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# Ozaki-Type Bi-Close-to-Convex and Bi-Concave Functions Involving a Modified Caputo's Fractional Operator Linked with a Three-Leaf Function

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**Abstract:** The focus of the present work is on the establishment and investigation of the coefficient estimates of two new subclasses of bi-close-to-convex functions and bi-concave functions; these are of an Ozaki type and involve a modified Caputo's fractional operator that is associated with three-leaf functions in the open unit disc. The classes are defined using the notion of subordination based on the previously established fractional integral operators and classes of starlike functions associated with a three-leaf function. For functions in these classes, the Fekete-Szegö inequalities and the initial coefficients, $|a_2|$ and $|a_3|$, are discussed. Several new implications of the findings are also highlighted as corollaries.

**Keywords:** analytic function; univalent function; bi-univalent function; bi-starlike and bi-convex function; coefficient bounds; convolution; Caputo's fractional derivative

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## 1. Introduction and Preliminaries

The investigation presented in this article deals with new classes of bi-univalent functions defined by applying the means of the geometric function theory combined with fractional calculus aspects. In order to get acquainted with the context of the research, let's begin by stating the basic notions.

In the open unit disc $\mathbf{\Delta} = \{\xi : |\xi| < 1\}$, let $\mathcal{A}$ symbolize the class of analytic functions of the following form:

$$\mathcal{G}(\xi) = \xi + \sum_{n\geq 2} a_n \xi^n, \quad \xi \in \Delta \tag{1}$$

normalized by the conditions $\mathcal{G}(0) = 0$ and $\mathcal{G}'(0) = 1$. Additionally, let $\mathcal{S} \subseteq \mathcal{A}$ be the class of all functions in $\mathbf{\Delta}$ that are univalent.

Several subclasses of $\mathcal{S}$, such as the starlike function, convex function, and close-to-convex functions have the geometrical conditions

$$\left(\Re\frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)}\right) > 0; \qquad \Re\left(\frac{(\xi\mathcal{G}'(\xi))'}{\mathcal{G}'(\xi)}\right) > 0; \qquad \Re\left(\frac{\mathcal{G}(\xi)}{\mathcal{G}(\xi)} > 0\right)$$

and many others have been worked on certain geometric properties discussed in the literature. Among the prominent and extensively studied subclasses of $\mathcal{S}$ are the class

$\mathcal{CV}(\alpha)$ of convex functions of the order of $\alpha$ $(0 \leq \alpha < 1)$, and $\mathcal{ST}(\alpha)$, the class of starlike functions of the order of $\alpha$ (for details, see [1]). The research into geometric function theory has been very active in recent years, and the typical problem in this field is studying a function made up of the combinations of the initial coefficients of the functions $f \in \mathcal{A}$. For a function in the class $\mathcal{S}$, it is well-known that $|a_n|$ is bounded by $n$. Moreover, the coefficient bounds give information about the geometric properties of those functions. For instance, the bound for the second coefficients of the class $\mathcal{S}$ gives the growth and distortion bounds for the class.

Let $\mathcal{G}_1, \mathcal{G}_2 \in \mathcal{A}$. The notation $\mathcal{G}_1(\xi) \prec \mathcal{G}_2(\xi)$ indicates that function $\mathcal{G}_1$ is subordinate to $\mathcal{G}_2$, provided that there exists $\varpi \in \mathcal{A}$, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$, so that $\mathcal{G}_1(\xi) = \mathcal{G}_2(\varpi(\xi))$.

The Koebe one-quarter theorem confirms that the image of $\Delta$ under every univalent function $\mathcal{G} \in \mathcal{A}$ comprises a disk of radius $\frac{1}{4}$. Thus, for each function $\mathcal{G} \in \mathcal{S}$, there is an inverse $\mathcal{G}^{-1}$ defined by

$$\mathcal{G}^{-1}(\mathcal{G}(\xi)) = \xi \ (\xi \in \mathbf{\Delta})$$
$$\text{and} \quad \mathcal{G}(\mathcal{G}^{-1}(w)) = w \ \ (|w| < r_0(\mathcal{G}); r_0(\mathcal{G}) \geq 1/4)$$

with

$$\mathcal{G}^{-1}(w) = \mathcal{H}(w) = w - a_2 w^2 + (-a_3 + 2a_2^2)w^3 - (a_4 - 5a_2 a_3 + 5a_2^3)w^4 + \cdots. \quad (2)$$

The family of all bi-univalent functions in $\mathbf{\Delta}$ denoted by $\Sigma$ is defined as

$$\Sigma = \left\{ \mathcal{G} \in \mathcal{A} : \mathcal{G}, \mathcal{G}^{-1} \in \mathcal{S} \right\}.$$

We review a number of functions in the $\Sigma$ family shown in Srivastava et al. [2],

$$\frac{\xi}{1-\xi}, \quad -\log(1-\xi) \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right),$$

with the inverses that relate to them:

$$\frac{w}{1+w}, \quad \frac{e^w - 1}{e^w} \text{ and } \quad \frac{e^{2w}-1}{e^{2w}+1}.$$

We can note that the family $\Sigma$ is not empty, though the Koebe function is not a member of $\Sigma$. Additionally, the functions $\xi - \frac{\xi^2}{2}$ and $\frac{\xi}{1-\xi^2}$ are not bi-univalent.

