM to investigate the synchronization behavior of an uncertain FOMNN. The results are expressed in LMI, which can be easily used in a real system.*

Remark 7. *When the bounds of the system parameter are unknown, the adaptive control method is under consideration.*

When the parameters of the system are determined, that is to say, for all $i=1,\cdots,n$, $\Delta c_i(t)=0$, then one can obtain the following result:

Corollary 1. *Under $A1-A2$, if there exist real numbers $\epsilon>0$, such that*

$$\begin{bmatrix} D+\frac{\epsilon}{2}\hat{A}\hat{A}^T & LI \\ \bullet & -2\epsilon I \end{bmatrix}<0, \tag{16}$$

where $D=diag(k_1-c_1,\cdots,k_n-c_n), \hat{A}=(a_{ij}^u)=\max\{|\hat{a}_{ij}|,|\check{a}_{ij}|\}, L^2=\max\{l_1^2,\cdots,l_n^2\}$. Then, System (8) is asymptotically stable based on the controller (9).

4. Numerical Examples

Two FOMNN models are enumerated to show the practicability of our theoretical analyses results in this section.

Example 1. *Consider the following FOMNN which is regarded as the master system:*

$$D^{\alpha}m(t) = -(C + \Delta C(t))m(t) + B(m(t))f(m(t)) + I,$$

where $\alpha = 0.95, m(t) = (m_1(t), m_2(t), m_3(t))^T$, $f(m) = (f_1(m_1(t)), f_2(m_2(t)), f_3(m_3(t)))^T$, $f_i(m_i) = 0.5(|1 + m_i| - |1 - m_i|)(i = 1, 2, 3)$.

$$C = \begin{pmatrix} 3.4 & 0 & 0 \\ 0 & 2.4 & 0 \\ 0 & 0 & 3.5 \end{pmatrix}, \Delta C(t) = \begin{pmatrix} -0.5sint & 0 & 0 \\ 0 & -0.5cost & 0 \\ 0 & 0 & 0.5sint \end{pmatrix}, I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$b_{11}(m_1) = \begin{cases} 1.1, & |m_1| < 1, \\ 1, & |m_1| \geq 1, \end{cases} b_{12}(m_2) = \begin{cases} 0.09, & |m_2| < 1, \\ 1, & |m_2| \geq 1, \end{cases} b_{13}(m_3) = \begin{cases} -1, & |m_3| < 1, \\ -1.5, & |m_3| \geq 1, \end{cases}$$

$$b_{21}(m_1) = \begin{cases} 0.1, & |m_1| < 1, \\ 0.2, & |m_1| \geq 1, \end{cases} b_{22}(m_2) = \begin{cases} 2, & |m_2| < 1, \\ 2.1, & |m_2| \geq 1, \end{cases} b_{23}(m_3) = \begin{cases} -1, & |m_3| < 1, \\ -2, & |m_3| \geq 1, \end{cases}$$

$$b_{31}(m_1) = \begin{cases} 3, & |m_1| < 1, \\ -3, & |m_1| \geq 1, \end{cases} b_{32}(m_2) = \begin{cases} -1, & |m_2| < 1, \\ -1.8, & |m_2| > 1, \end{cases} b_{33}(m_3) = \begin{cases} -2, & |m_3| < 1, \\ -1, & |m_3| > 1, \end{cases}$$

It is clear that $l_i = 1, \rho_i = 0.5, i = 1, 2$.

The slave system is presented by:

$$D^{\alpha}s(t) = -(C + \Delta C(t))s(t) + B(s(t))f(s(t)) + I + u(t),$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T$.

Using the Matlab Toolbox, one has $\epsilon_1 = 19.9003, \epsilon_2 = 2.1404, k_1 = k_2 = k_3 = -57.0497$. According to Theorem 1, the master–slave robust synchronization of the FOMNNs was synchronized, which was verified by the simulation results given by Figures 1–4. The master–slave system trajectories are depicted with initial conditions, taken as $m(0) = (0.4, 2.1, -0.9)^T, s(0) = (-2.6, -0.9, 1.8)^T$, in Figures 1 and 2. The evolution of the controller and the errors trajectories are shown in Figures 3 and 4.

