fractal and fractional

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Controllability of Mild Solution to Hilfer Fuzzy Fractional Differential Inclusion with Infinite Continuous Delay

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Abstract: This work investigates the solvability of the generalized Hilfer fractional inclusion associated with the solution set of a controlled system of minty type–fuzzy mixed quasi-hemivariational inequality (FMQHI). We explore the assumed inclusion via the infinite delay and the semi-group arguments in the area of solid continuity that sculpts the compactness area. The conformable Hilfer fractional time derivative, the theory of fuzzy sets, and the infinite delay arguments support the solution set's controllability. We explain the existence due to the convergence properties of Mittage–Leffler functions ($\mathbb{E}_{\alpha,\beta}$), that is, hatching the existing arguments according to FMQHI and the continuity of infinite delay, which has not been presented before. To prove the main results, we apply the Leray–Schauder nonlinear alternative thereom in the interpolation of Banach spaces. This problem seems to draw new extents on the controllability field of stochastic dynamic models.

Keywords: Hilfer fuzzy multi-valued operator; existence and uniqueness; controllability; infinite continuous delay; mild solution

MSC: 26A33; 34A08; 34A12

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1. Introduction

Strong theoretical visions via fractional calculus are some of the most significant ways to describe natural models. To our knowledge, fractional calculus is a vast field with many kinds of fractional differential operators and corresponding integrals. For examples, see [1–3]. One way to explain the importance of fractional calculus is by generating classical calculus, which is insufficient for modeling natural phenomena. The time-fractional operators substantially draw some fantastic results in the fractal topics field for modelings with memory. Here, we are interested in representing some medical and physical studies in [4–7]. In particular, see the usefulness of Hilfer fractional operators in papers on some different diseases [8–10].

The importance of controlled systems has arisen from Zadeh's work [11] with fuzzy sets used to understand the behavior of objects with a fractional grade of membership between zero and one. On the other hand, the controllability of fractional operators through the solvability region of mixed quasi-hemivariational inequalities has attracted attention to investigating and updating more results (in particular, with fuzzy sets). Here, we refer to [12,13]. In 2021, N. V. Hung [14] gave us strong and more worthwhile results on the generalization of Levitin–Polyak well-posedness for controlled systems of minty type–fuzzy mixed quasi-hemivariational inequalities (FMQHI). For more readings, it is worth looking into the engineering, mechanics, and economics literature as well, for example, [15–20].

Among the most robust theories that support stochastic modeling are differential inclusion theories and the continuous infinite delay ones. Many scientific teams have been conducting research on this topic in many different scientific fields. For examples, see [21,22].

Some researchers have presented several results by modeling with control, stochastic, delay, and memory systems. Many kinds are found in [12,23,24] and the references therein.

Y. Jiang et al. [13] prove the solvability theory of mild solution sets for multi-valued Caputo fractional differential initial problems with hemivariational inequality (HVI) with Clarke generalized directional derivatives.

$$\begin{aligned}
&D_c^{\alpha} x(t) \in Ax(t) + F(x(t), u(t)),\ t \in [0,b],\ \alpha \in [0,1] \\
&u(t) \in Solu(HVI), \\
&x(0) = \psi,
\end{aligned}$$

where A is the infinitesimal generator of a norm-continuous and uniformly bounded C_0 semi-group $\{K(\rho)\}_{\rho \geq 0}$ and F is a multi-valued map.

X. Pang et al. [25] presented the mild solution of Hilfer differential inclusion under the solvability constraints of variational–hemivariational inequality (VHVI).

$$\begin{aligned}
&D^{\nu,\mu} x(t) \in Ax(t) + (Rx)(t) + F(x(t), u(t)),\ t \in [0,b], \\
&u(t) \in Solu(VHVI), \\
&I_0^{(1-\nu)(1-\mu)} = x_0,\ \nu \in [0,1],\ \mu \in (0,1),
\end{aligned}$$

where A represents the infinitesimal generator of a norm-continuous and uniformly bounded C_0 semi-group $\{K(\rho)\}_{\rho \geq 0}$ and F is a multi-valued map. R is a history-dependent operator and for the order $(1-\nu)(1-\mu)$, I_0 defines the fractional order Riemann–Liouville integral.

N. T. V. Anh [26] focussed on the solvability of optimal control Caputo-fractional problems with HVI, Clarke-type subdifferentials and nonlocal initial conditions

$$\begin{aligned}
&D_c^{\alpha} z(\tau) \in Az(\tau) + F(z(\tau), w(\tau)) + \partial G(\tau,\ z(\tau),\ w(\tau)) + Bv(\tau), \\
&\tau \in [0,a],\ \alpha \in [0,1] \\
&w(t) \in Solu(HVI), \\
&z(0) = z_0 + \psi(z), \\
&v \in V_{ad},
\end{aligned}$$

where V_{ad} is an admissible control set of $v(.)$ and A denotes the infinitesimal generator of a norm-continuous and uniformly bounded C_0 semi-group $\{K(\tau)\}_{\tau \geq 0}$. F, G are single-valued maps and ∂G represents the Clarke-type generalized subdifferential operator of G. B is a bounded linear operator.

The new work comes to define the area that produces the data of fractional order derivatives with orders between zero and one associated with the one of fuzzy sets with grades with the same property. We suggest the fractional differential inclusion concerned with the generalized conformable Hilfer fractional operator depending on $\tau \in \mathbb{R}$ and $\alpha \in (0,1]$ with $\tau + \alpha \neq 0$ [2,27]. That will be under the effect of infinite continuous delay. The reason for choosing this derivative is apparent if we know its benefits in describing control and diffusive systems and its decent iterating behavior in the order data $\alpha \in [0,1]$. This fact was explored in [27] as a conformable fractional derivative. This type has the ability as a measure to show different straight lines and planes drawing specific curves and surfaces. A. Has et al. [28] have produced an excellent study on the physical and geometric implications of the conformable type of derivatives talking about the attainability of approximating the tangent, which is not available with the classical type. We can overcome this limitation through the use of substitutional tangents. In addition, conformable derivatives are definable even if the tangent plane is undefined. For a general vision, the conformable tangent aircraft is available for all points containing points with undefined derivatives. On the other hand, the Hilfer derivative was presented as a generalization of Hilfer–Hadamard, Hilfer–Katugampola, Caputo–Hadamard, Riemann–Liouville, Hadamard, Hilfer, Caputo, etc., into single-form derivatives that draw a massive field of natural applications. For more details, see [3].

Furthermore, we consider the mild solution set of the suggested inclusion in the solvability region of FMQHI of minty type endowed with the Clarke-type generalized directional derivative.

The problem considered here will be helpful in modeling heterogeneous natural systems with memory.

2. Setting of the Problem

Let $\mathcal{F}(E) = \{\omega \in E | \omega : E \to [0,1]\}$ be the family of all fuzzy sets over a given Banach space E. Then,

(i) $\varpi : E \to \mathcal{F}(E)$ is called fuzzy mapping, for all ϖ.

(ii) $\varpi(r)$ is fuzzy set on E, $\forall r \in E$ and consequently $\omega_{\varpi(r)}(\rho)$ denotes the membership function of ρ in $\varpi(r)$.

(iii) Define by the set $M_\gamma = \{w \in E | \omega_M(w) \geq \gamma\}$, the $\gamma-$ cut set of M for all $M \in \mathcal{F}(E)$, $\gamma \in [0,1]$.

