*fractal and fractional*

MDPI

*Article*

# Representations of Solutions of Time-Fractional Multi-Order Systems of Differential-Operator Equations

**Sabir Umarov**

Department of Mathematics, University of New Haven, 300 Boston Post Road, West Haven, CT 06516, USA; sumarov@newhaven.edu

**Abstract:** This paper is devoted to the general theory of systems of linear time-fractional differential-operator equations. The representation formulas for solutions of systems of ordinary differential equations with single (commensurate) fractional order is known through the matrix-valued Mittag-Leffler function. Multi-order (incommensurate) systems with rational components can be reduced to single-order systems, and, hence, representation formulas are also known. However, for arbitrary fractional multi-order (not necessarily with rational components) systems of differential equations, the representation formulas are still unknown, even in the case of fractional-order ordinary differential equations. In this paper, we obtain representation formulas for the solutions of arbitrary fractional multi-order systems of differential-operator equations. The existence and uniqueness theorems in appropriate topological vector spaces are also provided. Moreover, we introduce vector-indexed Mittag-Leffler functions and prove some of their properties.

**Keywords:** fractional derivatives; fractional-order systems of differential-operator equations; Cauchy problem; matrix-valued Mittag-Leffler function; representation of solution

check for updates

**Citation:** Umarov, S. Representations of Solutions of Time-Fractional Multi-Order Systems of Differential-Operator Equations. *Fractal Fract.* **2024**, *8*, 254. https://doi.org/10.3390/fractalfract8050254

Academic Editor: Ivanka Stamova

Received: 25 February 2024
Revised: 21 April 2024
Accepted: 24 April 2024
Published: 25 April 2024



## 1. Introduction

Let $X$ be a reflexive Banach space and $A : \mathcal{D} \to X$ a closed linear operator with a domain $\mathcal{D} \subset X$. Consider the systems of $m \geq 1$ time-fractional differential operator equations, which we will write in the form

$$D^{\beta}\mathcal{U}(t) = \mathcal{F}(A)\mathcal{U}(t) + \mathcal{H}(t), \quad t > t_0, \tag{1}$$

with the initial condition

$$B\,\mathcal{U}(t_0) = \Phi. \tag{2}$$

In Equation (1), $t > t_0$; $\mathcal{B} = (\beta_1, \dots, \beta_m)$, $0 < \beta_j \leq 1$, fractional orders of the system $\mathcal{U}(t) : [t_0, \infty) \to X \times \cdots \times X$, is an abstract vector-valued function with components $u_j(t)$, $j = 1, \dots, m$, to be found, and

$$D^{\mathcal{B}}\mathcal{U} = (D^{\beta_1}u_1(t), \dots, D^{\beta_m}u(t)). \tag{3}$$

Here, $D^{\beta_j}$, $j = 1, \dots, m$, is the fractional-order derivative of order $0 < \beta_j \leq 1$ in the sense of Riemann–Liouville or Caputo. The matrix-valued operator $\mathcal{F}(A)$ on the right-hand side of Equation (1) has the form

$$\mathcal{F}(A) = \begin{bmatrix} f_{11}(A) & \dots & f_{1m}(A) \\ \dots & & \\ f_{m1}(A) & \dots & f_{mm}(A) \end{bmatrix}; \tag{4}$$

$H(t) : [t_0, \infty] \to X \times \cdots \times X$ is a given vector-valued function satisfying some conditions clarified later. In initial condition (2), the operator $B$ depends on whether $D^{\mathcal{B}}$ is in the sense of Riemann–Liouville or Caputo and $\Phi$ is a given element of some topological vector space specified later. The operator $\mathcal{F}(A)$ has a matrix symbol $\mathcal{F}(z) \equiv \{f_{ki}(z)\}, k, j = 1, \ldots, m$, the entries of which may have singularities in the spectrum of the operator $A$. The exact definitions of operators $D^{\beta_j}$, $j = 1, \ldots, m$, and $f_{kj}(A)$, $k, j = 1, \ldots, m$, are given in Section 2.

It is well known (see, e.g., [1,2]) that if $\mathcal{B} = (1, \ldots, 1)$, then the solution is represented in the form

$$\mathcal{U}(t) = S(t, A)\Phi + \int_{t_0}^{t} S(t-\tau, A)H(\tau)d\tau,$$

where

$$S(t, A) = \exp(-t\mathcal{F}(A))$$

is *the solution operator*, which has an exponential form. It is also known [3–10] that, in the case of various fractional-order differential equations (not systems), the solution can be represented through the Mittag-Leffler (ML) function

$$E_{\beta,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \nu)}, \quad z \in \mathbb{C}, \tag{5}$$

which generalizes the exponential function. Namely, if $\beta = \nu = 1$, then we have $E_{\beta,\nu}(z) = \exp(z)$. In the case of systems, when $\mathcal{B}$ has equal components, i.e., $\beta_j = \beta, j = 1, \ldots, m$, a representation formula for the solution is obtained via the matrix-valued ML function [11–15]. Namely, the solution operator emerges in the form

$$S(t, A) = E_{\beta}(t^{\beta}\mathcal{F}(A)),$$

where $E_{\beta}(Z)$ for a matrix $Z$ is defined by

$$E_{\beta}(Z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\beta n + 1)} Z^n.$$

In the case where the components of the vector-order $\mathcal{B}$ in Equation (1) are rational, i.e., $\beta_j = p_j/q_j$, where $p_j$ and $q_j$ are co-prime numbers, the corresponding system can be reduced to a system with a scalar order $\beta \in (0, 1)$ [16–18]. However, the number of equations in the reduced system may increase significantly. Let $M$ be the lowest common multiple of numbers $q_1, \ldots, q_m$, and $\beta = 1/M$. Then, the number of equations in the reduced system becomes $N = M(\beta_1 + \cdots + \beta_m)$. For example, if the orders in the original system of four equations are $\beta_1 = \frac{1}{2}, \beta_2 = \frac{2}{3}, \beta_3 = \frac{1}{5}$, and $\beta_4 = \frac{6}{7}$, then $M = 210$ and $N = 210 \cdot (1/2 + 2/3 + 1/5 + 6/7) = 467$. Thus, the reduced system will contain 467 equations of order $\beta = \frac{1}{210}$, although, originally, we had only four equations in the system. Even numerical solutions of such reduced systems consume a significant amount of computing and time resources; thus, the method of reducing to a scalar-order system should be considered ineffective. Therefore, developing direct general techniques for the solution and qualitative analysis of systems of fractional-order differential equations with any positive real order is important.

The representation formula for the solution of fractional-order systems in the sense of Riemann–Liouville with equal orders $\beta_j = \beta, j = 1, \ldots, m$, (commensurate case) and constant matrix $\mathcal{F}$ was first obtained in [11]. The authors of [19] derived representations in the case of Riemann–Liouville, Caputo, and sequential Miller–Ross derivatives under the same conditions for orders and matrix $\mathcal{F}$. Applications to multi-term commensurate fractional-order ordinary differential equations, as well as various techniques for the calculation of the matrix-valued ML functions, are considered in [13]. In [16], the procedure for the reduction of incommensurate rational orders $\beta_j$ to the commensurate case is discussed. The authors of [17] use this technique to derive a representation formula for the solution. Note that, in these works, also $\mathcal{F}$ is a constant matrix—that is, the corresponding systems are of ordinary differential equations, and fractional derivatives are in the sense of Caputo. The representation formulas for a fractional multi-order system of pseudo-differential equations are found in [18], in both commensurate and incommensurate rational-order cases, for Riemann–Liouville and Caputo derivatives. Moreover, in this work, representation formulas are obtained for systems of arbitrary positive time-fractional-order pseudo-differential equations with upper or lower triangular matrix-valued pseudo-differential operators.

In the current paper, we obtain representation formulas for arbitrary multi-order $\mathcal{B}$ with real components (not necessarily rational) and arbitrary matrix-valued operator $\mathcal{F}(A)$. The results obtained in this paper are new even for time-fractional systems of linear ordinary differential equations. We also introduce more general ML functions, called *vector-indexed matrix-valued ML functions.* We show that the solution of systems (1) and (2) is represented through an operator-dependent matrix-valued ML function.

We note that systems of fractional-order ordinary and partial differential equations have rich applications. For example, they are used in the modeling of processes in biosystems [20–22], ecology [23,24], epidemiology [25,26], quantum systems [27–29], etc.

This paper is organized as follows. In Section 2, we provide some preliminary facts about the ML functions, including matrix-valued versions. To our knowledge, Lemmas 1 and 2 are new. Here, we also introduce the vector-indexed matrix-valued ML functions and study some of their properties used in this paper. This section introduces the basic topological vector spaces on which the corresponding matrix-valued operators with singular symbols act. In Sections 3 and 4, we formulate the main results. The representation formulas for the solution of the initial value problem (1), (2) are obtained in the general case: for arbitrary multi-order $\mathcal{B}$ and matrix-valued operator $\mathcal{F}(A)$. The main idea of the method used to obtain the representation formula is demonstrated for clarity, first in the case $m = 2$, and then for arbitrary $m \geq 2$. Note that some particular representation formulas were obtained in [18] in the case of systems of pseudo-differential operators. These results are also extended to the differential-operator case. Finally, in Section 5, we discuss some applications and examples.

## 2. Preliminaries

### *2.1. Fractional Derivatives*

By definition, the Riemann–Liouville fractional derivative of order $\beta \in (0,1)$ of a function $f(t)$ defined on $[0,\infty)$, is the integral

$$D_+^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^\beta}. \tag{6}$$

subject to existence, where $\Gamma(s)$ is Euler's gamma function. Similarly, if $0 < \beta < 1$, then the Caputo derivative is defined by the integral

$${}_aD_*^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^\beta}. \tag{7}$$

subject to existence.

The Laplace transforms of the Riemann–Liouville and Caputo derivatives are

$$L[D_+^\beta f](s) = s^\beta L[f](s) - (J^{1-\beta} f)(0), \tag{8}$$

$$L[D_*^\beta f](s) = s^\beta L[f](s) - f(0)s^{\beta-1}, \tag{9}$$

respectively. We will use these formulas in the vector form. Namely, for a vector-valued function $F(t) = (f_1(t), \dots, f_m(t))$, we have

$$\begin{aligned} L[D_+^{\mathcal{B}} F(t)](s) &= s^{\mathcal{B}} L[F(t)](s) - (\mathcal{J}^{1-\mathcal{B}} F)(0) \\ &= \Big( s^{\beta_1} L[f_1](s) - (J^{1-\beta_1} f_1)(0), \dots, \\ &\quad s^{\beta_m} L[f_m](s) - (J^{1-\beta_m} f_m)(0) \Big), \end{aligned} \tag{10}$$

where

$$(\mathcal{J}^{1-\mathcal{B}}\mathcal{F})(t) = \Big((J^{1-\beta_1}f_1)(t), \ldots, (J^{1-\beta_m}f_m)(t)\Big). \tag{11}$$

with fractional integrals

$$(J^{1-\beta_j}f)(t) = \frac{1}{\Gamma(1-\beta_j)}\int_0^t (t-\tau)^{\beta_j} f(\tau)d\tau, \quad j = 1, \ldots, m.$$

Similarly,

$$\begin{aligned} L[D_*^{\mathcal{B}}F(t)](s) &= s^{\mathcal{B}}L[F(t)](s) - F(0)s^{\mathcal{B}-1} \\ &= \Big(s^{\beta_1}L[f_1](s) - f_1(0)s^{\beta_1-1}, \ldots, \\ &\quad s^{\beta_m}L[f_m](s) - f_m(0)s^{\beta_m-1}\Big). \end{aligned} \tag{12}$$

In these formulas,

$$L[\mathcal{D}^{\mathcal{B}}F(t)](s) = \Big(L[\mathcal{D}^{\beta_1}f_1](s), \ldots, L[\mathcal{D}^{\beta_m}f_m](s)\Big)$$

for both operators $\mathcal{D} = D_+$ and $\mathcal{D} = D_*$.

### 2.2. Matrix-Valued Functions

Let $Z$ be a square matrix of size $m$ with the Jordan normal form

$$\mathbb{J} = M^{-1}ZM = \Lambda + N, \tag{13}$$

where $M$ is an invertible transformation matrix, $\Lambda$ is a diagonal matrix with eigenvalues on the diagonal and $N$ is the nilpotent matrix. Suppose that $J_\ell$, $\ell = 1, \ldots, L$, are Jordan blocks of $Z$ and $\|Z\|$ is the matrix norm of $Z$. Then, for a function $g(z)$, analytic in a neighborhood of $|z| \le \|Z\|$, one has the spectral representation

$$\begin{aligned} g(\mathbb{J}) &= \sum_{k=0}^{m-1}\frac{N^k}{k!}\int\limits_{\sigma(Z)} g^{(k)}(\lambda)dE_\lambda \\ &= \sum_{\ell=1}^{L}\sum_{k=0}^{m_\ell}\frac{N^k}{k!}g^{(k)}(\lambda_\ell)\mathbb{P}_{\lambda_\ell}, \end{aligned}$$

where the spectral measure $dE_\lambda$ is formed by projection operators $\mathbb{P}_{\lambda_\ell}$, determined by eigenvalues $\lambda_\ell, \ell = 1, \ldots, L$, of the matrix $Z$ of multiplicity $m_\ell, \ell = 1, \ldots, L$. In the explicit form, this means that

$$g(\mathbb{J}) = \begin{bmatrix} g(J_1) & 0 & \ldots & 0 \\ 0. & g(J_2) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0. & 0. & \ldots. & g(J_L) \end{bmatrix}, \tag{14}$$

where

$$g(J_\ell) = \begin{bmatrix} g(\lambda_\ell) & \frac{g'(\lambda_\ell)}{1!} & \frac{g''(\lambda_\ell)}{2!} & \ldots & \frac{g^{(m_\ell-2)}(\lambda_\ell)}{(m_\ell-2)!} & \frac{g^{(m_\ell-1)}(\lambda_\ell)}{(m_\ell-1)!} \\ 0 & g(\lambda_\ell) & \frac{g'(\lambda_\ell)}{1!} & \ldots & \frac{g^{(m_\ell-3)}(\lambda_\ell)}{(m_\ell-3)!} & \frac{g^{(m_\ell-2)}(\lambda_\ell)}{(m_\ell-2)!} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & g(\lambda_\ell) & \frac{g'(\lambda_\ell)}{1!} \\ 0. & .0. & .0. & \ldots & 0 & g(\lambda_\ell) \end{bmatrix}, \quad \ell = 1 \ldots, L. \tag{15}$$

It follows from (13) and (14) that

$$g(Z) = Mg(\mathbb{J})M^{-1}.$$

*2.3. Classical ML Functions*

The classical two-parameter ML function is defined by

$$E_{\beta,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \nu)}, \quad z \in \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers and parameters $\beta > 0, \nu > 0$. This function plays an important role in the theory of fractional-order differential equations. For various properties of the ML function, we refer the reader to sources [30,31] and the references therein. Here, we only mention some properties of $E_{\beta,\nu}(z)$ used in the current paper. The function $E_{\beta,\nu}(z)$ is an entire function of order $1/\beta$ and recovers the exponential function $\exp(z)$ when $\beta = \nu = 1$. It is known [30,31] that for $0 < \beta < 2,\ \nu \in \mathbb{C}$, the ML function $E_{\beta,\nu}(z)$ has asymptotic behavior

$$E_{\beta,\nu}(z) \sim \frac{1}{\beta} z^{(1-\nu)/\beta} \exp(z^{1/\beta}),\ |z| \to \infty, \quad \text{if } \frac{\beta\pi}{2} < |\arg(z)| < \min\{\pi, \beta\pi\}, \tag{16}$$

and

$$E_{\beta,\nu}(z) \sim 1/|z|, |z| \to \infty, \quad \text{if} \quad \min\{\pi, \beta\pi\} < |\arg(z)| \leq \pi. \tag{17}$$

For derivatives of $E_{\beta,\nu}(z^\beta)$, the following formulas are valid:

$$\frac{d^k}{dz^k}\left[z^{\nu-1}E_{\beta,\nu}(z^\beta)\right] = z^{\nu-k-1}E_{\beta,\nu-k}(z^\beta),\ Re(\nu) > k,\ k = 1, 2, \ldots. \tag{18}$$

Consider the function $E_{\beta,\nu}(\mu t^\beta)$, with a parameter $\mu \in \mathbb{C}$. This function plays an important role in the theory of fractional-order differential equations. For the Laplace transform of this function and its derivatives, the following formulas hold [3,10,30]:

$$L[t^{\nu-1}E_{\beta,\nu}(\mu t^\beta)](s) = \frac{s^{\beta-\nu}}{s^\beta - \mu}, \quad s > [Re(\mu)]^{1/\beta}, \tag{19}$$

$$L\left[\frac{t^{k\beta+\nu-1}}{k!}E^{(k)}_{\beta,\nu}(\mu t^\beta)\right](s) = \frac{s^{\beta-\nu}}{(s^\beta - \mu)^{k+1}}, \quad s > [Re(\mu)]^{1/\beta}, k = 1, 2, \ldots, \tag{20}$$

where $E^{(k)}_{\beta,\nu}(z) = \frac{d^k}{dz^k}E_{\beta,\nu}(z)$. In particular, if $\nu = 1$, then one obtains

$$L\left[\frac{t^{k\beta}}{k!}E^{(k)}_{\beta}(\mu t^\beta)\right](s) = \frac{s^{\beta-1}}{(s^\beta - \mu)^{k+1}}, \quad s > [Re(\mu)]^{1/\beta}, k = 0, 1, \ldots, \tag{21}$$

and if $\nu = \beta$ in (20),

$$L\left[\frac{t^{k\beta+\beta-1}}{k!}E^{(k)}_{\beta,\beta}(\mu t^\beta)\right](s) = \frac{1}{(s^\beta - \mu)^{k+1}}, \quad s > [Re(\mu)]^{1/\beta}, k = 0, 1, \ldots, \tag{22}$$

The convolution of functions $f(t),\ g(t),\ t \geq 0$, is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{23}$$

The following lemmas will be used in our further analysis.

**Lemma 1.** *For $0 < \beta_1 < \beta_2$, $\nu > 0$, and $k = 0, 1, \dots$, the following relations hold:*

*(a)*

$$\left(I - \mu J^{\beta}\right)^{k}\left[\frac{t^{k\beta+\nu-1}}{k!}E^{(k)}_{\beta,\nu}(\mu t^{\beta})\right] = J^{k\beta}E_{\beta,\nu}(\mu t^{\beta}), \tag{24}$$

*(b)*

$$\left(J^{\beta_2-\beta_1} - \mu J^{\beta_2}\right)^{k}\left[\frac{t^{k\beta_1+\nu-1}}{k!}E^{(k)}_{\beta_1,\nu}(\mu t^{\beta_1})\right] = J^{k\beta_2}E_{\beta_1,\nu}(\mu t^{\beta_1}), \tag{25}$$

*where $I$ is the identity operator and $J^{\beta}$ is the fractional integral of order $\beta$.*

**Proof.** (a) To prove this statement, we show that the Laplace transforms of both sides in (24) coincide. Indeed, applying the Laplace transform to the left side of (24), we have

$$\left(1 - \frac{\mu}{s^{\beta}}\right)^{k}\frac{s^{\beta-\nu}}{(s^{\beta}-\mu)^{k+1}} = \frac{s^{\beta-\nu}}{s^{k\beta}(s^{\beta}-\mu)}, \quad s > [Re(\mu)]^{1/\beta}.$$

This is obviously the Laplace transform of the right-hand side of (24), as well.