Example 2. *Consider 3-dimensional FOMNN with the following parameters as the master system:*

$$D^{\alpha}m_i(t) = -(c_i + \Delta c_i(t))m_i(t) + \sum_{j=1}^{3} b_{ij}(m_j(t))f_j(m_j(t)) + I_i, \quad i = 1, 2, 3.$$

where $\alpha = 0.95$*and*

$$b_{11}(m_1) = \begin{cases} 1.2, & |m_1| < 1, \\ 1.12, & |m_1| \geq 1, \end{cases} b_{12}(m_2) = \begin{cases} -1.75, & |m_2| < 1, \\ -1.72, & |m_2| \geq 1, \end{cases} b_{13}(m_3) = \begin{cases} 1.3, & |m_3| < 1, \\ 1.2, & |m_3| \geq 1, \end{cases}$$

$$b_{21}(m_1) = \begin{cases} -1.97, & |m_1| < 1, \\ -1.91, & |m_1| \geq 1, \end{cases} b_{22}(m_2) = \begin{cases} 1.71, & |m_2| < 1, \\ 1.5, & |m_2| \geq 1, \end{cases} b_{23}(m_3) = \begin{cases} -2.85, & |m_3| < 1, \\ -2.75, & |m_3| \geq 1, \end{cases}$$

$$b_{31}(m_1) = \begin{cases} -1.22, & |m_1| < 1, \\ -1.25, & |m_1| \geq 1, \end{cases} b_{32}(m_2) = \begin{cases} 2.15, & |m_2| < 1, \\ 2.25, & |m_2| > 1, \end{cases} b_{33}(m_3) = \begin{cases} 2.3, & |m_3| < 1, \\ 2.37, & |m_3| > 1, \end{cases}$$

$$C = \begin{pmatrix} 0.15 & 0 & 0 \\ 0 & 0.53 & 0 \\ 0 & 0 & 2.45 \end{pmatrix}, \Delta C(t) = \begin{pmatrix} 0.1sint & 0 & 0 \\ 0 & -0.1cost & 0 \\ 0 & 0 & 0.1sint \end{pmatrix}, I = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

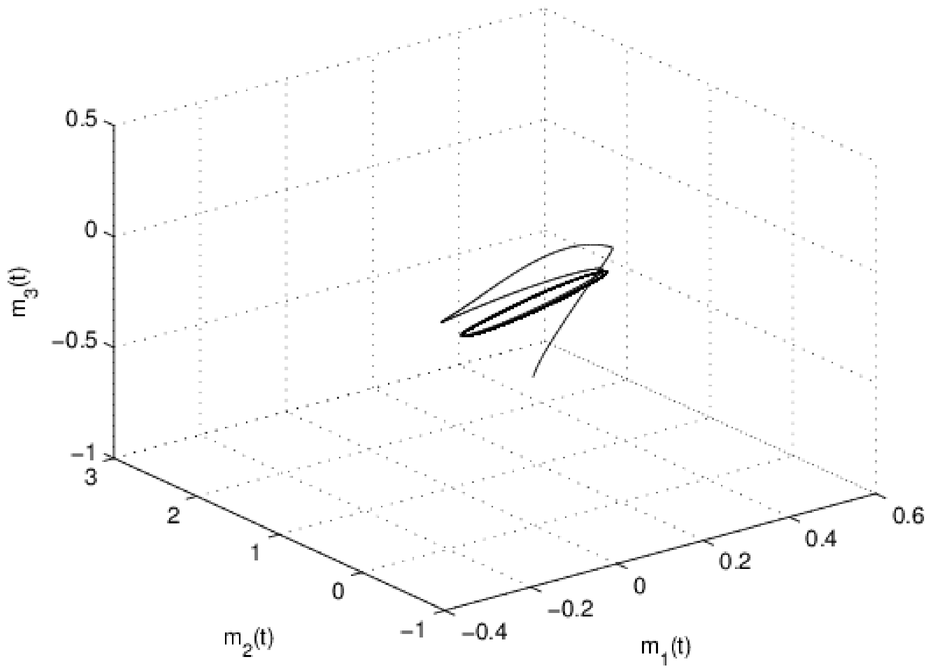

Figure 1. the master system trajectories.