Let W, C be two reflexive Banach spaces, $K \subset W$ be a nonempty closed subset, and $L(W,C)$ be the space of all linear continuous functions. Let Z be the control reflexive Banach space and $U \subset Z$ be the set of all admissible controls which is nonempty and closed. Let $S : K \to \mathcal{F}(K)$ and $P : K \to \mathcal{F}(L(W,C))$ be fuzzy mappings and $\jmath : K \times K \to \mathbb{R}$ be a given locally Lipschitz function. Let $h : L(W,C) \times K \times K \to \mathbb{R}$, satisfying

(1) $h(v, w_2 - w_1, u) = -h(v, w_1 - w_2, u)$,

(2) $h(v, w - w, u) = 0$,

for all $w, w_1, w_2 \in K$, $v \in L(W,C)$, $u \in U$ and let $f : K \times K \to \mathbb{R}$ be a function satisfying

$$f(w,w) = 0, \ \forall w \in K.$$

We want to study Hilfer fuzzy-type fractional differential inclusion defined by

$$\begin{aligned}
&{}^{\delta}_{GH}D^{\beta,\theta}_{a^+} x(t) \in \mathcal{A}x(t) + \Pi(t, x(t), x_t, \mathbb{H}^u), \qquad t \in [a,T],\ a > 0\ \theta, \beta \in [0,1], && (1)\\
&{}_{\delta}\mathcal{H}_a^{(1-\theta)(1-\beta)} x(a) = \frac{c\Gamma(\gamma)}{\Gamma(\omega+\gamma)} \left(\frac{t^\delta - a^\delta}{\delta} \right)^{\omega}, && (2)\\
&0 < \omega < 1,\ \beta + \omega = 1,\ \gamma = \beta + \theta(1-\beta)\\
&x(t) = \psi(t), \qquad t \in [a-\sigma, a], && (3)
\end{aligned}$$

where $\psi(a) = 0$, $\delta = \tau + \alpha$, $\tau \in \mathbb{R}$, $\alpha \in [0,1]$ and $\tau + \alpha \neq 0$,

$$x_t(r) = x(t+r), \qquad r \in [-\sigma, 0],\ \sigma \in [a,T),$$

${}^{\delta}_{GH}D^{\beta,\theta}_{a^+}$, ${}_{\delta}\mathcal{H}^{\gamma}_a$ denote the generalized Hilfer-type fractional derivative and integral, respectively, that their definitions are given later, in Section 3.2. $\mathcal{A}$ denotes a generator of compact C_0 semi-groups and $\mathbb{H}^u$ defines solutions collection of the minty type FMQHI-controlled system written as follows.

FMQHI: Find $w_1 \in K \cap S(w_1)_\beta$ such that

$$h(v, w_2 - w_1, u) + \jmath^0(w_1, w_2 - w_1) + f(w_2, w_1) \geq 0, \qquad \forall v \in P(w_2)_\theta,\ \forall w_2 \in S(w_1)_\beta, \quad (4)$$

where $\jmath^0$ denotes the generalized directional derivative of Clarke type for the function $\jmath$ at the point $w_1 \in K$ in the direction of $w_2 - w_1$ given by the relation

$$\jmath^0(u,v) = \lim_{k \to u} \sup_{\lambda \to 0^+} \frac{\jmath(k + \lambda v) - \jmath(k)}{\lambda},\ u = w_1,\ v = w_2 - w_1.$$

$u(t)$ is a control function and $S(w_1)_\beta$, $P(w_2)_\theta$ are defined, respectively, by

a: $S(w_1)_\beta = \left\{g \in K | \mu_{S(w_1)}(g) \geq \beta\right\}$,

b: $P(w_2)_\theta = \left\{g \in K | \mu_{P(w_2)}(g) \geq \theta\right\}$.

3. Preliminaries and Auxiliary Statements

The main results depend on five important concepts: Hilfer fractional operators, C_0 semi-groups, infinite delay, multi-valued operators, and FMQHI. So, we must present some basic definitions and facts related to these concepts.

3.1. FMQH Inequalities

From [14], we can see the following

Lemma 1. *Let $u \in U$ and $w_1 \in K \cap S(w_1)_\beta$. Then,*

$$\mathbb{H}^u = \{w_1 \in K \cap S(w_1)_\beta \| h(v, w_2 - w_1, u) + \jmath^0(w_1, w_2 - w_1) + f(w_2, w_1) \geq 0 \\ , \forall w_2 \in S(w_1)_\beta,\ v \in P(w_2)_\theta\}$$

is a nonempty set.

Proof. Since $w_1 \in K \cap S(w_1)_\beta$, then $w_1 \in K$ and $w_1 \in S(w_1)_\beta$. Taking $w_2 = w_1 = w$ implies that $h(v, w - w, u) = 0,$ *and* $f(w, w) = 0$. Since $\|\jmath^0(w, e)\| \leq L_e \|w\|$ if $e = 0$, then $\jmath^0(w, 0) = 0$. While $S(w_1)_\beta$ is nonempty and $w_2 = w_1 \in S(w_1)_\beta$ exists and satisfies (4), then $\mathbb{H}^u$ is a nonempty set. □

Definition 1. *Let $\mathbb{H}^u$ be the solution set of FMQHI. If*

(i) *$\mathbb{H}^u$ is nonempty,*

(ii) *every **LP** approximating sequence $\{x_n\}$ for FMQHI has a subsequence which converges to some points of $\mathbb{H}^u$,*

*then we say that FMQHI is **LP** well posed in the generalized sense.*

Let us define the approximate solution set of FMQHI by

$$\tilde{\mathbb{H}}^u(\epsilon) = \{x \in K \cap B(S(x)_\beta,\ \epsilon) |\ h(v, y - x, u) + \jmath^0(x, y - x) + f(y, x) + \epsilon \geq 0, \\ \forall y \in S(x)_\beta,\ \forall v \in P(y)_\theta\},$$

for arbitrary positive real numbers $\epsilon \geq 0$.

We can see clearly that $\forall\ \epsilon \geq 0$, $\tilde{\mathbb{H}}^u(0) = \mathbb{H}^u$ and $\mathbb{H}^u \subset \tilde{\mathbb{H}}^u(\epsilon)$.

The following Lemma has been proved in [29].

Lemma 2. *Consider that both Banach spaces $W,\ O$ are reflexive. Let $K \subset W$ be a nonempty closed subset and*

$$L(W,\ O) = \{\eta | \eta : W \to O,\ \eta \text{ is linear continuous operator}\}.$$

Define the control–reflexive Banach space by Z and assume the nonempty closed subset $U \subset Z$ to be the collection of admissible controls. Suppose two fuzzy mappings $S : K \to \mathcal{F}(K)$ and $P : K \to \mathcal{F}(L(W,\ O))$ and a locally Lipschitz function $\jmath$. Let both functions $h : L(W,\ O) \times K \times U \to \mathbb{R}$ and $f : K \times K \to \mathbb{R}$ be given. If

(i) *S is topologically closed on K and $w \to S(w)_\beta$ is l.s.c set-valued mapping with nonempty compact values on K;*

(ii) *P is l.s.c;*

(iii) *$\jmath$ is a locally Lipschitz function and f is u.s.c;*

(iv) *$\forall u \in U,\ h(,\,,\ u)$ is u.s.c.*

then, $\mathbb{H}^u$ is a compact set for all $u \in U$. Furthermore, $\tilde{\mathbb{H}}^u$ is u.s.c at 0 and for all $\epsilon > 0$, $\tilde{\mathbb{H}}^u(\epsilon)$ is compact.

Let $\tilde{S},\ \tilde{P}$ both be set-valued mappings defined, respectively, as follows

(i) $\tilde{S} : K \to \mathcal{F}(K)$ formed by

$$\tilde{S}(w) = S(w)_\beta,\ \forall w \in K, \tag{5}$$

(ii) $\tilde{P} : K \to \mathcal{F}(L(W, O))$ formed by

$$\tilde{P}(r) = P(r)_\theta,\ \forall r \in K. \tag{6}$$

Consequently, we have the following Lemma

Lemma 3. *Consider that both Banach spaces $W,\ O$ are reflexive. Let $K \subset W$ be a nonempty closed subset and*

$$L(W,\ O) = \{\eta | \eta : W \to O,\ \eta \text{ is linear continuous operator}\}.$$

Define the control reflexive Banach space by Z and assume the nonempty closed subset $U \subset Z$ to be the collection of admissible controls. Suppose two fuzzy mappings $S : K \to \mathcal{F}(K)$ and $P : K \to \mathcal{F}(L(W,\ O))$ and a locally Lipschitz function $\jmath$. Let both functions $h : L(W,\ O) \times K \times U \to \mathbb{R}$ and $f : K \times K \to \mathbb{R}$ be given. Suppose the following conditions

(i) $\tilde{S} : K \to \mathcal{F}(K)$ is a compact continuous set-valued mapping defined by (5);
(ii) $\tilde{P} : K \to \mathcal{F}(L(W, O))$ is an l.s.c set-valued mapping defined by (6);
(iii) $\jmath$ and f are, respectively, locally Lipschitz and u.s.c functions;
(iv) $h(,,u)$ is u.s.c for each $u \in U$.