(b) Similarly, the Laplace transform of the left-hand side of (25) is

$$\left(\frac{1}{s^{\beta_2-\beta_2}} - \frac{\mu}{s^{\beta_2}}\right)^{k}\frac{s^{\beta_1-\nu}}{(s^{\beta_1}-\mu)^{k+1}} = \frac{s^{\beta_1-\nu}}{s^{k\beta_2}(s^{\beta_1}-\mu)}, \quad s > [Re(\mu)]^{1/\beta_1},$$

which is the Laplace transform of the right-hand side of (25), as well. □

**Lemma 2.** *For any $\beta_1 > 0, \beta_2 > 0, \nu > 0$, and parameters $\mu_1, \mu_2 \in \mathbb{C}$, the following relations hold:*

*(i)*

$$(I - \mu_2 J^{\beta_2})^{-1}\left[t^{\nu-1}E_{\beta_1,\nu}(\mu_1 t^{\beta_1}) - \mu_1 J^{\nu}(t^{\beta_1-1}E_{\beta_1,\beta_1}(\mu_1 t^{\beta_1})\right] = t^{\nu-1}E_{\beta_2,\nu}(\mu_2 t^{\beta_2}), \tag{26}$$

*(ii)*

$$(I - \mu_2 J^{\beta_2})^{-1}J^{\beta_2}[t^{\nu-1}E_{\beta_1,\nu}(\mu_1 t^{\beta_1})] = \left(t^{\nu-1}E_{\beta_1,\nu}(\mu_1 t^{\beta_1})\right) * \left(t^{\beta_2-1}E_{\beta_2,\beta_2}(\mu_2 t^{\beta_2})\right), \tag{27}$$

*where "$*$" is the convolution operation.*

**Proof.** (i) Again, we show that the Laplace transforms of both sides in Equations (26) and (27) coincide. For the Laplace transform of the left-hand side of (26), we have

$$\begin{aligned}
&\mathcal{L}\left[(I - \mu_2 J^{\beta_2})^{-1}[t^{\nu-1}E_{\beta_1,\nu}(\mu_1 t^{\beta_1}) - \mu_1 J^{\nu}E_{\beta_1,\beta_1}(\mu_1 t^{\beta_1})]\right] \\
&= (1 - \frac{\mu_2}{s^{\beta_2}})^{-1}\Big(\frac{s^{\beta_1-\nu}}{s^{\beta_1}-\mu_1} - \frac{\mu_1}{s^{\nu}(s^{\beta_1}-\mu_1)}\Big) \\
&= \frac{s^{\beta_2}}{s^{\beta_2}-\mu_2}\,\frac{s^{\beta_1}-\mu_1}{s^{\nu}(s^{\beta_1}-\mu_1)} = \frac{s^{\beta_2-\nu}}{s^{\beta_2}-\mu_2},
\end{aligned}$$

where

$$s > \max\left\{[Re(\mu_1)]^{1/\beta_1}, [Re(\mu_2)]^{1/\beta_2}\right\}.$$

This is obviously the Laplace transform of the right-hand side of (26), as well.

(ii) Similarly, the Laplace transform of the left-hand side of (27) is

$$\begin{aligned}
&\mathcal{L}\left[(I - \mu_2 J^{\beta_2})^{-1}J^{\beta_2}[t^{\nu-1}E_{\beta_1,\nu}(\mu_1 t^{\beta_1})]\right] = (1 - \frac{\mu_2}{s^{\beta_2}})^{-1}\frac{s^{\beta_1-\nu}}{s^{\beta_2}(s^{\beta_1}-\mu_1)} \\
&= \frac{s^{\beta_2}}{s^{\beta_2}-\mu_2}\frac{s^{\beta_1-\nu}}{s^{\beta_2}(s^{\beta_1}-\mu_1)} = \frac{s^{\beta_1-\nu}}{s^{\beta_1}-\mu_1}\,\frac{1}{s^{\beta_2}-\mu_2}.
\end{aligned}$$

On the other hand, due to the convolution theorem, the Laplace transform of the right-hand side of (27) also results in the same expression. □

### 2.4. Matrix-Valued ML Functions

Since $E_{\beta,\nu}(z)$ is an entire function, in accordance with (14) and (15), for a matrix $Z$, one can introduce a matrix-valued version of the ML function as

$$\mathbb{E}_{\beta,\nu}(Z) = M\mathbb{E}_{\beta,\nu}(\mathbb{J})M^{-1} = M\mathbb{E}_{\beta,\nu}(\Lambda + N)M^{-1}, \tag{28}$$

where

$$\mathbb{E}_{\beta,\nu}(\mathbb{J}) = \begin{bmatrix} \mathbb{E}_{\beta,\nu}(J_1) & 0 & \dots & 0 \\ 0. & \mathbb{E}_{\beta,\nu}(J_2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0. & 0. & \dots. & \mathbb{E}_{\beta,\nu}(J_L) \end{bmatrix}, \tag{29}$$

with the block matrices

$$\mathbb{E}_{\beta,\nu}(J_\ell) = \begin{bmatrix} E_{\beta,\nu}(\lambda_\ell) & \frac{E'_{\beta,\nu}(\lambda_\ell)}{1!} & \dots & \frac{E^{(m_\ell-2)}_{\beta,\nu}(\lambda_\ell)}{(m_\ell-2)!} & \frac{E^{(m_\ell-1)}_{\beta,\nu}(\lambda_\ell)}{(m_\ell-1)!} \\ 0 & E_{\beta,\nu}(\lambda_\ell) & \dots & \frac{E^{(m_\ell-3)}_{\beta,\nu}(\lambda_\ell)}{(m_\ell-3)!} & \frac{E^{(m_\ell-2)}_{\beta,\nu}(\lambda_\ell)}{(m_\ell-2)!} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E_{\beta,\nu}(\lambda_\ell) & \frac{E'_{\beta,\nu}(\lambda_\ell)}{1!} \\ 0 & 0 & \dots & 0 & E_{\beta,\nu}(\lambda_\ell) \end{bmatrix}, \tag{30}$$

corresponding to (algebraic) eigenvalues $\lambda_\ell, \ell = 1\dots, L$, of the matrix $Z$. It is not difficult to verify that, using Formulas (19), (20), and (30), one obtains the Laplace transforms of the matrix-valued function $t^{\nu-1}\mathbb{E}_{\beta,\nu}(t^\beta J_\ell)$ :

$$L[t^{\nu-1}\mathbb{E}_{\beta,\nu}(t^\beta J_\ell)](s) = \begin{bmatrix} \frac{s^{\beta-\nu}}{s^\beta-\lambda_\ell} & \frac{s^{\beta-\nu}}{(s^\beta-\lambda_\ell)^2} & \dots & \frac{s^{\beta-\nu}}{(s^\beta-\lambda_\ell)^{m_\ell}} \\ 0 & \frac{s^{\beta-\nu}}{s^\beta-\lambda_\ell} & \dots & \frac{s^{\beta-\nu}}{(s^\beta-\lambda_\ell)^{m_\ell-1}} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{s^{\beta-\nu}}{s^\beta-\lambda_\ell} \end{bmatrix}, \quad \ell = 1,\dots L. \tag{31}$$

### 2.5. Vector-Indexed Matrix-Valued ML Functions

Let $\mathcal{B} = (\beta_1,\dots,\beta_m)$ and $\mathcal{V} = (\nu_1,\dots,\nu_m)$ be vector indices with components $\beta_j > 0,\ \nu_j > 0,\ j = 1,\dots,m$. For a diagonal matrix $D$ with diagonal entries $d_1,\dots,d_m$, we use the notation

$$D = \mathrm{diag}(d_1,\dots,d_m).$$

**Definition 1.** *Let $Z$ be a square matrix of size $m\times m$ with complex entries. A vector-indexed matrix-valued ML function denoted by $E_{\mathcal{B},\mathcal{V}}(z)$, is defined by*

$$E_{\mathcal{B},\mathcal{V}}(Z) = \sum_{n=0}^{\infty}\Big(I\Gamma(n\mathcal{B}+\mathcal{V})\Big)^{-1} Z^n, \tag{32}$$

*where $I\Gamma(n\mathcal{B}+\mathcal{V}) = diag\Big(\Gamma(n\beta_1+\nu_1),\dots,n\Gamma(\beta_m+\nu_m)\Big)$.*

The vector-indexed matrix-valued ML function $E_{\mathcal{B},\mathcal{V}}(Z)$ generalizes the classical and above-considered matrix-valued ML functions. Below are some examples.

1. Let $m = 1$ and $\beta_1 = \beta, \nu_1 = \nu$. Then, we obtain the classical two-parameter ML function $E_{\beta,\nu}(z)$.

2. Let $\mathcal{B} = (\beta, \ldots, \beta)$ and $\mathcal{V} = (\nu_1, \ldots, \nu)$. Then, we obtain the matrix-valued ML function [13,30]
$$E_{\beta,\nu}(Z) = \sum_{n=0}^{\infty} \frac{Z^n}{\Gamma(n\beta+\nu)}. \tag{33}$$
Indeed, in this case,
$$\begin{aligned}\Big(I\Gamma(n\mathcal{B}+\mathcal{V})\Big)^{-1} Z^n &= \Big[\mathrm{diag}(\Gamma(n\beta+\nu), \ldots, \Gamma(n\beta+\nu))\Big]^{-1} Z^n \\ &= \mathrm{diag}\left(\frac{1}{\Gamma(n\beta+\nu)}, \ldots, \frac{1}{\Gamma(n\beta+\nu)}\right) Z^n \\ &= \frac{1}{\Gamma(n\beta+\nu)} Z^n.\end{aligned}$$
Therefore, in this case, (32) reduces to (33).

Let $\lambda_j, j = 1, \ldots, L$, be eigenvalues of (algebraic) multiplicity $m_j$ of the matrix $Z = M(\Lambda+N)M^{-1}$, and let $J_\ell$ be the Jordan block of the Jordan canonical form $\Lambda + N$ corresponding to $\lambda_j$. Then, it is not difficult to see that

$$\mathbb{E}_{\mathcal{B},\mathcal{V}}(\Lambda+N) = \begin{bmatrix} \mathbb{E}_{\mathcal{B}_1,\mathcal{V}_1}(J_1) & 0 & \cdots & 0 \\ 0 & \mathbb{E}_{\mathcal{B}_2,\mathcal{V}_2}(J_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathbb{E}_{\mathcal{B}_L,\mathcal{V}_L}(J_L) \end{bmatrix}, \tag{34}$$

where $\mathcal{B}_\ell = (\beta_{M_\ell+1}, \ldots, \beta_{M_\ell+m_\ell})$, $\mathcal{V}_\ell = (\nu_{M_\ell+1}, \ldots, \nu_{M_\ell+m_\ell})$, and $M_\ell = m_1 + \cdots + m_{\ell-1}$, with blocks

$$\mathbb{E}_{\mathcal{B}_\ell,\mathcal{V}_\ell}(J_\ell) = \begin{bmatrix} E_{\beta_{M_\ell}+1,\nu_{M_\ell}+1}(\lambda_\ell) & \frac{E'_{\beta_{M_\ell}+1,\nu_{M_\ell}+1}(\lambda_\ell)}{1!} & \cdots & \frac{E^{(m_\ell-1)}_{\beta_{M_\ell}+1,\nu_{M_\ell}+1}(\lambda_\ell)}{(m_\ell-1)!} \\ 0 & E_{\beta_{M_\ell}+2,\nu_{M_\ell}+2}(\lambda_\ell) & \cdots & \frac{E^{(m_\ell-2)}_{\beta_{M_\ell}+2,\nu_{M_\ell}+2}(\lambda_\ell)}{(m_\ell-2)!} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_{\beta_{M_\ell}+m_\ell,\nu_{M_\ell}+m_\ell}(\lambda_\ell) \end{bmatrix}. \tag{35}$$

Now, suppose that

$$\mathcal{B} = (\underbrace{\beta_1, \ldots, \beta_1}_{m_1 \text{ times}}, \underbrace{\beta_2, \ldots, \beta_2}_{m_2 \text{ times}}, \ldots, \underbrace{\beta_L, \ldots, \beta_L}_{m_L \text{ times}}),$$

and

$$\mathcal{V} = (\underbrace{\nu_1, \ldots, \nu_1}_{m_1 \text{ times}}, \underbrace{\nu_2, \ldots, \nu_2}_{m_2 \text{ times}}, \ldots, \underbrace{\nu_L, \ldots, \nu_L}_{m_L \text{ times}}).$$

Then, it follows from (35) that, for each $\ell = 1, \ldots, L$,

$$\begin{aligned}&\mathbb{E}_{\mathcal{B}_\ell,\mathcal{V}_\ell}(It^{\mathcal{B}_\ell} J_\ell) \\ &= \begin{bmatrix} E_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell}) & \frac{t^{\beta_\ell} E'_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell})}{1!} & \cdots & \frac{t^{(m_\ell-2)\beta_\ell} E^{(m_\ell-2)}_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell})}{(m_\ell-2)!} & \frac{t^{(m_\ell-1)\beta_\ell} E^{(m_\ell-1)}_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta}_\ell)}{(m_\ell-1)!} \\ 0 & E_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell}) & \cdots & \frac{t^{(m_\ell-3)\beta} E^{(m_\ell-3)}_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell})}{(m_\ell-3)!} & \frac{t^{(m_\ell-2)\beta} E^{(m_\ell-2)}_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell})}{(m_\ell-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell}) & \frac{t^{\beta_\ell} E'_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell})}{1!} \\ 0 & 0 & \cdots & 0 & E_{\beta_\ell,\nu_\ell}(\lambda_\ell t^{\beta_\ell}) \end{bmatrix},\end{aligned} \tag{36}$$

where $It^{\mathcal{B}_\ell} = \text{diag}(t^{\beta_\ell}, \ldots, t^{\beta_\ell})$ is a diagonal matrix of size $m_\ell \times m_\ell$.

The latter implies

$$L[It^{\mathcal{B}_\ell - \mathcal{V}_\ell}\mathbb{E}_{\mathcal{B}_\ell,\mathcal{V}_\ell}(t^{\mathcal{B}_\ell}J_\ell)](s) = \begin{bmatrix} \frac{s^{\beta_\ell-\nu_\ell}}{s^{\beta_\ell}-\lambda_\ell} & \frac{s^{\beta_\ell-\nu_\ell}}{(s^{\beta_\ell}-\lambda_\ell)^2} & \cdots & \frac{s^{\beta_\ell-\nu_\ell}}{(s^{\beta_\ell}-\lambda_\ell)^{m_\ell}} \\ 0 & \frac{s^{\beta_\ell-\nu_\ell}}{s^{\beta_\ell}-\lambda_\ell} & \cdots & \frac{s^{\beta_\ell-\nu_\ell}}{(s^{\beta_\ell}-\lambda_\ell)^{m_\ell-1}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{s^{\beta-1}}{s^{\beta_\ell}-\lambda_\ell} \end{bmatrix}, \quad \ell = 1, \ldots L. \tag{37}$$

We note that, in general, $M\mathbb{E}_{\mathcal{B},\mathcal{V}}(\Lambda + N)M^{-1}$ is not the same as $E_{\mathcal{B},\mathcal{V}}(Z)$ unless vector indices $\mathcal{B}$ and $\mathcal{V}$ have equal components. Indeed, using the equality

$$Z^n = M(\Lambda + N)^n M^{-1},$$

one obtains

$$E_{\mathcal{B},\mathcal{V}}(Z) = \sum_{n=0}^{\infty} \Big(I\Gamma(n\mathcal{B} + \mathcal{V})\Big)^{-1} M\,(\Lambda + N)^n\, M^{-1} \neq M\mathbb{E}_{\mathcal{B},\mathcal{V}}(\Lambda + N)M^{-1},$$

since matrices $\Big(I\Gamma(n\mathcal{B} + \mathcal{V})\Big)^{-1}$ and $M$ do not commute. It is not difficult to verify that these matrices commute if and only if vectors $\mathcal{B}$ and $\mathcal{V}$ have equal components. In this case, the following theorem holds.

**Theorem 1.** *Let $\mathcal{B} = (\beta, \ldots, \beta)$, $0 < \beta \leq 1$, and $\mathcal{V} = (\nu, \ldots, \nu), \nu > 0$. Then,*

1. *for the matrix-valued ML function $E_{\mathcal{B},\mathcal{V}}(Z)$, the following representation is valid*

$$E_{\mathcal{B},\mathcal{V}}(Z) = ME_{\mathcal{B},\mathcal{V}}(\Lambda + N)M^{-1};$$

2. *the following Laplace transform formula holds*

$$\mathcal{L}[It^{\mathcal{V}-\mathbf{1}}E_{\mathcal{B},\mathcal{V}}(It^{\mathcal{B}}Z)](s) = Is^{\mathcal{B}-\mathcal{V}}(Is^{\mathcal{B}} - Z)^{-1}, \tag{38}$$

*where $\mathcal{V} - \mathbf{1} = (\nu - 1, \ldots, \nu - 1)$ and $It^{\mathcal{B}} = diag(t^\beta, \ldots, t^\beta)$.*

**Proof.** We need to prove only part 2. Using the definition (32) of the ML function, we have

$$\mathcal{L}\Big[It^{\mathcal{V}-\mathbf{1}}E_{\mathcal{B},\mathcal{V}}(It^{\mathcal{B}}Z)\Big](s) = \sum_{n=0}^{\infty} \Big(I\Gamma(n\mathcal{B} + \mathcal{V})\Big)^{-1} \mathcal{L}\Big[It^{n\mathcal{B}+\mathcal{V}-\mathbf{1}}\Big]Z^n, \tag{39}$$

since matrices $It^{\mathcal{V}-\mathbf{1}}$ and $I\Gamma(n\mathcal{B} + \mathcal{V})\Big)^{-1}$ commute under the conditions to $\mathcal{B}$ and $\mathcal{V}$. Further, using the well-known relation $\mathcal{L}[t^\rho](s) = \Gamma(\rho + 1)/s^{\rho+1}$, we obtain

$$\mathcal{L}\Big[It^{\mathcal{V}-\mathbf{1}}E_{\mathcal{B},\mathcal{V}}(It^{\mathcal{B}}Z)\Big](s) = Is^{-\mathcal{V}} \sum_{n=0}^{\infty} \Big[Is^{-n\mathcal{B}}Z^n\Big] = Is^{\mathcal{B}-\mathcal{V}}(Is^{\mathcal{B}} - Z)^{-1}, \tag{40}$$

completing the proof. □

*2.6. Matrix-Valued Operators with Singular Symbols*

In this section, we describe matrix-valued operators $\mathcal{F}(A)$ on the right-hand side of system (1). Let $A$ be a closed linear operator with a domain $\mathcal{D}(A)$ dense in a reflexive Banach space $X$ and a nonempty spectrum $\sigma(A) \subset \mathbb{C}$. Assume that the entry $f_{jk}(A)$ of the matrix-valued operator $\mathcal{F}(A)$ has the symbol $f_{jk}(z)$, $z \in \mathbb{C}$, analytic in an open connected domain $G \subset \mathbb{C}$. If $\sigma(A)$ is bounded and $G$ contains $\sigma(A)$, then one can define the operator $f_{jk}(A)$ as (see, e.g., [1,32])

$$f_{jk}(A) = \int_\gamma f_{jk}(\zeta)\mathcal{R}(\zeta, A)d\zeta, \quad j,k = 1,\dots,m, \tag{41}$$

where $\gamma$ is a contour in $G$ containing $\sigma(A)$, and $\mathcal{R}(\zeta, A)$, $\zeta \in \mathbb{C} \setminus \sigma(A)$, is the resolvent operator of $A$. Representation (41) is not valid if $f_{jk}(z)$ has singularities on the spectrum $\sigma(A)$.