Bi-starlike functions of the order $\alpha(0 < \alpha \leq 1)$ denoted by $\mathcal{S}^*_\Sigma(\alpha)$ and bi-convex functions of the order $\alpha$ denoted by $\mathcal{CV}_\Sigma(\alpha)$ were presented by Brannan and Taha [3]. The first two Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, were shown to have non-sharp estimates for each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{CV}_\Sigma(\alpha)$ [3,4]. Unfortunately, there is still an unresolved problem for each of the Taylor-Maclaurin coefficients $|a_n| (n \in \mathbb{N} \setminus \{1,2\})$. After studying many interesting subclasses of $\Sigma$, a number of authors (see [2,5–12] and references cited therein) came to the conclusion that the estimations of the first two Taylor-Maclaurin coefficients, $|a_2|$ and $|a_3|$, are not sharp.

*Modified Caputo's Fractional Operator*

Certain aspects of fractional calculus have been included in the studies pertaining to geometric function theory. Fractional calculus is an area of mathematics that is derived from the traditional definition of calculus, which includes integral and derivative operators. In the same way, the exponents that are integer numbers are the source of fractional exponents. The development of fractional calculus began with a hypothetical question posed in a 1695 letter from G F A de L'Hospital to G W Leibniz: "What if the order of the derivative $\frac{d^n}{dt^n}$ such that $n = \frac{1}{2}$." He replied, "It will lead to a paradox, from which useful consequences will be drawn one day." (See [13,14] for additional details). Several mathematicians, including Riemann, Liouville, L Euler, Letnikov, Grunwald, Marchuad, Weyl, Riesz, Caputo, Abel, and others, were inspired by the discussion between the two well-known mathematicians and worked to expand upon, generalize, and formulate the theory of non-integer orders [15–17]. Numerous domains, including physics, mechanics, engineering, and biology [18], use fractional derivatives and fractional integrals. In the field of geometric function theory, the definitions provided by Owa (1978) [19] are relevant, and we refer to them in this investigation.

**Definition 1** ([19])**.** *Assume that the function $\mathcal{G}$ is analytic in a simply connected region of the $\xi-$ plane that contains the origin.*

*The fractional integral of $\mathcal{G}$ of the order $\tau$ is defined by*

$$\mathcal{D}_{\xi}^{-\tau}\mathcal{G}(\xi) = \frac{1}{\Gamma(\tau)}\int_{0}^{\xi}\frac{\mathcal{G}(t)}{(\xi-t)^{1-\tau}}dt, \quad \tau > 0, \tag{3}$$

*and the fractional derivatives of $\mathcal{G}$ of the order $\tau$, is defined by*

$$\mathcal{D}_{\xi}^{\tau}\mathcal{G}(\xi) = \frac{1}{\Gamma(1-\tau)}\frac{d}{d\xi}\int_{0}^{\xi}\frac{\mathcal{G}(t)}{(\xi-t)^{\tau}}dt, \quad 0 \le \tau < 1, \tag{4}$$

*where the multiplicity of $(\xi-t)^{1-\tau}$ and $(\xi-t)^{-\tau}$ is removed by requiring $log(\xi-t)$ to be real when $\xi-t>0$.*

**Definition 2** ([19])**.** *The fractional derivative of $\mathcal{G}$ of the order $n+\tau$ is defined by*

$$\mathcal{D}_{\xi}^{n+\tau}\mathcal{G}(\xi) = \frac{d^n}{d\xi^n}\mathcal{D}_{\xi}^{\tau}\mathcal{G}(\xi), \;\; 0 \le \tau < 1\;;\, n \in \mathbb{N}_0. \tag{5}$$

Using the aforementioned definitions and their established expansions concerning fractional derivatives and fractional integrals, Srivastava and Owa [20] constructed the following operator:

$$\Omega^{\varrho} : \mathcal{A} \to \mathcal{A},$$

$$\Omega^{\varrho}\mathcal{G}(\xi) = \Gamma(2-\varrho)\xi^{\varrho}\mathcal{D}_{\xi}^{\varrho}\mathcal{G}(\xi) = \xi + \sum_{n\ge 2}\Phi(n,\varrho)a_n\xi^n,$$

where

$$\Phi(n,\varrho) = \frac{\Gamma(n+1)\Gamma(2-\varrho)}{\Gamma(n+1-\varrho),}$$

and $\varrho \in \mathbb{R}; \varrho \neq 2,3,4,\cdots$.

For $\mathcal{G} \in \mathcal{A}$ and various choices of $\varrho$, we obtain a different operator. We mention the following:

$$\Omega^{0}\mathcal{G}(\xi) = \mathcal{G}(\xi) = z + \sum_{n=2}^{\infty} a_n z^n \tag{6}$$

$$\Omega^{1}\mathcal{G}(\xi) = \xi\mathcal{G}^{'}(\xi) = \xi + \sum_{n\ge 2} n a_n \xi^n \tag{7}$$

$$\Omega^{j}\mathcal{G}(\xi) = \Omega(\Omega^{j-1}\mathcal{G}(\xi)) = \xi + \sum_{n\ge 2} n^j a_n \xi^n, \;\; j = 1,2,3,\ldots \tag{8}$$

which is known as a Sălăgean operator (Sălăgean, 1983) [21]. Additionally,

$$\Omega^{-1}\mathcal{G}(\xi) = \frac{2}{\xi}\int_0^{\xi}\mathcal{G}(t)dt = \xi + \sum_{n\ge 2}\left(\frac{2}{n+1}\right)a_n\xi^n$$

and

$$\Omega^{-j}\mathcal{G}(\xi) = \Omega^{-1}(\Omega^{-j+1}\mathcal{G}(\xi)) = \xi + \sum_{n\geq 2}\left(\frac{2}{n+1}\right)^{j} a_n\xi^n, (j = 1, 2, 3, ...) \tag{9}$$