*Then, the sufficient and necessary condition for FMQHI to be **LP** well-posed in the generalized sense is that $\mathbb{H}^u$ is a nonempty set.*

Proof. See [14]. □

Definition 2. *For FMQHI-controlled systems, we say that $g : K \to \mathbb{R}$ is a gap function if*

(i) $g(w) \geq 0,\ \forall w \in \tilde{S}(w)$;
(ii) The two sentences $g(w) = 0$ and $w \in \mathbb{H}^u$ are equivalent.

Lemma 4. *Suppose that $\tilde{S}$ and $\tilde{P}$ have compact values in a neighborhood of the reference point. The function $g : D(K) \to \mathbb{R}$, where $D(K) = \bigcup_{w \in K} D(w) = \bigcup_{w \in K} \{w \in K : w \in \tilde{S}(w)\}$ defined by*

$$g(w) = \max_{r \in \tilde{S}(r)} \max_{v \in \tilde{P}(r)} \{h(v,\ w - r, u) - \jmath^0(w,\ r - w) - f(r,\ w)\}. \tag{7}$$

is a gap function for FMQHI-controlled systems. Moreover, the sentences $g(w^*) = 0$ and $w^* \in \mathbb{H}^u$ are equivalent.

Proof. See [14]. □

Lemma 5. *For FMQHI, g is continuous in K if*

(i) $\tilde{S}$ and $\tilde{P}$ are compact continuous set-valued maps;
(ii) f is continuous;
(iii) J is a locally Lipschitz function;
(iv) h is continuous, $\forall u \in U$.

Proof. See [14]. □

Lemma 6. *Assume that $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a real-valued function satisfying*

$$\phi(r, s) \geq 0,\ \forall r,\ s \geq 0, \phi(0, 0) = 0; \tag{8}$$

$$s_n \to 0,\ r_n \geq 0,\ \phi(r_n,\ s_n) \to 0 \text{ imply } r_n \to 0. \tag{9}$$

We can find a function ϕ satisfying (8) and (9) for which

$$|g(x)| \geq \phi(d(x, \mathbb{H}^u), d(x, \tilde{S}(x))), \forall x \in K. \tag{10}$$

*if $\mathbb{H}^u$ is **LP** well-posed in the generalized sense.*

Proof. See [14]. □

3.2. Fractional Calculus

Definition 3 (Conformable Integrable Function). *Let $[a_1, a_2] \subset [0, \infty)$, $0 < \alpha \leq 1$ and $\tau \in \mathbb{R}$ with $\tau + \alpha \neq 0$. Let $x \in L_\alpha[a_1, a_2] = \{x(t) : \int_{a_1}^{a_2} x(\rho) d_\alpha \rho < \infty\}$ where $d_\alpha \rho = d(\rho^\alpha)$. Then, the operator $K_{a_1}^{\tau,\alpha} : L_\alpha[a_1, a_2] \to \mathbb{R}$ given by*

$$K_{a_1}^{\tau,\alpha} = \int_{a_1}^{t} x(\rho) \rho^\tau d_\alpha \rho$$

represents a conformable fractional integral.

Definition 4 (Generalized Conformable (GC) Integrable Function). *For an order $\beta > 0$, the left-side GC fractional integral ${}_\delta\mathcal{H}_{a_1^+}^{\beta}$ with $0 < \alpha \leq 1$, $\tau \in \mathbb{R}$ and $\delta = \tau + \alpha \neq 0$ is defined by*

$${}_\delta\mathcal{H}_{a_1^+}^{\beta}(x)(t) = \frac{1}{\Gamma(\beta)} \int_{a_1}^{t} \left(\frac{t^\delta - \rho^\delta}{\delta} \right)^{\beta - 1} \rho^{\delta - 1} x(\rho) d\rho,$$

for all conformable type integrable functions x on the interval $[a_1, a_2] \subset [0, \infty)$.

Definition 5 (Generalized Hilfer-type (GH) fractional derivative). *Let $\beta \in (0,1)$, $\theta \in [0,1]$, $\tau \in \mathbb{R}$ and $0 < \alpha \leq 1$ such that $\delta = \tau + \alpha \neq 0$. For a conformable integrable function x on the interval $[a_1, a_2] \subset [0, \infty]$, the left-side GH fractional derivative operator of order β and type θ is defined by*

$${}_{GH}^{\delta}D_{a_1^+}^{\beta,\theta}(x)(t) = \left[{}_\delta\mathcal{H}_{a_1^+}^{\theta(1-\beta)} \left(t^{1-\delta} \frac{d}{dt} \right) {}_\delta\mathcal{H}_{a_1^+}^{(1-\theta)(1-\beta)} \right](x)(t).$$

Lemma 7. *Let β, θ, τ, α, δ, and x all be defined as in Definition 5. Then, we have the following statements*

(1) *For all $\nu > 0$,*

$${}_\delta\mathcal{H}_{a_1^+}^{\beta} \left(\frac{t^\delta - a^\delta}{\delta} \right)^{\nu - 1} = \frac{\Gamma(\nu)}{\Gamma(\nu + \beta)} \left(\frac{t^\delta - a^\delta}{\delta} \right)^{\nu + \beta - 1};$$

(2) *for $x \in C^1[a_1, a_2]$,*

$${}_{GH}^{\delta}D_{a_1^+}^{\beta,\theta} \, {}_\delta\mathcal{H}_{a_1^+}^{\beta}(x)(t) = x(t)$$

(3) *for $x \in C^1[a_1, a_2]$,*

$${}_\delta\mathcal{H}_{a_1^+}^{\beta} \, {}_{GH}^{\delta}D_{a_1^+}^{\beta,\theta}(x)(t) = x(t) - \left(\frac{t^\delta - a^\delta}{\delta} \right)^{\gamma - 1} {}_\delta\mathcal{H}_{a}^{(1-\theta)(1-\beta)} x(a),$$

where $\gamma = \beta + \theta(1 - \beta)$

Proof. In [1]: Lemma 2 and Theorems 5 and 7, take $\psi(t) = \frac{t^{(\alpha+\tau)}}{\alpha+\tau}$, $n = 1$. Then, we obtain the statements above. □

For more details, see [1–3,27,30].

3.3. Banach and Phase Banach Spaces

Here, we give some properties of Banach and phase spaces that help explore the solvability of the inclusion problem (1)–(3) with infinite delay.

3.3.1. Processes on Banach Spaces

According to both articles [31,32], the space

$$L^p[a,b] = \left\{\omega(t) \middle| \int_a^b |\omega(s)|^p ds < \infty\right\}, \quad 1 \le p < \infty$$

is a Banach space introduced with the norm

$$\|\omega\|_p = \left(\int_a^b |\omega(s)|^p ds\right)^{\frac{1}{p}}.$$

and $C[a,b] = \{\omega(t) | \, \omega(t) : [a,b] \to \mathbb{R}, |\omega(t)| \le M, \ for\ some\ M\}$ is endowed with the norm

$$\|\omega\|_C = \sup_{a<s<b} |\omega(s)| < \infty.$$

Accordingly, we have the next theorem

Theorem 1 ([31]). *Consider $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

(i) **Holder Inequality.** *If $\omega \in L^p$ and $\omega^* \in L^q$. Then, $\omega\omega^* \in L^1$ and*

$$\|\omega\omega^*\|_{L^1} \le \|\omega\|_{L^p} \|\omega^*\|_{L^q}.$$

(ii) **Minkowski Inequality.** *If ω, $\omega^* \in L^p$. Then, $\omega + \omega^* \in L^p$ and*

$$\|\omega + \omega^*\|_{L^p} \le \|\omega\|_{L^p} + \|\omega^*\|_{L^p}.$$

(iii) **Embedding Theorem.** *If Ω has a finite positive measure and $q \le p$. Then, $L_P(\Omega) \subseteq L_q(\Omega)$ and*

$$\|\omega\|_{L^q} \le [\mu(\Omega)]^{\frac{1}{r}} \|\omega\|_{L^p}, \ r > 0 \ for\ which\ \frac{1}{q} - \frac{1}{p} = \frac{1}{r}.$$

(iv) $\lim_{p\to\infty} \|\omega\|_{L^p} = \|\omega\|_{L^\infty} = \|\omega\|_\infty = \sup_{t\in\Omega} |\omega(t)| = \|\omega\|_{C(\Omega)}$.

