fractal and fractional

MDPI

Article

# Leader-Following Formation Control for Discrete-Time Fractional Stochastic Multi-Agent Systems by Event-Triggered Strategy

**Jiawei Wu , Yongguang Yu and Guojian Ren ***

School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China; 19118005@bjtu.edu.cn (J.W.); ygyu@bjtu.edu.cn (Y.Y.)
* Correspondence: gjren@bjtu.edu.cn

**Abstract:** Fractional differential equations, which are non-local and can better describe memory and genetic properties, are widely used to describe various physical, chemical, and biological phenomena. Therefore, the multi-agent systems based on discrete-time fractional stochastic models are established. First, some followers are selected for pinning control. In order to save resources and energy, an event-triggered based control mechanism is proposed. Second, under this control mechanism, sufficient conditions on the interaction graph and the fractional derivative order such that formation control can be achieved are given. Additionally, influenced by noise, the multi-agent system completes formation control in the mean square. In addition to that, these results are equally applicable to the discrete-time fractional formation problem without noise. Finally, the example of numerical simulation is given to prove the correctness of the results.

**Keywords:** formation control; stochastic systems; discrete-time fractional systems; multi-agent systems; event-triggered strategy



**Citation:** Wu, J.; Yu, Y.; Ren, G. Leader-Following Formation Control for Discrete-Time Fractional Stochastic Multi-Agent Systems by Event-Triggered Strategy. *Fractal Fract.* **2024**, *8*, 246. https://doi.org/10.3390/fractalfract8050246

Academic Editor: David Kubanek

Received: 22 January 2024
Revised: 14 April 2024
Accepted: 17 April 2024
Published: 23 April 2024



## 1. Introduction

In recent years, multi-agent systems have been instrumental in the field of robotics, involving multiple agents with autonomous decision-making and collaborative abilities. These systems enable the collective accomplishment of complex tasks through communication and cooperation. They are applied in diverse areas like unmanned aerial vehicle formations, robot collaborations, autonomous vehicles, and distributed sensor systems [1,2].

The key issue of multi-agent systems is the formation and adaptation in their development. Formation refers to a group of agents maintaining specific relative positions and exhibiting coordinated behavior while performing tasks. These formations can be either static, with agents maintaining fixed relative positions, or dynamic, allowing agents to flexibly adjust their positions during task execution. Solving formation problems is crucial for enhancing system efficiency, robustness, and collaborative capabilities. Solving the formation problem in multi-agent systems presents several challenges. First, determining the optimal formation configuration that satisfies task constraints and objectives is a complex optimization problem [3,4]. It involves considering factors such as agent capabilities [5], communication constraints [6], obstacle avoidance [7], and task-specific requirements [8]. Second, ensuring robustness and fault tolerance in formation is crucial, as agents may face uncertainties, failures, or communication disruptions during task execution [9,10]. Developing algorithms and strategies that can handle these uncertainties and maintain formation integrity is a significant research challenge. Finally, scalability is another important consideration, as the number of agents and the complexity of tasks increase. Efficient algorithms and communication protocols are required to handle large-scale multi-agent systems. Therefore, it is highly useful to study the problem of formation control in multi-agent systems.

Recently, the exploration of formation control in fractional systems has garnered considerable interest due to their capacity to capture intricate dynamics with memory and long-range dependencies [11,12]. Fractional systems, characterized by non-integer derivatives, have demonstrated promising applications across diverse domains, including control systems [13], signal processing [14], and optimization. By extending fractional dynamics to multi-agent systems, the formation control challenge in fractional multi-agent systems has emerged as a captivating and sought-after research area. Unlike conventional integer-order systems, fractional-order dynamics introduce memory effects and long-range interactions among agents, allowing for more sophisticated and nuanced formation control strategies [15–17]. Various control techniques have been employed and refined in the realm of fractional formation control, such as sliding-mode control [18,19], double-integrator control [20], and both-and-observer control [21]. Thus, building on these prior studies and the benefits of fractional-order control, the domain of fractional multi-agent system formation holds the potential to revolutionize numerous applications.

In practical scenarios, the prevalence of attacks and uncertainties necessitates considering the impact of the resulting perturbations on systems. As a result, many studies have utilized stochastic differential equations with noise terms to model systems. While the research on continuous-time stochastic systems has been significant, the control strategies proposed for continuous systems are often too idealized for direct application in real-world settings. Consequently, discrete-time stochastic systems have garnered considerable attention due to their enhanced realism and practicality. The researchers in Ref. [22] delved into finite-time $H_\infty$ state estimation for discrete time-delayed stochastic systems using communication protocols. Additionally, Ref. [23] explored recursive fusion estimation for stochastic discrete-time-varying complex networks under stochastic communication protocols. To address energy consumption concerns in scenarios with limited resources, energy-efficient controllers are in high demand. Moreover, the fixed allocation of bandwidth can constrain communication between nodes in networks. To conserve resources and bandwidth, event-triggered control mechanisms are widely implemented in discrete-time stochastic systems. In Ref. [24], an event-triggered law was developed for discrete stochastic systems to ensure system stability. Ref. [25] investigated event-triggered control problems in the presence of packet loss. Furthermore, novel approaches for addressing event-triggered control problems in discrete-time stochastic systems were presented in the context of the pinning synchronization problem in Ref. [26] and the quasi-synchronization in probability problem in Ref. [27]. Compared to systems with continuous-time dynamics, discrete-time systems are more computationally implementable. Additionally, many real-world systems are not accurately captured by continuous dynamics. References [28–30] have explored intriguing questions concerning the consensus stability of discrete-time multi-agent systems. Notably, significant progress has been made in event-triggered control for integer discrete-time stochastic systems. This paper extends event-triggered control from integer discrete-time stochastic systems to fractional systems with time delay, with results applicable to both types of systems.

Inspired by the above, this paper investigates leader-following formation control for discrete-time fractional stochastic multi-agent systems by the event-triggered strategy. In summary, the contributions of this paper can be outlined as:

(1) In contrast with the existing discrete-time literature [31,32], a novel fractional stochastic discrete-time multi-agent system model is introduced.
(2) Unlike the traditional discrete-time fractional stochastic systems [33,34], this paper uses a control law with an event-triggering mechanism, which conserves resources and saves bandwidth.
(3) Pinning controllers and the corresponding sufficient conditions are provided to achieve mean square pinning consensus in the fractional stochastic networks.
(4) Differently from the traditional synchronous control problems [27,35,36], the formation control proposed is structurally more flexible and expandable. When the ratio

formation structure is taken as a zero vector, it is the ordinary asymptotic synchronization problem.

The paper is structured as follows: Section 2 introduces some preliminaries. Section 3 presents the system model and main results. Section 4 illustrates a numerical example. Finally, the conclusion is presented in Section 5.

## 2. Preliminaries

**Definition 1** ([37])**.** *The discrete-time difference operator of order $\alpha$ of function $f(\cdot)$ is defined by*

$$\nabla_T^\alpha f(k) = \frac{1}{T^\alpha}\sum_{r=0}^{[k/T]}(-1)^r\begin{pmatrix}\alpha\\ r\end{pmatrix}f(k-rT), \alpha > 0,$$

*where 0 is the initial time. Take $T = 1$ for convenience, then $\nabla^\alpha f(k) = \sum_{r=0}^{k}(-1)^r\begin{pmatrix}\alpha\\ r\end{pmatrix}f(k-r)$.*

**Lemma 1** ([37])**.** *For $\alpha \in (0,1)$, the function $w_r^\alpha = (-1)^{r+1}\begin{pmatrix}\alpha\\ r\end{pmatrix}$ satisfies the properties $w_0^\alpha = -1, w_{r-1}^\alpha > w_r^\alpha$ for $r \in \mathbb{N}^+ \setminus \{1\}, \lim_{r\to+\infty} w_r^\alpha = 0$ and $\sum_{r=1}^{+\infty} w_r^\alpha = 1$.*

**Lemma 2** ([38])**.** *Consider a sequence of nonnegative random variables $\{V(k)\}_{k\geq 0}$ with $\mathbb{E}\{V(0)\} < \infty$. Let*

$$\mathbb{E}\{V(k+1)\} \leq (1-c_1(k))\mathbb{E}V(k) + c_2(k)$$

*where*

$$\begin{aligned} &c_1(k) \geq 0, c_2(k) \geq 0, \forall k\\ &\sum_{k=0}^{\infty} c_2(k) < \infty, \sum_{k=0}^{\infty} c_1(k) = \infty\\ &\lim_{k\to\infty}\frac{c_2(k)}{c_1(k)} = 0. \end{aligned}$$