This is called a Libera integral operator and was generalized by Bernardi (1969) [22], given by

$$\frac{1+\nu}{\xi^{\nu}}\int_0^{\xi} t^{\nu-1}\mathcal{G}(t)dt = \xi + \sum_{n\geq 2}\left(\frac{1+\nu}{n+1}\right)a_n\xi^n, \quad \nu = 1, 2, 3, \ldots$$

commonly known as Bernardi integral operator.

We examine Caputo's definition (Caplinger and Causey, 1973) [23] of the fractional-order derivative throughout this paper, assuming that

$$\mathcal{D}^{\alpha}\mathcal{G}(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{\mathcal{G}^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \tag{10}$$

where $n-1 < \Re(\alpha) \leq n, n \in \mathbb{N}$. Additionally, as $\alpha$ is the starting value of the function $\mathcal{G}$, it can be real or even complex.

Salah and Darusin (2004) [24]defined the following operator

$$\mathcal{C}_{\tau}^{\vartheta} f(z) = \frac{\Gamma(2+\vartheta-\tau)}{\Gamma(\vartheta-\tau)} z^{\tau-\vartheta}\int_0^z \frac{\Omega^{\vartheta} f(t)}{(z-t)^{\tau+1-\vartheta}}dt \tag{11}$$

where $\vartheta$ (real number) and $(\vartheta - 1 < \tau < \vartheta < 2)$. Simple, straightforward computations for $\mathcal{G} \in \mathcal{A}$ give

$$\begin{aligned} \mathcal{C}_{\tau}^{\vartheta}\mathcal{G}(\xi) &= \xi + \sum_{n\geq 2}\frac{\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)(\Gamma(n+1))^2}{\Gamma(n-\vartheta+1)\Gamma(n+\vartheta-\tau+1)}a_n\xi^n \quad \xi \in \mathbf{\Delta} \\ &= \xi + \sum_{n\geq 2}\Xi_n(\vartheta,\tau)a_n\xi^n \quad \xi \in \mathbf{\Delta} \end{aligned} \tag{12}$$

where

$$\Xi_n(\vartheta,\tau) = \frac{\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)(\Gamma(n+1))^2}{\Gamma(n-\vartheta+1)\Gamma(n+\vartheta-\tau+1)}. \tag{13}$$

Furthermore, note that $\mathcal{C}_0^0\mathcal{G}(\xi) = \mathcal{G}(\xi)$ and $\mathcal{C}_1^1\mathcal{G}(\xi) = \xi\mathcal{G}'(\xi)$.
Several scholars have recently examined the subclass of starlike functions $\mathcal{S}^*(\Lambda)$ using the following criteria:

$$\mathcal{S}^*(\Lambda) = \left\{\mathcal{G} \in \mathcal{A} : \frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)} \prec \Lambda(\xi)\right\}, \tag{14}$$

where $\Lambda(\zeta) = (1+\xi)/(1-\xi)$. Recently, the notion of subordination has been used to develop several analytic function classes based on the geometrical interpretation of their image domains, such as the right half plane, circular disc, oval, and petal, conic domain, generalized conic and leaf-like domains, by varying $\Lambda$ in (14). Here are just a few of them:

1. Cho et al. [25] fixed $\Lambda(\xi) = 1 + \sin\xi$, and Mendiratta et al. [26] considered $\Lambda(\xi) = e^{\xi}$ and discussed the class $\mathcal{S}$ for certain geometric properties and radii problems.
2. Sharma et al. [27] considered $\Lambda(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2$, which is a petal-shaped domain, and Wani and Swaminathan [28] fixed $\Lambda(\xi) = 1 + \xi - \frac{1}{3}\xi^3$, which maps $\mathbf{\Delta}$ onto the interior of the two-cusped-kidney-shaped region and discussed applications of the general coefficient problem for some subclasses of $\mathcal{S}$.
3. Assuming $\Lambda(\xi) = \sqrt{1+\xi}$, Sokól [29] developed a new class that is bounded by the lemniscate of Bernoulli in the right half plane.

4. By fixing $\Lambda(\xi) = \xi + \sqrt{1+\xi^2}$, which maps $\mathbf{\Delta}$ to a crescent-shaped region, the initial Taylor coefficients for subclasses $\mathcal{S}$ were introduced and discussed by Raina and Sokól [30].

All these above subclasses of starlike functions have been extensively studied for initial coefficient bounds, Fekete-Szegö inequalities, and Hankel inequalities. Lately, Gandhi [31] defined the class of starlike functions connected with three-leaf functions as

$$\mathcal{S}^*_{3\mathcal{L}} = \left\{ \mathcal{G} \in \mathcal{A} : \frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)} \prec \Lambda(\xi) = 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4, \xi \in \mathbf{\Delta} \right\}$$

and studied certain subclasses of analytic functions defined by the subordination to the three-leaf function. Motivated by the study on bi-univalent functions (see [2,8–12] and references cited therein), in Sections 2 and 3 of this article, using the modified Caputo's fractional operator, we introduced two new subclasses, namely Ozaki-type bi-close-to-convex functions and bi-concave functions, in the open unit disc, as given in Definitions 4 and 5 by subordinating this to the three-leaf function, respectively. For functions in these classes, the initial coefficients $|a_2|$ and $|a_3|$ are established, and we discuss the bounds on Fekete-Szegö results, which have not been studied so far for the function classes related to the three-leaf function.