In the case that $f$ has singular points in the spectrum $\sigma(A)$ of the operator $A$, the corresponding operator $f(A)$ can be constructed as follows. Denote by $\text{sing}(f)$ the set of singular points of $f$ on $\sigma(A)$. Let $\mathbb{D}$ be an open set in $\mathbb{C}$ containing $\sigma(A)$. In particular, if $\sigma(A) = \mathbb{C}$, then $\mathbb{D} = \mathbb{C}$ as well. Consider an open set $G \subset \mathbb{D} \setminus \text{sing}(f)$. Let $0 < r \leq +\infty$ and $\mu < r$. Denote by $X_\mu$ the set of elements $x \in \cap_{k\geq 1}\mathcal{D}(A^k)$ satisfying the inequalities $\|A^k x\| \leq C\mu^k \|x\|$ for all $k = 1,2,...$, with a constant $C > 0$ not depending on $k$. A sequence of elements $x_n \in X_\mu,\ n = 1,2,...$, is said to converge to an element $x_0 \in X_\mu$ if $\|x_n - x_0\| \to 0$, as $n \to \infty$. It is easy to see that $X_{\mu_1} \subset X_{\mu_2}$, if $\mu_1 < \mu_2$, and this inclusion is continuous. Denote by $X_{A,r}$ the inductive limit of spaces $X_\mu$ as $\mu \to r$, i.e.,

$$X_{A,r} = \operatorname*{ind\text{-}lim}_{\mu \to r} X_\mu,$$

meaning that $X_{A,r} = \cup_{0<\mu<r} X_\mu$ with the strongest topology. For basic notions of topological vector spaces including inductive and projective limits, we refer the reader to [33]. The space $X_{A,r}$ is called a space of exponential vectors of type $r$ (see, e.g., [34,35]) associated with the operator $A$.

Let $A_\lambda = A - \lambda I$, where $\lambda \in G$, and denote by $X_{A,G}$ the space whose elements are locally finite sums of the elements in $X_{A_\lambda,r}$, $r < \text{dist}(\lambda, \partial G)$, with the corresponding topology. Here, $\text{dist}(\lambda, \partial G)$ is the minimal distance between the point $\lambda$ and the boundary of the domain $G$. By definition, any $u \in X_{A,G}$ has a representation

$$u = \sum_{k=1}^{m_u} u_{\lambda_k}, \quad u_{\lambda_k} \in X_{A_{\lambda_k},r},$$

where $\lambda_k \in G$, and $m_u$ is a finite number.

Now, we can define operators $f(A)$ with symbols $f(z)$ analytic in the domain $G$. Recall that $f(z)$ may have singular points on the spectrum $\sigma(A)$, but $G$ does not contain singularities of $f(z)$. As an analytic function in $G$, $f(z)$ has the Taylor expansion

$$f_\lambda(z) = \sum_{n=0}^\infty \frac{f^{(n)}(\lambda)}{n!}(z-\lambda)^n, \quad \lambda \in G,$$

convergent in any open disc $|z - \lambda| < r$, where $r < \text{dist}(\lambda, \partial G)$. Therefore, the operator $f_\lambda(A)$ defined as

$$f_\lambda(A)u_\lambda = \sum_{n=0}^\infty \frac{f^{(n)}(\lambda)}{n!} A_\lambda^n u_\lambda \tag{42}$$

on elements $u_\lambda \in X_{A_\lambda,r}$ is well defined. Indeed, we have

$$\begin{aligned} \|f_\lambda(A)u_\lambda\| &\leq C\sum_{n=0}^\infty \frac{|f^{(n)}(\lambda)|}{n!}\|A_\lambda^n u_\lambda\| \\ &\leq C\|u_\lambda\| \sum_{n=0}^\infty \frac{|f^{(n)}(\lambda)|}{n!}\mu^n < \infty, \quad \mu < r. \end{aligned} \tag{43}$$

Finally, for an arbitrary $u \in X_{A,G}$ with the representation

$$u = \sum_{\lambda \in G} u_\lambda, \quad u_\lambda \in X_{A_\lambda, r}, \tag{44}$$

the operator $f(A)$ is defined by the formula

$$f(A)u = \sum_{\lambda \in G} f_\lambda(A) u_\lambda, \tag{45}$$

where $f_\lambda(A)u_\lambda$ is defined in (42). Using estimate (43) and representation (45), it is easy to show that the operator $f(A)$ is well defined on the space $X_{A,G}$.

Further, suppose that there exists a one-parameter family of bounded invertible operators $E_\lambda : X \to X$ such that

$$A_\lambda = E_\lambda A E_\lambda^{-1}, \ \lambda \in G. \tag{46}$$

**Example 1.** *Let $X = L_2 \equiv L_2(\mathbb{R})$ and $A = -i\frac{d}{dx} : L_2 \to L_2$ with domain $\mathcal{D}(A) = \{v \in L_2 : Av \in L_2\}$. Then, for operators $E_\lambda : v(x) \to e^{i\lambda x}v(x)$, we have*

$$\begin{aligned} AE_\lambda v(x) &= -i\frac{d}{dx}(e^{i\lambda x}v(x)) = \lambda e^{i\lambda x}v(x) - ie^{i\lambda x}\frac{dv}{dx} \\ &= \lambda E_\lambda v(x) + E_\lambda Av(x), \quad x \in \mathbb{R}, \end{aligned}$$

*obtaining (46).*

It follows from (46) that

$$A_\lambda^n = E_\lambda A^n E_\lambda^{-1},$$

for all $n = 1, 2, \ldots$, yielding

$$f(A)u = \sum_{\lambda \in G} \sum_{n=0}^{\infty} \frac{f^{(n)}(\lambda)}{n!} U_\lambda A^n U_\lambda^{-1} u_\lambda. \tag{47}$$

Recall that, here, the sum with respect to $\lambda$ is finite.

The operator $f(A)$ defined in (45) maps $X_{A,G}$ to itself. Namely, the mapping

$$f(A) : X_{A,G} \to X_{A,G}$$

is continuous. Indeed, let $u \in X_{A,G}$ have a representation $u = \sum_\lambda u_\lambda,\ u_\lambda \in X_{A_\lambda,r}$. Then, for $f(A)u$, we have the estimate

$$\begin{aligned} \|A_\lambda^k f_\lambda(A) u_\lambda\| &\le \sum_{n=0}^{\infty} \frac{|f^n(\lambda)|}{n!} \|(A - \lambda I)^n A_\lambda^k u_\lambda\| \\ &\le \max_{|z-\lambda|<r} |f(z)| \|A_\lambda^k u_\lambda\| \le C\mu^k \|u_\lambda\|, \end{aligned} \tag{48}$$

with some constant $C > 0$ and $\mu < r$. The latter means that $f_\lambda(A)u_\lambda \in X_{A_\lambda,r}$. Therefore, $f(A)u$ has a representation $\sum_\lambda v_\lambda$, where $v_\lambda = f_\lambda(A)u_\lambda \in X_{A_\lambda,r}$, implying $f(A)u \in X_{A,G}$. The estimate (48) also implies the continuity of the mapping $f(A)$ in the topology of $X_{A,G}$.

**Remark 1.** *If the spectrum of the operator $A$ is discrete, then $X_{A_\lambda,r}$ consists of all linear combinations of eigenvectors and associated eigenvectors corresponding to eigenvalues $\lambda_k$ in the disc $|\lambda - \lambda_k| < r$, and the space $X_{A,G}$ is their locally finite sum.*

Finally, it follows from the construction above that a matrix-valued operator $\mathcal{F}(A)$ with the matrix symbol $\mathcal{F}(z) = \{f_{kj}(z),\ k, j = 1, \ldots, m\}$, analytic in the domain $G$, is well defined on elements of the direct product space

$$\mathcal{X}_{A,G} = X_{A,G} \underbrace{\otimes \cdots \otimes}_{m \text{ times}} X_{A,G},$$

with the corresponding direct product topology. Moreover, the mapping

$$\mathcal{F}(A) : \mathcal{X}_{A,G} \to \mathcal{X}_{A,G} \tag{49}$$

is continuous.

We note that the space $\mathcal{X}_{A,G}$ is relatively narrow. For example, if $A = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n})$ acting in the space $L_2(\mathbb{R}^n)$, then the corresponding space $\mathcal{X}_{A,G}$ is the direct product of the space of functions analytic in $G \subset \mathbb{R}^n$. However, the duality construction allows us to expand the introduced spaces and consider wider classes of fractional-order systems. Let $X^*$ denote the dual of $X$, and $A^* : X^* \to X^*$ be the operator adjoint to $A$. We denote by $X'_{A^*,G^*}$ the space of linear continuous functionals defined on $X_{A,G}$, with respect to weak convergence. In other words, $X'_{A^*,G^*}$ is the projective limit of spaces $X'_{A^*_\lambda,r}$, which are dual to $X_{A_\lambda,r}$ with the coarsest topology. Continuing the example $A = (-i\frac{\partial}{\partial x_1}, \ldots, -i\frac{\partial}{\partial x_n})$, now, one can see that the corresponding space $X'_{A^*,G^*}$ gives rise to the space of analytic functionals (Sato's hyperfunctions; see, e.g., [36]).

For an analytic matrix symbol $\mathcal{F}(z)$ defined on $G^* = \{z \in \mathbb{C} : \bar{z} \in G\}$, we define a matrix-valued operator $\mathcal{F}(A^*)$ as follows:

$$< \mathcal{F}(A^*)u^*, v > = < u^*, \mathcal{F}^T(A)v >, \ \ \forall v \in \mathcal{X}_{A,G}, \tag{50}$$

where $\mathcal{F}^T(A)$ is the matrix-valued operator with the symbol $\mathcal{F}(z)$ analytic in $G$, and $u^*$ is an element of the space $\mathcal{X}'_{A^*,G^*}$, dual to $\mathcal{X}_{A,G}$. By construction, as a dual to the space of the direct product, the space $\mathcal{X}'_{A^*,G^*}$ represents the direct sum

$$\mathcal{X}'_{A^*,G^*} = X'_{A^*,G^*} \underbrace{\oplus \cdots \oplus}_{m \text{ times}} X'_{A^*,G^*},$$

with the corresponding topology. It follows from (49) that the mapping

$$\mathcal{F}(A^*) : \mathcal{X}'_{A^*,G^*} \to \mathcal{X}'_{A^*,G^*} \tag{51}$$

is continuous. Indeed, assume that a sequence $u_n^* \in \mathcal{X}'_{A^*,G^*}$ converges to 0 in the topology of $\mathcal{X}'_{A^*,G^*}$. Then, for arbitrary $v \in \mathcal{X}_{A,G}$, we have

$$< \mathcal{F}(A^*)u_n^*, v > = < u_n^*, \mathcal{F}^T(A)v > = < u_n^*, w >,$$

where $w = \mathcal{F}^T(A)v \in \mathcal{X}_{A,G}$ due to (49). Hence, $\mathcal{F}(A^*)u_n^* \to 0$, as $n \to \infty$, in the topology of $\mathcal{X}'_{A^*,G^*}$, obtaining the continuity of mapping (51).

## 3. Main Results

Below, we derive representation formulas for the solutions of fractional-order systems of differential-operator equations. We demonstrate the derivation in the case of the Caputo fractional derivative. For the sake of clarity, we start with the case $m = 2$ and then the general case. The case of the Riemann–Liouville fractional derivative can be treated similarly (see Section 4).

*3.1. Fractional Multi-Order Systems of Differential-Operator Equations: $m = 2$*

In this section, we demonstrate the formal method of obtaining the representation formula for the solution of time-fractional arbitrary multi-order systems of differential-operator equations in the particular case of two equations. Namely, consider the system

$$D_*^{\mathcal{B}}\mathcal{U}(t) = \mathbb{F}(A)\mathcal{U}(t) + \mathcal{H}(t), \tag{52}$$

where $\mathcal{B} = (\beta_1, \beta_2), 0 < \beta_1 < \beta_2 \leq 1$, $\mathcal{H}(t) = (h_1(t), h_2(t))$ is a given vector-valued function, and

$$\mathbb{F}(A) = \begin{bmatrix} f_{11}(A) & f_{12}(A) \\ f_{21}(A) & f_{22}(A) \end{bmatrix}, \tag{53}$$

with the initial condition

$$\mathcal{U}(0) = \Phi = (\varphi_1, \varphi_2), \tag{54}$$

where $\Phi \in \mathcal{X}_{A,G}$. We assume that $G$ does not contain the roots of the equation

$$\Delta(z) = f_{11}(z)f_{22}(z) - f_{21}(z)f_{12}(z)) = 0.$$

To find entries of the solution operator $\mathcal{S}(t,z)$, we consider the homogeneous counterpart of system (52), writing it in the explicit form

$$\begin{cases} D_*^{\beta_1}u_1(t) &= f_{11}(A)u_1(t) + f_{12}(A)u_2(t), \\ D_*^{\beta_2}u_2(t) &= f_{21}(A)u_1(t) + f_{22}(A)u_2(t). \end{cases}$$

Applying the Laplace transform and replacing $A$ with the parameter $z$, we have

$$\begin{cases} (Is^{\beta_1} - f_{11}(z))\mathcal{L}[u_1](s) - f_{12}(z)\mathcal{L}[u_2](s) &= Is^{\beta_1 - 1}\varphi_1, \\ -f_{21}(z)\mathcal{L}[u_1](s) + (Is^{\beta_2} - f_{22}(z))\mathcal{L}[u_2](s) &= Is^{\beta_2 - 1}\varphi_2. \end{cases} \tag{55}$$

The solution of system (55) is

$$\mathcal{L}[u_1](s) = \frac{1}{\Psi(s,z)}\Big(p_1(s,z)\varphi_1 + q_1(s,z)\varphi_2\Big) \quad z \in G, s > r_*(z) \tag{56}$$

$$\mathcal{L}[u_2](s) = \frac{1}{\Psi(s,z)}\Big(q_2(s,z)\varphi_1 + p_2(s,z)\varphi_2\Big), \quad z \in G, s > r_*(z). \tag{57}$$

where

$$\Psi(s,z) = s^{\beta_1+\beta_2} - s^{\beta_2}f_{11}(z) - s^{\beta_1}f_{22}(z) + \Delta(z), \tag{58}$$

$$p_1(s,z) = s^{\beta_1+\beta_2-1} - s^{\beta_1-1}f_{22}(z), \quad q_1(s,z) = s^{\beta_2-1}f_{12}(z), \tag{59}$$

$$p_2(s,z) = s^{\beta_1+\beta_2-1} - s^{\beta_2-1}f_{11}(z), \quad q_2(s,z) = s^{\beta_1-1}f_{21}(z), \tag{60}$$

and $r_*(z)$ is the real part of the roots of the equation $\Psi(s,z) = 0$. This solution is uniquely defined, since, by assumption, $G \cap Q = \emptyset$, where $Q = \{z : \Psi(s,z) = 0\}$. We have

$$\begin{aligned}\frac{1}{\Psi(s,z)} &= \frac{1}{s^{\beta_2}\Big(s^{\beta_1} - f_{11}(z) - f_{22}(z)s^{\beta_1-\beta_2} + \Delta(z)s^{-\beta_2}\Big)} \\ &= \frac{1}{s^{\beta_2}\Big(s^{\beta_1} - f_{11}(z)\Big)\Big(1 - \frac{f_{22}(z)s^{\beta_1-\beta_2} - \Delta(z)s^{-\beta_2}}{s^{\beta_1}-f_{11}(z)}\Big)} \\ &= \sum_{k=0}^{\infty} \frac{\big(s^{\beta_1-\beta_2}f_{22}(z) - s^{-\beta_2}\Delta(z)\big)^k}{s^{\beta_2}\big(s^{\beta_1} - f_{11}(z)\big)^{k+1}} \\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j} f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j} \frac{s^{-k\beta_2+j\beta_1-\beta_2}}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}. \end{aligned} \tag{61}$$

For $s > r_*(z)$ large enough, the inequality

$$\left| \frac{s^{-k\beta_2+j\beta_1-\beta_2}}{s^{\beta_1} - f_{11}(z)} \right| < 1$$

is verified, and, therefore, the series in (61) is convergent. Now, for the solution of system (55), we have

$$\mathcal{L}[u_1](s) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j} f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j} \frac{s^{-k\beta_2+j\beta_1-\beta_2}}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}[p_1(s,z)\varphi_1 + q_1(s,z)\varphi_2], \tag{62}$$

$$\mathcal{L}[u_2](s) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j} f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j} \frac{s^{-k\beta_2+j\beta_1-\beta_2}}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}[q_2(s,z)\varphi_1 + p_2(s,z)\varphi_2]. \tag{63}$$

Consider the expressions

$$P_1(s,z) = \frac{s^{-k\beta_2+j\beta_1-\beta_2}p_1(s,z)}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}, \quad Q_1(s,z) = \frac{s^{-k\beta_2+j\beta_1-\beta_2}q_1(s,z)}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}},$$

and

$$P_2(s,z) = \frac{s^{-k\beta_2+j\beta_1-\beta_2}p_2(s,z)}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}, \quad Q_2(s,z) = \frac{s^{-k\beta_2+j\beta_1-\beta_2}q_2(s,z)}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}.$$

