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Non-Polynomial Collocation Spectral Scheme for Systems of Nonlinear Caputo–Hadamard Differential Equations

Mahmoud A. Zaky [1,2,*], Ibrahem G. Ameen [3], Mohammed Babatin [1], Ali Akgül [4,5], Magda Hammad [6] and António M. Lopes [7,*]

[1] Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), P.O. Box 65892, Riyadh 11566, Saudi Arabia; mmbabatin@imamu.edu.sa
[2] Department of Applied Mathematics, Physics Research Institute, National Research Centre, Dokki, Cairo 12622, Egypt
[3] Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo 11884, Egypt; ibrahem_gamal.201@azhar.edu.eg
[4] Department of Mathematics, Art and Science Faculty, Siirt University, Siirt 56100, Turkey; aliakgul00727@gmail.com
[5] Department of Computer Science and Mathematics, Lebanese American University, Beirut 102, Lebanon
[6] Department of Mathematics and Information Science, Faculty of Science, Beni-Suef University, Beni-Suef 62514, Egypt; magda.hammad@science.bsu.edu.eg
[7] LAETA/INEGI, Faculty of Engineering, University of Porto, Rua Dr. Roberto Frias, 4200-465 Porto, Portugal
* Correspondence: mibrahimm@imamu.edu.sa (M.A.Z.); aml@fe.up.pt (A.M.L.)

Abstract: In this paper, we provide a collocation spectral scheme for systems of nonlinear Caputo–Hadamard differential equations. Since the Caputo–Hadamard operators contain logarithmic kernels, their solutions can not be well approximated using the usual spectral methods that are classical polynomial-based schemes. Hence, we construct a non-polynomial spectral collocation scheme, describe its effective implementation, and derive its convergence analysis in both L^2 and L^∞. In addition, we provide numerical results to support our theoretical analysis.

Keywords: orthogonal functions; spectral methods; error analysis; Caputo–Hadamard derivative

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1. Introduction

Fractional differential equations represent an extension of classical ones by replacing the standard derivative with a fractional order one. Models incorporating fractional derivatives have garnered substantial attention in recent years due to their capacity to provide a more precise portrayal of various real-world phenomena [1,2]. In mathematical treatises on fractional differential equations, the Riemann–Liouville approach is typically employed, while in applications, the Caputo definition is frequently encountered. In comparison to these two types of fractional operators, the Hadamard fractional calculus, introduced in 1892 by Hadamard [3], has not received much attention. It combines elements of the Caputo derivative and the Hadamard fractional integral. Indeed, the Caputo–Hadamard (C-H) derivative also merits further investigation. Since the integral kernel of the Hadamard operators is defined in terms of logarithmic functions, the Hadamard fractional derivative is a suitable choice for modeling ultraslow diffusion processes. The authors of [4,5] utilized a logarithmic function to represent ultraslow diffusion, demonstrating that the Hadamard derivative is more effective than the Riemann–Liouville or Caputo derivatives in characterizing ultraslow diffusion.

Recent research on C-H equations focused on fundamental theoretical aspects. Kilbas et al. [6,7] provided detailed descriptions and mathematical treatments of fractional calculus in the Hadamard sense. He et al. [8] discussed the stability of C-H equations and Hadamard systems without delay. The necessary conditions of solvability, the existence, and the uniqueness of the solution for the nonlinear C-H differential equations were investigated in [9,10]. The modified Laplace transform and its inverse were used in [11] to construct the mild solutions of the Hadamard-type fractional Fokker–Planck equation. Ou et al. [12] adopted the modified Laplace transform and the well-known finite Fourier sine transform to obtain the analytical solution of C-H fractional diffusion-wave equations with initial singularity. Muthaiah et al. [13] studied the existence, uniqueness, and Hyers–Ulam stability of nonlinear systems of C-H fractional differential equations with nonlocal integral and multipoint boundary conditions.

All of the above-mentioned studies are concerned with theoretical aspects of C-H fractional equations. However, studies on numerical methods for nonlinear C-H fractional differential equations are still in their early stages. As a result, we study the following nonlinear system of C-H differential equations:

$$\begin{cases} {}^{CH}_{\rho}D^{\kappa}_{t}\varphi_1(t) = g_1(t,\varphi_1(t),\ldots,\varphi_M(t)),\ \rho < t < L, \\ {}^{CH}_{\rho}D^{\kappa}_{t}\varphi_2(t) = g_2(t,\varphi_1(t),\ldots,\varphi_M(t)),\ \rho < t < L, \\ \vdots \\ {}^{CH}_{\rho}D^{\kappa}_{t}\varphi_M(t) = g_M(t,\varphi_1(t),\ldots,\varphi_M(t)),\ \rho < t < L, \\ \varphi_i(\rho) = \varphi_{i\rho},\ i = 1,2,\ldots,M, \qquad \rho \in (0,t),\ \kappa \in (0,1), \end{cases} \tag{1}$$

where $g_i : [\rho, L] \times \mathbb{R}^M \to \mathbb{R}$ are given continuous functions, and the C-H derivative ${}^{CH}_{\rho}D^{\kappa}_{t}$ of order $0 < \kappa < 1$ is given by (3).