*Then, $V(k)$ converges to zero in the mean square, i.e.,*

$$\lim_{k\to\infty}\mathbb{E}V(k) = 0.$$

**Lemma 3** ([39])**.** *Given a convergent series $\{l_m \mid l_m \in \mathbb{R}, m = 1,2,\ldots\}$, then $\left(\sum_{m=1}^{+\infty} l_m z_m\right)^T Q\left(\sum_{m=1}^{+\infty} l_m z_m\right) \leq \left(\sum_{m=1}^{+\infty} l_m\right)\sum_{m=1}^{+\infty} l_m z_m^T Q z_m$ where $z_m \in \mathbb{R}^n, Q \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix.*

## 3. Problem Formulation

Consider the following discrete-time fractional stochastic multi-agent systems composed of the leader

$$\nabla^\alpha x_0(k+1) = 0, \tag{1}$$

and $N$ followers

$$\nabla^\alpha x_i(k+1) = \sigma(k)\zeta(k) + u_i(t), i = 1,2,\ldots,N, \tag{2}$$

where $k \in \mathbb{N}, \alpha \in (0,1), x_i(k) = [x_{i1}(k), x_{i2}(k),\ldots,x_{in}(k)]^T \in \mathbb{R}^n$ is the state of the $i$ th agent at time $k$; $\zeta(k) = [\zeta_1(k),\zeta_2(k),\ldots,\zeta_m(k)]^T \in \mathbb{R}^m$ are system noises on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathrm{P})$ satisfying the following statistical properties:

$$\begin{aligned} &E\{\zeta_i(k)\} = 0,\\ &E\left\{\| \zeta_i(k) \|^2\right\} = \mu^2,\\ &E\{\zeta_i(k)\zeta_j(k)\} = 0, i \neq j, i,j = 1,2,\ldots,m \end{aligned} \tag{3}$$

and $\sigma(\cdot):\mathbb{R}\to\mathbb{R}^{n\times m}$ is continuous nonlinear noise intensity function.

**Assumption 1.** *Assume noise intensity function $\sigma(k)$ satisfied*

$$\sum_{k=1}^{\infty}\parallel \sigma(k)\parallel^2<+\infty,\lim_{k\to+\infty}\parallel \sigma(k)\parallel=0.$$

Defining a formation using a vector $h=\left[h_1^T,\dots,h_N^T\right]^T\in\mathbb{R}^{Nn}$, where $h_i\in\mathbb{R}^n$ represents the desired relative position of agent $i$ with respect to the leader, the leader-following system is deemed to have achieved formation $h$ in a mean square sense if

$$\lim_{k\to\infty}\mathbb{E}\parallel x_i(k)-x_0(k)-h_i\parallel=0, i=1,2,...,N. \tag{4}$$

For this purpose, the control law is designed as follows:

$$u_i(k)=c\sum_{j=1}^{N}g_{ij}\Gamma\left(x_j(k)-h_j\right)-d_i\Gamma(x_i(k)-x_0(k)-h_i)-\sum_{r=1}^{k+1}\omega_r^{\alpha}h_i+h_i, \tag{5}$$

where $c>0$ represents the coupling strength. The matrix $\Gamma=\mathrm{diag}\{\gamma_1,\gamma_2,\dots,\gamma_n\}$ is the internal linking matrix with $\gamma_i>0$. The matrix $G=\left[g_{ij}\right]\in\mathbb{R}^{N\times N}$ denotes the coupling configuration matrix, where $g_{ii}=-\sum_{j=1,j\neq i}^{N}g_{ij}$. If agent $i$ can transmit information to agent $j$, then $g_{ij}>0$, ensuring $\sum_{j=1}^{N}g_{ij}=0$ for all $i=1,2,\dots,N$. A directed path from agent $i_1$ to $i_p$ is represented by a sequence of edges $(i_1,i_2),(i_2,i_3),\dots,\left(i_{p-1},i_p\right)$. Drawing inspiration from pinning control concepts, only a small subset of agents are selected for control. Without loss of generality, control actions are applied to the first $l$ agents. This implies $d_i>0$ for $i=1,\dots,l$, and $d_i=0$ for $i=l+1,\dots,N$.

According to Definition 1 and Equation (5), then Equations (1) and (2) can be rewritten as

$$\begin{aligned}
x_0(k+1)=&\sum_{r=1}^{k+1}\omega_r^{\alpha}x_0(k+1-r)\\
x_i(k+1)=&\sum_{r=1}^{k+1}\omega_r^{\alpha}x_i(k+1-r)+\sigma(k)\zeta(k)+c\sum_{j=1}^{N}g_{ij}\Gamma\left(x_j(k)-h_j\right)\\
&-d_i\Gamma(x_i(k)-x_0(k)-h_i)-\sum_{r=1}^{k+1}\omega_r^{\alpha}h_i+h_i,
\end{aligned} \tag{6}$$

where the function $w_r^{\alpha}=(-1)^{r+1}\begin{pmatrix}\alpha\\ r\end{pmatrix}$ represents a heavy-tailed discretization following a power-law distribution. In the context of a multi-agent system modeled by a discrete-time difference equation with specific "time-delays", where $x_i(k+1)$ relies on all historical states $\{x_i(0),x_i(1),\dots,x_i(k)\}$, it is essential to consider that the impact of historical states on $x_i(k+1)$ may diminish as the time-delay increases. In this scenario, the function $\omega_r^{\alpha}$ is selected as the weight function.

**Remark 1.** *Unlike in the literature [40–42] in the original text on system modeling, which investigates integer-order time delay for discrete systems, this paper considers fractional-order distributed time delay. The propagation delays are distributed over a period of time and may influence the control system to oscillate more. Hence, it is mandatory to investigate the formation control systems with fractional-order distributed delay. Additionally, differing from the literature [32,34] in the original text on system modeling, the systems mentioned in this paper add noise term $\zeta(k)$, which considers the effect of disturbance in real life, having real significance.*

To minimize the frequency of controller updates and communication overhead, a distributed event-triggered strategy is employed for the system described in Equation (6).

Assuming that the first $l$ nodes in the network are under control ($1 \le l < N$), a pinning event-triggered control system is formulated as:

$$\begin{aligned} x_i(k+1) =& \sum_{r=1}^{k+1} \omega_r^\alpha x_i(k+1-r) + c\sum_{j=1}^{N} g_{ij}\Gamma\left(x_j\left(k_s^i\right) - h_j\right) \\ &+\sigma(k)\zeta(k) + h_i - \sum_{r=1}^{k+1} \omega_r^\alpha h_i - d_i\Gamma\left(x_i\left(k_s^i\right) - x_0\left(k_s^i\right) - h_i\right), \\ &k \in \left[k_s^i, k_{s+1}^i\right), i = 1,2,\dots,l, \\ x_i(k+1) =& \sum_{r=1}^{k+1} \omega_r^\alpha x_i(k+1-r) + h_i - \sum_{r=1}^{k+1} \omega_r^\alpha h_i + c\sum_{j=1}^{N} g_{ij}\Gamma\left(x_j\left(k_s^i\right) - h_i\right) + \sigma(k)\zeta(k), \\ &k \in \left[k_s^i, k_{s+1}^i\right), i = l+1, l+2,\dots,N, \end{aligned} \tag{7}$$

where the time sequence $k_0^i, k_1^i, \dots, k_s^i, k_{s+1}^i, \dots$ represents the event triggering time sequence for agent $i$, which is generated by the following event-triggered mechanism.