## 2. Ozaki-Type Bi-Close-to-Convex Function

The class $\mathcal{K}$, knonw as close-to-convex functions, was first formally introduced by Kaplan [32] in 1952. Ozaki [33] had already considered these functions in 1935 for $\mathcal{A}$, satisfying the following condition

$$\Re\left(1 + \frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)}\right) > -\frac{1}{2}, \quad \xi \in \mathbf{\Delta}. \tag{15}$$

It follows from the original definition of Kaplan [5] that functions satisfying (15) are close-to-convex and are, therefore, members of $\mathcal{S}$. These functions are known to be univalent and close-to-convex. Lately, Kargar and Ebadian [34] proposed the following as the generalization of Ozaki's condition (for more details, see [35,36]):

**Definition 3** ([34])**.** *For $-\frac{1}{2} < \wp \leq \frac{1}{2}$, and let $\mathcal{G} \in \mathcal{A}$ be locally univalent. Then, $\mathcal{G} \in \mathcal{A}$ is called an Ozaki close-to-convex function in $\mathbf{\Delta}$ if*

$$\Re\left(1 + \frac{\xi\mathcal{G}'(\xi)}{\mathcal{G}(\xi)}\right) > \frac{1}{2} - \wp, \quad \xi \in \mathbf{\Delta}.$$

Now, we define the Ozaki-type bi-close-to-convex function:

**Definition 4.** *The family $OCV_\Sigma(\wp;\vartheta)$ contains all the functions $\mathcal{G} \in \Sigma$ if the below subordinations are satisfied:*

$$\frac{2\wp-1}{2\wp+1} + \frac{2}{2\wp+1}\left(\frac{((\xi\mathcal{C}^\vartheta_\tau\mathcal{G}(\xi))')'}{(\mathcal{C}^\vartheta_\tau\mathcal{G}(\xi))'}\right) \prec \Lambda(\xi)$$

*and*

$$\frac{2\wp-1}{2\wp+1} + \frac{2}{2\wp+1}\left(\frac{((w\mathcal{C}^\vartheta_\tau\mathcal{H}(w))')'}{(\mathcal{C}^\vartheta_\tau\mathcal{H}(w))'}\right) \prec \Lambda(w),$$

*where $\frac{1}{2} \leq \wp \leq 1$ and $\mathcal{H}(w) = \mathcal{G}^{-1}(w)$.*

**Remark 1.** *By fixing $\wp = \frac{1}{2}$, the family $OCV_\Sigma(\wp;\vartheta) = CV_\Sigma(\vartheta)$ contains all the functions $\mathcal{G} \in \Sigma$ if it satisfies the following:*

$$\left(\frac{((\xi \mathcal{C}_\tau^\vartheta \mathcal{G}(\xi))')'}{(\mathcal{C}_\tau^\vartheta \mathcal{G}(\xi))'}\right) \prec \Lambda(\xi) \quad \text{and} \quad \left(\frac{((w \mathcal{C}_\tau^\vartheta \mathcal{H}(w))')'}{(\mathcal{C}_\tau^\vartheta \mathcal{H}(w))'}\right) \prec \Lambda(w),$$

*where $\mathcal{H}(w) = \mathcal{G}^{-1}(w)$.*

In order to derive our main results, we need the following lemma:

**Lemma 1** ([37])**.** *Let $\mathcal{P}$ be the family of all functions, $h$, which are analytic in $\mathbf{\Delta}$ with $\Re(h(\xi)) > 0$ and are given by*

$$h(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots, \qquad (\xi \in \Delta)$$

*then*

$$|p_k| \leq 2, \forall\, k.$$

For the sake of brevity in notation, we let

$$\Xi_2 = \frac{4\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)}{\Gamma(3+\vartheta-\tau)\Gamma(1-\vartheta)}, \qquad \Xi_3 = \frac{36\Gamma(2+\vartheta-\tau)\Gamma(2-\vartheta)}{\Gamma(4+\vartheta-\tau)\Gamma(4-\vartheta)} \tag{16}$$

and

$$\Lambda(\xi) := 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4 \tag{17}$$

unless otherwise stated.

**Theorem 1.** *Let $\mathcal{G} \in OCV_\Sigma(\wp;\vartheta)$ $(\frac{1}{2} \leq \wp \leq 1)$ and $\mathcal{G}$ have the form (1). Then,*

$$|a_2| \leqq \min\left\{\frac{\sqrt{2}(2\wp+1)}{5\Xi_2}, \frac{\sqrt{2}(2\wp+1)}{\sqrt{5\{2(2\wp+1)(3\Xi_3 - 2\Xi_2^2) + 5\Xi_2^2\}}}\right\}$$

*and*

$$|a_3| \leqq \min\left\{\frac{2\wp+1}{15\Xi_3} + \frac{2(2\wp+1)^2}{25\Xi_2^2}, \frac{(2\wp+1)}{15\Xi_3} + \frac{2(2\wp+1)^2}{5\{2(2\wp+1)(3\Xi_3 - 2\Xi_2^2) + 5\Xi_2^2\}}\right\}$$