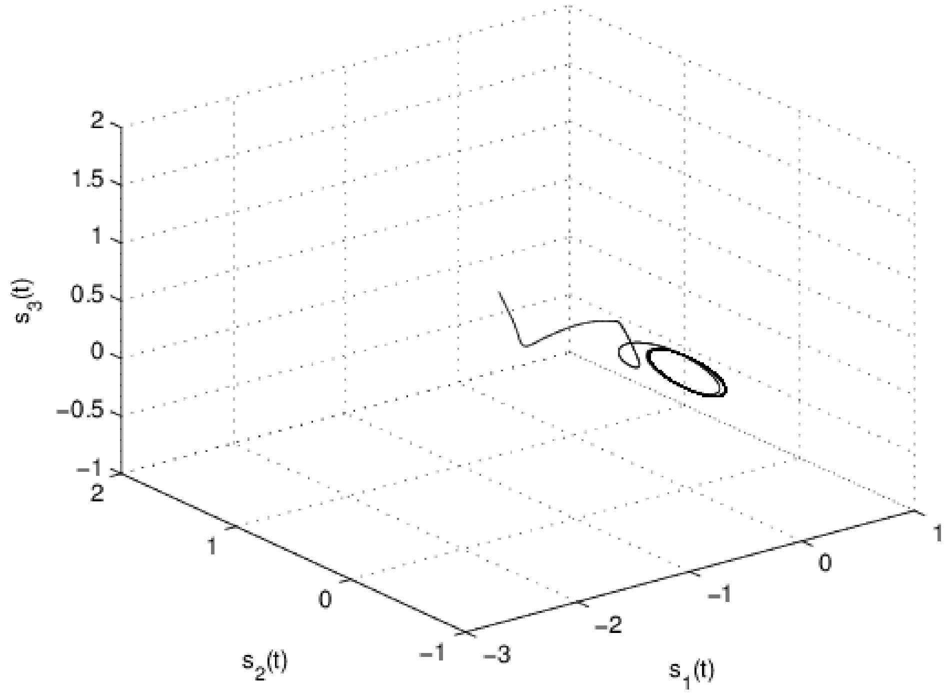

Figure 2. the slave system trajectories.

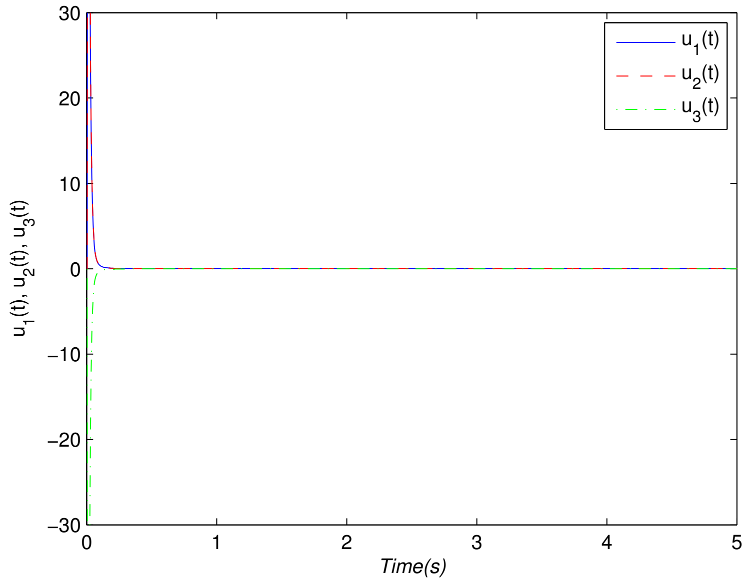

Figure 3. Time evolution of the controller.