**Remark 2.** *The pinning control adopted in this paper only requires controlling a subset of nodes to achieve control over the entire system. Compared to non-pinning control [43], the control strategy proposed in this paper is simpler and more efficient. In addition, unlike traditional continuous control [44], event-triggered control only transmits and updates information when triggered, saving resources and bandwidth.*

Let the combined measurement of the systems Equation (5) be

$$q_i(k) = c\sum_{j=1}^{N} g_{ij}\Gamma\left(x_j(k) - h_i\right) - d_i\Gamma(x_i(k) - x_0(k) - h_i), \tag{8}$$

for $i = 1, 2, ..., N$, and the measurement error be

$$e_i(k) = q_i\left(k_s^i\right) - q_i(k), i = 1,2,\dots,N \tag{9}$$

The sequence $k_0^i, k_1^i, \dots, k_s^i, k_{s+1}^i, \dots$ for agent $i$ is determined by the triggering condition specified in Theorem 1:

$$\phi_i(e_i(k), q_i(k)) = 0, i = 1,2,\dots,N. \tag{10}$$

## 4. Main Result

Let $y_i(k) = x_i(k) - x_0(k) - h_i, i = 1,2,\dots,N$. From Equation (7),

$$\begin{aligned} y_i(k+1) =& \sum_{r=1}^{k+1} \omega_r^\alpha y_i(k+1-r) + \sigma(k)\zeta(k) + c\sum_{j=1}^{N} g_{ij}\Gamma\left(x_j\left(k_s^i\right) - h_j\right) - d_i\Gamma y_i\left(k_s^i\right), \\ &k \in \left[k_s^i, k_{s+1}^i\right), i = 1,2,\dots,l, \\ y_i(k+1) =& \sum_{r=1}^{k+1} \omega_r^\alpha y_i(k+1-r) + \sigma(k)\zeta(k) + c\sum_{j=1}^{N} g_{ij}\Gamma\left(x_j\left(k_s^i\right) - h_i\right) \\ &k \in \left[k_s^i, k_{s+1}^i\right), i = l+1, l+2,\dots,N. \end{aligned} \tag{11}$$

Let $f_r^\alpha \triangleq \omega_{r+1}^\alpha$, then

$$\begin{aligned}\sum_{r=1}^{k+1}\omega_r^\alpha x_i(k+1-r) &= \omega_1^\alpha x_i(k)+\sum_{r=2}^{k+1}\omega_r^\alpha x_i(k+1-r)\\ &= \alpha x_i(k)+\sum_{r=1}^{k}\omega_{r+1}^\alpha x_i(k-r) = \alpha x_i(k)+\sum_{r=1}^{k} f_r^\alpha x_i(k-r).\end{aligned} \tag{12}$$

By the gefinition of $q_i(k)$ and $e_i(k)$, we obtain

$$\begin{aligned}y_i(k+1) =& q_i(k)+e_i(k)+\sigma(k)\zeta(k)+\alpha y_i(k)+\sum_{r=1}^{k} f_r^\alpha y_i(k-r),\\ & k\in\left[k_s^i,k_{s+1}^i\right), i=1,2,\ldots,N,\end{aligned} \tag{13}$$

where $d_i = 0, i = l+1, l+2, ..., N$.

Denote

$$\begin{aligned}y(k) &= \left(y_1^{\mathrm{T}}(k), y_2^{\mathrm{T}}(k),\ldots,y_N^{\mathrm{T}}(k)\right)^{\mathrm{T}},\\ e(k) &= \left(e_1^{\mathrm{T}}(k), e_2^{\mathrm{T}}(k),\ldots,e_N^{\mathrm{T}}(k)\right)^{\mathrm{T}},\end{aligned} \tag{14}$$

and $q(k) = \left(q_1^{\mathrm{T}}(k), q_2^{\mathrm{T}}(k),\ldots,q_N^{\mathrm{T}}(k)\right)^{\mathrm{T}}$, then

$$q(k) = ((cG-D)\otimes\Gamma)y(k) \tag{15}$$

where $D = \mathrm{diag}\{d_1, d_2, \ldots, d_l, 0, \ldots, 0\}$.

Thus,

$$\begin{aligned}y(k+1) =& ((cG-D)\otimes\Gamma)y(k)+e(k)+\alpha y(k)+\sum_{r=1}^{k} f_r^\alpha y(k-r)+(1_N\otimes\sigma(k))\zeta(k)\\ =& [\alpha I_{n\times N}+(cG-D)\otimes\Gamma]y(k)+e(k)+\sum_{r=1}^{k} f_r^\alpha y(k-r)+(1_N\otimes\sigma(k))\zeta(k).\end{aligned} \tag{16}$$

**Theorem 1.** *Under Assumption 1, the system Equation (6) will realize formation control if there exists* $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *such that*

$$H \triangleq (1-\eta_2)I_{n\times N} - 2\alpha\eta_1\left(c\frac{G+G^T}{2}-D\right)\otimes\Gamma - \eta_1(cG-D)^T(cG-D)\otimes\Gamma\Gamma > 0, \tag{17}$$

*where* $\eta_1 = 1+\varepsilon_1+\varepsilon_2$, $\eta_2 = \eta_1\alpha^2 + 4(1-\alpha)\left(1+\frac{1}{\varepsilon_1}+\varepsilon_3\right)$ *and the triggering condition as follows*

$$\phi_i(e_i(k), q_i(k)) = \parallel e_i(k)\parallel^2 - \frac{(1-\nu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)}\parallel q_i(k)\parallel^2 \le 0, i = 1,2,\ldots,N \tag{18}$$

*where* $\eta_3 = 1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}$, $A = ((cG-D)\otimes\Gamma)^T((cG-D)\otimes\Gamma)$ *and* $0 < \nu_i < 1$.

**Proof.** Construct the Lyapunov function as follows:

$$V(k) = V_1(k)+V_2(k)+V_3(k), \tag{19}$$

with

$$\begin{aligned}V_1(k) &= y^T(k)y(k),\\ V_2(k) &= \sum_{r=1}^{+\infty} f_r^\alpha \sum_{j=k-r}^{k-1} y^T(j)y(j),\\ V_3(k) &= \sum_{r=k+1}^{+\infty} f_r^\alpha \sum_{j=k-r}^{k-1} y^T(j)y(j),\end{aligned} \tag{20}$$

where $V_2(k), V_3(k) < +\infty$ according to Lemma 1. Then

$$\begin{aligned}
&\mathbb{E}[\Delta V_1(k)]\\
=&\mathbb{E}\Big\{y^T(k)\Big(\alpha I_{n\times N} + (cG-D)^T \otimes \Gamma\Big)\,[(\alpha I_{n\times N} + (cG-D)\otimes\Gamma)y(k)\\
&+2\sum_{r=1}^{k} f_r^{\alpha} y(k-r) + 2(1_N \otimes \sigma(k))\,\zeta(k) + 2e(k)]\\
&+\left(\sum_{r=1}^{k} f_r^{\alpha} y(k-r)\right)^T \left(\sum_{r=1}^{k} f_r^{\alpha} y(k-r) + 2(1_N\otimes\sigma(k))\zeta(k) + 2e(k)\right)\\
&+e^T(k)[e(k) + 2(1_N\otimes\sigma(k))\zeta(k)] + \zeta^T(k)(1_N\otimes\sigma(k))^T(1_N\otimes\sigma(k))\zeta(t)\Big\}.
\end{aligned} \tag{21}$$

Fromthepropertiesofthestochasticprocess$\{\zeta(k), k \geq 0\}$ in Equation (3), then

$$\begin{aligned}
&\mathbb{E}[\Delta V_1(k)]\\
=&\mathbb{E}\Big\{y^T(k)\Big(\alpha I_{n\times N} + (cG-D)^T \otimes \Gamma\Big)\,[(\alpha I_{n\times N} + (cG-D)\otimes\Gamma)y(k)\\
&+2\sum_{r=1}^{k} f_r^{\alpha}\, y(k-r) + 2e(k)] + \left(\sum_{r=1}^{k} f_r^{\alpha} y(k-r)\right)^T \left(\sum_{r=1}^{k} f_r^{\alpha} y(k-r)\right.\\
&\left.+2e(k)\right) + e^T(k)e(k) + \zeta^T(k)(1_N\otimes\sigma(t))^T(1_N\otimes\sigma(t))\zeta(k)\Big\}\\
\leq&\mathbb{E}\Big\{y^T(k)[(1+\varepsilon_2+\varepsilon_3)\,\Big(\alpha I_{n\times N} + (cG-D)^T\otimes\Gamma\Big)\\
&\times(\alpha I_{n\times N} + (cG-D)\otimes\Gamma) - I_{n\times N}]y(k)\\
&+\left(1+\frac{1}{\varepsilon_1}+\varepsilon_3\right)\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right)^T\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right)\\
&+\left(1+\frac{1}{\varepsilon_1}+\varepsilon_3\right)\left(\sum_{r=k+1}^{+\infty} f_r^{\alpha} y(k-r)\right)^T\left(\sum_{r=k+1}^{+\infty} f_r^{\alpha} y(k-r)\right)\\
&+e^T(k)e(k) + \zeta^T(k)(1_N\otimes\sigma(t))^T(1_N\otimes\sigma(t))\zeta(k)\Big\}.
\end{aligned} \tag{22}$$