Definition 6. *Let W, W_0, and W_1 be given Banach spaces. Then,*

(a) **Compatible couple of Banach Spaces** *consists of two Banach spaces W_0 and W_1 continuously embedded in the same Housdroff topological vector space V. The spaces $W_0 \cap W_1$ and $W_0 + W_1$ are both Banach spaces equipped, respectively, with norms*

- $\|x\|_{W_0 \cap W_1} = \max\big(\|x\|_{W_0},\ \|x\|_{W_1}\big)$
- $\|x\|_{W_0 + W_1} = \inf\{\|x_0\|_{W_0} + \|x_1\|_{W_1},\ x = x_0 + x_1,\ x_0 \in W_0,\ and\ x_1 \in W_1\}$

(b) **Interpolation** *is the family of all intermediate spaces W between W_0 and W_1 in the sense that*

$$W_0 \cap W_1 \subset W \subset W_0 + W_1,$$

where the two included maps are continuous.

Remark 1. *We can understand that:*

- *The couple $\big(L^\infty, L^1\big)(\mathbb{R})$ is a compatible couple since L^∞ and L^1 are both embedded in the space of measurable functions on the real line, equipped with topology convergence in measure;*

- *For all* $1 < p < \infty$*, the spaces* $L^p(\mathbb{R})$ *are intermediate spaces between* $L^\infty(\mathbb{R})$ *and* $L^1(\mathbb{R})$*. Hence,*
$$L^{1,\infty}(\mathbb{R}) = L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^p(\mathbb{R}) \subset L^\infty(\mathbb{R}) + L^1(\mathbb{R}).$$

3.3.2. Phase Banach Space

From [33], denote by $\mathcal{B}$ the space of all continuous function mapping $[-\sigma, 0]$ to $\mathbb{R}$. For $-\infty < a < T$, let $x : [a-\sigma, T] \to \mathbb{R}$ be defined in $(a-\sigma, T)$ and continuous on $[a, T]$. For all $r \in [-\sigma, 0]$, $t \in [a, T]$, define $x_t : C[-\sigma, 0] \to \mathbb{R}$ by $x_t(r) = x(t+r)$, $\forall$. Note that x_t translates x from $[t-\sigma, t]$ back to $[-\sigma, 0]$ and $x_a = x|_{[a-\sigma,a]}$.

Definition 7. *The set* $\mathcal{B}$ *is said to be admissible whenever there exist two constants* $A_1,\ A_2 \geq 0$ *and a continuous function* $N : [0,\infty) \to [0,\infty)$ *such that if* $x : [a-\sigma, T] \to \mathbb{R}$ *is defined in* $(a-\sigma, T)$ *and continuous on* $[a, T)$ *with* $x_a \in \mathcal{B}$*, then for all* $t \in [a, T]$ *the following statements all hold:*

(a_1) $x_t \in \mathcal{B}$*;*

(a_2) x_t *is continuous in* t *with respect to* $\|.\|_{\mathcal{B}}$*;*

(a_3) $\|x_t\|_{\mathcal{B}} \leq A_1 \max_{s\in[a,t]} |x(s)| + N(t-a)\|x_a\|_{\mathcal{B}}$*, and* $N(t) \to 0$ *as* $t \to \infty$*;*

(a_4) $|v(0)| \leq A_2\|v\|_{\mathcal{B}}$ *for all* $v \in \mathcal{B}$*.*

Remark 2. *In* (a_2) *let* $A_3 > 0$ *be given. We can see for all* $r \in [-\sigma, 0]$, $A_3 > 0$*, and* $t \in [a-\sigma, T]$ *that* $s = t - r \in [a, T]$*, which implies the following:*

$$\begin{aligned}
&x(t) = x((t-r)+r) = x_{t-r}(r), \\
&|x(t)| = |x_{t-r}(r)| \leq A_3\|x_s\|_{\mathcal{B}}, \qquad s = t-r \in [a, T] \\
&\|x(s)\|_{[a,T]} \leq \|x(t)\|_{[a-\sigma,T]} \leq A_3\|x_s\|_{\mathcal{B}}
\end{aligned}$$

3.4. *Multi-Valued Mappings*

Here, we introduce some facts about multi-valued mappings and their properties. These facts are confirmed in [34–38].

Consider that we have two Banach spaces $(W, \|.\|)$ and $(O, \|.\|)$. We say that $\phi : W \to P_{cl}(W)$ is convex (closed) multi-valued mapping if $\phi(w)$ is convex (closed) for all $w \in W$. If $\phi(B)$ is relatively compact for every $B \in P_b(W)$, then ϕ is completely continuous.

ϕ is said to be upper semi-continuous if $E \subset W; \phi^{-1}(E)$ is a closed subset of W for each closed subset (i.e., the set $\{w \in W : \phi(w) \subseteq H\}$ is open whenever $H \subset W$ is open). In contrast, it is lower semi-continuous if $\forall Z \subset W; \phi^{-1}(Z)$ is an open subset of W. By another meaning, ϕ is lower semi-continuous whenever the set $\{w \in W : \phi(w) \cap H \neq \emptyset\}$ is open for all open sets $H \subset W$.

We say that a multi-valued map $\phi : [0, \tau] \to P_{cl}(W)$ is measurable if for every $w \in W$, the function $s \to d(w, A(s)) = \inf\{d(w, a) : a \in \phi(s)\}$ is an $\mathcal{L}-$measurable function.

Given $U, V \in P_{cl}(W)$, the Pompeiu–Housdorff distance of U, V is defined by

$$h(U, V) = H_d(U, V) = d_H(U, V) = \max\left\{\sup_{u\in U} d(u, V), \sup_{v\in V} d(U, v)\right\}.$$

Moreover, the diameter distance of V is given by

$$\hat{\delta}(V) = \sup_{v_1, v_2 \in V} d(v_1, v_2).$$

Note that there exists $M > 0$ such that $\hat{\delta}(V) \leq M$ if V is bounded.

Suppose we adopt ϕ as a nonempty compact valued completely continuous function. In that case, [ϕ is upper semi-continuous] is equivalent to [ϕ has a closed graph (i.e., if $\nu_n \to \nu_*$ *and* $y_n \to y_*$, then $y_n \in \phi(\nu_n)$ implies $y_* \in \phi(\nu_*)$)].

Definition 8. *Consider a multi-valued map $\Theta : [a,b] \times \mathbb{R}^n \to P(\mathbb{R})$. Then, Θ is said to be a Caratheodory if*

(1) $\tau \to \Theta(\tau, \{v_i\})$ *is measurable,* $\forall v_i \in \mathbb{R}, n \in \mathbb{N}$.

(2) $(\{v_i\}) \to \Theta(\tau, \{v_i\})$ *a.e* $\tau \in [a,b]$ *is upper semi-continuous.*

Adding to the assumptions (1) and (2), the map Θ is L^1- Caratheodory if for each $k > 0$, there exists $\phi_k \in L^\infty[a,b]$ satisfying $\sup_{\tau\geq 0}|\phi_k(\tau)| < +\infty$ and $\phi_k > 0$ and a nondecreasing map Ł $\in L^1[a,b]$ for which

$$\|\Theta(\tau, \{v_i\})\| = \sup\{|\theta| : \theta(\tau) \in \Theta(\tau, \{v_i\})\} \leq \phi_k(\tau)Ł(\{|v_i\|\}),$$

for all $\|v_i\| < k,\ i = 1, \ldots, n,\ n \in \mathbb{N},\ \tau \in [a,b]$.

Lemma 8 ([39] (pp. 781–786))**.** *Let Ω be a Banach space,*

$$\Theta : [0,L] \times \Omega \to P_{cp,cv}(\Omega)$$

be a L^1-Caratheodory multi-valued map and P be a continuous and linear map from $L^1([0,L]\Omega)$ to $C([0,L],\Omega)$. Then, the operator:

$$P \circ S_\Theta : C([0,L],\Omega) \to P_{cp,cv}(C([0,L],\Omega)),$$

such that:

$$y \mapsto (P \circ S_\Theta)(y) = P(S_{\Theta,y})$$

is an operator with closed graph in $C([0,L],\Omega) \times C([0,L],\Omega)$.