*where $\Xi_2$ and $\Xi_3$ are as assumed as per (16).*

**Proof.** Define the functions $p(\xi)$ and $q(\xi)$ by

$$p(\xi) := \frac{1+u(\xi)}{1-u(\xi)} = 1 + b_1\xi + b_2\xi^2 + \cdots$$

and

$$q(\xi) := \frac{1+v(\xi)}{1-v(\xi)} = 1 + c_1\xi + c_2\xi^2 + \cdots.$$

Then, $p(\xi)$ and $q(\xi)$ are analytic in $\Delta$ with $p(0) = 1 = q(0)$. It follows that

$$u(\xi) := \frac{p(\xi)-1}{p(\xi)+1} = \frac{1}{2}\left[b_1\xi + \left(b_2 - \frac{b_1^2}{2}\right)\xi^2 + \cdots\right]$$

and

$$v(\xi) := \frac{q(\xi)-1}{q(\xi)+1} = \frac{1}{2}\left[c_1\xi + \left(c_2 - \frac{c_1^2}{2}\right)\xi^2 + \cdots\right].$$

Since $u, v : \Delta \to \Delta$, the functions $p, q$ have a positive real part, and $|b_i| \leq 2$ and $|c_i| \leq 2$ for each $i$.

Now,

$$\begin{aligned} \Lambda(u(\xi)) &= 1 + \frac{4}{5}u(\xi) + \frac{1}{5}(u(\xi))^4 \\ &= 1 + \frac{2}{5}b_1\xi + \left(\frac{2}{5}b_2 - \frac{1}{5}b_1^2\right)\xi^2 + \left(\frac{1}{10}b_1^3 - \frac{2}{5}b_2b_1 + \frac{2}{5}b_3\right)\xi^3 + \cdots. \end{aligned} \tag{18}$$

and

$$\Lambda(v(w)) = 1 + \frac{2}{5}c_1w + \left(\frac{2}{5}c_2 - \frac{1}{5}c_1^2\right)w^2 + \left(\frac{1}{10}c_1^3 - \frac{2}{5}c_2b_1 + \frac{2}{5}c_3\right)w^3 + \cdots. \tag{19}$$

Assume that $\mathcal{G} \in \mathcal{OCV}_\Sigma(\wp;\vartheta)$ and $\mathcal{H} = \mathcal{G}^{-1}$. Specifically, there exist holomorphic functions $u, v : \Delta \longrightarrow \Delta$, hence,

$$\frac{2\wp - 1}{2\wp + 1} + \frac{2}{2\wp + 1}\left(\frac{((\xi\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi))')'}{(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi))'}\right) = \Lambda(u(\xi)), \quad \text{where} \quad \xi \in \Delta, \tag{20}$$

and

$$\frac{2\wp - 1}{2\wp + 1} + \frac{2}{2\wp + 1}\left(\frac{((w\mathcal{C}_\tau^\vartheta\mathcal{H}(w))')'}{(\mathcal{C}_\tau^\vartheta\mathcal{H}(w))'}\right) = \Lambda(v(w)), \quad \text{where} \quad w \in \Delta. \tag{21}$$

From (20) and (21), we deduce that

$$\begin{aligned} \frac{2\wp - 1}{2\wp + 1} + \frac{2}{2\wp + 1}\left(\frac{((z\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi))')'}{(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi))'}\right) &= 1 + \tfrac{4}{5}u(\xi) + \tfrac{1}{5}(u(\xi))^4 \\ &= 1 + \tfrac{2}{5}b_1\xi + \left(\tfrac{2}{5}b_2 - \tfrac{1}{5}b_1^2\right)\xi^2 + \left(\tfrac{1}{10}b_1^3 - \tfrac{2}{5}b_2b_1 + \tfrac{2}{5}b_3\right)\xi^3 + \cdots, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \frac{2\wp - 1}{2\wp + 1} + \frac{2}{2\wp + 1}\left(\frac{((z\mathcal{C}_\tau^\vartheta g(w))')'}{(\mathcal{C}_\tau^\vartheta g(w))'}\right) &= 1 + \tfrac{4}{5}v(w) + \tfrac{1}{5}(v(w))^4 \\ &= 1 + \tfrac{2}{5}c_1w + \left(\tfrac{2}{5}c_2 - \tfrac{1}{5}c_1^2\right)w^2 + \left(\tfrac{1}{10}c_1^3 - \tfrac{2}{5}c_2b_1 + \tfrac{2}{5}c_3\right)w^3 + \cdots. \end{aligned} \tag{23}$$

Equating the coefficients in (22) and (23) yields

$$\frac{4}{2\wp + 1}\Xi_2a_2 = \frac{2}{5}b_1, \tag{24}$$

$$\frac{12}{2\wp + 1}\Xi_3a_3 - \frac{8}{2\wp + 1}\Xi_2^2a_2^2 = \frac{2}{5}b_2 - \frac{1}{5}b_1^2, \tag{25}$$

$$-\frac{4}{2\wp + 1}\Xi_2a_2 = \frac{2}{5}c_1 \tag{26}$$

and

$$\frac{12}{2\wp + 1}(2a_2^2 - a_3)\Xi_3 - \frac{8}{2\wp + 1}\Xi_2^2a_2^2 = \frac{2}{5}c_2 - \frac{1}{5}c_1^2. \tag{27}$$