Moreover,

$$\begin{aligned}
&\mathbb{E}[\Delta V_2(k)]\\
=&\mathbb{E}\left[\sum_{r=1}^{+\infty} f_r^{\alpha} \sum_{j=k+1-r}^{k} y^T(j)y(j) - \sum_{r=1}^{+\infty} f_r^{\alpha} \sum_{j=k-r}^{k} y^T(j)y(j)\right]\\
=&\mathbb{E}\left[\sum_{r=1}^{+\infty} f_r^{\alpha} y^T(k)y(k) - \sum_{r=1}^{+\infty} f_r^{\alpha} y^T(k-r)y(k-r)\right].
\end{aligned} \tag{23}$$

According to Lemma 1, then $\sum_{r=1}^{+\infty} f_r^{\alpha} = 1-\alpha \in (0,1)$. From Lemma 3, then

$$\sum_{r=1}^{+\infty} f_r^{\alpha} y^T(k-r)y(k-r) \geq \frac{1}{1-\alpha}\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right)^T\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right) \tag{24}$$

Thus,

$$\mathbb{E}[\Delta V_2(k)] \leq \mathbb{E}\left[y^T(k)y(k) - \frac{1}{1-\alpha}\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right)^T\left(\sum_{r=1}^{+\infty} f_r^{\alpha} y(k-r)\right)\right]. \tag{25}$$

Furthermore,

$$\begin{aligned}
&\mathbb{E}[\Delta V_3(k)] \\
=&\mathbb{E}\left[\sum_{r=k+2}^{+\infty} f_r^\alpha \sum_{j=k+1-r}^{k} y^T(j)y(j) - \sum_{r=k+1}^{+\infty} f_r^\alpha \sum_{j=k-r}^{k-1} y^T(j)y(j)\right] \\
\leq&\mathbb{E}\left[\sum_{r=k+1}^{+\infty} f_r^\alpha y^T(k)y(k) - \sum_{r=k+1}^{+\infty} f_r^\alpha y^T(k-r)y(k-r)\right] \\
\leq&\mathbb{E}\left[y^T(k)y(k) - \frac{1}{1-\alpha}\left(\sum_{r=k+1}^{+\infty} f_r^\alpha y(k-r)\right)^T\left(\sum_{r=k+1}^{+\infty} f_r^\alpha y(k-r)\right)\right].
\end{aligned} \tag{26}$$

Combining Equations (19)–(26) gives

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
=&\mathbb{E}\{\Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k)\} \\
\leq&\mathbb{E}\left\{-y^{\mathrm{T}}(k)Hy(k) + e^{\mathrm{T}}(k)e(k)\right\} \\
\leq&-\lambda_{\min}(H)\mathbb{E}\left\{\| y(k) \|^2\right\} + \mathbb{E}\left\{\| e(k) \|^2\right\} + Nm\mu^2\| \sigma(t) \|^2.
\end{aligned} \tag{27}$$

Then, from Equation (18),

$$\begin{aligned}
\| e(k) \|^2 \leq& \sum_{i=1}^{N} \| e_i(k) \|^2 \\
\leq& \frac{(1-\nu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} \sum_{i=1}^{N} \| q_i(k) \|^2 \\
=& \frac{(1-\nu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} \| q(k) \|^2 \\
=& \frac{(1-\nu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} \| ((cW - \gamma(k)D) \otimes \Gamma)y(k) \|^2 \\
\leq& \frac{(1-\nu)\lambda_{\min}(H)}{\eta_3} \| y(k) \|^2,
\end{aligned} \tag{28}$$

where $\nu = \max\{\nu_i\}$. Therefore,

$$\mathbb{E}\{\Delta V(k)\} \leq -\varepsilon\mathbb{E}\left\{\| y(k) \|^2\right\} + Nm\mu^2\| \sigma(t) \|^2. \tag{29}$$

where $\varepsilon = \nu\lambda_{\min}(H) > 0$.

Then, from Assumption 1, we can obtain $Nm\mu^2\| \sigma(k) \|^2 > 0$, $Nm\sum_{k=1}^{\infty} \| \sigma(k) \|^2 < +\infty$, $\sum_{k=1}^{\infty} \varepsilon = +\infty$ and $\lim_{k\to+\infty} \frac{\sigma(k)}{\varepsilon} = 0$, which satisfies the condition of Lemma 2. Thus, from Lemma 2, $\lim_{k\to\infty} \mathbb{E}\| x_i(k) - x_0(k) - h_i \| = 0, \forall i = 1,2,\ldots,N$. □

## 5. Simulation

The directed communication graph given by Figure 1 is composed of four agents. The communication topology shown in Figure 1 has a spanning tree, with agent 1 as the root node. Let $N = 4, n = 3, \alpha = 0.88, \Gamma = \mathrm{diag}\{1,1,1\}, d_1 = d_2 = d_3 = d_4 = 0.2, c = 0.02, \sigma(k) = \frac{3}{(k+1)^2}$, and the initial position is as follows:

$$\begin{aligned}
&x_0(0) = \begin{pmatrix} 1.1 & -0.9 & 0.3 \end{pmatrix}^T \\
&x_1(0) = \begin{pmatrix} 1.8 & 0.9 & -1.1 \end{pmatrix}^T, x_2(0) = \begin{pmatrix} 0.7 & -0.6 & -1.1 \end{pmatrix}^T, \\
&x_3(0) = \begin{pmatrix} -0.9 & 1.1 & -1.4 \end{pmatrix}^T, x_4(0) = \begin{pmatrix} 1.4 & -0.8 & -0.6 \end{pmatrix}^T.
\end{aligned}$$

Choose the formation vector

$$h = \begin{pmatrix} 3 & -3 & 5 & -5 \\ -3 & 3 & 4 & -4 \\ -3 & 3 & 3 & -3 \end{pmatrix}$$

for the leader-following system. For the triggering condition (17),

$$\| e_i(k) \|^2 \leq \frac{(1-\nu_i)\lambda_{\min}(H)}{\eta_3 \lambda_{\max}(A)} \| q_i(k) \|^2, i = 1, 2, \ldots, N$$

It can be calculated that $H = \begin{bmatrix} -3.7600 & 0.0048 & 0.0056 & 0 \\ 0.0048 & -3.7475 & 0.0056 & 0 \\ 0.0056 & 0.0056 & -3.7475 & 0.0048 \\ 0 & 0 & 0.0048 & -3.7360 \end{bmatrix} \otimes I_3 < 0$, which satisfies the condition of Theorem 1. Note that the definitions of $H$ and $A$ are independent of $\mu_i$, so regardless of the sizes of $H$ and $A$, it is always possible to find three sets suitable $\mu_i \in (0,1)$ and $\eta_3$ such that $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3 \lambda_{\max}(A)} = 0.75$, $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3 \lambda_{\max}(A)} = 0.9$, and $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3 \lambda_{\max}(A)} = 1.05$, respectively. Therefore, we can assume that the event-triggering conditions used in this simulation are $\| e_i(k) \|^2 \leq 0.75 \| q_i(k) \|$, $\| e_i(k) \|^2 \leq 0.9 \| q_i(k) \|$, and $\| e_i(k) \|^2 \leq 1.05 \| q_i(k) \|$, separately. Obviously, from Figure 2, it can be observed that the state of the error system tends to zero over time, indicating that systems (1) and (2) achieve formation control under the requirements of Theorem 1. Figures 3–5 show the moments of updating the control law; it can be seen that as the trigger parameter increases, the number of controller updates becomes fewer, revealing that the larger the threshold value, the fewer the number of updates to the controller. Thus, event-triggered control reduces the update frequency compared to continuous control, which requires real-time monitoring of agent information, saving the resources needed for monitoring and interacting information. Figure 6 demonstrates that the followers eventually stabilize at a fixed position and maintain a constant relative position with the leader, thus achieving formation control. Figures 7–10 depict their specific positions in space, showing that the followers stabilize over time, indicating successful formation control.