Further, let

$$\nu_{kj} = k\beta_2 - j\beta_1.$$

Since $\beta_2 > \beta_1$ and $k \geq j$, we have $\nu_{kj} > 0$, if $k \geq 1$, and $\nu_{00} = 0$. Taking this into account, we obtain

$$P_1(s,z) = \frac{1}{s^{\nu_{kj}}}\frac{s^{\beta_1-1}}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}} - \frac{f_{22}(z)}{s^{\nu_{kj}+\beta_2}}\frac{s^{\beta_1-1}}{\Big(s^{\beta_1} - f_{11}(z)\Big)^{k+1}}.$$

Now, taking the inverse Laplace transform, due to formula (21), we obtain

$$\mathcal{L}^{-1}[P_1(\cdot,z)](t) = J^{\nu_{kj}}\frac{t^{k\beta_1}}{k!}E_{\beta_1}^{(k)}(t^{\beta_1}f_{11}(z)) - f_{22}(z)J^{\nu_{kj}+\beta_2}\frac{t^{k\beta_1}}{k!}E_{\beta_1}^{(k)}(t^{\beta_1}f_{11}(z)).$$

Similarly,

$$\mathcal{L}^{-1}[Q_1(\cdot,z)](t) = f_{12}(z)J^{\nu_{kj}+1}\frac{t^{k\beta_1+\beta_1-1}}{k!}E_{\beta_1,\beta_1}^{(k)}(t^{\beta_1}f_{11}(z)),$$

$$\mathcal{L}^{-1}[P_2(\cdot,z)](t) = J^{\nu_{kj}}\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z)) - f_{11}(z)J^{\nu_{kj}+1}\frac{t^{k\beta_1+\beta_1-1}}{k!}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big].$$

and

$$\mathcal{L}^{-1}[Q_2(\cdot,z)](t) = f_{2,1}(z)J^{\nu_{kj}+\beta_2}\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z)),$$

It follows from (62) and (63) that the entries $S_{jl}(t,z), j,l = 1,2,$ of the matrix symbol $\mathcal{S}(t,z)$ have representations

$$\begin{aligned}S_{11}(t,z) &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j}\Big[J^{\nu_{kj}}\Big(t^{k\beta_1}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big)\\&\quad - f_{22}(z)J^{\nu_{kj}+\beta_2}\Big(t^{k\beta_1}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big)\Big]\\&= \Big(I - f_{22}(z)J^{\beta_2}\Big)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^{k}\Big[\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big],\end{aligned} \tag{64}$$

$$\begin{aligned}S_{12}(t,z) &= f_{12}(z)\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j}J^{\nu_{kj}+1}\Big(t^{k\beta_1+\beta_1-1}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big)\\&= f_{12}(z)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^{k}J\Big(\frac{t^{k\beta_1+\beta_1-1}}{k!}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big),\end{aligned} \tag{65}$$

$$\begin{aligned}S_{21}(t,z) &= f_{21}(z)\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j}J^{\nu_{kj}+\beta_2}\Big(t^{k\beta_1}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big)\\&= f_{21}(z)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^{k}J^{\beta_2}\Big(\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big),\end{aligned} \tag{66}$$

$$\begin{aligned}S_{22}(t,z) &= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}f_{22}^{j}(z)\Big(-\Delta(z)\Big)^{k-j}\Big[J^{\nu_{kj}}\Big(t^{k\beta_1}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\Big)\\&\quad - f_{11}(z)J^{\nu_{kj}+1}\Big(t^{k\beta_1+\beta_1-1}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big)\Big],\\&= \sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^{k}\Big[\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))\\&\quad - f_{11}(z)J\Big(\frac{t^{k\beta_1+\beta_1-1}}{k!}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big)\Big].\end{aligned} \tag{67}$$

Thus, we prove the following theorem.

**Theorem 2.** *The formal solution to system* (52), (54) *has the representation*

$$\mathcal{U}(t) = \mathcal{S}(t,A)\Phi + \int_0^t \mathcal{S}(t-\tau,A)D_+^{1-\mathcal{B}}\mathcal{H}(\tau)d\tau$$

*where $\mathcal{S}(t,A)$ is the matrix-valued solution operator with the matrix symbol $\mathcal{S}(t,z)$, the entries of which are defined in* (64)–(67).

**Theorem 3.** *Let $f_{12}(z) = 0$. Then, the solution operator $\mathcal{S}(t, A)$ has the matrix symbol with entries*

$$S_{11}(t,z) = E_{\beta_1}(t^{\beta_1} f_{11}(z)), \quad S_{12}(t,z) = 0, \tag{68}$$

$$S_{21}(t,z) = f_{21}(z)\Big(E_{\beta_1}(t^{\beta_1} f_{11}(z))\Big) * \Big(t^{\beta_2-1} E_{\beta_2,\beta_2}(t^{\beta_2} f_{22}(z))\Big), \tag{69}$$

$$S_{22}(t,z) = E_{\beta_2}(t^{\beta_2} f_{22}(z)), \tag{70}$$

*where "$*$" is the convolution operation.*

**Proof.** The fact that $S_{12}(t,z) = 0$ obviously follows from (65). Now, we show the equality for $S_{11}(t,z)$. First, we notice that $f_{12}(z) = 0$ implies $\Delta(z) = f_{11}(z) f_{22}(z)$. Taking this fact into account and utilizing the semigroup property [3,10] of the fractional integration operator $J^{\beta}$, we can express $S_{11}(t,z)$ in the form

$$\begin{aligned} S_{11}(t,z) &= \Big(I - f_{22}(z) J^{\beta_2}\Big) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1} - f_{11}(z) f_{22}(z) J^{\beta_2}\Big)^k \Big[\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(t^{\beta_1} f_{11}(z))\Big] \\ &= \Big(I - f_{22}(z) J^{\beta_2}\Big) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1}\Big)^k \Big(I - f_{11} J^{\beta_1}\Big)^k \Big[\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(t^{\beta_1} f_{11}(z))\Big], \end{aligned} \tag{71}$$

where $I$ is the identity operator. Now, due to Lemma 1 with $\nu = 1$ and $\mu = f_{11}(z)$, one has

$$\Big(I - f_{11}(z) J^{\beta_1}\Big)^k \Big[\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(t^{\beta_1} f_{11}(z))\Big] = J^{k\beta_1} E_{\beta_1}(t^{\beta_1} f_{11}(z)), \tag{72}$$

valid for all $k = 0, 1, \ldots$. Thus, (71) reduces to

$$\begin{aligned} S_{11}(t,z) &= \Big(I - f_{22}(z) J^{\beta_2}\Big) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1}\Big)^k J^{k\beta_1} E_{\beta_1}(t^{\beta_1} f_{11}(z)) \\ &= \Big(I - f_{22}(z) J^{\beta_2}\Big) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1} J^{\beta_1}\Big)^k E_{\beta_1}(t^{\beta_1} f_{11}(z)), \end{aligned} \tag{73}$$

Since

$$\sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2}\Big)^k = \Big(I - f_{22}(z) J^{\beta_2}\Big)^{-1},$$

it follows from (73) the desired equality for $S_{11}(t,z)$ in (68).

Now, we show the validity of (69). Taking into account $\Delta(z) = f_{11}(z) f_{22}(z)$, we rewrite $S_{21}(t,z)$ in (66) in the form

$$\begin{aligned} S_{21}(t,z) &= f_{21}(z) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1} - f_{11}(z) f_{22}(z) J^{\beta_2}\Big)^k J^{\beta_2} \Big(\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(t^{\beta_1} f_{11}(z))\Big) \\ &= f_{21}(z) \sum_{k=0}^{\infty} \Big(f_{22}(z) J^{\beta_2-\beta_1}\Big)^k J^{\beta_2} \Big(I - f_{11} J^{\beta_1}\Big)^k \Big(\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(t^{\beta_1} f_{11}(z))\Big), \end{aligned}$$

Using the relation (72), we have

$$S_{21}(t,z) = f_{21}(z)\Big(I - f_{22}(z) J^{\beta_2}\Big)^{-1} J^{\beta_2} E_{\beta_1}(t^{\beta_1} f_{11}(z)).$$

Further, due to Lemma 2 (see Equation (27)) with $\mu_1 = f_{11}(z)$ and $\mu_2 = f_{22}(z)$, we obtain relation (69).

Similarly, $S_{22}(t,z)$ in (67) can be written as

$$S_{22}(t,z) = \sum_{k=0}^{\infty} \Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^k \Big[\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))$$
$$- f_{11}(z)J\Big(\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big)\Big]$$
$$= \sum_{k=0}^{\infty} \Big(f_{22}(z)J^{\beta_2-\beta_1}\Big)^k \Big(I - f_{11}J^{\beta_1}\Big)^k \Big(\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}(t^{\beta_1}f_{11}(z))$$
$$- f_{11}(z)J\frac{t^{k\beta_1+\beta_1-1}}{k!}E^{(k)}_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big).$$

Finally, using Lemma 1, we obtain

$$S_{22}(t,z) = \Big(I - f_{22}(z)J^{\beta_2}\Big)^{-1}\Big(E_{\beta_1}(t^{\beta_1}f_{11}(z)) - f_{11}(z)JE_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z))\Big)$$
$$= E_{\beta_2}(t^{\beta_2}f_{22}(z)).$$

In the last step, we used relation (27) with $\mu_1 = f_{11}(z)$ and $\mu_2 = f_{22}(z)$. $\square$

**Remark 2.** *Theorem 3 states that the representation formula presented in Theorem 2 coincides with the representation formula obtained in [18] for the solution of fractional-order systems with a lower triangular matrix-valued operator.*

*3.2. Fractional Multi-Order Systems of Differential-Operator Equations: $m \geq 2$*

The method demonstrated in the previous section for $m = 2$ works, in fact, for an arbitrary number of equations. To derive the solution operator, consider the homogeneous system

$$D^{\mathcal{B}}_{*}\mathcal{U}(t) = \mathcal{F}(A)\mathcal{U}(t), \quad t > 0, \tag{74}$$

with the initial condition

$$\mathcal{U}(0) = \Phi \in \mathcal{X}_{A,G}. \tag{75}$$

Here, $\mathcal{B} = (\beta_1, \ldots, \beta_m), 0 < \beta_j \leq 1,\ j = 1, \ldots, m$, is an arbitrary vector order. We can assume, without loss of generality, that $\beta_1 = \min\{\beta_1, \ldots, \beta_m\}$. Applying the Laplace transform and replacing $A$ with a parameter $z \in G$, we obtain

$$(Is^{\mathcal{B}} - \mathcal{F}(z))\mathcal{L}[\mathcal{U}](s) = Is^{\mathcal{B}-\mathbf{1}}\Phi, \tag{76}$$

where $Is^{\mathcal{B}} = \mathrm{diag}(s^{\beta_1}, \ldots, s^{\beta_m})$. This is a system of linear algebraic equations dependent on parameters $s \in \mathbb{C}$ and $z \in G$. The determinant of the matrix on the left has the structure

$$\Psi(s,z) = \det(Is^{\mathcal{B}} - \mathcal{F}(z)) = s^{|\beta|} + \mathcal{G}(s,z) + (-1)^m\Delta(z),$$

where $|\beta| = \beta_1 + \cdots + \beta_m$, $\Delta(z) = \det\mathcal{F}(z)$ and $\mathcal{G}(s,z)$ has the form

$$\mathcal{G}(s,z) = g_1(z)s^{\beta_1} + \cdots + g_m(z)s^{\beta_m} + g_{12}s^{\beta_1+\beta_2} + g_{13}s^{\beta_1+\beta_3} + \cdots + g(z)s^{\beta_2+\cdots+\beta_m}. \tag{77}$$

In other words, $\mathcal{G}(s,z)$ is the sum of functions of the form

$$\sum_{\alpha} g_\alpha(z)s^{\beta_{\alpha_p}+\cdots+\beta_{\alpha_q}},$$

where $\alpha = (\alpha_p, \ldots, \alpha_q)$, $1 \leq p < q \leq m$, is a multi-index taking values in the subsets of the set $\{1, \ldots, m\}$ except $(1, \ldots, m)$; the function $g_\alpha(z)$ is the sum and multiplication combination of entrees $f_{kj}(z)$ of the matrix symbol $\mathcal{F}(z)$. Let $r_*$ be the real part of the largest root of the equation $\Psi(s,z) = 0$. Then, for $s > r_*$, system (76) has a unique solution

$$\mathcal{L}[\mathcal{U}](s) = (Is^{\mathcal{B}} - \mathcal{F}(z))^{-1} Is^{\mathcal{B}-\mathbf{1}} \Phi, \tag{78}$$

It follows that the solution operator has the matrix-valued symbol

$$\mathcal{S}(t,z) = \mathcal{L}^{-1}_{s\to t}\Big[(Is^{\mathcal{B}} - \mathcal{F}(z))^{-1} Is^{\mathcal{B}-\mathbf{1}}\Big]. \tag{79}$$

The components of the solution have the structure (as an implication of the well-known Cramer's rule)

$$\mathcal{L}[u_j](s) = \frac{\mathcal{P}_j(s,z)}{\Psi(s,z)}, \quad j = 1, \ldots, m,$$

where $u_j = u_j(t)$ is the $j$-th component of $\mathcal{U}(t)$ and $\mathcal{P}_j(s,z)$ is the determinant of the matrix obtained by replacing the $j$-th column of the matrix $Is^{\mathcal{B}} - \mathbb{F}(z)$ with the vector $Is^{\mathcal{B}-\mathbf{1}}\Phi$. The latter can be rewritten in the form

$$\mathcal{L}[u_j](s) = \sum_{l=1}^{m} \frac{p_{jl}(s,z)}{\Psi(s,z)} \varphi_l, \quad j = 1, \ldots, m, \tag{80}$$

where $\varphi_l, l = 1, \ldots, m$, are components of $\Phi$, and functions $p_{jl}(s,z)$ have form (77). Let $\beta_* = \beta_2 + \cdots + \beta_m$. We have

$$\frac{1}{\Psi(s,z)} = \frac{1}{s^{\beta_*}\Big(s^{\beta_1} + g(z) - g(s,z)s^{-\beta_*} + (-1)^m \Delta(z) s^{-\beta_*}\Big)}$$

where $g(z)$ is the coefficient in the term $g(z)s^{\beta_2+\cdots+\beta_m}$ in (77) and

$$g(s,z) = -G(s,z) + g(z)s^{\beta_2+\cdots+\beta_m}.$$

Further, similar to the case $m = 2$, we represent $1/\Psi(s,z)$ in the infinite functional series form

$$\begin{aligned} \frac{1}{\Psi(s,z)} &= \frac{1}{s^{\beta_*}\Big(s^{\beta_1} + g(z)\Big)\Big(1 - \frac{g(s,z)s^{-\beta_*} + (-1)^m \Delta(z) s^{-\beta_*}}{s^{\beta_1} + g(z)}\Big)} \\ &= \sum_{k=0}^{\infty} \frac{\Big(g(s,z) + (-1)^m \Delta(z)\Big)^k s^{-k\beta_* - \beta_*}}{\Big(s^{\beta_1} + g(z)\Big)^{k+1}}. \end{aligned} \tag{81}$$

For $s > r_*(z)$ large enough, the inequality

$$\left| \frac{g(s,z)s^{-\beta_*} + (-1)^m \Delta(z) s^{-\beta_*}}{s^{\beta_1} + g(z)} \right| < 1$$

is valid, and, therefore, the series in (81) is convergent.

Further, it follows from (80) that

$$u_j(t) = \sum_{l=0}^{m} \mathcal{L}^{-1}\left[\frac{p_{jl}(s,A)}{\Psi(s,z)}\right] \varphi_j, \quad j = 1, \ldots, m.$$

Hence, the matrix symbol of the solution operator $\mathcal{S}(t,A)$ has entries

$$s_{jl}(t,z) = \mathcal{L}^{-1}_{s\to t}\left[\frac{p_{jl}(s,z)}{\Psi(s,z)}\right](t), \quad j,l = 1, \ldots, m.$$

Now, using (81), we have

$$s_{jl}(s,z) = \mathcal{L}[s_{jl}(t,z)](s) = \sum_{k=0}^{\infty} \frac{p_{jl}(s,z)\Big(g(s,z)+(-1)^m\Delta(z)\Big)^k s^{-k\beta_*-\beta_*}}{\Big(s^{\beta_1}+g(z)\Big)^{k+1}}.$$

Taking into account the fact that $g(s,z)$ and functions $p_{jl}(s,z)$ can be represented in form (77), we can write the expressions $p_{jl}(s,z)(g(s,z)+(-1)^m\Delta(z))^k,\ j,l=1,\dots,m$, as

$$\sum_{\alpha} Q_{\alpha,j,l,k}(z)s^{\gamma_{\alpha,j,l,k}}, \tag{82}$$

where $Q_{\alpha,j,l,k}(z)$ are the sum and product combinations of $f_{j,k}(z)$ and exponents $\gamma_{\beta,j,l,k}$ depend on the sum combinations of $\beta_1,\dots,\beta_m$ and their multiples. Therefore,

$$s_{jl}(s,z) = \sum_{k=0}^{\infty}\sum_{\alpha} Q_{\alpha,j,l,k}(z)\frac{s^{\gamma_{\alpha,j,l,k}-(k+1)\beta_*}}{\Big(s^{\beta_1}+g(z)\Big)^{k+1}}.$$

Further, let $\nu_{\alpha,j,l,k} = (k+1)\beta_* + \beta_1 - 1 - \gamma_{\alpha,j,l,k}$. By construction, $\nu_{\alpha,j,l,k} \geq 0$ for all indices $\alpha, j, l, k$. Then,

$$\begin{aligned} s_{jl}(t,z) &= \sum_{k=0}^{\infty}\sum_{\alpha} Q_{\alpha,j,l,k}(z)\mathcal{L}^{-1}_{s\to t}\left[\frac{s^{\beta_1-1}}{s^{\nu_{\alpha,j,l,k}}\Big(s^{\beta_1}+g(z)\Big)^{k+1}}\right] \\ &= \sum_{k=0}^{\infty}\sum_{\alpha} Q_{\alpha,j,l,k}(z)J^{\nu_{\alpha,j,l,k}}\left[\frac{t^{k\beta_1}}{k!}E^{(k)}_{\beta_1}\Big(-t^{\beta_1}g(z)\Big)\right], \quad j,l=1,\dots,m. \end{aligned} \tag{83}$$

Hence, the solution to Cauchy problem (74), (75) has the form

$$\mathcal{U}(t) = \mathcal{S}(t,A)\Phi, \tag{84}$$

where $\mathcal{S}(t,A)$ is the matrix-valued solution operator with the matrix symbol $\mathcal{S}(t,z)$ defined by (83).