From (24) and (26), we have

$$b_1 = -c_1 \tag{28}$$

and

$$\begin{aligned}\frac{32}{(2\wp+1)^2}\Xi_2^2a_2^2 &= \frac{8}{25}(b_1^2+c_1^2)\\ \frac{100}{(2\wp+1)^2}\Xi_2^2a_2^2 &= b_1^2+c_1^2 &(29)\\ a_2^2 &= \frac{(2\wp+1)^2}{100\Xi_2^2}(b_1^2+c_1^2). &(30)\end{aligned}$$

According to triangular inequality, we have

$$|a_2|^2 \leq \frac{|2\wp+1|^2}{|100\Xi_2^2|}(|b_1|^2+|c_1|^2)$$

By applying Lemma 1 for the coefficients $b_1, c_1$, in (30) , we obtain

$$|a_2| \leq \frac{\sqrt{2}(2\wp+1)}{5\Xi_2}.$$

If we add (25) to (27), we obtain

$$\frac{8(3\Xi_3-2\Xi_2^2)}{2\wp+1}a_2^2 = \frac{2}{5}(b_2+c_2) - \frac{1}{5}(b_1^2+c_1^2). \tag{31}$$

By substituting from (29) the value of $b_1^2+c_1^2$ in the relation (31), we deduce that

$$a_2^2 = \frac{(2\wp+1)^2(b_2+c_2)}{10\{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2\}}. \tag{32}$$

By applying Lemma 1 for the coefficients $b_2, c_2$ in (32), we obtain

$$|a_2| \leqq \frac{\sqrt{2}(2\wp+1)}{\sqrt{5\{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2\}}}.$$

By subtracting (25) from the relation (27) and applying $b_1^2 = c_1^2$, (28) we obtain $|a_3|$. Hence, this yields

$$\begin{aligned}\frac{24\Xi_3}{2\wp+1}(a_3-a_2^2) &= \frac{2}{5}(b_2-c_2),\\ a_3 &= \frac{(2\wp+1)(b_2-c_2)}{60\Xi_3}+a_2^2 &(33)\end{aligned}$$

By applying Lemma 1 to the coefficients $b_2, c_2$, and then by using (30) in (33), we have

$$|a_3| \leqq \frac{2\wp+1}{15\Xi_3} + \frac{2(2\wp+1)^2}{25\Xi_2^2}.$$

Additionally, from (32) and (33), we get

$$a_3 = \frac{(2\wp+1)(b_2-c_2)}{60\Xi_3} + \frac{(2\wp+1)^2(b_2+c_2)}{10\{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2\}}$$

and we have

$$|a_3| \leqq \frac{(2\wp+1)}{15\Xi_3} + \frac{2(2\wp+1)^2}{5\{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2\}}.$$

□

*Fekete-Szegö Problem*

By utilizing $a_2^2$ and $a_3$ values and motivated by Zaprawa's recent work [38], as given in the below lemma, we prove the Fekete-Szegö problem for $\mathcal{G} \in \mathcal{OCV}_\Sigma(\wp;\vartheta)$ in the following theorem.

**Lemma 2** ([38])**.** *Let $l_1, l_2 \in \mathbb{R}$ and $p_1, p_2 \in \mathbb{C}$. If $|p_1|, |p_2| < \zeta$, then*

$$|(l_1+l_2)p_1 + (l_1-l_2)p_2| \leq \begin{cases} 2|l_1|\zeta & , \quad |l_1| \geq |l_2| \\ 2|l_2|\zeta & , \quad |l_1| \leq |l_2| \end{cases}$$

**Theorem 2.** *For $\frac{1}{2} \leq \wp \leq 1; \hbar \in R$, and let $\mathcal{G} \in OCV_\Sigma(\wp;\vartheta)$ be of the form (1). Then,*

$$\left|a_3 - \hbar a_2^2\right| \leq \begin{cases} \frac{2\wp+1}{15\Xi_3}, & 0 \leqq |\phi(\hbar,\wp)| \leq \frac{2\wp+1}{6\Xi_3}, \\ \\ \frac{2}{5}|\phi(\hbar,\xi)|, & |\phi(\hbar,\wp)| \geq \frac{2\wp+1}{6\Xi_3}. \end{cases}$$

**Proof.** It follows from (32) and (33) that

$$\begin{aligned} a_3 - \hbar a_2^2 &= \frac{(2\wp+1)(b_2-c_2)}{60\Xi_3} + (1-\hbar)a_2^2 \\ &= \frac{(2\wp+1)(b_2-c_2)}{60\Xi_3} + (1-\hbar)\frac{(2\wp+1)^2(b_2+c_2)}{10\{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2\}} \\ &= \frac{1}{10}\left[\left(\phi(\hbar,\wp) + \frac{2\wp+1}{6\Xi_3}\right)b_2 + \left(\phi(\hbar,\wp) - \frac{2\wp+1}{6\Xi_3}\right)c_2\right], \end{aligned}$$

where

$$\phi(\hbar,\wp) = \frac{(2\wp+1)^2(1-\hbar)}{2(2\wp+1)(3\Xi_3-2\Xi_2^2)+5\Xi_2^2}.$$