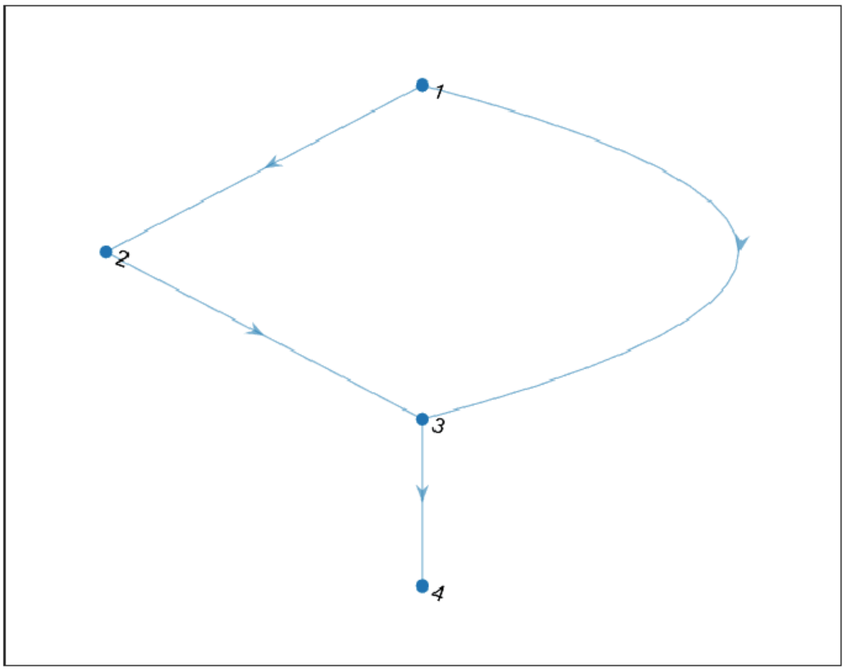


**Figure 1.** Communication graph of multi-agent system.

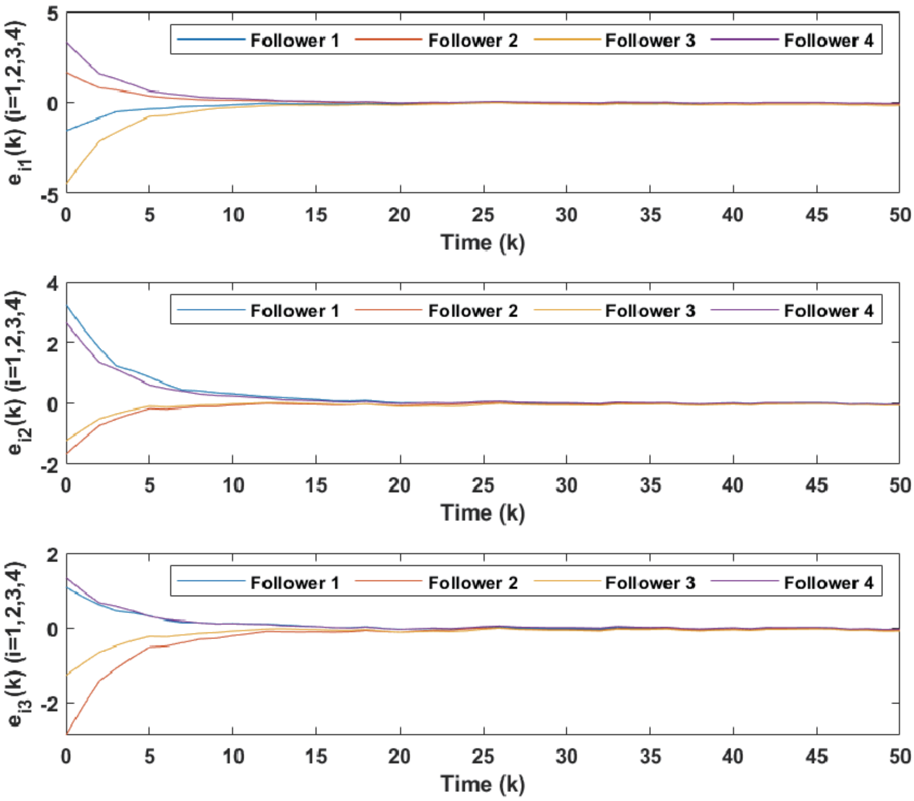


**Figure 2.** Estimation error trajectories of agents over time.

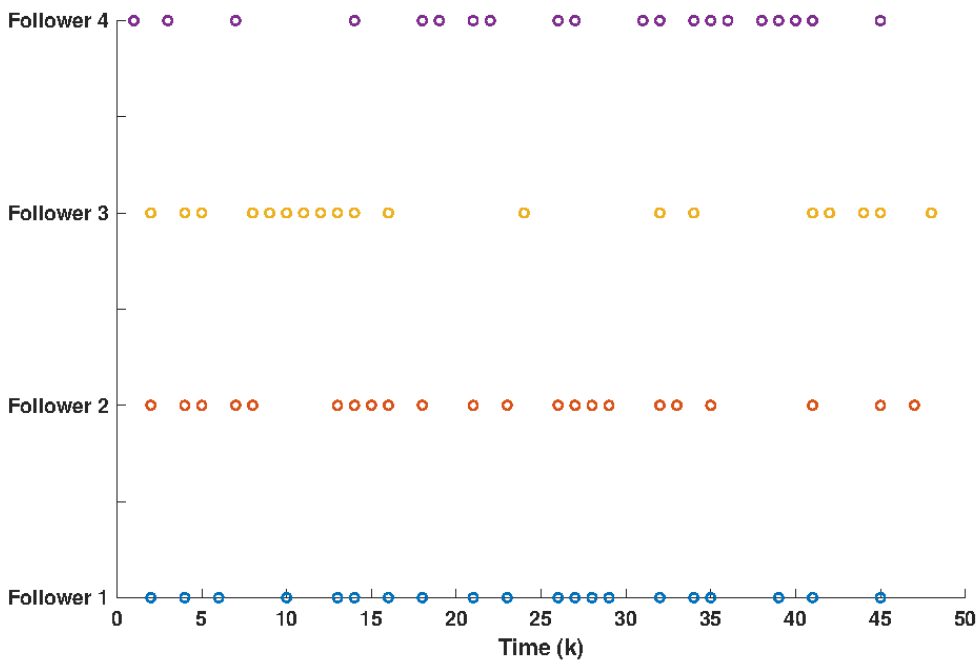


**Figure 3.** Update times of event-trigger controllers for $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} = 0.75$.

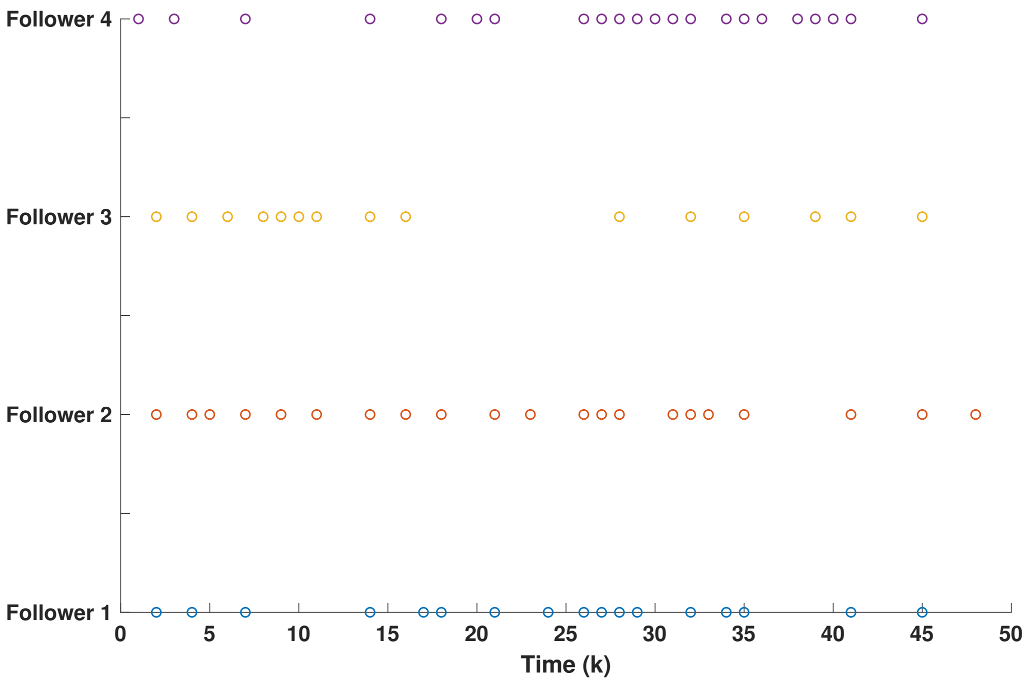


**Figure 4.** Update times of event-trigger controllers for $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} = 0.9$.