**Theorem 4.** *Let $\mathcal{B} = (\beta_1,\dots,\beta_m)$, where $\beta_j \in (0,1), j=1,\dots,m$, arbitrary numbers and $\beta_1 = \min\{\beta_1,\dots,\beta_m\}$. Let $A$ be a closed operator defined on a Banach space $X$, and the set $G$ satisfies the condition $G \cap Q_0 = \emptyset$, where $Q_0 = \{z : \Psi(s,z)=0\}$, $\Phi \in \mathcal{X}_{A,G}$, $H(t) \in AC[\mathbb{R}_+;\mathcal{X}_{A,G}]$, and $D_+^{1-\mathcal{B}}H(\tau,x) \in C[\mathbb{R}_+;\mathcal{X}_{A,G}]$.*

*Then, for any $T>0$, Cauchy problem (1), (2) has a unique solution $U(t) \in C^\infty[(0,T];\mathcal{X}_{A,G}] \cap C[[0,T];\mathcal{X}_{A,G}]$, having the representation*

$$U(t) = S(t,A)\Phi + \int_0^t S(t-\tau,A)D_+^{1-\mathcal{B}}H(\tau)d\tau, \quad t>0, \tag{85}$$

*where $S(t,A)$ is the solution matrix operator with the matrix symbol $\mathcal{S}(t,z)$ defined in (83).*

**Proof.** The representation (85) follows from (84) and the fractional Duhamel principle [10,37]. Let

$$V(t) = S(t,A)\Phi, \tag{86}$$

$$W(t) = \int_0^t S(t-\tau,A)D_+^{1-\mathcal{B}}H(\tau)d\tau, \quad t \geq 0. \tag{87}$$

It follows from (49) that $V(t) \in \mathcal{X}_{A,G}$ for every fixed $t \geq 0$, continuous on $[0,T]$, and infinitely differentiable on $(0,T)$ in the topology of $\mathcal{X}_{A,G}$ due to the construction of the solution operator $S(t,A)$. Similarly, by virtue of the continuity in $\mathcal{X}_{A,G}$ of operators with symbols analytic in $G$ (see (49)), for each fixed $t$, we have $W(t) \in \mathcal{X}_{A,G}$ for each fixed $t \in [0,T]$. The continuity of $W(t)$ on $[0,T]$ in the variable $t$ and its infinite differentiability on $(0,T)$ follow from the construction of the solution operator $S(t,D)$ in the standard way. $\square$

**Theorem 5.** *Let $X$ be a reflexive Banach space with the conjugate $X^*$, $A$ be a closed operator with a domain $\mathcal{D} \subset X$, and $\mathcal{F}(A)$ be a matrix operator with the symbol $\mathcal{F}(z)$ continuous on $G$ and satisfying condition $G \cup Q_0 = \emptyset$, where $Q_0$ is defined in (115). Assume that $\Psi \in \mathcal{X}'_{A^*,G^*}$, $H(t) \in AC[\mathbb{R}_+;\mathcal{X}'_{A^*,G^*}]$, and $D_+^{1-\mathcal{B}}H(t) \in C[\mathbb{R}_+;\mathcal{X}'_{A^*,G^*}]$. Then, for any $T > 0$, Cauchy problem*

$$D_*^{\mathcal{B}}V(t) = \mathcal{F}(A^*)V(t) + H(t), \quad t > 0, \tag{88}$$

$$V(0) = \Psi, \tag{89}$$

*has a unique solution $V(t) \in C^\infty[(0,T];\mathcal{X}'_{A^*,G^*}] \cap C[[0,T];\mathcal{X}'_{A^*,G^*}]$, having the representation*

$$V(t) = S(t,A^*)\Psi + \int_0^t S(t-\tau,A^*)D_+^{1-\mathcal{B}}H(\tau)d\tau, \quad t > 0, \tag{90}$$

*where $S(t,A^*)$ is the operator with the matrix symbol $\mathcal{S}(t,z)$ defined in (158).*

**Proof.** We note that elements $D_*^{\mathcal{B}}V(t)$ and $\mathcal{F}(A^*)V(t)$ belong to the space $\mathcal{X}'_{A^*,G^*}$ if $V(t) \in \mathcal{X}'_{A^*,G^*}$ for each fixed $t \geq 0$. This fact follows from the definition of the fractional derivative $D_*^{\mathcal{B}}$ and Theorem 4.

We show that $V(t)$ defined in (90) satisfies the following conditions:

$$\langle D_*^{\mathcal{B}}V(t), \Phi\rangle = \langle V(t), \mathcal{F}^T(A)\Phi\rangle + \langle H(t), \Phi\rangle, \quad t > 0, \tag{91}$$

$$\langle V(0), \Phi\rangle = \langle \Psi(x), \Phi\rangle, \tag{92}$$

where $\mathcal{F}^T(A)$ is the conjugate transpose of $\mathcal{F}(A^*)$, for an arbitrary element $\Phi$ in the space $\mathcal{X}_{A,G}$. Indeed, to prove this fact, let us first assume that $H(t) = 0$ (as an element of $\mathcal{X}'_{A^*,G^*}$) for all $t \geq 0$. Then, (91) takes the form

$$\left\langle \left[D_*^{\mathcal{B}}S(t,A^*) - \mathcal{F}(A^*)\right]V(t), \Phi\right\rangle = \left\langle V(t), \left[D_*^{\mathcal{B}}S(t,A) - \mathcal{F}(A)\right]^T\Phi\right\rangle = 0, \tag{93}$$

for each fixed $t > 0$. The operator $S(t,A)$ is constructed so that $D_*^{\mathcal{B}}S(t,A) - \mathcal{F}(A) = 0$, which implies $[D_*^{\mathcal{B}}S(t,A) - \mathcal{F}(A)]^T = 0$, as well. Indeed, if $V(t)$ is a solution to Equation (74), then it follows from representation (84) that $D_*^{\mathcal{B}}V(t) = D_*^{\mathcal{B}}S(t,A)\Phi = \mathcal{F}(A)\Phi$ for any fixed $\Phi \in \mathcal{X}_{A,G}$. This implies the equality $D_*^{\mathcal{B}}S(t,A) = \mathcal{F}(A)$. Thus, Equation (91) is valid for all $\Phi \in \mathcal{X}_{A,G}$.

Further, it follows from the construction of the operator $\mathcal{S}(t,A)$ that the symbol $\mathcal{S}(t,z)$ at $t = 0$ reduces to the identity matrix. Therefore, the operator corresponding to the matrix symbol $\mathcal{S}(0,z)$ is the identity operator. Hence, $V(0) = S(0,A^*)\Phi = \Phi$. Thus, equality (92) is also verified.

In the general case, for non-zero $H(t)$, representation (90) is an implication of the fractional Duhamel principle [10,37]. $\square$

**Remark 3.** *Obtaining closed-form representations for $Q_{\alpha,j,l,k}$ and $v_{\alpha,j,l,k}$ in the general case is possible, but it is very cumbersome. In the case of $m = 2$, see (64)–(67).*

*3.3. Fractional Multi-Order Systems of Differential-Operator Equations with Triangular Matrix-Valued Operators*

If the matrix symbol $\mathcal{F}(z)$ in system (1) is a lower or upper triangular matrix, then representation (83) is significantly simplified. See Theorem 3 in the case of a lower triangular matrix for $m = 2$. In this section, we derive representation formulas for the solution for arbitrary $m \geq 2$. For fractional systems of pseudo-differential equations with lower or upper triangular matrix symbols, the representation formulas are presented in paper [18].

Assume that $\mathcal{B}$ is an arbitrary multi-order with components $\beta_k \in (0,1)$, $k = 1, \ldots, m$ and $\mathcal{F}(z)$ is a lower triangular matrix symbol. Then, system (76) takes the form

$$\begin{bmatrix} s^{\beta_1} - f_{11}(z) & 0 & \ldots & 0 \\ -f_{21}(z) & s^{\beta_2} - f_{22}(z) & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ -f_{m1}(z) & -f_{m2}(z) & \ldots & s^{\beta_m} - f_{mm}(z) \end{bmatrix} \begin{bmatrix} L[u_1](s) \\ L[u_2](s) \\ \ldots \\ L[u_m](s) \end{bmatrix} = \begin{bmatrix} s^{\beta_1 - 1}\varphi_1 \\ s^{\beta_2 - 1}\varphi_2 \\ \ldots \\ s^{\beta_m - 1}\varphi_m \end{bmatrix}.$$

The latter implies the following recurrent relations:

$$L[u_1](s) = \frac{s^{\beta_1 - 1}}{s^{\beta_1} - f_{11}(z)}\varphi_1, \tag{94}$$

$$L[u_k](s) = \frac{s^{\beta_k - 1}}{s^{\beta_k} - f_{11}(z)}\varphi_k + \sum_{j=1}^{k-1} \frac{f_{kj}(z)}{s^{\beta_k} - f_{kk}(z)} L[u_j](s), \quad k = 2, \ldots, m. \tag{95}$$

In order to represent the solution (95) through $\varphi_1, \ldots, \varphi_m$, we introduce the following notations. Let $\mathcal{T} = \{k, k-1, \ldots, k-j\}$, where $k$ and $j$ are integers satisfying conditions $1 \leq k \leq m$ and $1 \leq j < k$, respectively. By $\mathcal{T}_l$, $1 \leq l \leq j-1$, we denote the set of subsets of $\mathcal{T}$ such that, from $\mathcal{T}$, exactly $l$ numbers except $k$ and $k-j$ are removed. Then, for $L[u_k](s)$, $k = 2, \ldots, m$, in (95), we have

$$L[u_k](s) = \frac{s^{\beta_k - 1}}{s^{\beta_k} - f_{11}(z)}\varphi_k + \sum_{j=1}^{k-1}\sum_{l=1}^{j-1} \frac{P_{kjl}(z) s^{\beta_{k-j} - 1}}{\prod\limits_{\tau \in \mathcal{T}_l} \left(s^{\beta_\tau} - f_{\tau\tau}(z)\right)} \varphi_{k-j}, \quad k = 2, \ldots, m, \tag{96}$$

where $P_{kjl}$, $k = 2, \ldots, m$, $j = 1, \ldots, k-1$, are the multiplication and sum combinations of functions $f_{\tau\mu}$, $\tau = 2, \ldots, k$, $\mu = 1, \ldots, \nu - 1$. Now, making use of formula (19) and the convolution formula for the Laplace transform, it follows from (94) and (96) that

$$u_1(t) = E_{\beta_1}(f_{11}(A)t^{\beta_1})\varphi_1, \tag{97}$$

$$\begin{aligned} u_k(t) &= E_{\beta_k}(f_{kk}(A)t^{\beta_k})\varphi_k \\ &+ \sum_{j=1}^{k-1}\sum_{l=1}^{j-1} \left[ P_{kjl}(A) E_{\beta_{k-j}}\left(f_{k-jk-j}(A)t^{\beta_{k-j}}\right) * \left( * \prod_{\substack{\tau \in \mathcal{T}_l \\ \tau \neq j}} t^{\beta_\tau - 1} E_{\beta_\tau, \beta_\tau}(f_{\tau\tau}(A)t^{\beta_\tau}) \right) \right] \varphi_{k-j}, \\ & k = 2, \ldots, m, \end{aligned} \tag{98}$$

where $J^\gamma$ is the fractional integration operator (see (11)) of order $\gamma > 0$, "$*$" is the convolution operation, and "$* \prod$" is the convolution product. Thus, the solution of homogeneous Cauchy problem (74), (75) has the representation

$$\mathcal{U}(t) = \mathcal{S}(t, A)\Phi, \tag{99}$$

where $\mathcal{S}(t, A)$ is the solution matrix operator with the matrix symbol $\mathcal{S}(t, z)$ with entries

$$s_{kj}(t,z) = \begin{cases} 0, & \text{if } j > k, \\ E_{\beta_k}(f_{kk}(z)t^{\beta_k}), & \text{if } j = k, \\ \sum\limits_{l=1}^{j-1} P_{kjl}(z)E_{\beta_j}(f_{jj}(z)t^{\beta_j}) * \Big( * \prod\limits_{\substack{\tau \in \mathcal{T}_l \\ \tau \neq k-j}} t^{\beta_\tau - 1}E_{\beta_\tau,\beta_\tau}(f_{\tau\tau}(z)t^{\beta_\tau})\Big), & \text{if } j < k. \end{cases} \tag{100}$$

Similarly, if the matrix symbol $\mathcal{F}(z)$ is upper triangular, then the components of the solution $\mathcal{U}(t)$ take the form

$$u_m(t) = E_{\beta_m}(f_{11}(A)t^{\beta_m})\varphi_m, \tag{101}$$

$$\begin{aligned} u_k(t) &= E_{\beta_k}(f_{kk}(A)t^{\beta_k})\varphi_k \\ &+ \sum_{j=k+1}^{m}\sum_{l=1}^{m-j-1}\left[Q_{kjl}(A)E_{\beta_j}(f_{jj}(A)t^{\beta_j}) * \Big( * \prod_{\substack{\nu \in \mathcal{P}_l \\ \nu \neq j}} t^{\beta_\nu - 1}E_{\beta_\nu,\beta_\nu}(f_{\nu\nu}(A)t^{\beta_\nu})\Big)\right]\varphi_j, \\ &\qquad k = m-1,\dots,1, \end{aligned} \tag{102}$$

where $Q_{kj}(A)$ are operators obtained from $f_{\nu\mu}(A), \nu = k,\dots,m-1, \mu = k+1,\dots,m,$ via the product and sum combinations and $\mathcal{P}_l$ is the set of subsets of the set $\mathcal{P} = \{k, k+1,\dots,m\}$ such that exactly $l$ $(1 \leq l \leq m-k-1)$ numbers except $k$ and $m$ are removed from $\mathcal{P}$. In turn, the matrix symbol of the solution operator in representation formula (99) takes the form

$$s_{kj}(t,z) = \begin{cases} 0, & \text{if } j < k, \\ E_{\beta_k}(f_{kk}(z)t^{\beta_k}), & \text{if } j = k, \\ \sum\limits_{l=1}^{m-j-1} Q_{kjl}(z)E_{\beta_j}(f_{jj}(z)t^{\beta_j}) * \Big( * \prod\limits_{\substack{\nu \in \mathcal{P}_l \\ \nu \neq j}} t^{\beta_\nu - 1}E_{\beta_\nu,\beta_\nu}(f_{\nu\nu}(z)t^{\beta_\nu})\Big), & \text{if } j > k. \end{cases} \tag{103}$$

### *3.4. Commensurate and Non-Commensurate Rational-Order Systems*

If all components of the vector-order $\mathcal{B}$ are equal, then the transformation of the matrix symbol $\mathcal{F}(z)$ to the Jordan canonical form can be effectively utilized in the derivation of representation formulas for the solution. Theorem 1 serves as an important mathematical basis for such an approach. If all components of $\mathcal{B}$ are rational (not necessarily being equal), then this case can be reduced to the case with equal components, but at the cost of an increased number of equations (see [18]). We note that both these cases were presented in [18] for time-fractional systems of pseudo-differential equations. Below, applying the same technique presented in [18], but not providing explicit details, we generalize it for fractional-order systems of differential-operator equations, which significantly expands the scope of application.

Let $\mathcal{F}(A)$ be the matrix-valued operator with the matrix symbol $\mathcal{F}(z)$, $z \in G$, whose Jordan normal form is

$$M^{-1}(z)\mathcal{F}(z)M(z) = \Lambda(z) + N = \begin{bmatrix} \mathbb{J}_1(z) & 0 & \dots & 0 \\ 0 & \mathbb{J}_2(z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbb{J}_L(z) \end{bmatrix}, \tag{104}$$

where $M(z)$ is a transformation matrix invertible at each $z \in G$, and $\mathbb{J}_\ell(z)$, $\ell = 1,\dots,L$, are Jordan blocks corresponding to eigenvalues $\lambda_1(z),\dots,\lambda_L(z)$ of the matrix

$$\mathcal{F}(z) = M(z)\Big(\Lambda(z) + N\Big)M^{-1}(z).$$

First, we derive a representation formula for the solution operator of the initial value problem for system (1) in the homogeneous case

$$D_*^{\mathcal{B}} U(t) = \mathcal{F}(A)U(t), \quad t > 0, \tag{105}$$

$$U(0) = \Phi, \tag{106}$$

or due to (104) equivalently

$$D_*^{\mathcal{B}} U(t) = M(A)\Big(\Lambda(A) + N\Big)M^{-1}(A)U(t), \quad t > 0, \tag{107}$$

$$U(0) = \Phi, \tag{108}$$

where $M(A)$ is the transformation matrix operator with the matrix symbol $M(z)$.