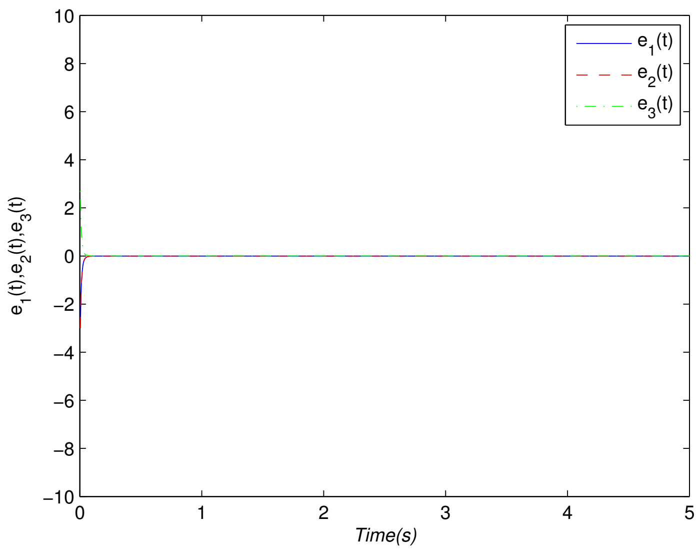

Figure 4. Time evolution of the errors trajectories.

The neuron activation function was chosen as $f_j(m_j) = tanh(|m_j|), j = 1,2,3$. By a standard operating procedure, one has $l_i = 1, \rho_i = 0.1, i = 1,2,3$.

The slave system is given as follows:

$$D^{\alpha}s_i(t) = -(c_i + \Delta c_i(t))s_i(t) + \sum_{j=1}^{3} b_{ij}(s_j(t))f_j(s_j(t)) + u_i, \quad i = 1,2,3.$$

Using the Matlab Toolbox, one has $\epsilon_1 = 20.2365, \epsilon_2 = 1.3258, k_1 = k_2 = k_3 = -56.919$. According to Theorem 1, the master–slave robust synchronization of the FOMNNs was synchronized, which was verified by the simulation results given by Figures 5–8. The master–slave system trajectories are depicted with initial conditions, taken as $m(0) = (1.35, -0.75, 0.43)^T, s(0) = (-2.70, 1.90, -1.80)^T$, in Figures 5 and 6. The evolution of the controller and the errors' trajectories are shown in the Figures 7 and 8.

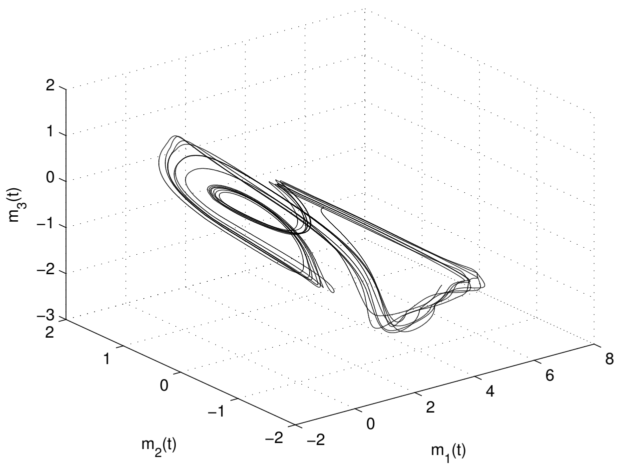

Figure 5. The master system trajectories.