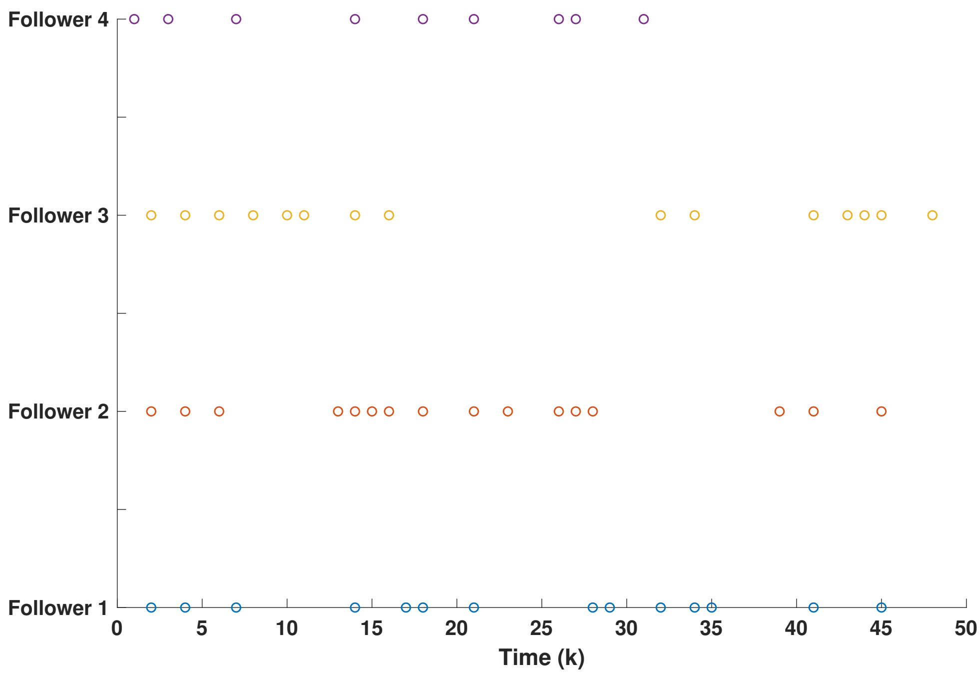


**Figure 5.** Update times of event-trigger controllers for $\frac{(1-\mu_i)\lambda_{\min}(H)}{\eta_3\lambda_{\max}(A)} = 1.05$.