In order to solve problem (107), (108), we use the operator approach, considering the following system of ordinary fractional-order differential equations dependent on the parameter $z \in G$ :

$$D_*^{\mathcal{B}} U(t,z) = M(z)\Big(\Lambda(z) + N\Big)M^{-1}(z)U(t,z), \quad t > 0,\ z \in G, \tag{109}$$

$$U(0,z) = \Psi, \tag{110}$$

assuming that $\Psi$ is a vector of length $m$. Since all components of $\mathcal{B}$ are equal, the matrix-valued operator $ID_*^{\mathcal{B}} = \mathrm{diag}(D_*^{\beta}, \ldots, D_*^{\beta})$ commutes with $M(z)$, and therefore system (109) can be expressed as

$$M(z)\ ID_*^{\mathcal{B}}\ M^{-1}(z)U(t,z) = M(z)\Big(\Lambda(z) + N\Big)M^{-1}(z)U(t,z), \quad t > 0,\ z \in G, \tag{111}$$

Denote $V(t,z) = M^{-1}(z)U(t,z)$. Then, we have the Cauchy problem

$$D_*^{\mathcal{B}} V(t,z) = \Big(\Lambda(z) + N\Big)V(t,z), \quad t > 0,\ z \in G, \tag{112}$$

$$V(0,z) = M(z)\Psi. \tag{113}$$

Now, applying the Laplace transform in the vector form (12) to both sides of system (112), we obtain

$$Is^{\mathcal{B}}\mathcal{L}[V](s) = Is^{\mathcal{B}-1}M(z)\Psi + \Big(\Lambda(z) + N\Big)\mathcal{L}[V](s), \quad s > 0,\ \xi \in G.$$

where $Is^{\mathcal{B}}$, $Is^{\mathcal{B}-1}$ are diagonal matrices with diagonal entries $s^{\beta_j}$, $s^{\beta_j - 1}$ $j = 1, \ldots, m$, respectively. The latter implies a system of algebraic equations with parameter $z \in G$ :

$$\Big[Is^{\mathcal{B}} - \Big(\Lambda(z) - N\Big)\Big]L[V](s) = Is^{\mathcal{B}-1}M(z)\Psi. \tag{114}$$

Let

$$Q_0 = \{z \in \mathbb{C} : \det\Big[Is^{\mathcal{B}} - \Big(\Lambda(z) - N\Big)\Big] = 0, \forall s > 0\}. \tag{115}$$

If $G \cap Q_0 = \emptyset$, then system (114) has a unique solution

$$L[V](s) = \mathcal{N}(s,z)M(z)\Psi, \tag{116}$$

where

$$\mathcal{N}(s,z) = \Big[Is^{\mathcal{B}} - \Big(\Lambda(z) - N\Big)\Big]^{-1} Is^{\mathcal{B}-1}. \tag{117}$$

This matrix has the block diagonal form

$$\mathcal{N}(s,z) = \begin{bmatrix} \mathbb{S}_1(s,z) & 0 & \dots & 0 \\ 0 & \mathbb{S}_2(s,z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbb{S}_L(s,z) \end{bmatrix},$$

with the blocks $\mathbb{S}_\ell(s,z), \ell = 1,\dots,L$, of size $m_\ell$ corresponding to the eigenvalue $\lambda_\ell$ of multiplicity $m_\ell$ :

$$\mathbb{S}_\ell = \begin{bmatrix} \frac{s^{\beta-1}}{s^\beta-\lambda_\ell(z)} & \frac{s^{\beta-1}}{(s^\beta-\lambda_\ell(z))^2} & \dots & \frac{s^{\beta-1}}{(s^\beta-\lambda_\ell(z))^{m_\ell}} \\ 0 & \frac{s^{\beta-1}}{s^\beta-\lambda_\ell(z)} & \dots & \frac{s^{\beta-1}}{(s^\beta-\lambda_\ell(z))^{m_\ell-1}} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{s^{\beta-1}}{s^\beta-\lambda_\ell(z)} \end{bmatrix}, \quad \ell = 1,\dots L.$$

Further, applying the inverse Laplace transform, taking into account (19) and (21), and returning to $U(t,z) = M(z)V(t,z)$, in accordance with Theorem 1, we have

$$U(t,z) = E_{\mathcal{B}}(It^{\mathcal{B}}\mathcal{F}(z))\Phi = M(z)\mathbb{E}_{\mathcal{B}}\Big((\Lambda(z)+N)t^{\mathcal{B}}\Big)M^{-1}(z)\Phi, \quad t>0, \tag{118}$$

where $\mathbb{E}_{\mathcal{B}}\Big((\Lambda(z)+N)t^{\mathcal{B}}\Big)$ is the block diagonal matrix of the form

$$\mathbb{E}_{\mathcal{B}}\Big((\Lambda(z)+N)t^{\mathcal{B}}\Big) = \begin{bmatrix} \mathbb{E}_\beta(t^\beta J_1(z)) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \mathbb{E}_\beta(t^\beta J_L(z)) \end{bmatrix}, \tag{119}$$

with blocks

$$\mathbb{E}_\beta(t^\beta J_\ell(z)) = \begin{bmatrix} E_\beta(\lambda_\ell(z)t^\beta) & \frac{t^\beta E'_\beta(\lambda_\ell(z)t^\beta)}{1!} & \dots & \frac{t^{(m_\ell-1)\beta}E_\beta^{(m_\ell-1)}(\lambda_\ell(z)t^\beta)}{(m_\ell-1)!} \\ 0 & E_\beta(\lambda_\ell(z)t^\beta) & \dots & \frac{t^{(m_\ell-2)\beta}E_\beta^{(m_\ell-2)}(\lambda_\ell(z)t^\beta)}{(m_\ell-2)!} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & E_\beta(\lambda_\ell(z)t^\beta) \end{bmatrix}, \tag{120}$$

for $\ell = 1,\dots,L$.

Thus, the solution of problem (105)-(106) has the representation

$$U(t) = S(t,A)\Phi, \quad t>0, \tag{121}$$

where $S(t,A)$ is the solution matrix operator with the matrix symbol

$$\mathcal{S}(t,z) = E_{\mathcal{B}}(It^{\mathcal{B}}\mathcal{F}(z)) = M(z)\,\mathbb{E}_{\mathcal{B}}\Big(t^\beta(\Lambda(z)+N)\Big)\,M^{-1}(z), \quad t>0,\ z\in G. \tag{122}$$

Now, let us consider the incommensurate case $\mathcal{B} = (\beta_1,\dots,\beta_m)$, with rational components $\beta_j = q_j/p_j \in (0,1)$, where $p_j$, $q_j$ are positive co-prime integers. Let $p$ be the least common divisor of numbers $p_1,\dots,p_m$. Then, one can write $\beta_j$ as $\beta_j = n_j/p$, where $n_j = (q_jp)/p_j$ is an integer. Therefore, the operator $D_*^{\mathcal{B}}$ can be presented in the form

$$D_*^{\mathcal{B}} = (D_*^{\beta_1},\dots,D_*^{\beta_m}) = \Big((D_*^{\frac{1}{p}})^{n_1},\dots,(D_*^{\frac{1}{p}})^{n_m}\Big).$$

It follows that, for each $j = 1, \dots, m$, we have

$$D_*^{\beta_j} u_j = D_*^{\frac{1}{p}} \underbrace{\circ \cdots \circ}_{n_j \text{ times}} D_*^{\frac{1}{p}} u_j.$$

Introduce a vector function $\mathbb{U}(t)$ of length $N = n_1 + \cdots + n_m$ :

$$\mathbb{U}(t) = (u_1(t), u_1^1(t), \dots, u_1^{n_1-1}(t), \dots, u_m(t), u_m^1(t), \dots, u_m^{n_m-1}),$$

where $u_j^1 = D_*^{\frac{1}{p}} u_j, \dots, u_j^{n_j} = D_*^{\frac{1}{p}} u_j^{n_j-1}$, $j = 1, \dots, m$. Now, system (105) can be reduced to a system of $N$ equations with equal fractional-order $1/p$ derivatives on the left-hand side. The reduced system has the form

$$D_*^{\frac{1}{p}} \mathbb{U}(t) = \mathbb{F}(A)\mathbb{U}(t), \tag{123}$$

where $\mathbb{F}(A)$ is the matrix operator with the matrix symbol

$$\mathbb{F}(z) = \begin{bmatrix} \mathbb{F}_{11}(z) & \mathbb{F}_{12}(z) & \dots & \mathbb{F}_{1m}(z) \\ \mathbb{F}_{21}(z) & \mathbb{F}_{22}(z) & \dots & \mathbb{F}_{2m}(z) \\ \dots & \dots. & \dots. & \\ \mathbb{F}_{m1}(z) & \mathbb{F}_{m2}(z) & \dots & \mathbb{F}_{mm}(z) \end{bmatrix}$$

whose diagonal block matrices are of sizes $n_j \times n_j$

$$\mathbb{F}_{jj}(z) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ f_{jj}(z) & 0 & 0 & \dots & 0 \end{bmatrix}, \quad j = 1, \dots, m,$$

and non-diagonal block matrices are of sizes $n_j \times n_k$

$$\mathbb{F}_{jk}(z) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 \\ f_{jk}(z) & 0 & 0 & \dots & 0 \end{bmatrix}, \quad j, k = 1, \dots, m.$$

The initial condition for system (123) takes the form

$$\mathbb{U}(0) = (\varphi_1, \underbrace{0, \dots, 0}_{n_1-1 \text{ times}}, \varphi_2, \underbrace{0, \dots, 0}_{n_2-1 \text{ times}}, \dots, \varphi_m, \underbrace{0, \dots, 0}_{n_m-1 \text{ times}}). \tag{124}$$

Now, we derive a representation formula for the solution of Cauchy problem (105), (106). We notice that that the characteristic polynomial of $\mathbb{F}(z)$,

$$P_N(\lambda, z) = \det(I\lambda - \mathbb{F}(z)), \quad \lambda \in \mathbb{C}, \ z \in G,$$

and the function $h(s, z) = \det(Is^{\mathcal{B}} - \mathcal{F}(z))$ are connected through the relationship

$$h(s, z) = P_N(s^{\frac{1}{p}}, z), \quad z \in G. \tag{125}$$

Let $\lambda_1(z), \dots, \lambda_L(z)$ be eigenvalues of the matrix $\mathbb{F}(z)$ with respective multiplicities $\mu_1, \dots, \mu_L$. Then,

$$P_N(\lambda, z) = \prod_{\ell=1}^{L} (\lambda - \lambda_\ell(z))^{\mu_\ell},$$

which implies

$$\det\left(Is^{\mathcal{B}} - \mathcal{F}(z)\right) = \prod_{\ell=1}^{L} \left(s^{\frac{1}{p}} - \lambda_\ell(z)\right)^{\mu_\ell}.$$

Similarly, we can write the determinant of the matrix $\mathcal{F}_j(s,z)$, obtained by replacing the $j$-th column of the matrix $Is^{\mathcal{B}} - \mathcal{F}(z)$ with the column vector $Is^{\mathcal{B}-1}\Phi$, in the form

$$\det(\mathcal{F}_j(s,z)) = \sum_{k=1}^{m} P_{jk}(s^{\frac{1}{p}}, z) s^{\beta_k - 1} \varphi_k,$$

where $P_{jk}(\lambda, z)$ is a polynomial in the variable $\lambda$ of order $N - n_j$. Hence, for the $j$-th component $u_j(t)$ of the Laplace transform of the solution vector $\mathcal{U}(t)$, we have

$$\begin{aligned} L[u_j](s) &= \frac{\det(\mathcal{F}_j(s,z))}{\det\left(Is^{\mathcal{B}} - \mathcal{F}(z)\right)} \\ &= \sum_{k=1}^{m} \frac{P_{jk}(s^{\frac{1}{p}}, z)}{\prod_{\ell=1}^{L}\left(s^{\frac{1}{p}} - \lambda_\ell(z)\right)^{\mu_\ell}} s^{\beta_k - 1} \varphi_k, \end{aligned} \tag{126}$$

Further, using the partial fraction decomposition

$$\frac{P_{jk}(s^{\frac{1}{p}}, z)}{\prod_{\ell=1}^{L}\left(s^{\frac{1}{p}} - \lambda_\ell(z)\right)^{\mu_\ell}} = \sum_{\ell=1}^{L} \sum_{\nu=1}^{\mu_\ell} \frac{C_{\ell\nu}^{jk}(z)}{\left(s^{\frac{1}{p}} - \lambda_\ell(z)\right)^{\nu}},$$

where $C_{\ell\nu}^{jk}(z)$ do not depend on $s$, Equation (126) can be expressed as

$$L[u_j](s) = \sum_{k=1}^{m} \sum_{\ell=1}^{L} \sum_{\nu=1}^{\mu_L} C_{\ell\nu}^{jk} \frac{s^{\beta_k - 1}}{\left(s^{\frac{1}{p}} - \lambda_\ell(z)\right)^{\nu}} \varphi_k.$$

Inverting the latter and using formula (20), we obtain

$$u_j(t) = \sum_{k=1}^{m} \sum_{\ell=1}^{L} \sum_{\nu=1}^{\mu_L} \frac{C_{\ell\nu}^{jk}\, t^{\frac{\nu}{p} - \beta_k}}{(\nu - 1)!} E_{\frac{1}{p}, \frac{1}{p} - \beta_k + 1}^{(\nu-1)}\left(t^{\frac{1}{p}} \lambda_\ell(z)\right) \varphi_k, \quad j = 1, \dots, m.$$

It follows that the solution operator $S(t, A)$ has the matrix symbol $\mathcal{S}(t, z)$, whose entries are

$$s_{jk}(t,z) = \sum_{\ell=1}^{L} \sum_{\nu=1}^{\mu_L} \frac{C_{\ell\nu}^{jk}\, t^{\frac{\nu}{p} - \beta_k}}{(\nu - 1)!} E_{\frac{1}{p}, \frac{1}{p} - \beta_k + 1}^{(\nu-1)}\left(t^{\frac{1}{p}} \lambda_\ell(z)\right), \quad j, k = 1, \dots, m. \tag{127}$$

Summarizing, we obtain the following theorem on formal representations of the solutions.

**Theorem 6.** *Let the eigenvalues $\lambda_\ell(\xi)$, $\ell = 1, \dots, L$, of the matrix symbol $\mathcal{F}(\xi)$ of the matrix-valued operator $\mathcal{F}(A)$ in system (1) have respective multiplicities $m_\ell, \ell = 1, \dots, L$, where $m_1 + \cdots + m_L = m$. Then, the solution to system (1), (2) has the representation*

$$\mathcal{U}(t) = \mathcal{S}(t, A)\Phi + \int_0^t \mathcal{S}(t-\tau, A) D_+^{1-\mathcal{B}} \mathcal{H}(\tau) d\tau$$

*where $\mathcal{S}(t, A)$ is the matrix-valued solution operator with the matrix symbol $\mathcal{S}(t, z)$ defined*

(a) *in (122) if $\mathcal{B} = (\beta, \dots, \beta)$ with equal components $\beta \in (0, 1]$, and*

(b) *in (127) if $\mathcal{B} = (\beta_1, \dots, \beta_m)$ with rational components $\beta_j = q_j/p_j \in (0, 1]$.*

## 4. The Riemann–Liouville Case

Similar results hold in the case that the fractional derivatives in system (1) are in the Riemann–Liouville sense. Therefore, below, we briefly formulate the corresponding assertions.

Consider the initial value problem

$$D_+^{\mathcal{B}} \mathcal{U}(t) = \mathcal{F}(A)\mathcal{U}(t) + H(t), \quad t > 0, \tag{128}$$

$$(\mathcal{J}^{1-\mathcal{B}} \mathcal{U})(0) = \Phi, \tag{129}$$

where $D_+^{\mathcal{B}} \mathcal{U}(t) = (D_+^{\beta_1} u_1(t), \dots, D_+^{\beta_m} u_m(t))$, and the matrix operator $\mathcal{F}(A)$, vector-valued elements $H(t)$, and $\Phi$ are specified below. Using the same technique that was used in the case of Caputo derivatives, one can show that, in the case of Riemann–Liouville derivatives, the solution matrix operator $\mathcal{S}_+(t, A)$ has the symbol

$$\mathcal{S}_+(t, z) = L_{s \to t}^{-1} \left\{ \left[ I s^{\mathcal{B}} - \mathcal{F}(z) \right]^{-1} \right\}, \quad t \geq 0, \ z \in G \subset \mathbb{C}. \tag{130}$$

The following theorems hold.

**Theorem 7.** *Let $A$ be a closed operator defined on a Banach space $X$, and the set $G$ satisfies the condition $G \cap Q_0 = \emptyset$, where $Q_0$ is defined in (115), $\Phi \in \mathrm{Exp}_{A,G}$, and $H(t) \in AC[\mathbb{R}_+; \mathrm{Exp}_{A,G}]$.*

*Then, for any $T > 0$, Cauchy problem (128) and (129) has a unique solution $U(t) \in C^\infty[(0, T]; \mathrm{Exp}_{A,G}] \cap C[[0, T]; \mathrm{Exp}_{A,G}]$, having the representation*

$$U(t) = \mathcal{S}_+(t, A)\Phi + \int_0^t \mathcal{S}_+(t-\tau, A) H(\tau) d\tau, \quad t > 0, \tag{131}$$

*where $\mathcal{S}_+(t, A)$ is the solution matrix operator with the matrix symbol $\mathcal{S}_+(t, z)$ defined in (130).*

**Theorem 8.** *Let $X$ be a reflexive Banach space with the conjugate $X^*$, $A$ be a closed operator with a domain $\mathcal{D} \subset X$, and $\mathcal{F}(A)$ be a matrix operator with the symbol $\mathcal{F}(z)$ continuous on $G$ and satisfying condition $G \cup Q_0 = \emptyset$, where $Q_0$ is defined in (115). Assume that $\Psi \in \mathcal{E}'_{A^*,G^*}$, and $H(t) \in AC[\mathbb{R}_+; \mathcal{E}'_{A^*,G^*}]$.*

*Then, for any $T > 0$, Cauchy problem*

$$D_+^{\mathcal{B}} V(t) = \mathcal{F}(A^*) V(t) + H(t), \quad t > 0, \tag{132}$$

$$\mathcal{J}^{1-\mathcal{B}} V(0) = \Psi, \tag{133}$$

*has a unique solution $V(t) \in C^\infty[(0, T]; \mathcal{E}'_{A^*,G^*}] \cap C[[0, T]; \mathcal{E}'_{A^*,G^*}]$, having the representation*

$$V(t) = \mathcal{S}_+(t, A^*)\Psi + \int_0^t \mathcal{S}_+(t-\tau, A^*) H(\tau) d\tau, \quad t > 0, \tag{134}$$

*where $\mathcal{S}_+(t, A^*)$ is the operator with the matrix symbol $\mathcal{S}_+(t, z)$ defined in (130).*

Concerning representation formulas for the solution in the case of Riemann–Liouville derivatives, their derivation is similar to the Caputo derivative case. Therefore, we demonstrate the detailed derivation of the representation formula only in the case $m = 2$ with arbitrary multi-order $\mathcal{B}$ and matrix symbol $\mathcal{F}(z)$, providing the final result for arbitrary $m \geq 2$.

### 4.1. The Case $m = 2$ and $\beta_1 \neq \beta_2$

Consider the system

$$D_+^{\mathcal{B}}\mathcal{U}(t) = \mathbb{F}(A)\mathcal{U}(t) + \mathcal{H}(t), \tag{135}$$

where $\mathcal{B} = (\beta_1, \beta_2), 0 < \beta_1 < \beta_2 \leq 1$, $\mathcal{H}(t) = (h_1(t), h_2(t))$ is a given vector function, and

$$\mathbb{F}(A) = \begin{bmatrix} f_{11}(A) & f_{12}(A) \\ f_{21}(A) & f_{22}(A) \end{bmatrix}, \tag{136}$$

with the initial condition

$$(J^{1-\mathcal{B}}\mathcal{U})(0) = \Phi = (\varphi_1, \varphi_2), \tag{137}$$

where $\Phi \in \mathcal{X}_{A,G}$. We assume that $G$ does not contain the roots of the equation

$$\Delta(z) = f_{11}(z)f_{22}(z) - f_{21}(z)f_{12}(z)) = 0.$$

To find entries of the solution operator $\mathcal{S}(t,z)$, we consider the homogeneous counterpart of system (52), writing it in the explicit form

$$\begin{cases} D_*^{\beta_1}u_1(t) & = f_{11}(A)u_1(t) + f_{12}(A)u_2(t), \\ D_*^{\beta_2}u_2(t) & = f_{21}(A)u_1(t) + f_{22}(A)u_2(t). \end{cases}$$

Applying the Laplace transform and replacing $A$ with the parameter $z$, we have

$$\begin{cases} (Is^{\beta_1} - f_{11}(z))\mathcal{L}[u_1](s) - f_{12}(z)\mathcal{L}[u_2](s) = \varphi_1, \\ -f_{21}(z)\mathcal{L}[u_1](s) + (Is^{\beta_2} - f_{22}(z))\mathcal{L}[u_2](s) = \varphi_2. \end{cases} \tag{138}$$

The solution of system (55) is

$$\mathcal{L}[u_1](s) = \frac{1}{\Psi(s,z)}\Big(p_1(s,z)\varphi_1 + q_1(s,z)\varphi_2\Big) \quad z \in G, s > r_*(z) \tag{139}$$

$$\mathcal{L}[u_2](s) = \frac{1}{\Psi(s,z)}\Big(q_2(s,z)\varphi_1 + p_2(s,z)\varphi_2\Big), \quad z \in G, s > r_*(z). \tag{140}$$

where $\Psi(s,z)$ is defined in (58) and

$$p_1(s,z) = s^{\beta_2} - f_{22}(z), \quad q_1(s,z) = f_{12}(z), \tag{141}$$
$$p_2(s,z) = s^{\beta_1} - f_{11}(z), \quad q_2(s,z) = f_{21}(z), \tag{142}$$

and $r_*(z)$ is the real part of the roots of the equation $\Psi(s,z) = 0$. This solution is uniquely defined, since, by assumption, $G \cap Q = \emptyset$, where $Q = \{z : \Psi(s,z) = 0\}$.