Here,

$$S_{\Theta,y} = \Big\{\theta \in L^1([0,L],\mathbb{R}) : \theta(\tau) \in \Theta(\tau, y(\tau))\Big\}.$$

Theorem 2 (**Leray–Schauder Nonlinear Alternative Type** [40] (p. 169), [41] (p. 188))**.** *Assuming that Σ is Banach space, E is a convex closed subset of Σ, and Ω is an open subset of E with $0 \in \Omega$. If $\Psi : \overline{\Omega} \to P_{cp,cv}(E)$ is an upper semi-continuous multi-compact map, then either*

(i) *there exists $\omega \in \partial\Omega,\ \rho \in (0,1)$ such that $\omega \in \rho\Psi(\omega)$, or*

(ii) *there exists a fixed point $\omega \in \overline{\Omega}$.*

3.5. Auxiliary Statements

Lemma 9. *Take the function $\eta(t) \in \Pi(t, x(t), x_t, \mathbb{H}^u)$ for which we have*

$$ {}^{\delta}_{GH}D^{\beta,\theta}_{a^+}x(t) = \mathcal{A}x(t) + \eta(t), \qquad t \in [a,T],\ a > 0\ \alpha, \beta \in [0,1], \tag{11}$$

$$ {}_{\delta}\mathcal{H}^{(1-\theta)(1-\beta)}_a x(a) = \frac{c\Gamma(\gamma)}{\Gamma(\omega+\gamma)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\omega}, \tag{12}$$

$$\omega \in (0,1),\ \omega + \beta = 1.$$

Then, the unique conformable solution is given by

$$x(t) = \Gamma(\xi)x_0(t)\mathbb{E}_{\beta,\xi}\left(\mathcal{A}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta}\right) + \int_0^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1}\mathbb{E}_{\beta,\beta}\left(\mathcal{A}\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta}\right)\eta(\rho)d\rho^\delta, \tag{13}$$

where

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)}$$

and

$$x_0(t) = \frac{c}{\Gamma(\xi)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}, \ \xi = \gamma + \omega.$$

Proof. By applying ${}_\delta\mathcal{H}_a^\beta$ to both sides of (11) and applying condition (12), one has

$$x(t) = \frac{c}{\Gamma(\xi)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1} + {}_\delta\mathcal{H}_a^\beta[\mathcal{A}x(t) + \eta(t)]$$

Take

$$x_0(t) = \frac{c}{\Gamma(\xi)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}.$$

So, we obtain

$$x(t) = x_0(t) + {}_\delta\mathcal{H}_a^\beta[\mathcal{A}x](t) + {}_\delta\mathcal{H}_a^\beta\eta(t).$$

The proof is similar to the proof of the solution in [30]: Theorem 4.

Now, to obtain the conformable solution, define the operator $\Im$ by

$$\Im(x)(t) = x_0(t) + {}_\delta\mathcal{H}_a^\beta[\mathcal{A}x](t) + {}_\delta\mathcal{H}_a^\beta\eta(t).$$

Accordingly, define the sequence $(x_k)_{k\geq 1}$ by

$$x_k(t) = \Im x_{k-1}(t).$$

Hence, we obtain the general formula

$$\begin{aligned} x_k(t) =& c\sum_{j=1}^{k+1}\frac{\mathcal{A}^{j-1}}{\Gamma(\beta j + \omega + \theta(1-\beta))}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta j+\omega+\theta(1-\beta)-1} \\ &+ \int_a^t\sum_{j=1}^{k}\frac{\mathcal{A}^{j-1}}{\Gamma(\beta j)}\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta j-1}\eta(\rho)d\rho^\delta. \end{aligned}$$

Take the limit as $k \to \infty$ and apply the changing $j \to j+1$; we have

$$\begin{aligned} x(t) =& c\sum_{j=0}^{\infty}\frac{\mathcal{A}^{j}}{\Gamma(\beta j + \xi)}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta j+\xi-1} \\ &+ \int_a^t\sum_{j=0}^{\infty}\frac{\mathcal{A}^{j}}{\Gamma(\beta j+\beta)}\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta j+\beta-1}\eta(\rho)d\rho^\delta \\ &= \Gamma(\xi)x_0(t)\mathbb{E}_{\beta,\xi}\left(\mathcal{A}\left(\frac{t^\delta - a^\delta}{\delta}\right)^\beta\right) \\ &+ \int_0^t\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1}\mathbb{E}_{\beta,\beta}\left(\mathcal{A}\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^\beta\right)\eta(\rho)d\rho^\delta. \end{aligned}$$

□

Now, define the set-valued map $\overline{S_{\Pi,x}^{u,(1,\infty)}}$ such as

$$\overline{S_{\Pi,x}^{u,(1,\infty)}} = \left\{\eta(t) \in L^{1,\infty}[a,T] |\ \eta(t) \in \Pi(t,x,x_t,\mathbb{H}^u)\right\},$$

and define the linear operator $\Delta_\eta : L^{1,\infty}[a,T] \to P\big(L^{1,\infty}[a,T]\big)$ for all $\eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$ by

$$\Delta_\eta(t) = cQ^{\xi}_{\beta}\Big(t^\delta - a^\delta\Big) + \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\Big(t^\delta - \rho^\delta\Big)\eta(\rho)d\rho^\delta, \qquad \forall t \in [a,T], \tag{14}$$

$$Q^{\xi}_{\beta}\Big(t^\delta - a^\delta\Big) = \left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1} \mathbb{E}_{\beta,\xi}\left(\mathcal{A}(t)\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta}\right), \tag{15}$$

$$\hat{Q}_\beta\Big(t^\delta - \rho^\delta\Big) = \mathbb{E}_{\beta,\beta}\left(\mathcal{A}(t)\left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta}\right). \tag{16}$$

After that, define the operator $\underline{\Delta}^{\psi}_{\eta} : L^{1,\infty}[a-\sigma,T] \to P\big(L^{1,\infty}[a-\sigma,T]\big)$ by

$$\underline{\Delta}^{\psi}_{\eta}(t) = \begin{cases} \psi(t), & t \in [a-\sigma, a], \\ \Delta_\eta(t), & t \in [a,T] \end{cases} \tag{17}$$

where $\eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$ and then define the operator $\aleph : K \to P\big(L^{1,\infty}[a-\sigma,T]\big)$ such as

$$\aleph(x)(t) = \left\{e(t) \in L^{1,\infty}[a-\sigma,T] | e(t) = \Delta^{\psi}_{\eta}(t),\ \eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}},\ \psi \in L^{1,\infty}[a-\sigma,a]\right\}. \tag{18}$$

Hence,

$$\aleph_J(x)(t) = \left\{e_J(t) \in L^{1,\infty}[a,T] | e_J(t) = \Delta_\eta(t),\ \eta(t) \in \overline{S^{u,(1,\infty)}_{\Pi,x}}\right\}. \tag{19}$$

Proposition 1. *Let $0 < \beta < 1,\ 0 \le \theta,\ \alpha \le 1,\ \tau \in \mathbb{R}$ be given and define $\gamma,\ \nu,\ \delta$, respectively, by $\gamma = \beta + \theta(1-\beta)$, $\xi = \omega + \gamma$, $\nu = \xi - \beta$ and $\delta = \alpha + \tau \neq 0$. Then, the following statement is satisfied*

$${}_{\delta}\mathcal{H}^{\nu}_{a^+}\left[\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\Big(t^\delta - a^\delta\Big)\right] = Q^{\xi}_{\beta}\Big(t^\delta - a^\delta\Big)$$

Proof.

$$\begin{aligned}
L.H.S &= {}_{\delta}\mathcal{H}^{\nu}_{a^+}\left[\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\Big(t^\delta - a^\delta\Big)\right] \\
&= {}_{\delta}\mathcal{H}^{\nu}_{a^+}\left[\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta-1} \mathbb{E}_{\beta,\beta}\left(\mathcal{A}(t)\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta}\right)\right] \\
&= \frac{1}{\Gamma(\nu)} \sum_{j=0}^{\infty} \frac{\mathcal{A}^j}{\Gamma(\beta j + \beta)} \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\nu-1} \left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta j + \beta - 1} \rho^{\delta-1} d\rho \\
&= \sum_{j=0}^{\infty} \frac{\mathcal{A}^j}{\Gamma(\beta j + \beta)} \frac{\Gamma(\beta j + \beta)}{\Gamma(\beta j + \beta + \nu)} \left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta j + \beta + \nu - 1} \\
&= \left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1} \sum_{j=0}^{\infty} \frac{\left[\mathcal{A}\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta}\right]^j}{\Gamma(\beta j + \xi)} \\
&= R.H.S
\end{aligned}$$

□

Now, since $\mathcal{A}$ is a generator of compact C_0 semi-groups, there exists $M_\beta > 0$ such that $\left\|\mathbb{E}_{\beta,\beta}\right\| \leq M_\beta$ and consequently we have the following proposition.