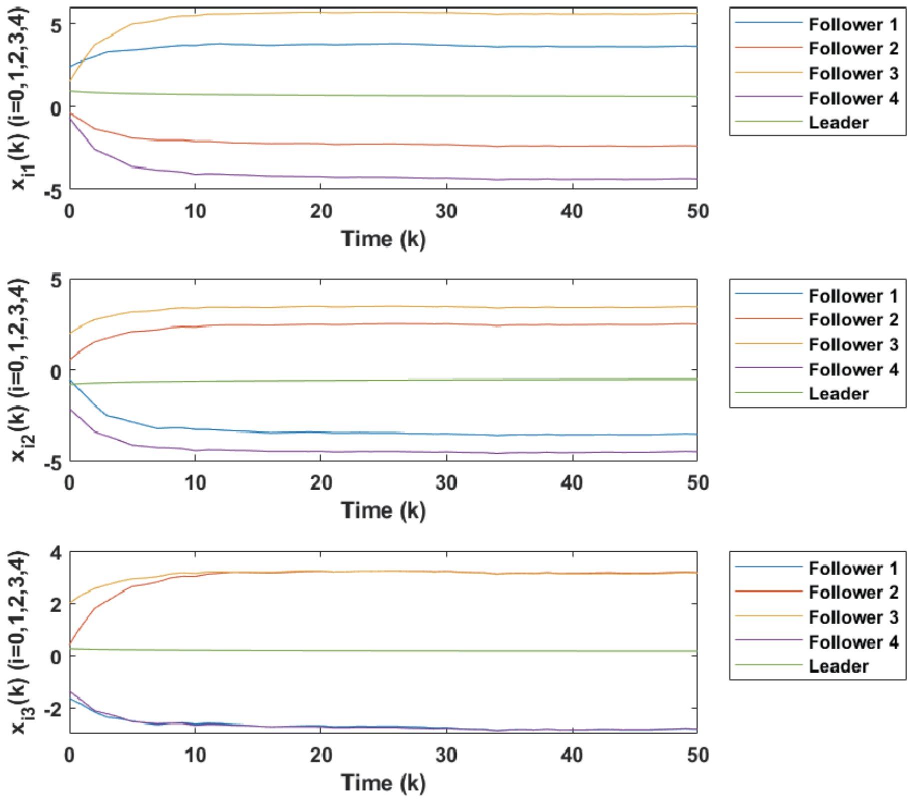


**Figure 6.** The state trajectories of followers and leader.

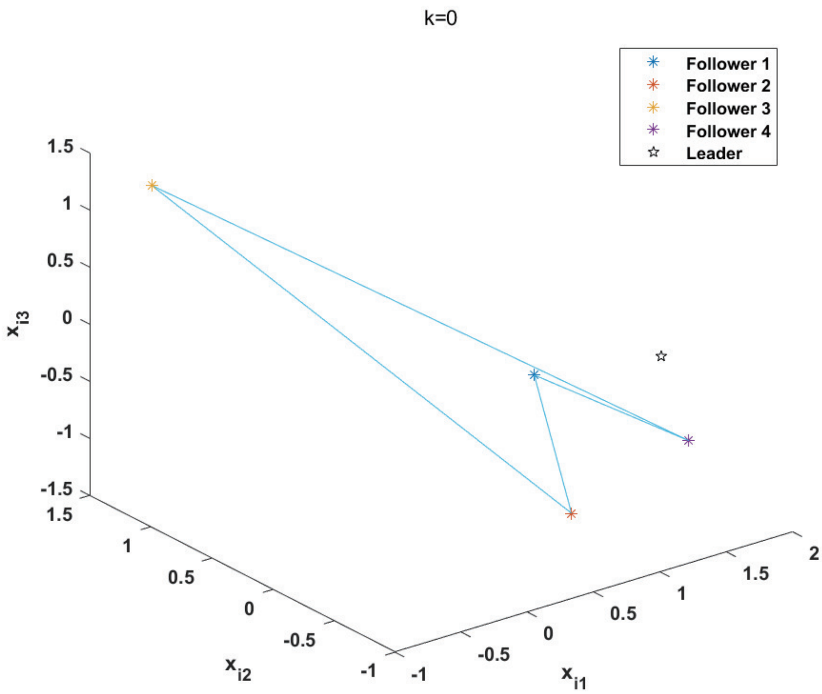


**Figure 7.** Position trajectories at $k = 0$.

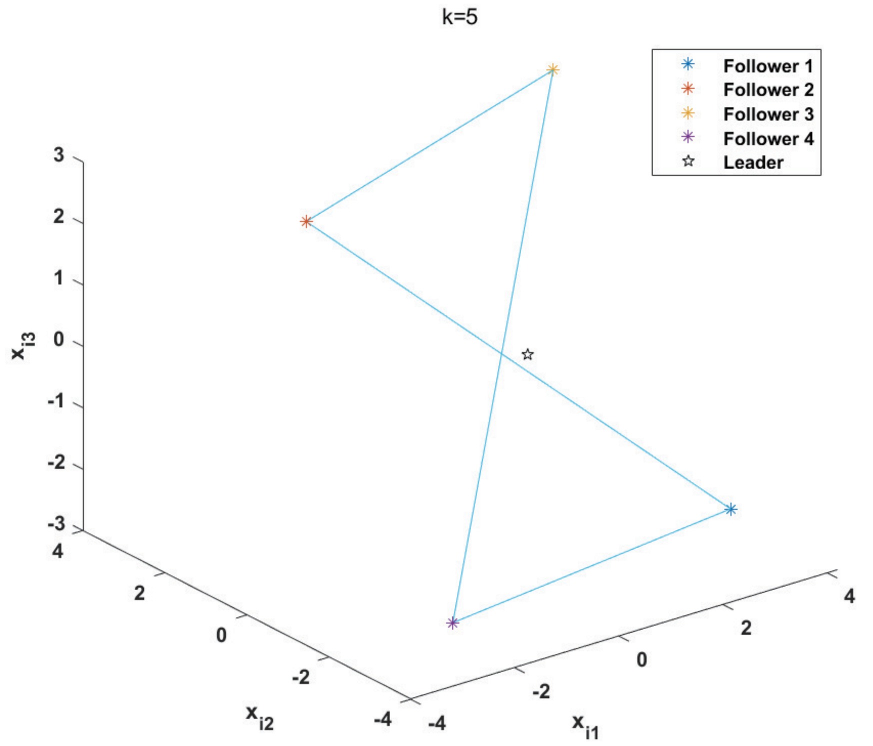


**Figure 8.** Position trajectories at $k = 5$.

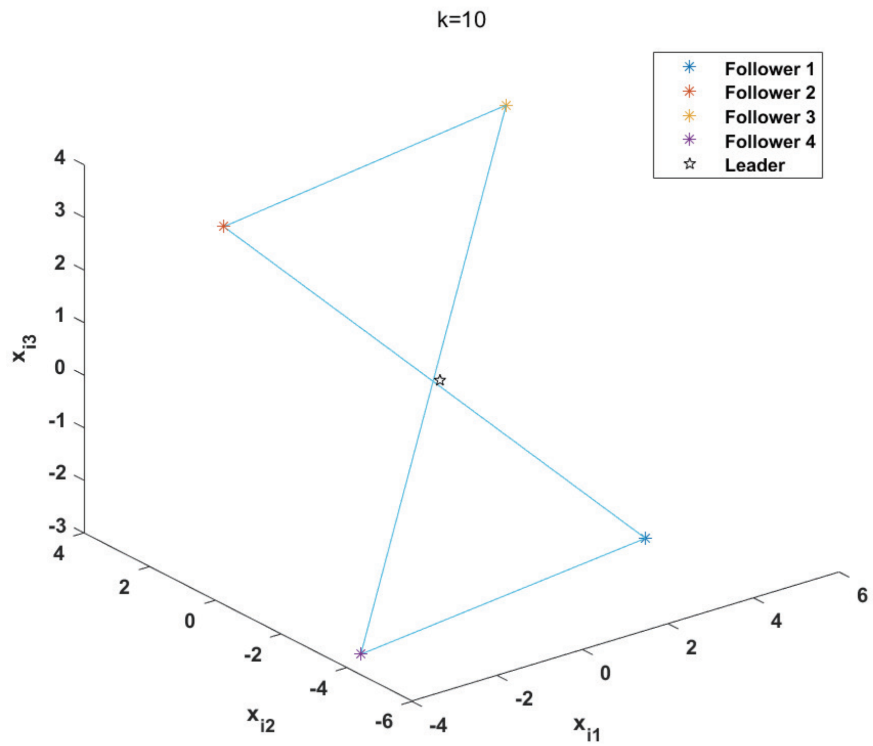


**Figure 9.** Position trajectories at $k = 10$.

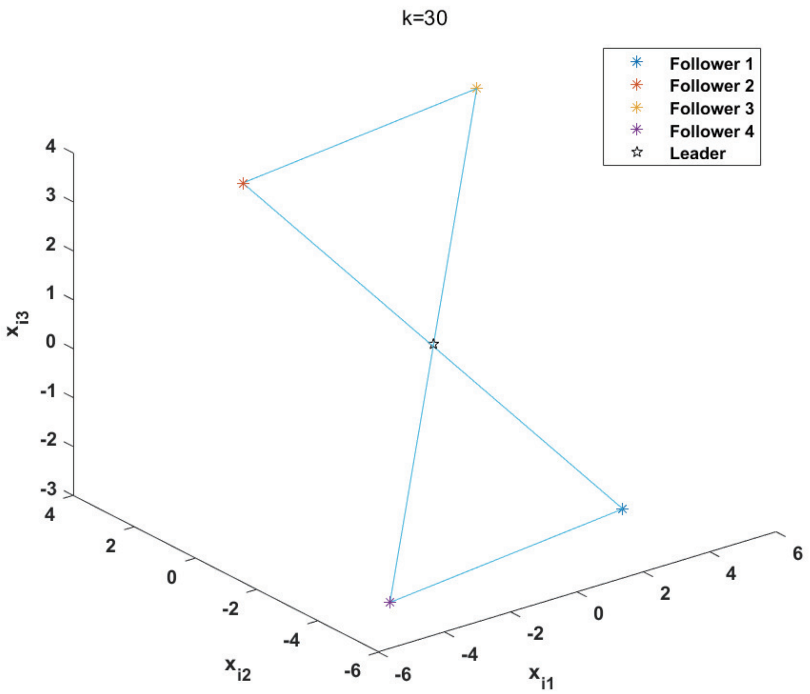


**Figure 10.** Position trajectories at $k = 30$.

## 6. Conclusions

We studied leader-following formation control for discrete-time fractional stochastic multi-agent systems by an event-triggered strategy. First, a new discrete-time fractional stochastic system is modeled, which with stochastic terms can better characterize systems operating under the influence of disturbances and cases of restricted communication. Second, an event-triggered control law is proposed for controlling partial followers, with the aim of making all the agents eventually reach their expected positions and achieve a certain shape of motion. Finally, the theory of Lyapunov's energy equation is employed to prove that under the impact of disturbance, the formation error eventually tends to zero in the mean square, and formation control can be achieved. However, the formation formed in this way can only achieve translational motion in the end, so the affine formation control with higher maneuverability and flexibility will be studied in the future.

**Author Contributions:** Writing—original draft, J.W.; Writing—review & editing, J.W., Y.Y. and G.R.; Funding acquisition, G.R. and Y.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work is funded by the National Natural Science Foundation of China (No. 62173027) and the Talent Fund of Beijing Jiaotong University (Grant No. 2023XKRC009).

**Data Availability Statement:** The data are available upon request.

**Acknowledgments:** I am grateful for the valuable comments and observations of the reviewers of this study, which allowed for a substantial improvement of this work.

**Conflicts of Interest:** The author declares no conflicts of interest.

## References

1. Olfati, S.R.; Fax, J.A.; Murray, R.M. Consensus and Cooperation in Networked Multi-Agent Systems. *Proc. IEEE* **2007**, *95*, 215–233. [CrossRef]
2. Olfati, S. Flocking for multi-agent dynamic systems: Algorithms and theory. *IEEE Trans. Autom. Control* **2006**, *51*, 401–420. [CrossRef]
3. Wu, B.F.; Peng, Z.X.; Wen, G.G.; Yang, S.C.; Huang, T.W. Sampling-Optimization-Based Adaptive Formation Tracking with Trajectory Protection. *IEEE Syst. J.* **2023**, *17*, 6449–6459.
4. Liu, Y.; Li, Y.M. Application of Inverse Optimal Formation Control for Euler-Lagrange Systems. *IEEE Trans. Intell. Transp. Syst.* **2023**, *24*, 5655–5662. [CrossRef]