Introduce $\rho_{jk} = j(\beta_2 - \beta_1) + (k-j)\beta_2$. Obviously, $\rho_{00} = 0$ and $\rho_{jk} > 0$ if $k > 0$, $0 \leq j \leq k$. The entries $s_{jl}^+(t,z), j,l = 1,2,$ of the matrix symbol $\mathcal{S}_+(t,z)$ have representations

$$s_{11}^+(t,z) = \Big(I - f_{22}(z)J^{\beta_2}\Big)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1} - \Delta(z)J^{\beta_2}\Big)^k \mathbb{W}_{\beta_1,\beta_1}(t,z), \tag{143}$$

$$s_{12}^{+}(t,z)=f_{12}(z)\Big(I-f_{22}(z)J^{\beta_2}\Big)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1}-\Delta(z)J^{\beta_2}\Big)^{k}J^{\beta_2}\mathbb{W}_{\beta_1,\beta_1}(t,z), \tag{144}$$

$$s_{21}^{+}(t,z)=f_{21}(z)\Big(I-f_{22}(z)J^{\beta_2}\Big)\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1}-\Delta(z)J^{\beta_2}\Big)^{k}J^{\beta_2}\mathbb{W}_{\beta_1,\beta_1}(t,z), \tag{145}$$

$$s_{22}^{+}(t,z)=\sum_{k=0}^{\infty}\Big(f_{22}(z)J^{\beta_2-\beta_1}-\Delta(z)J^{\beta_2}\Big)^{k}\Big[\mathbb{W}_{\beta_1,\beta_2}(t,z)-f_{11}(z)J^{\beta_2}\mathbb{W}_{\beta_1,\beta_1}(t,z)\Big], \tag{146}$$

where

$$\mathbb{W}_{\beta_1,\beta_j}(t,z)=\frac{t^{k\beta_i+\beta_j-1}}{k!}E^{(k)}_{\beta_1,\beta_j}(t^{\beta_1}f_{11}(z)), \quad j=1,2.$$

**Theorem 9.** *The solution to system* (128), (129) *has the representation*

$$\mathcal{U}(t)=\mathcal{S}_{+}(t,A)\Phi+\int_0^t\mathcal{S}_{+}(t-\tau,A)D_{+}^{1-\mathcal{B}}\mathcal{H}(\tau)d\tau$$

*where $\mathcal{S}_{+}(t,A)$ is the matrix-valued solution operator with the matrix symbol $\mathcal{S}_{+}(t,z)$, the entries of which are defined in* (143)–(146).

**Theorem 10.** *Let $f_{12}(z)=0$. Then, the symbol of the solution operator has entries*

$$s_{11}^{+}(t,z)=t^{\beta_1-1}E_{\beta_1,\beta_1}(t^{\beta_1}f_{11}(z)), \quad S_{12}(t,z)=0, \tag{147}$$
$$s_{21}^{+}(t,z)=f_{21}(t,z)\Big(t^{\beta_1-1}E_{\beta_1}(t^{\beta_1}f_{11}(z))\Big)*\Big(t^{\beta_2-1}E_{\beta_2,\beta_2}(t^{\beta_2}f_{22}(z))\Big), \tag{148}$$
$$s_{22}^{+}(t,z)=t^{\beta_2-1}E_{\beta_2,\beta_2}(t^{\beta_2}f_{22}(z)), \tag{149}$$

*where "$*$" is the convolution operation.*

*The proof* is similar to the proof of Theorem 3.

### *4.2. The Case $m\geq 2$ and Arbitrary $\mathcal{B}$*

In the case of arbitrary $m\geq 2$, following the same approach demonstrated in Section 3.2, one can derive the representation formula. Namely, instead of Equation (76), one has

$$(Is^{\mathcal{B}}-\mathcal{F}(z))\mathcal{L}[\mathcal{U}](s)=\Phi, \tag{150}$$

and, therefore, instead of Equation (79), one obtains

$$\mathcal{S}(t,z)=\mathcal{L}^{-1}_{s\to t}\Big[(Is^{\mathcal{B}}-\mathcal{F}(z))^{-1}\Big], \tag{151}$$

the symbol of the solution operator. The entries of the latter are

$$\begin{aligned} s_{jl}(t,z)&=\sum_{k=0}^{\infty}\sum_{\alpha}R_{\alpha,j,l,k}(z)\mathcal{L}^{-1}_{s\to t}\left[\frac{1}{s^{\nu_{\alpha,j,l,k}}\Big(s^{\beta_1}+g(z)\Big)^{k+1}}\right]\\ &=\sum_{k=0}^{\infty}\sum_{\alpha}R_{\alpha,j,l,k}(z)J^{\nu_{\alpha,j,l,k}}\left[\frac{t^{k\beta_1+\beta_1-1}}{k!}E^{(k)}_{\beta_1,\beta_1}\Big(-t^{\beta_1}g(z)\Big)\right], \quad j,l=1,\dots,m, \end{aligned} \tag{152}$$

where $R_{\alpha,j,l,k}(z)$ are the sum and product combinations of the entries of $\mathcal{F}(z)$.

### 4.3. The Case $m \geq 2$ and $\mathcal{B} = (\beta, \dots, \beta)$

If the components of $\mathcal{B}$ are equal, then, again, we can use the Jordan normal form to derive a representation formula for the solution. In this case, following the method used in Section 3.4, we consider the system of ordinary fractional-order differential equations depending on the parameter $z \in G$

$$D_+^{\mathcal{B}} V(t,z) = \Big(\Lambda(z) + N\Big) V(t,z), \quad t > 0, \tag{153}$$

where the symbol $\Lambda(z) + N$ of the matrix operator $\Lambda(A) + N$ has a representation in the Jordan block form (104). Then, for the Laplace transform of $V(t)$, we obtain a linear system of algebraic equations

$$\Big(Is^{\mathcal{B}} - \Lambda(z) - N\Big) L[V](s) = M^{-1}\Phi. \tag{154}$$

If $Q_0 \cap G = \emptyset$, then the latter has a unique solution represented through the inverse matrix $\mathcal{N}(s,z) = \Big(Is^{\mathcal{B}} - \Lambda(z) - N\Big)^{-1}$, which has the block diagonal form

$$\mathbb{S}(s,z) = \begin{bmatrix} \mathbb{S}_1(s,z) & 0 & \dots & 0 \\ 0 & \mathbb{S}_2(s,z) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbb{S}_L(s,z) \end{bmatrix},$$

with the blocks $\mathbb{S}_\ell(s,z), \ell = 1, \dots, L$, of size $m_\ell$ corresponding to the eigenvalue $\lambda_\ell$ of multiplicity $m_\ell$ :

$$\mathbb{S}_\ell = \begin{bmatrix} \frac{1}{s^\beta - \lambda_\ell(z)} & \frac{1}{(s^\beta - \lambda_\ell(z))^2} & \dots & \frac{1}{(s^\beta - \lambda_\ell(z))^{m_\ell}} \\ 0 & \frac{1}{s^\beta - \lambda_\ell(z)} & \dots & \frac{1}{(s^\beta - \lambda_\ell(z))^{m_\ell - 1}} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{s^\beta - \lambda_\ell(z)} \end{bmatrix}, \quad \ell = 1, \dots L.$$

Now, using formula (22), one can find the inverse Laplace transform of each entry of matrices $\mathbb{S}_\ell$, $\ell = 1, \dots, L$. Hence, the solution matrix operator has the block matrix representation

$$\mathcal{S}(t, A) = M(A)\mathbb{G}_{\mathcal{B}}\Big(\big(\Lambda(A) + N\big)t^{\mathcal{B}}\Big)M^{-1}(A),$$

where the block matrix operator $\mathbb{G}_{\mathcal{B}}\Big(\big(\Lambda(A) + N\big)t^{\mathcal{B}}\Big)$ has the matrix symbol

$$\mathbb{G}_{\mathcal{B}}\Big(\big(\Lambda(z) + N\big)t^{\mathcal{B}}\Big) = \begin{bmatrix} \mathbb{G}_\beta(t^\beta J_1(z)) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \mathbb{G}_\beta(t^\beta J_L(z)) \end{bmatrix}, \tag{155}$$

with blocks

$$\mathbb{G}_\beta(t^\beta J_\ell(z)) = \begin{bmatrix} t^{\beta-1}E_{\beta,\beta}(\lambda_\ell(z)t^\beta) & \frac{t^{2\beta-1}E'_{\beta,\beta}(\lambda_\ell(z)t^\beta)}{1!} & \dots & \frac{t^{(m_\ell-1)\beta-1}E^{(m_\ell-1)}_{\beta,\beta}(\lambda_\ell(z)t^\beta)}{(m_\ell-1)!} \\ 0 & t^{\beta-1}E_{\beta,\beta}(\lambda_\ell(z)t^\beta) & \dots & \frac{t^{(m_\ell-2)\beta-1}E^{(m_\ell-2)}_{\beta,\beta}(\lambda_\ell(z)t^\beta)}{(m_\ell-2)!} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & t^{\beta-1}E_{\beta,\beta}(\lambda_\ell(z)t^\beta) \end{bmatrix}, \tag{156}$$

for $\ell = 1, \dots, L$.

Concluding, the solution matrix operator has the representation

$$U(t) = S(t, A)\Phi, \quad t > 0, \tag{157}$$

where $S(t, A)$ is the solution matrix operator with the matrix symbol

$$\mathcal{S}(t, z) = M(z)\, \mathbb{G}_{\mathcal{B}}\Big(\big(\Lambda(z) + N\big)t^{\mathcal{B}}\Big)\, M^{-1}(z), \quad t > 0,\ z \in G. \tag{158}$$

## 5. Some Applications and Examples

**Example 2.** *Time-fractional systems of ordinary differential equations.*

Consider the following initial-value problem for a time-fractional system of ordinary differential equations

$$\begin{aligned} D_*^{\mathcal{B}} U(t) &= \mathbb{A}\, U(t) + H(t), \quad t > 0, \\ U(0) &= U_0, \end{aligned}$$

where $\mathbb{A}$ is a (constant) $m \times m$ matrix, $U_0 = (u_1^0, \dots, u_m^0) \in \mathbb{R}^m$, and $H(t)$ is an absolute continuous vector function. The theorems obtained above are applicable to this case with the proper interpretation.

Let $\mathcal{B} = (\beta_1, \dots, \beta_m)$, $0 < \beta_j \le 1$, be arbitrary numbers, $a_{jk}$, $j, k = 1, \dots, m$, be entries of the matrix $\mathbb{A}$, and $d = \det(\mathbb{A})$. Suppose that $\beta_1 = \min\{\beta_1, \dots, \beta_m\}$. Consider first the corresponding homogeneous system

$$D_*^{\mathcal{B}} U(t) = \mathbb{A}\, U(t), \quad U(0) = U_0.$$

Define the function

$$\Psi(s) = \det(Is^{\beta} - \mathbb{F}) = s^{\beta_1 + \dots + \beta_m} + g s^{\beta_2 + \dots + \beta_m} + \dots.$$

The solution to the latter has the form $U(t) = S(t)U_0$, where $S(t), t \ge 0$, is the solution matrix. The components of $S(t)$ due to Theorem 4 have entries

$$s_{jl}(t) = \sum_{k=0}^{\infty} \sum_{\alpha} Q_{\alpha,j,l,k} J^{\nu_{\alpha,j,l,k}} \left[ \frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}\Big( - g t^{\beta_1}\Big) \right], \quad j, l = 1, \dots, m, \tag{159}$$

where $Q_{\alpha,j,l,k}$ are defined similarly to $Q_{\alpha,j,l,k}(z)$ in (82), replacing $f_{kj}(z)$ with $a_{kj}$. In particular, if $m = 2$, then

$$s_{11}(t) = \Big(I - a_{22}J^{\beta_2}\Big) \sum_{k=0}^{\infty} \Big(a_{22}J^{\beta_2 - \beta_1} - dJ^{\beta_2}\Big)^k \Big[\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(a_{11}t^{\beta_1})\Big],$$

$$s_{12}(t) = a_{12} \sum_{k=0}^{\infty} \Big(a_{22}J^{\beta_2 - \beta_1} - dJ^{\beta_2}\Big)^k J\Big(\frac{t^{k\beta_1 + \beta_1 - 1}}{k!} E_{\beta_1, \beta_1}^{(k)}(a_{11}t^{\beta_1})\Big),$$

$$s_{21}(t) = a_{21} \sum_{k=0}^{\infty} \Big(a_{22}J^{\beta_2 - \beta_1} - dJ^{\beta_2}\Big)^k J^{\beta_2}\Big(\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(a_{11}t^{\beta_1})\Big),$$

$$s_{22}(t) = \sum_{k=0}^{\infty} \Big(a_{22}J^{\beta_2 - \beta_1} - dJ^{\beta_2}\Big)^k \Big[\frac{t^{k\beta_1}}{k!} E_{\beta_1}^{(k)}(a_{11}t^{\beta_1}) - a_{11}J\Big(\frac{t^{k\beta_1 + \beta_1 - 1}}{k!} E_{\beta_1, \beta_1}^{(k)}(t^{\beta_1}a_{11})\Big)\Big].$$

Additionally, if $a_{12} = 0$, then

$$s_{11}(t) = E_{\beta_1}(a_{11}t^{\beta_1}), \quad s_{12}(t) = 0,$$

$$s_{21}(t) = a_{21}E_{\beta_1}(a_{11}t^{\beta_1}) * (t^{\beta_2-1}E_{\beta_1,\beta_2}(a_22t^{\beta_2})), \quad s_{22} = E_{\beta_2}(a_{22}t^{\beta_2}).$$

**Example 3.** *Blood alcohol level problem.*

The authors of paper [38] considered the following blood alcohol problem using fractional-order derivatives in the sense of Caputo–Djrbashian:

$$\begin{cases} D_*^{\alpha}A(t) = -\kappa_1 A(t), \\ D_*^{\beta}B(t) = \kappa_1 A(t) - \kappa_2 B(t), \end{cases} \tag{160}$$

where $A$ represents the concentration of alcohol in the stomach and $B$ is the concentration of alcohol in the blood, and $\kappa_1, \ \kappa_2$ are some real constants, which indicate transition rates. The initial conditions are given by

$$A(0) = A_0, \ B(0) = B_0. \ (B_0 = 0 \text{ if initially there is no alcohol in the blood.})$$

This problem can be presented as $D_*^{\mathcal{B}}\mathcal{U}(t) = \mathcal{F}\mathcal{U}(t)$, where $\mathcal{U}(t) = (A(t), B(t))$, $\mathcal{B} = (\alpha, \beta)$ and

$$\mathcal{F} = \begin{bmatrix} -\kappa_1 & 0 \\ \kappa_1 & -\kappa_2 \end{bmatrix}.$$

In accordance with Theorem 2, the solution $\mathcal{U}(t)$ has the representation

$$A(t) = A_0 E_{\alpha}(-\kappa_1 t^{\alpha}), \tag{161}$$

$$B(t) = \kappa_1 A_0 \Big( E_{\alpha}(-\kappa_1 t^{\alpha}) \Big) * \Big( t^{\beta-1}E_{\beta,\beta}(-\kappa_2 t^{\beta}) \Big) + B_0 E_{\beta}(-\kappa_2 t^{\beta}). \tag{162}$$

We note that the authors of [38] found the solution in the form $A(t) = A_0 E_{\alpha}(-\kappa_1 t^{\alpha})$ and (with $B_0 = 0$)

$$B(t) = \beta \int_0^t \kappa_1 A_0 E_{\alpha}(-\kappa_1 (t-s)^{\alpha}) s^{\beta-1} E'_{\beta}(-\kappa_2 s^{\beta}) ds,$$

which is the same as (161), (162), due to the equality $\beta E'_{\beta}(z) = E_{\beta,\beta}(z)$.

**Example 4.** *Fractional-order systems for a relativistically free particle.*

The wave function of a relativistically free particle of mass $m$ is described by the Klein–Gordon equation

$$\left( \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2} \right) \Psi(t,x) = 0, \ x \in \mathbb{R}^3,$$

where $c$ is the speed of light, $\hbar$ is Planck's constant, and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, the gradient operator. Dirac's equation for relativistically free particles, in fact, is a system of the form

$$i\hbar \frac{\partial \Psi(x,t)}{\partial t} = (ic\hbar \boldsymbol{\alpha} \cdot \nabla + \boldsymbol{\beta} mc^2)\Psi(x,t),$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $4 \times 4$ matrices satisfying certain conditions, and $\Psi(x,t)$ is a multi-component wave function. Using the adopted units $c = \hbar = 1$, the latter can be written in the explicit form [39]

$$i \begin{bmatrix} \frac{\partial \psi_1}{\partial t} \\ \frac{\partial \psi_2}{\partial t} \\ \frac{\partial \psi_3}{\partial t} \\ \frac{\partial \psi_4}{\partial t} \end{bmatrix} = \begin{bmatrix} m & 0 & -i\frac{\partial}{\partial x_3} & i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \\ 0 & m & -i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ -i\frac{\partial}{\partial x_3} & -i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} & -m & 0 \\ -i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} & i\frac{\partial}{\partial x_3} & 0 & -m \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \tag{163}$$

Multiplying this by $-i$, we can rewrite system (163) in the form (1)

$$\frac{\partial \Psi(t,x)}{\partial t} = \mathcal{F}(A)\Psi(t,x), \quad t > 0, \ x \in \mathbb{R}^3, \tag{164}$$

where $\mathcal{F}(A)$, $A = -i\nabla = -i(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, is the matrix-valued operator with the symbol

$$\mathcal{F}(\xi) = \begin{bmatrix} -im & 0 & \xi_3 & i\xi_1 + \xi_2 \\ 0 & -im & -i\xi_1 + \xi_2 & \xi_3 \\ -i\xi_3 & -i\xi_1 - \xi_2 & im & 0 \\ -i\xi_1 + \xi_2 & i\xi_3 & 0 & im \end{bmatrix}, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \tag{165}$$

Replacing $\partial/\partial t$ on the left of (164) with $D_*^\gamma$, we obtain a Dirac-like fractional-order system. Note that some Dirac-like systems are considered in papers [27,28,40,41].