Proposition 2. *Let $0 < \beta < 1$, $0 \leq \theta$, $\alpha \leq 1$, $\tau \in \mathbb{R}$ be given and define γ, ν, δ, respectively, by $\gamma = \beta + \theta(1-\beta)$, $\xi = \omega + \gamma$, $\nu = \xi - \beta$ and $\delta = \alpha + \tau \neq 0$. Then, the following statement is valid.*

$$\left\|Q_\beta^\xi\left(t^\delta - a^\delta\right)\right\| \leq \frac{M_\beta \Gamma(\beta)}{\Gamma(\xi)} \sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right|$$

Proof.

$$\begin{aligned}\left\|Q_\beta^\xi\left(t^\delta - a^\delta\right)\right\| &\leq {}_\delta\mathcal{H}_{a^+}^\nu\left[\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta-1}\left|\hat{Q}_\beta\left(t^\delta - a^\delta\right)\right|\right] \\ &\leq M_\beta\, {}_\delta\mathcal{H}_{a^+}^\nu\left[\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\beta-1}\right]\end{aligned}$$

Using [Lemma 7: (1)], one has

$$\left\|Q_\beta^\xi\left(t^\delta - a^\delta\right)\right\| \leq \frac{M_\beta \Gamma(\beta)}{\Gamma(\xi)} \sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right|.$$

□

Define the statement
(P_R): For some $x \in L^p[a-\sigma, T]$, we have

$$\max\left\{\|x\|_{[a,T]}^{1,\infty},\ \|\psi\|_{\mathcal{B}}\right\} \leq R,$$

where

$$\|x\|_{[a-\sigma,T]}^{1,\infty} = \max\{\|x\|_{[a-\sigma,a]}^{1,\infty},\ \|x\|_{[a,T]}^{1,\infty}\}.$$

Then, define the set K by

$$K = \{x \in \mathcal{B} \cap L^p[a-\sigma, T] |\ x\ satisfies\ (P_R)\}.$$

It is clear that K is closed in $L^{1,\infty}[a-\sigma, T] \preceq L^p[a-\sigma, T]$ and in the phase Banach space $\mathcal{B}$.

Consider the following hypotheses.

$(\mathcal{J}_1)$ The mappings $\tilde{S}$, $\tilde{P}$, $\jmath$ and f satisfy all conditions given in Lemmas 2 and 3;

$(\mathcal{J}_2)$ $\Pi : [a,T] \times K \times \mathcal{B} \times \mathbb{H}^u \to P_{cp,cv}(\mathbb{R})$ is L^p- Caratheodory multi-valued mapping satisfying the below condition;
For each $R > 0$ there exist ϕ_R, $\hat{\phi}_R \in L^\infty([a,T], \mathbb{R}_+)$ and non-decreasing functions $Ł_1$, $Ł_2$, *and* $Ł_3 \in L^1([a,T], \mathbb{R})$ such that

$$\|\Pi\|^{1,\infty} \leq \phi_R(t)\left[Ł_1\left(\|x\|^{1,\infty}\right) + Ł_2(\|x_t\|_{\mathcal{B}})\right] + \hat{\phi}_R(t)Ł_3(\hat{\delta}(\mathbb{H}^u)),$$

for all $\|x\| \leq R$ and $\mathbb{H}^u$ is compact;

$(\mathcal{J}_3)$ The mappings $\tilde{S}$, $\tilde{P}$, $\jmath$ and f satisfy all conditions given in Lemmas 2, 4 and 5;

$(\mathcal{J}_4)$ $\Pi : [a,T] \times K \times \mathcal{B} \times \mathbb{H}^u \to P_{cp,cv}(\mathbb{R})$ is L^p- Caratheodory multi-valued mapping satisfying the below condition.
For each $R > 0$, there exist ϕ_R, $\hat{\phi}_R \in L^\infty([a,T], \mathbb{R}_+)$ and non-decreasing functions L_1, L_2, and $L_3 \in L^1([a,T], \mathbb{R})$ such that

$$\|\Pi\|^{1,\infty} \leq \phi_R(t)\Big[L_1\Big(\|x\|^{1,\infty}\Big) + L_2(\|x_t\|_{\mathcal{B}})\Big] + \hat{\phi}_R(t)L_3(\phi(\kappa_1,\kappa_2)),$$

for all $\|x\| \leq R$ and $\mathbb{H}^u$ is compact, where ϕ is defined by (8)–(10) in Lemma 6 with $\kappa_1 = d(x,\mathbb{H}^u)$ and $\kappa_2 = d(x,\tilde{S}(x))$.

Then, for all $t \in [a,T],\ a > 0$ we have the following propositions

Proposition 3. *Let $x \in K$ be given. The operator $\aleph_J(x)(t)$ is convex if $(\mathcal{J}_2)$ holds.*

Proof. Let $e_J^1,\ e_J^2 \in \aleph_J(x)(t)$; then, there exist $\eta_1,\ \eta_2 \in \overline{S_{\Pi,x}^{u,(1,\infty)}}$ subject to

$$e_J^i = \Delta\eta_i, \qquad i = 1,2,$$

where Δ_η is defined by (14)–(16). Let $\lambda \in [0,1]$ be given. Then, by the linearty of Δ_η, we obtain

$$\begin{aligned}\lambda e_J^1 + (1-\lambda)e_J^2 =& \lambda\Delta_{\eta_1} + (1-\lambda)\Delta_{\eta_2} \\ =& \Delta(\lambda\eta_1 + (1-\lambda)\eta_2).\end{aligned}$$

By $(\mathcal{J}_2)$, $\lambda\eta_1 + (1-\lambda)\eta_2 \in \overline{S_{\Pi,x}^{u,(1,\infty)}}$ and then $\lambda e_J^1 + (1-\lambda)e_J^2 \in \aleph_J(x)(t)$ which completes the result. □

Proposition 4. *Let $x \in K$ with $x_a = \psi$ be given. The operator $\aleph_J(x)(t)$ is completely continuous if $(\mathcal{J}_1)$ and $(\mathcal{J}_2)$ are fulfilled and so are (a_1)–(a_4).*

Proof. To show that $\aleph_J(x)(t)$ is equicontinuous, we should prove that $\aleph_J(x)(t)$ is bounded and relatively compact on bounded subsets.

Step 1: Let $x \in K$ and $e_J \in \aleph_J(x)$; then,

$$\begin{aligned}|e_J(t)| =& |\Delta_\eta(t)| \\ \leq& |c|\left|Q_\beta^\xi\left(t^\delta - a^\delta\right)\right| + \left|\int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1} \hat{Q}_\beta\left(t^\delta - \rho^\delta\right)\eta(\rho)d\rho^\delta\right|.\end{aligned}$$

By using Propositions 1 and 2 and Lemma 7:(1) we have

$$\begin{aligned}|e_J(t)| \leq& |c|\frac{M_\beta\Gamma(\beta)}{\Gamma(\xi)}\sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right| \\ &+ \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1}\left|\hat{Q}_\beta\left(t^\delta - \rho^\delta\right)\eta(\rho)\right|d\rho^\delta \\ &\leq |c|\frac{M_\beta\Gamma(\beta)}{\Gamma(\xi)}\sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right| \\ &+ \int_a^t \left(\frac{t^\delta - \rho^\delta}{\delta}\right)^{\beta-1}|\eta(\rho)|d\rho^\delta \\ &\leq |c|\frac{M_\beta\Gamma(\beta)}{\Gamma(\xi)}\sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right| \\ &+ \frac{M_\beta\Gamma(\beta)}{\Gamma(\beta+1)}\left(\frac{T^\delta - a^\delta}{\delta}\right)^{\beta} G_0\left(R,\hat{\delta}(\mathbb{H}^u)\right) \\ &\leq M_\beta\Gamma(\beta)G\left(\delta,\xi,\beta,R,\hat{\delta}(\mathbb{H}^u)\right),\end{aligned}$$

where

$$G_0\big(R,\hat{\delta}(\mathbb{H}^u)\big) = \|\phi_R\|[Ł_1(R) + Ł_2((A_1+N^*)R)] + \|\hat{\phi}_R\|Ł_3\big(\hat{\delta}(\mathbb{H}^u)\big);$$

$$G\big(\delta,\xi,\beta,R,\hat{\delta}(\mathbb{H}^u)\big) = \frac{|c|}{\Gamma(\xi)}\sup_{t\in[a,T]}\left|\left(\frac{t^\delta - a^\delta}{\delta}\right)^{\xi-1}\right| + \frac{1}{\Gamma(\beta+1)}\left(\frac{T^\delta-a^\delta}{\delta}\right)^{\beta} G_0\big(R,\hat{\delta}(\mathbb{H}^u)\big)$$

and $N^* = \sup_{t\in[a,T]}|N(t-a)|$.