5. Huang, F.P.; Duan, M.M.; Su, H.F.; Zhu, S.Y. Distributed Optimal Formation Control of Second-Order Multiagent Systems with Obstacle Avoidance. *IEEE Control Syst. Lett.* **2023**, *7*, 2647–2652. [CrossRef]
6. Zhang, J.X.; Fu, Y.; Fu, J. Adaptive Finite-Time Optimal Formation Control for Second-Order Nonlinear Multiagent Systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2023**, *53*, 6132–6144. [CrossRef]
7. Wu, W.; Tong, S.C. Collision-Free Adaptive Fuzzy Formation Control for Stochastic Nonlinear Multiagent Systems. *IEEE Trans. Syst. Man, Cybern. Syst.* **2023**, *53*, 5454–5465. [CrossRef]
8. Su, H.; Zhang, B.L.; Zhou, J.; Xue, J.; Zheng, Y.S.; Ma, H. Collision-Risk-Based Event-Triggered Optimal Formation Control for Mobile Multiagent Systems Under Incomplete Information Conditions. *IEEE Trans. Syst. Man, Cybern. Syst.* **2023**, *53*, 4888–4898. [CrossRef]
9. Zhao, J.N.; Li, X.M.; Yu, X.; Wang, H.S. Finite-Time Cooperative Control for Bearing-Defined Leader-Following Formation of Multiple Double-Integrators. *IEEE Trans. Cybern.* **2022**, *52*, 13363–13372. [CrossRef] [PubMed]
10. Cheng, W.L.; Zhang, K.; Jiang, B. Fixed-Time Fault-Tolerant Formation Control for a Cooperative Heterogeneous Multiagent System With Prescribed Performance. *IEEE Trans. Syst. Man Cybern. Syst.* **2023**, *53*, 462–474. [CrossRef]
11. Gao, Z.Y.; Zhang, H.J.; Wang, Y.C.; Mu, Y.F. Time-varying output formation-containment control for homogeneous/heterogeneous descriptor fractional-order multi-agent. *Inf. Sci.* **2021**, *567*, 146–166. [CrossRef]
12. Cajo, R.; Guinaldo, M.; Fabregas, E.; Dormido, S.; Plaza, D.; De Keyser, R.; Ionescu, C. Distributed Formation Control for Multiagent Systems Using a Fractional-Order Proportional–Integral Structure. *IEEE Trans. Control Syst. Technol.* **2021**, *29*, 2738–2745. [CrossRef]
13. Liu, L.; Li, P.; Chen, W.; Qin, K.Y.; Qi, L. Distributed formation control of fractional-order multi-agent systems with relative damping and nonuniform time-delays. *ISA Trans.* **2019**, *93*, 189–198. [CrossRef] [PubMed]
14. Cao, Y.C.; Ren, W. Distributed formation control for fractional-order systems: Dynamic interaction and absolute/relative damping. *Syst. Control Lett.* **2021**, *59*, 233–240. [CrossRef]
15. Wang, Y.G.; Zhang, J.; Wu, H.Q. Distributed Adaptive Mittag–Leffler Formation Control for Second-Order Fractional Multi-Agent Systems via Event-Triggered Control Strategy. *Fractal Fract.* **2022**, *6*, 380. [CrossRef]
16. Zamani, H.; Johari Majd, V.; Khandani, K. Formation tracking control of fractional-order multi-agent systems with fixed-time convergence. *Proc. Inst. Mech. Eng. Part I J. Syst. Control Eng.* **2022**, *236*, 1618–1629. [CrossRef]
17. Wu, F.; Liu, M.; Feng, Z.Y.; Cao, X.B. Fractional-order sliding mode attitude coordinated control for spacecraft formation flying with unreliable wireless communication. *IET Control Theory Appl.* **2023**, *17*, 368–380. [CrossRef]
18. Meng, X.; Jiang, B.P.; Hamid, R.K.; Gao, C.C. Leader–follower sliding mode formation control of fractional-order multi-agent systems: A dynamic event-triggered mechanism. *Neurocomputing* **2023**, *69*, 126691. [CrossRef]
19. Zamani, H.; Khandani, K.; Majd, V.J. Fixed-time sliding-mode distributed consensus and formation control of disturbed fractional-order multi-agent systems. *ISA Trans.* **2023**, *138*, 37–48.
20. Jun, L.; Li, P.; Qi, L.; Chen, W.; Qin, K.Y. Distributed formation control of double-integrator fractional-order multi-agent systems with relative damping and nonuniform time-delays. *J. Frankl. Inst.* **2019**, *356*, 5122–5150.
21. Gong, Y.F.; Wen, G.G.; Peng, Z.X.; Huang, T.W.; Chen, Y.W. Observer-Based Time-Varying Formation Control of Fractional-Order Multi-Agent Systems With General Linear Dynamics. *IEEE Trans. Circuits Syst. II Express Briefs* **2020**, *67*, 82–86. [CrossRef]
22. Wan, X.B.; Wang, Z.D.; Han, Q.L.; Wu, M. Finite-Time $H_\infty$ State Estimation for Discrete Time-Delayed Genetic Regulatory Networks Under Stochastic Communication Protocols. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2018**, *65*, 3481–3491. [CrossRef]
23. Alsaadi, F.E.; Wang, Z.D.; Wang, D.; Alsaadi, F.E.; Alsaade, F.W. Recursive fusion estimation for stochastic discrete time-varying complex networks under stochastic communication protocol: The state-saturated case. *Inf. Fusion* **2020**, *60*, 11–19. [CrossRef]
24. Wang, M.; Wang, Z.D.; Chen, Y.; Sheng, W.G. Event-based adaptive neural tracking control for discrete-time stochastic nonlinear systems: A triggering threshold compensation strategy. *IEEE Trans. Neural Netw. Learn. Syst.* **2019**, *31*, 1968–1981. [CrossRef]
25. Hu, Z.P.; Shi, P.; Zhang, J.; Deng, F.Q. Control of discrete-time stochastic systems with packet loss by event-triggered approach. *IEEE Trans. Syst. Man Cybern. Syst.* **2018**, *51*, 755–764. [CrossRef]
26. Li, B.; Wang, Z.D.; Ma, L.F. An event-triggered pinning control approach to synchronization of discrete-time stochastic complex dynamical networks. *IEEE Trans. Neural Netw. Learn. Syst.* **2018**, *29*, 5812–5822. [CrossRef]
27. Ding, D.R.; Wang, Z.D.; Shen, B.; Wei, G.L. Event-triggered consensus control for discrete-time stochastic multi-agent systems: The input-to-state stability in probability. *Automatica* **2015**, *62*, 284–291. [CrossRef]
28. Lin, P.; Jia, Y.M. Consensus of second-order discrete-time multi-agent systems with nonuniform time-delays and dynamically changing topologies. *Automatica* **2009**, *45*, 2154–2158. [CrossRef]
29. Gao, Y.P.; Ma, J.W.; Zuo, M.; Jiang, T.Q.; Du, J.P. EConsensus of discrete-time second-order agents with time-varying topology and time-varying delays. *J. Frankl. Inst.* **2012**, *349*, 2598–2608. [CrossRef]
30. Xu, X.L.; Chen, S.Y.; Huang, W.; Gao, L.X. Leader-following consensus of discrete-time multi-agent systems with observer-based protocols. *Neurocomputing* **2013**, *118*, 334–341. [CrossRef]
31. Liu, Z.X.; Li, Y.B.; Wang, F.Y.; Chen, Z.Q. Reduced-order observer-based leader-following formation control for discrete-time linear multi-agent systems. *IEEE/CAA J. Autom. Sin.* **2020**, *8*, 1715–1723. [CrossRef]
32. Ren, G.J.; Yu, Y.G.; Wei, J.M.; Xu, C.H. Pinning event-triggered control for stochastic discrete-time complex networks with time-varying delay. *IET Control Theory Appl.* **2019**, *13*, 2207–2216. [CrossRef]

33. Yuan, X.L.; Ren, G.J.; Yu, Y.G.; Sun, W.J. Mean-square pinning control of fractional stochastic discrete-time complex networks. *J. Frankl. Inst.* **2022**, *359*, 2663–2680. [CrossRef]
34. Yang, D.S.; Yu, Y.G.; Hu, W.; Yuan, X.L.; Ren, G.J. Mean Square Asymptotic Stability of Discrete-Time Fractional Order Stochastic Neural Networks with Multiple Time-Varying Delays. *Neural Process. Lett.* **2023**, *55*, 9247–9268. [CrossRef]
35. Yang, H.Y.; Zhang, H.; Wang, Z.P.; Yan, H.C. Reliable leader-following consensus of discrete-time semi-markovian jump multi-agent systems. *IEEE Trans. Netw. Sci. Eng.* **2023**, *10*, 3505–3518. [CrossRef]
36. Zhang, R.H.; Zhang, Y.Y.; Zong, X.F. Stochastic leader-following consensus of discrete-time nonlinear multi-agent systems with multiplicative noises. *J. Frankl. Inst.* **2022**, *359*, 7753–7774. [CrossRef]
37. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998; pp. 22–26.
38. Li, T.; Zhang, J.F. Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises. *IEEE Trans. Autom. Control* **2010**, *55*, 2043–2057. [CrossRef]
39. Li, Z.; Zhang, W.B.; Fang, J.A.; He, G. Synchronisation of discrete-time complex networks with delayed heterogeneous impulses. *IET Control Theory Appl.* **2015**, *9*, 2648–2656. [CrossRef]
40. Liu, P.; Kong, M.X.; Zeng, Z.G. Projective Synchronization Analysis of Fractional-Order Neural Networks With Mixed Time Delays. *IEEE Trans. Cybern.* **2022**, *52*, 6798–6808. [CrossRef]
41. Si, L.Z.; Xiao, M.; Jiang, G.P.; Cheng, Z.S.; Song, Q.K.; Cao, J.D. Dynamics of Fractional-Order Neural Networks With Discrete and Distributed Delays. *IEEE Access* **2020**, *8*, 46071–46080. [CrossRef]
42. Lv, S.S.; Pan, M.; Li, X.G.; Cai, W.Y.; Lan, T.Y.; Li, B.Q. Consensus Control of Fractional-Order Multi-Agent Systems With Time Delays via Fractional-Order Iterative Learning Control. *IEEE Access* **2019**, *7*, 159731–159742. [CrossRef]
43. Liu, J.; Yang, G.B.; Zhou, N.; Qin, K.Y.; Chen, B.D.; Wu, Y.H.; Choi, K.S. Event-triggered consensus control based on maximum correntropy criterion for discrete-time multi-agent systems. *Neurocomputing* **2023**, *545*, 126323. [CrossRef]
44. Zhuang, J.W.; Peng, S.G.; Wang, Y.H. Exponential consensus of stochastic discrete multi-agent systems under DoS attacks via periodically intermittent control: An impulsive framework. *Appl. Math. Comput.* **2022**, *433*, 127389. [CrossRef]