Thus, consider the system

$$D_*^\gamma \Psi(t,x) = \mathcal{F}(-i\nabla)\Psi(t,x),$$

where $1/2 < \gamma \leq 1$ and $\mathcal{F}(-i\nabla)$ has the symbol $\mathcal{F}(\xi_2, \xi_2, \xi_3)$ defined in (165). The matrix $\mathcal{F}(z)$ has eigenvalues

$$\lambda_{1-4}(\xi) = \pm\sqrt{-m^2 \pm i\sqrt{(\xi_1^2 + \xi_2^2)^2 + \xi_3^4}}.$$

If $\xi \neq 0$, then all eigenvalues are of multiplicity one, and hence diagonalizable. Consequently, there exists an invertible matrix $M(\xi)$, such that $\mathcal{F}(\xi) = M(\xi)\Lambda(\xi)M^{-1}(\xi)$, with $\Lambda(\xi) = \mathrm{diag}(\lambda_1(\xi), \ldots, \lambda_4(\xi))$. Thus, in accordance with Theorem 6, the solution is represented in the form $\Psi(t,x) = \mathcal{S}(t, -i\nabla)\Psi(0,x)$, where the solution pseudo-differential operator $\mathcal{S}(t, -i\nabla)$ has the matrix symbol

$$\mathcal{S}(t,\xi) = M(\xi)\mathrm{diag}\Big(E_\gamma(t^\gamma)\lambda_1(\xi)), \ldots, E_\gamma(t^\gamma\lambda_4(\xi))\Big)M^{-1}(\xi),$$

and components $\psi_k(x), k = 1, \ldots, 4$, of $\Psi(0,x)$ have Fourier transforms with compact supports in $G = \mathbb{R}^3 \setminus \{0\}$.

**Example 5.** *A commensurate system of pseudo-differential equations.*

Let the matrix-valued operator $\mathcal{F}(A)$ be given by the matrix symbol

$$\mathcal{F}(z) = \begin{bmatrix} -|z|^2 & a^2\bar{z} \\ z & -|z|^2 \end{bmatrix}, \quad z \neq 0.$$

Consider the system with the matrix-valued operator on the right corresponding to the symbol $\mathcal{F}(z)$ :

$$D_*^{\mathcal{B}}\mathcal{U}(t) = \begin{bmatrix} -A^2 & a^2 A^* \\ A & -A^2 \end{bmatrix}\mathcal{U}(t), \quad t > 0, \tag{166}$$

with $\mathcal{B} = (\beta, \beta)$, $0 < \beta \leq 1$, $A^*$ as the adjoint of $A$, and the initial condition

$$\mathcal{U}(0) = (\varphi_1, \varphi_2). \tag{167}$$

Assume that $a \neq 0$. Then, one can easily verify that the eigenvalues of $\mathcal{F}(z)$ are $\lambda(z) = |z|^2 \pm a|z|$. Therefore, $\mathcal{F}(z) = M(z)\Lambda(z)M^{-1}(z)$, where

$$\Lambda(z) = \begin{bmatrix} |z|^2 - a|z| & 0 \\ 0 & |z|^2 + a|z| \end{bmatrix}, \quad M(z) = \begin{bmatrix} a\bar{z} & a\bar{z} \\ -|z| & |z| \end{bmatrix}.$$

Then, due to Theorem 6, the solution operator $\mathcal{S}(t, A)$ has the matrix symbol

$$\mathcal{S}(t,z) = \begin{bmatrix} a\bar{z} & a\bar{z} \\ -|z| & |z| \end{bmatrix} \begin{bmatrix} E_\beta((-|z|^2 - a|z|)t^{\beta-1}) & 0 \\ 0 & E_\beta((-|z|^2 + a|z|)t^{\beta-1}) \end{bmatrix} \begin{bmatrix} \frac{1}{2a\bar{z}} & -\frac{1}{2|z|} \\ \frac{1}{2a\bar{z}} & \frac{1}{2|z|} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\Big(E^-(z,t) + E^+(z,t)\Big) & -\frac{a\bar{z}}{2|z|}\Big(E^-(z,t) - E^+(z,t)\Big) \\ -\frac{|z|}{2a\bar{z}}\Big(E^-(z,t) - E^+(z,t)\Big) & \frac{1}{2}\Big(E^-(z,t) + \frac{1}{2}E^+(z,t)\Big) \end{bmatrix},$$

where $E^{\pm}(z,t) = E_\beta\Big((-|z|^2 \pm a|z|)t^{\beta-1}\Big)$.

Suppose that $A = \sqrt{-\Delta}$, where $\Delta$ is the Laplace operator, with the domain

$$Dom(A) = H^1(\mathbb{R}^n) = \{f \in L_2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^{1/2} |F[f](\xi)|^2 d\xi < \infty\}.$$

Here, $F[f]$ is the Fourier transform of $f(x)$. It is known that the spectrum of $A$ is the positive semi-axis, and hence we can accept $G = [0, \infty)$. Then, the symbol $\mathcal{S}(t,z)$ simplifies to

$$\mathcal{S}(t,z) = \begin{bmatrix} \frac{1}{2}\Big(E^-(z,t) + E^+(z,t)\Big) & -\frac{a}{2}\Big(E^-(z,t) - E^+(z,t)\Big) \\ -\frac{1}{2a}\Big(E^-(z,t) - E^+(z,t)\Big) & \frac{1}{2}\Big(E^-(z,t) + \frac{1}{2}E^+(z,t)\Big) \end{bmatrix},$$

If $a = 0$, then $\mathcal{F}(z)$ has a double eigenvalue $\lambda(z) = |z|^2$ and has the Jordan form $\mathcal{F}(z) = M(z)\Big(\Lambda(z) + N\Big)M^{-1}(z)$, where

$$\Lambda(z) = \begin{bmatrix} |z|^2 & 0 \\ 0 & |z|^2 \end{bmatrix}, \quad N(z) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M(z) = \begin{bmatrix} 0 & 1/z \\ 1 & 1 \end{bmatrix}.$$

Assume for simplicity that $\mathcal{B} = (1/2, 1/2)$. Then, in accordance with Theorem 6, the solution operator $S(t, A)$ has the matrix symbol

$$\begin{aligned} \mathcal{S}(t,z) &= \begin{bmatrix} 0 & 1/z \\ 1 & 1 \end{bmatrix} \begin{bmatrix} E_{1/2}(-|z|^2 t^{1/2}) & t^{1/2}E'_{1/2}(-|z|^2 t^{1/2}) \\ 0 & E_{1/2}(-|z|^2 t^{1/2}) \end{bmatrix} \begin{bmatrix} -z & 1 \\ z & 0 \end{bmatrix} \\ &= \begin{bmatrix} E_{1/2}(-|z|^2 t^{1/2}) & 0 \\ z t^{1/2} E'_{1/2}(-|z|^2 t^{1/2}) & E_{1/2}(-|z|^2 t^{1/2}) \end{bmatrix}. \end{aligned}$$

Thus, the solution $\mathcal{U}(t)$ has components

$$u_1(t) = E_{1/2}(t^{1/2}\Delta)\varphi_1, \tag{168}$$

$$u_2(t) = t^{1/2}\sqrt{-\Delta}\, E'_{1/2}(t^{1/2}\Delta)\varphi_1 + E_{1/2}(t^{1/2}\Delta)\varphi_2. \tag{169}$$

**Example 6.** *An incommensurate system of pseudo-differential equations.*

Now, assume that $\mathcal{B} = (1/2, 1/3)$, $A = \sqrt{-\Delta}$, and $\mathcal{F}(z)$ is as in Example 5 with $a = 0$. Then, since $\beta_1 = 1/2$ and $\beta_2 = 1/3$ are rational numbers, we can use the technique described in Section 3.4 and reduce problem (166), (167) to a system of five equations with equal orders $\beta_j^* = 1/6, j = 1, \ldots, 5$. The reduced system has the form

$$\begin{bmatrix} D_*^{1/6} U_1(t,x) \\ D_*^{1/6} U_2(t,x) \\ D_*^{1/6} U_3(t,x) \\ D_*^{1/6} U_4(t,x) \\ D_*^{1/6} U_5(t,x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \sqrt{-\Delta} & 1 & 0 & \Delta & 0 \end{bmatrix} \begin{bmatrix} U_1(t,x) \\ U_2(t,x) \\ U_3(t,x) \\ U_4(t,x) \\ U_5(t,x) \end{bmatrix}, \quad t > 0, \ x \in \mathbb{R}^n, \tag{170}$$

with the initial condition

$$\Big(U_1(0,x), U_2(0,x), U_3(0,x), U_4(0,x), U_5(0,x)\Big) = \Big(\varphi_1(x), 0, 0, \varphi_2(x), 0\Big).$$

The components $U_1(t,x) = u_1(t,x)$ and $U_4(t,x) = u_2(t,x)$ correspond to the solution of (166), (167) with $\mathcal{B} = (1/2, 1/3)$. Applying the Fourier and Laplace transforms, we can transform system (170) into the following algebraic equations:

$$\begin{aligned}
s^{1/6} L[F[U_1]](s,z) &= L[F[U_2]](s,z) + s^{-5/6} F[\varphi_1](z), \\
s^{1/6} L[F[U_2]](s,z) &= L[F[U_3]](s,z), \\
s^{1/6} L[F[U_3]](s,z) &= -|z|^2 L[F[U_1]](s,z), \\
s^{1/6} L[F[U_4]](s,z) &= L[F[U_5]](s,z) + s^{-5/6} F[\varphi_2](z), \\
s^{1/6} L[F[U_5]](s,z) &= z L[F[U_1]](s,z) - |z|^2 L[F[U_4]](s,z).
\end{aligned}$$

It follows that

$$L[F[U_1]](s,z) = \frac{s^{-1/2} F[\varphi_1](z)}{s^{1/2} + |z|^2},$$

and

$$L[F[U_4]](s,z) = z \frac{s^{-1/2} F[\varphi_1](z)}{(s^{1/2} + |z|^2)(s^{1/3} + |z|^2)} + \frac{s^{-2/3} F[\varphi_2](z)}{s^{1/3} + |z|^2}.$$

Thus, the solution $\mathcal{U}(t,x)$ of problem (166), (167) in accordance with Theorem 6 has the following components:

$$\begin{aligned}
u_1(t,x) &= E_{1/2}(\Delta t^{1/2}) \varphi_1(x), \\
u_2(t,x) &= \sqrt{-\Delta} \Big( E_{1/2}(\Delta t^{1/2}) * t^{-2/3} E_{1/3,1/3}(\Delta t^{1/3}) \Big) \varphi_1(x) + E_{1/3}(\Delta t^{1/3}) \varphi_2(x).
\end{aligned}$$

Note that Theorem 3 is applicable to this problem as well, resulting in the same solution.

## 6. Discussion

In this paper, the representation formula for the solution of the Cauchy problem for arbitrary time-fractional multi-order linear systems of differential-operator equations is obtained. Heretofore, the representation formulas were known in particular cases of commensurate multi-order and in incommensurate multi-order cases with rational components. The latter can be reduced to the former. The results obtained in this paper are new, even in the case of linear time-fractional multi-order ordinary differential equations with a constant matrix. The existence of a solution and its uniqueness is proven in some appropriate topological vector spaces. Examples illustrating the obtained results are provided.

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Acknowledgments:** The author is grateful to the anonymous reviewers for their insightful comments, which significantly improved the content of this work. The author acknowledges support from the Ministry of Higher Education, Science and Innovation of the Republic of Uzbekistan, Grant No. F-FA-2021-424.

**Conflicts of Interest:** The author declares no conflicts of interest.

## Abbreviations

The following abbreviation is used in this manuscript:

ML Mittag-Leffler

## References

1. Krein, S.G. *Linear Differential Equations in Banach Space*; AMS: Providence, RI, USA, 1971.
2. Vasil'yev, V.V.; Piskarev, S.I. Differential equations in Banach spaces I. Theory of cosine operator functions II. *J. Math. Sci.* **2004,** *122*, 3055–3174. [CrossRef]
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach Science Publishers: New York, NY, USA; London, UK, 1993.
4. Bazhlekova, E. Fractional Evolution Equations in Banach Spaces. Ph.D. Dissertation, Technische Universiteit Eindhoven, Eindhoven, The Netherlands, 2001.
5. Kostin, V.A. The Cauchy problem for an abstract differential equation with fractional derivatives. *Russ. Dokl. Math.* **1993**, *46*, 316–319.
6. Kochubei, A.N. A Cauchy problem for evolution equations of fractional order. *Differ. Equ.* **1989**, *25,* 967–974.
7. Gorenflo, R.; Luchko, Y.F.; Zabreiko, R.R. On solvability of linear fractional differential equations in Banach spaces. *Fract. Calc. Appl. Anal.* **1999**, *2,* 163–176.
8. Eidelman, S.D.; Kochubei, A.N. Cauchy problem for fractional diffusion equations. *J. Differ. Equ.* **2004**, *199*, 211–255. [CrossRef]
9. Kilbas, A.A.; Srivastawa, H.M.; Trijillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science: Amsterdam, The Netherlands, 2006.
10. Umarov, S. *Introduction to Fractional and Pseudo-Differential Equations with Singular Symbols*; Springer: Berlin/Heidelberg, Germany, 2015.
11. Veber, V.K. The structure of general solution of the system $y(\alpha) = Ay, 0 < \alpha \leq 1$. *Trudy Kirgiz. Gos. Univ. Ser. Mat. Nauk* **1976,** *11*, 26–32. (In Russian)
12. Matychyn, I.; Onyshchenko, V. Matrix Mittag-Leffler function in fractional systems and its computation. *Bull. Pol. Acad. Sci. Techn. Sci.* **2018**, *66*, 495–500.
13. Garrappa, R.; Popolizio, M. Computing the matrix Mittag-Leffler function with applications to fractional Calculus. *J. Sci. Comput.* **2018**, *77*, 129–153. [CrossRef]
14. Daftardar-Gejji, V.; Babakhani, A. Analysis of a system of fractional differential equations. *J. Math. Anal. Appl.* **2004**, *293*, 511–522. [CrossRef]
15. Umarov, S.; Ashurov, R.; Chen, Y. On a method of solution of systems of fractional pseudo-differential equations. *Fract. Calc. Appl. Anal.* **2021,** *24*, 254–277. [CrossRef] [PubMed]
16. Deng, W.; Li, C.; Guo, Q. Analysis of fractional differential equations with multi-orders. *Fractals* **2007**, *15*, 173–182. [CrossRef]
17. Odibat, Z. Analytic study on linear systems of fractional differential equations. *Comp. Math. Appl.* **2010**, *59*, 1170–1183. [CrossRef]
18. Umarov, S. Representations of solutions of systems of time-fractional pseudo-differential equations. *Frac. Calc. Appl. Anal.* **2024**, *2024*, 616–651. [CrossRef]
19. Chikriy, A.A.; Matichin, I.I. Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann-Liouville, Caputo and Miller-Ross. *J. Autom. Inf. Sci.* **2008**, *40*, 1–11.
20. Das, S.; Gupta, P.K. A mathematical model on fractional Lotka-Volterra equations. *J. Theor. Biol.* **2011**, *277*, 1–6. [CrossRef]
21. Rihan, F. Numerical modeling of fractional-order biological systems. *Abstract Appl. Anal.* **2013**, *2013,* 816803. [CrossRef]
22. Guo, C.; Fang, S. Stability and approximate analytic solutions of the fractional Lotka-Volterra equations for three competitors. *Adv. Differ. Equat.* **2016**, *2016*, 219. [CrossRef]
23. Khan, N.A.; Razzaq, O.A.; Mondal, S.P.; Rubbab, Q. Fractional order ecological system for complexities of interacting species with harvesting threshold in imprecise environment. *Adv. Differ. Equat.* **2019**, *2019*, 405. [CrossRef]
24. Rana, S.; Bhattacharya, S.; Pal, J.; Guerekata, G.; Chattopadhyay, J. Paradox of enrichment: A fractional differential approach with memory. *Phys. A Stat. Mech. Appl.* **2013**, *392*, 3610–3621. [CrossRef] [PubMed]
25. Zeb, A.; Zaman, G.; Chohan, M.I.; Momani, S.; Erturk, V.S. Analytic numeric solution for SIRC epidemic model in fractional order. *Asian J. Math. Appl.* **2013**, *2013*, ama0058.
26. Islam, R.; Pease, A.; Medina, D.; Oraby, T. Integer versus fractional order SEIR deterministic and stochastic models of measles. *Int. J. Environ. Res. Public Health* **2020**, *17*, 2014. [CrossRef] [PubMed]
27. Pierantozzi, T. Fractional evolution Dirac-like equations: Some properties and a discrete Von Neumann-type analysis. *J. Comp. Appl. Anal.* **2009**, *224*, 284–295. [CrossRef]

28. Ercan, A. On the fractional Dirac systems with non-singular operators. *Therm. Sci.* **2019**, *23*, 2159–2168. [CrossRef]
29. Bernstein, S. Fractional Dirac operator. Noncommutative Analysis. *Oper. Theor. Appl.* **2016**, *252*, 27–41.
30. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S.V. Mittag-Leffler Functions. In *Related Topics and Applications*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 2020.
31. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Mittag-Leffler functions and their applications. *J. Appl. Math.* **2011**, *2011*, 298628. [CrossRef]
32. Dunford, N.; Schwartz, J. *Linear Operators*; Interscience Publishers: Geneva, Switzerland, 1964.
33. Robertson, A.P.; Robertson, W.J. *Topological Vector Spaces*; Cambridge University Press: London, UK, 1964.
34. Radyno, Y.V. *Linear Equations and Bornology*; BSU: Minsk, Belarus, 1982. (In Russian)
35. Umarov, S. Nonlocal boundary value problems for pseudo-differential and differential operator equations II. *Differ. Equat.* **1998**, *34*, 374–381.
36. Kaneko, A. On the structure of hyperfunctions with compact supports. *Proc. Jpn. Acad.* **1971**, *47*, 956–958. [CrossRef] [PubMed]
37. Umarov, S. Fractional Duhamel principle. In *Handbook of Fractional Calculus*; De-Gruyter: Berlin, Germany, 2019.
38. Almeida, R.; Bastos, N.R.O.; Monteiro, M.T.T. Modeling some real phenomena by fractional differential equations. *Math. Meth. Appl. Sci.* **2016**, *39*, 4846–4855. [CrossRef]
39. Collas, P.; Klein, D. *The Dirac Equation in Curved Spacetime: A Guide for Calculations*; Springer: Berlin/Heidelberg, Germany, 2019.
40. Vazquez, L.; Mendes, R.V. Fractionally coupled solutions of the diffusion equation. *Appl. Math. Comp.* **2003**, *141*, 125–130. [CrossRef]
41. Blackledge, J.; Babajanov, B. The fractional Schrödinger-Klein-Gordon equation and intermediate relativism. *Math. Aeterna* **2013**, *3*, 601–615.