Step 2: Suppose that $t_1,\ t_2 \in [a,T]$ such that $t_1 < t_2$ with $t_1 \to t_2$ and take $e_J \in \aleph_j(x),\ x \in K$

$$\begin{aligned}
|e_J(t_2) - e_J(t_1)| &= |\Delta_\eta(t_2) - \Delta_\eta(t_1)| \\
&\leq |c|\left|Q_\beta^\xi\left(t_2^\delta - a^\delta\right) - Q_\beta^\xi\left(t_1^\delta - a^\delta\right)\right| \equiv (I_1) \\
&+\left|\int_a^{t_1}\left(\frac{t_2^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\hat{Q}_\beta\left(t_2^\delta-\rho^\delta\right)\eta(\rho)d\rho^\delta - I\right| \equiv (I_2) \\
&+\left|I - \int_a^{t_1}\left(\frac{t_1^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\hat{Q}_\beta\left(t_1^\delta-\rho^\delta\right)\eta(\rho)d\rho^\delta\right| \equiv (I_3) \\
&+\left|\int_{t_1}^{t_2}\left(\frac{t_2^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\hat{Q}_\beta\left(t_2^\delta-\rho^\delta\right)\eta(\rho)d\rho^\delta\right| \equiv (I_4),
\end{aligned}$$

where

$$I = \int_a^{t_1}\left(\frac{t_1^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\hat{Q}_\beta\left(t_2^\delta-\rho^\delta\right)\eta(\rho)d\rho^\delta.$$

It is easy to understand that $I_1 \to 0$ as $t_1 \to t_2$ since Q_β^ξ is strongly continuous in $[a,T]$.
For I_2, we have

$$\begin{aligned}
I_2 =& \left|\int_a^{t_1}\left[\left(\frac{t_2^\delta-\rho^\delta}{\delta}\right)^{\beta-1} - \left(\frac{t_1^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\right]\hat{Q}_\beta\left(t_2^\delta-\rho^\delta\right)\eta(\rho)d\rho^\delta\right| \\
&\leq \frac{M_\beta}{\beta}G_0\big(R,\hat{\delta}(\mathbb{H}^u)\big)\left[\left(\frac{t_2^\delta-t_1^\delta}{\delta}\right)^\beta - \left(\frac{t_2^\delta-a^\delta}{\delta}\right)^\beta + \left(\frac{t_1^\delta-a^\delta}{\delta}\right)^\beta\right].
\end{aligned}$$

Hence, $I_2 \to 0$ as $t_1 \to t_2$

Since $\hat{Q}_\beta$ is also strongly continuous in $[a,T]$ and

$$I_3 \leq \int_a^{t_1}\left(\frac{t_1^\delta-\rho^\delta}{\delta}\right)^{\beta-1}\left|\hat{Q}_\beta\left(t_2^\delta-\rho^\delta\right) - \hat{Q}_\beta\left(t_1^\delta-\rho^\delta\right)\right|\|\eta(\rho)\|d\rho^\delta,$$

then we can see that $I_3 \to 0$ as $t_1 \to t_2$.

Finally,

$$I_4 \leq \frac{M_\beta}{\beta}G_0\big(R,\hat{\delta}(\mathbb{H}^u)\big)\left[\left(\frac{t_2^\delta-a^\delta}{\delta}\right)^\beta - \left(\frac{t_1^\delta-a^\delta}{\delta}\right)^\beta\right],$$

which shows that $I_4 \to 0$ as $t_1 \to t_2$.

Because of that, $I_1,\ I_2,\ I_3$ and $I_4 \to 0$ as $t_1 \to t_2$, then we obtain the result $|e_J(t_2) - e_J(t_1)| \to 0$ as $t_1 \to t_2$.

According to Steps 1 and 2, we conclude that $\aleph_J(x)(t)$ is completely continuous. □

Proposition 5. *Let $x \in K$ with $x_a = \psi$ be given. The operator $\aleph_J(x)(t)$ is upper semi-continuous if $(\mathcal{J}_1)$, $(\mathcal{J}_2)$ and (a_1)–(a_4) are satisfied.*

Proof. Since $\aleph_J(x)(t)$ is completely continuous, it is enough to claim that it has a closed graph to obtain the upper semi-continuity of $\aleph_J(x)(t)$. Let $x_n \in K,\ x_n \to x^*,\ e_J^n \in \aleph_J(x_n)$ and $e_J^n \to e_J^*$. If $e_J^n \in \aleph_J(x_n)$, there exists $\eta_n \in \overline{S^{u,(1,\infty)}_{\Pi,x_n}}$ such that $e_J^n = \Delta_{\eta_n}$. Using the linearity of Δ and Lemma 8 shows that Δ has a closed graph. Thus, $e_J^n = \Delta_{\eta_n} \to \Delta_{\eta^*},\ \eta^* \in \overline{S^{u,(1,\infty)}_{\Pi,x^*}}$. Take $e_J^* = \Delta_{\eta^*}$, then we obtain $e_J^* \in \aleph_J(x^*)$ which tends to the upper semi-continuity of $\aleph_J(x)(t)$. $\square$

4. Main Results

Theorem 3. *Consider that hypothesis $(\mathcal{J}_1)$, $(\mathcal{J}_2)$ and (a_1)–(a_4) are valid. Then, problem (1)–(3) has at least one solution in K if the following condition holds*

$$\frac{R}{\psi^* + M_\beta \Gamma(\beta) G\big(\delta, \xi, \beta, R, \hat{\delta}(\mathbb{H}^u)\big)} \geq 1,$$

where $x_a = \psi$

Proof. To obtain the suggested result, we follow all arguments given in Lemma 8 and Theorem 2 for the operator $\aleph(K)$ over the closed convex subset K.

Step 1: Let $x \in K,\ \lambda \in [0,1],\ e_1,\ e_2 \in \aleph(x)(t)$ and $t \in [a-\sigma, T]$. We want to claim that $\lambda e_1 + (1-\lambda)e_2 \in \aleph(x)$. So, since $e_1,\ e_2 \in \aleph(x)(t)$ implies the existence of two elements, $\eta_i \in \overline{S^{u,(1,\infty)}_{\Pi,x}}$ such that

$$e_i = \underline{\Delta}^{\psi}_{\eta_i}(t),\ i = 1,2,$$

where $\underline{\Delta}^{\psi}_{\eta_i}(t)$ is defined by (14)–(17). Due to the linearty of Δ, we can see the linearty of $\underline{\Delta}$ and by using Proposition 3 and the convexity of $\overline{S^{u,(1,\infty)}_{\Pi,x}}$, we have

$$\begin{aligned} \lambda e_1 + (1-\lambda)e_2 &= \lambda \underline{\Delta}^{\psi}_{\eta_1}(t) + (1-\lambda)\underline{\Delta}^{\psi}_{\eta_2}(t) \\ &= \underline{\Delta}^{(\lambda+(1-\lambda))\psi}_{\lambda\eta_1+(1-\lambda)\eta_2}(t) \\ &= \underline{\Delta}^{\psi}_{\lambda\eta_1+(1-\lambda)\eta_2}(t) \in \aleph(x)(t), \end{aligned}$$

We can understand the proof since Δ is convex due to Proposition 3.

Step 2: To show that that is completely continuous in K, we need to prove that $\aleph : K \to K$ and is equicontinuous.