According to Lemma 2, we obtain

$$\left|a_3 - \hbar a_2^2\right| \leqq \begin{cases} \frac{2\wp+1}{15\Xi_3}, & 0 \leq |\phi(\hbar,\wp)| \leq \frac{2\wp+1}{6\Xi_3}, \\ \\ \frac{2}{5}|\phi(\hbar,\xi)|, & |\phi(\hbar,\wp)| \geq \frac{2\wp+1}{6\Xi_3}. \end{cases}$$

□

By fixing $\hbar = 1$ in Theorem 2, we obtain the following result:

**Corollary 1.** *If $\mathcal{G} \in OCV_\Sigma(\wp;\vartheta)$ is as given in (1), then*

$$\left|a_3 - a_2^2\right| \leqq \frac{2\wp+1}{15\Xi_3}.$$

## 3. A New Class of Bi-Concave Functions

The function $\mathcal{G} : \Delta \to \mathbb{C}$ is said to belong to the family of concave functions $\mathcal{CV}_0(\alpha)$, if $\mathcal{G}$ satisfies the following conditions:

1. $\mathcal{G} \in \mathcal{A}$ with $\mathcal{G}(0) = \mathcal{G}'(0) - 1 = 0$;
2. $\mathcal{G}$ maps $\Delta$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex;
3. The opening angle of $\mathcal{G}(\Delta)$ at $\infty$ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class of concave univalent functions is referred to by $\mathcal{CV}_0(\alpha)$ (see [39,40]) if the following inequality holds:

$$\Re\left(1+\frac{\xi\,\mathcal{G}''(\xi)}{\mathcal{G}'(\xi)}\right)<0,\quad (\xi\in\Delta).$$

Bhowmik et al. [41] showed that the function $\mathcal{G}\in\mathcal{A}$ maps $\Delta$ onto a concave domain of angle $\pi\alpha$ if and only if $\Re(P_{\mathcal{G}}(\xi))>0$, where

$$P_{\mathcal{G}}(\xi)=\frac{2}{\alpha-1}\left[\frac{(\alpha+1)(1+\xi)}{2(1-\xi)}-1-\frac{\xi\,\mathcal{G}''(\xi)}{\mathcal{G}'(\xi)}\right].$$

Numerous studies have been conducted on the fundamental subclasses of concave univalent functions (see [39,42–47]).

Motivated by the mentioned results, this is the first time we define a new subclass of bi-concave functions $\mathcal{BCV}_\tau^\vartheta(\lambda)$ associated with the three-leaf domain:

**Definition 5.** *Let $\mathcal{G}\in\mathcal{A}$ have the form (1) if it is said to be in the class $\mathcal{BCV}_\tau^\vartheta(\lambda)$ if it satisfies the following conditions:*

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\xi)}{2(1-\xi)}-1-\frac{\xi\,\left(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi)\right)''}{\left(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi)\right)'}\right]\prec\Lambda(\xi),\tag{34}$$

*and*

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+w)}{2(1-w)}-1-\frac{w\,\left(\mathcal{C}_\tau^\vartheta\mathcal{H}(w)\right)''}{\left(\mathcal{C}_\tau^\vartheta\mathcal{H}(w)\right)'}\right]\prec\Lambda(w),\tag{35}$$

*with $\lambda\in(1,2]$ and $\mathcal{H}(w)=\mathcal{G}^{-1}(w)$.*

**Theorem 3.** *Let $\mathcal{G}\in\mathcal{A}$, as in the form (1). If $\mathcal{G}\in\mathcal{BCV}_\tau^\vartheta(\lambda)$, then*

$$|a_2|\le\min\left\{\sqrt{\frac{13\lambda^2+30\lambda+17}{20\Xi_2^2}+\frac{(\lambda-1)^2}{25\Xi_2^2}};\ \sqrt{\frac{\lambda}{|2\Xi_2^2-3\Xi_3|}}\right\};$$

*and*

$$|a_3|\le\min\left\{\frac{13\lambda^2+30\lambda+17}{20\Xi_2^2}+\frac{4(\lambda-1)^2}{25\Xi_2^2}+\frac{\lambda-1}{6\Xi_2},\ \frac{\lambda}{(2\Xi_2^2-3\Xi_3)}+\frac{\lambda-1}{6\Xi_3}\right\},\tag{36}$$

*where $\lambda\in(1,2]$, $\Xi_2$ and $\Xi_3$ are as assumed as per (16).*

**Proof.** If $\mathcal{G}\in\mathcal{BCV}_\tau^\vartheta(\lambda)$ from (34) and (35), it follows that

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+\xi)}{2(1-\xi)}-1-\frac{\xi\,\left(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi)\right)''}{\left(\mathcal{C}_\tau^\vartheta\mathcal{G}(\xi)\right)'}\right]=\Lambda(u(\xi)),\tag{37}$$

and

$$\frac{2}{\lambda-1}\left[\frac{(1+\lambda)(1+w)}{2(1-w)}-1-\frac{w\,\left(\mathcal{C}_\tau^\vartheta\mathcal{H}(w)\right)''}{\left(\mathcal{C}_\tau^\vartheta\mathcal{H}(w)\right)'}\right]=\Lambda(v(w)).\tag{38}$$