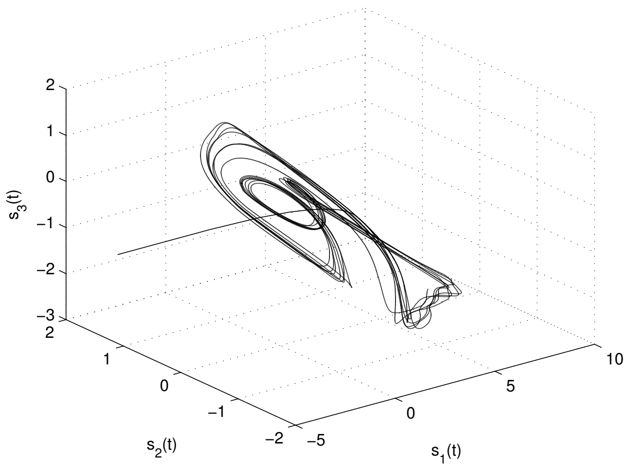

Figure 6. The slave system trajectories.

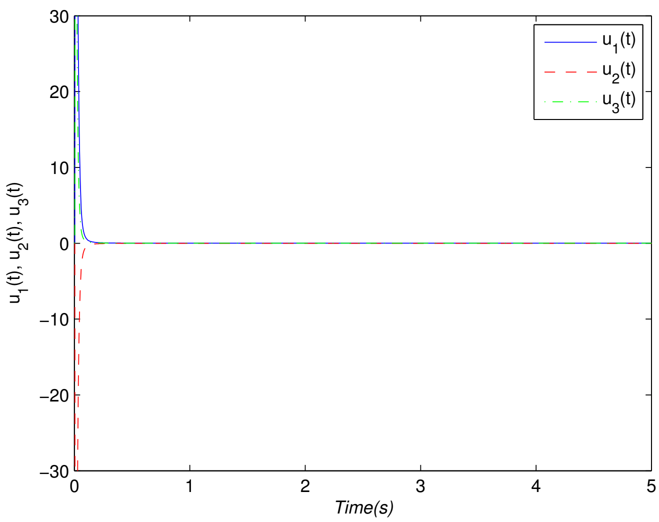

Figure 7. Time evolution of the controller.

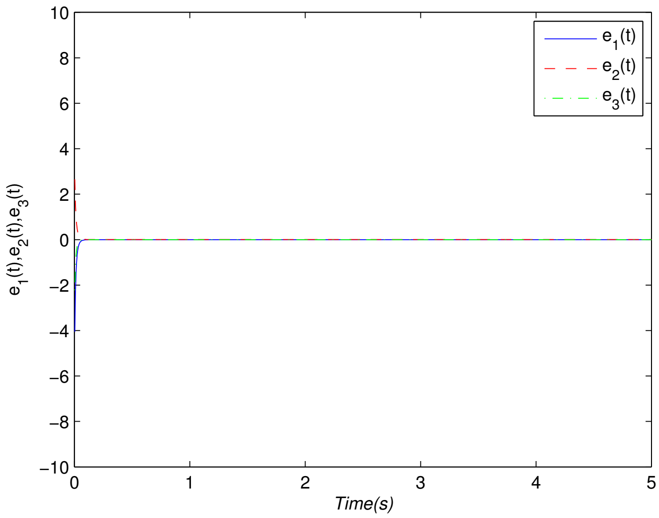

Figure 8. Time evolution of the errors trajectories.

5. Conclusions

In this study, a general kind of FOMNN with parameter uncertainty are studied. Different from the traditional bounded real uncertainty, norm bounded uncertainty is studied for the first time. Using CFDEMM and LMI, new results which ensure the master–slave robust synchronization of FOMNN are presented in LMI forms which are easily realized using Matlab toolbox. In the end, the efficiency of the derived criteria are validated by two classical FOMNN examples. In real world systems, time-delay and impulsive effects are common and inevitable phenomenon in the process of signal transmission and reception. Meanwhile, finite-time synchronization or fixed-time synchronization behavior of the FOMNN are more valuable than the traditional synchronization behavior. Therefore, finite-time or fixed-time dynamical behavior of the FOMNN with delay and impulsive effect will be further discussed in our future work.

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