(l_1) Let $x \in K,\ e(t) \in \aleph(x)(t)$. By using Proposition 4 Step 1, one has

$$\begin{aligned} |e(t)| &= \left|\underline{\Delta}^{\psi}_{\eta}(t)\right| \\ &\leq \|\psi\|_{\mathcal{B}} + |\Delta_\eta(t)|_{[a,T]} \\ &\leq \psi^* + M_\beta \Gamma(\beta) G\big(\delta, \xi, \beta, R, \hat{\delta}(\mathbb{H}^u)\big) \leq R. \end{aligned}$$

(l_2) Let $t_1,\ t_2 \in [a-\sigma, T],\ t_1 < t_2$ with $t_1 \to t_2$

Case 1: If $t_1,\ t_2 \in [a-\sigma, a]$, then by continuity of ψ in $[a-\sigma, a]$ we obtain

$$|e(t_2) - e(t_1)| = |\psi(t_2) - \psi(t_1)| \to 0, \qquad \text{as } t_1 \to t_2.$$

Case 2: If $t_1,\ t_2 \in [a, T]$, then by using Proposition 4 Step 2, one has

$$|e(t_2) - e(t_1)| \leq |\aleph_J(x)(t_2) - \aleph_J(x)(t_1)| \to 0 \qquad \text{as } t_1 \to t_2.$$

Case 3: If $t_1 \in [a-\sigma, a]$, $t_2 \in [a, T]$ and $t_1 \to t_2$, then there exists $0 < \epsilon \to 0$ such that t_1, $t_2 \in (a-\epsilon, a+\epsilon)$ which implies that t_1, $t_2 \to a$. According to Case 1 and Case 2, we have

$$|e(t_2) - e(t_1)| \le |e(t_2) - e(a)|_{(a,a+\epsilon)} + |e(a) - e(t_1)|_{(a-\epsilon,a)} \to 0$$

as t_1, $t_2 \to a$.

By (l_1) and (l_2), we conclude that $\aleph$ is completely continuous in K.

Step 3: We still need to explore that $\aleph$ has a closed graph to see the upper semi-continuity of $\aleph$. In the vision of Proposition 5 and the continuity of ψ, we understand the upper semi-continuity of $\aleph$.

Step 4: For the set K, we choose

$$R = \psi^* + M_\beta \Gamma(\beta) G\big(\delta, \xi, \beta, R, \hat{\delta}(\mathbb{H}^u)\big) + 1.$$

By Theorem 2 and Step 1–Step 4, we conclude the solvability of problem (1)–(3). □

Theorem 4. *Consider that hypotheses $(\mathcal{J}_2)$–$(\mathcal{J}_4)$ and (a_1)–(a_4) are satisfied. Then, problem (1)–(3) has at least one solution in K if the following condition is valid.*

$$\frac{R}{\psi^* + M_\beta \Gamma(\beta) G(\delta, \xi, \beta, R, g^*)} \ge 1,$$

where $g^* = \|g\|$,

$$G_0(R, g^*) = \|\phi_R\|[L_1(R) + L_2((A_1 + N^*)R)] + \|\hat{\phi}_R\| L_3(g^*);$$
$$G(\delta, \xi, \beta, R, g^*) = \frac{|c|}{\Gamma(\xi)} \sup_{t\in[a,T]} \left| \left(\frac{t^\delta - a^\delta}{\delta} \right)^{\xi-1} \right| + \frac{1}{\Gamma(\beta+1)} \left(\frac{T^\delta - a^\delta}{\delta} \right)^{\beta} G_0(R, g^*)$$

and $x_a = \psi$.

Proof. Similarly to the proof of Theorem 3, we take

$$R = \psi^* + M_\beta \Gamma(\beta) G(\delta, \xi, \beta, R, g^*) + 1.$$

□

5. Applications

Example 1. *Consider that $\mathcal{J}_1$ holds and*

$$\Pi(t, x, x_t, \mathbb{H}^u) = \left[\int_{-\sigma}^{0} B_i(t, r) x_t(r) dr \right]_{i=1}^{\infty} + \chi_{\mathbb{H}^u}(x);$$
$$\sum_{i=1}^{\infty} |B_i(t, r)| x_t(r) dr \le 1 \tag{20}$$

and

$$\chi_{\mathbb{H}^u}(x) = \begin{cases} 1, & x \in \mathbb{H}^u; \\ 0, & x \notin \mathbb{H}^u \end{cases} \tag{21}$$

Hence, if $\mathcal{B} = C[-\sigma, r]$, one has

$$\begin{aligned}|\Pi| &\le \|x_t\|_{\mathcal{B}} \int_{-\sigma}^{0} |B_i(t,r)dr| + 1 \\ &\le A_1 \|x\|^{1,\infty} + N(t-a)\|x_a\|_{\mathcal{B}} + 1. \\ &\le \|x\|^{1,\infty} + N^* \|\psi\|_{\mathcal{B}} + 1 \\ &\le (A_1 + N^*)R + 1\end{aligned}$$

Take $\phi_R = \hat{\phi}_R = 1$, $Ł_1 = 0$, $Ł_2 = I$(identity map) and $Ł_3 = 1$. So, we obtain $\Gamma_0(R, \mathbb{H}^u) = (A_1 + N^*)R + 1$.

Furthermore, assume that $\beta = 1 \to \omega = 0$ tends to $\xi = \gamma = 1$, which implies

$$\mathbb{E}_{\beta,\xi}(z) = \mathbb{E}_{\beta,\beta}(z) = \mathbb{E}_{1,1}(z) = \exp(z).$$

If we take $z = \mathcal{A}(t)\left(\frac{t^\delta - \rho^\delta}{\delta}\right) = -\lambda\left(\frac{t^\delta - \rho^\delta}{\delta}\right)$, $\lambda \in \mathbb{R}^+$, then we obtain

$$\exp(z) \le 1 = M_1, \ \forall \rho \in [a, t].$$

Moreover,

$$\begin{aligned}G\big(\delta, \xi, \beta, R, \hat{\delta}(\mathbb{H}^u)\big) &= G\big(\delta, 1, 1, R, \hat{\delta}(\mathbb{H}^u)\big) \\ &= |c| + \left(\frac{T^\delta - \rho^\delta}{\delta}\right) G_0\big(R, \hat{\delta}(\mathbb{H}^u)\big)\end{aligned}.$$

Take $R = \psi^* + G\big(\delta, 1, 1, R, \hat{\delta}(\mathbb{H}^u)\big) + 1$; then, by Theorem 3 the problem (1)–(3) associated with (20) and (21) has at least one solution.

Example 2. *Consider that $\mathcal{J}_3$ holds and*

$$\Pi(t, x, x_t, \mathbb{H}^u) = \left[\int_{-\sigma}^{0} B_i(t,r) x_t(r) dr\right]_{i=1}^{\infty} + W_{\mathbb{H}^u}(x);$$

$$\sum_{i=1}^{\infty} |B_i(t,r)| x_t(r) dr \le 1 \tag{22}$$

and

$$W_{\mathbb{H}^u}(x) = \begin{cases} \inf |g(x)|, & x \in \mathbb{H}^u; \\ 0, & x \notin \mathbb{H}^u \end{cases} \tag{23}$$

and similarly β, z in Example 1. Then, if $\mathcal{B} = C[-\sigma, r]$, one has

$$\begin{aligned}|\Pi| &\le \|x_t\|_{\mathcal{B}} \int_{-\sigma}^{0} |B_i(t,r)dr| + g^* \\ &\le A_1 \|x\|^{1,\infty} + N(t-a)\|x_a\|_{\mathcal{B}} + g^*. \\ &\le \|x\|^{1,\infty} + N^* \|\psi\|_{\mathcal{B}} + g^* \\ &\le (A_1 + N^*)R + g^*\end{aligned}$$

Take $\phi_R = \hat{\phi}_R = 1$, $L_1 = 0$, $L_2 = L_3 = I$(identity map); we obtain

$$\Gamma_0(R, g^*) = (A_1 + N^*)R + g^*.$$

Moreover,

$$G(\delta,\xi,\beta,R,g^*) = G\big(\delta,1,1,R,\hat{\delta}(\mathbb{H}^u)\big)$$
$$= |c| + \left(\frac{T^\delta - \rho^\delta}{\delta}\right) G_0(R,g^*)^{\cdot}$$

Take $R = \psi^* + G(\delta,1,1,R,g^*) + 1$; *then, by Theorem 4 the problem (1)–(3) associated with (22) and (23) has at least one solution.*

6. Conclusions

This article is devoted to the mild solution of Hilfer fractional inclusion with infinite delay. The solution set intersects with the solution set of FMQHI. We present two theorems according to Lemmas 1–6. We proved these theorems due to the compactness in interpolation of Banach spaces. We first look at the properties of the solution set in Propositions 1–5. After that, we apply the Leray–Schauder Nonlinear Alternative Theorem with phase Banach space rules to the suggested solution set. Finally, we presented some examples related to the proven theorems. We hope to study the stability of this model in subsequent work using the Ulam–Mittage–Lefller test.

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