By equalizing the coefficients of $\xi$ and $w$ in (37) and (38), it is obvious that

$$\frac{2[(1+\lambda)-2\Xi_2 a_2]}{\lambda-1}=\frac{2}{5}b_1, \tag{39}$$

$$\frac{2\left[(1+\lambda)+4\Xi_2^2 a_2^2-6\Xi_3 a_3\right]}{\lambda-1}=\frac{2}{5}b_2-\frac{1}{5}b_1^2, \tag{40}$$

$$-\frac{2[(1+\lambda)-2\Xi_2 a_2]}{\lambda-1}=\frac{2}{5}c_1, \tag{41}$$

and

$$\frac{2\left[(\lambda+1)+4\Xi_2^2 a_2^2-6\Xi_3\left(2a_2^2-a_3\right)\right]}{\lambda-1}=\frac{2}{5}c_2-\frac{1}{5}c_1^2. \tag{42}$$

By using (39) and (41), we obtain

$$b_1=-c_1, \tag{43}$$

and from (39), we can write

$$\begin{aligned} a_2 &= \frac{(\lambda+1)}{2\Xi_2}-\frac{(\lambda-1)}{10\Xi_2}b_1. \\ |a_2| &\leq \frac{(\lambda+1)}{2\Xi_2}+\frac{|\lambda-1|}{5\Xi_2} \end{aligned} \tag{44}$$

When squaring (39) and (41) after adding the relations, we obtain

$$a_2^2=-\frac{3(\lambda+1)^2}{4\Xi_2^2}+\frac{\left(\lambda^2-1\right)}{10\Xi_2^2}+\frac{(\lambda-1)^2\left(b_1^2+c_1^2\right)}{200\Xi_2^2}. \tag{45}$$

$$|a_2^2|\leq\frac{13\lambda^2+30\lambda+17}{20\Xi_2^2}+\frac{(\lambda-1)^2}{25\Xi_2^2}. \tag{46}$$

Thus, we get

$$|a_2|\leq\sqrt{\frac{13\lambda^2+30\lambda+17}{20\Xi_2^2}+\frac{(\lambda-1)^2}{25\Xi_2^2}},$$

By adding (40) and (42), we have

$$a_2^2=\frac{1}{8\left(2\Xi_2^2-3\Xi_3\right)}((\lambda-1)(b_2+c_2)-4(\lambda+1)). \tag{47}$$

$$|a_2^2| \leq \frac{1}{8|2\Xi_2^2-3\Xi_3|}|(\lambda-1)(b_2+c_2)-4(\lambda+1)| \tag{48}$$

$$= \frac{\lambda}{|2\Xi_2^2-3\Xi_3|}, \tag{49}$$

which gives the bound for $|a_2|$, as we asserted in our theorem.

In order to find the bound for $|a_3|$, by subtracting (42) from (40), we obtain

$$a_3=a_2^2-\frac{(\lambda-1)(b_2-c_2)}{24\Xi_2}. \tag{50}$$

Additionally, upon substituting the value of $a_2^2$ in view of (45) and (47) in (50), we obtain

$$a_3=-\frac{13\lambda^2+30\lambda+17}{20\Xi_2^2}+\frac{(\lambda-1)^2\left(b_1^2+c_1^2\right)}{200\Xi_2^2}-\frac{(\lambda-1)(b_2-c_2)}{24\Xi_2}, \tag{51}$$

and

$$a_3 = \frac{1}{8\left(2\Xi_2^2 - 3\Xi_3\right)}\left((\lambda-1)(b_2+c_2) - 4(\lambda+1)\right) - \tfrac{(\lambda-1)(b_2-c_2)}{24\Xi_3}. \tag{52}$$

By taking the absolute value of (51) and (52), we obtain

$$|a_3| \le \frac{13\lambda^2 + 30\lambda + 17}{20\Xi_2^2} + \frac{4(\lambda-1)^2}{25\Xi_2^2} + \frac{|\lambda-1|}{6\Xi_2},$$

and

$$|a_3| \le \frac{\lambda}{\left(2\Xi_2^2 - 3\Xi_3\right)} + \frac{\lambda-1}{6\Xi_3}.$$

This concludes the theorem's proof. □

## 4. Conclusions

The research described here deals with two novel subclasses of Ozaki-type bi-close-to-convex functions ($\mathcal{OCV}_\Sigma(\wp;\vartheta)$) and bi-concave functions ($\mathcal{BCV}_\tau^\vartheta(\lambda)$) that involve three-leaf functions in the open unit disc and modified Caputo's fractional operator. This is the first time we have obtained the initial coefficients $|a_2|$ and $|a_3|$ for the functions that are part of the classes $\mathcal{OCV}_\Sigma(\wp;\vartheta)$ and $\mathcal{BCV}_\tau^\vartheta(\lambda)$. In addition, the Fekete-Szegö inequalities have been studied for $\mathcal{G} \in \mathcal{OCV}_\Sigma(\wp;\vartheta)$. By fixing the parameters, a number of novel consequences from the findings can be highlighted as corollaries. These subclasses of functions have not been previously examined in the literature in relation to any of the geometrical domains mentioned in [10,25–31] associated with modified Caputo's fractional operator.

We conclude that one can determine new results for various new subclasses of $\mathcal{OCV}_\Sigma(\wp;\vartheta)$ and $\mathcal{BCV}_\tau^\vartheta(\lambda)$ by selecting a specific $\Lambda$ related to the series associated with telephone numbers, Van der Pol Numbers (VPNs), Gregory coefficients, and rational functions (see [11,12,48]). In conclusion, considering the tremendous amount of research embedding $q$-calculus into geometric function theory (see, for example, [49] and the references stated therein), this research could inspire future $q$-calculus developments for the above-mentioned classes by applying some obvious parametric and argument variations.

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