Because fractional derivatives have a complex form, it is sometimes necessary to establish an appropriate numerical scheme to approximate them [14–16], which considerably enhances the efficiency of the actual calculation process. In this paper, we provide a collocation spectral scheme for systems of nonlinear C-H differential equations. Since the C-H operators contain logarithmic kernels, their solutions can not be well approximated using the usual spectral methods that are classical polynomial-based methods. Hence, our contribution in this paper is to construct a non-polynomial spectral collocation scheme, describe its effective implementation, and derive its convergence analysis for solving systems of nonlinear C-H differential equations. In addition, we provide numerical results to support our theoretical analysis.

The outline of this paper is as follows: In Section 2, we introduce some necessary definitions and preliminary concepts. In Section 3, we construct the spectral collocation scheme. In Section 4, we provide some auxiliary lemmas. In Section 5, we discuss the convergence and provide some numerical results. In Section 6, we introduce two numerical examples. In Section 7, we summarize the conclusions.

2. Preliminaries

In this section, some relevant properties of the C-H fractional calculus and the logarithmic Jacobi (*log J*) approximation are presented.

Definition 1. *The C-H fractional integral with order $\kappa > 0$ is defined as [17]*

$${}_{\rho}J^{\kappa}_{z}\varphi(z) = \frac{1}{\Gamma(\kappa)}\int_{\rho}^{z}\beta^{\kappa-1}(z,w)\varphi(w)\frac{dw}{w},\ z > \rho > 0, \tag{2}$$

where $\beta(z,w) = \log(\frac{z}{w})$.

Definition 2. *The C-H fractional differential operator of order $0 < \kappa < 1$ is given as [18]*

$${}^{CH}_{\rho}D^{\kappa}_{z}\varphi(z) = \frac{1}{\Gamma(1-\kappa)}\int_{\rho}^{z}\beta^{-\kappa}(z,w)\,\varphi'(w)dw. \tag{3}$$

Definition 3. *Let $\kappa, \eta > -1$, $I := [\rho, \rho\, e]$, and $\rho > 0$. The log J functions of order p are given by [19]*

$$\begin{aligned}\mathscr{R}_p^{\kappa,\eta,\rho}(z) &= \mathscr{R}_p^{\kappa,\eta}\Big(\beta^2(z,\rho)-1\Big), \quad \eta,\kappa > -1,\ \rho > 0,\ \forall z \in I \\ &= \frac{\Gamma(p+\kappa+1)}{p!\Gamma(p+\kappa+\eta+1)}\sum_{k=0}^{p}\begin{pmatrix} p \\ k \end{pmatrix}\frac{\Gamma(p+k+\kappa+\eta+1)}{\Gamma(k+\kappa+1)}(\beta(z,\rho)-1)^k,\end{aligned} \tag{4}$$

where $\mathscr{R}_p^{\kappa,\eta}(z)$ *is the Jacobi polynomial defined as*

$$\mathscr{R}_p^{\kappa,\eta}(z) = \frac{\Gamma(p+\kappa+1)}{\Gamma(p+1+\kappa+\eta)p!}\sum_{k=0}^{p}\begin{pmatrix} p \\ k \end{pmatrix}\frac{\Gamma(p+k+\kappa+\eta+1)}{\Gamma(k+\kappa+1)}\left(\frac{z-1}{2}\right)^k.$$

We define the space of logarithmic functions of order s by

$$P_s^{\log}(\Omega) := span\Big\{1, \beta(z,\rho), \beta(z,\rho)^2, \dots, \beta(z,\rho)^s\Big\},$$

where $\Omega = [\rho, +\infty),\ \rho > 0$. Let

$$\chi^{\kappa,\eta,\rho}(z) := z^{-1}\beta(z,\rho)^{\eta}(1-\beta(z,\rho))^{\kappa}. \tag{5}$$

We denote by $L^2_{\chi^{\kappa,\eta,\rho}}(I)$ the weighted L^2 space with the following inner product and norm,

$$(\varphi,\phi)_{\chi^{\kappa,\eta,\rho}} = \int_I \varphi(z)\phi(z)\chi^{\kappa,\eta,\rho}(z)dz, \qquad \|\varphi\|_{\chi^{\kappa,\eta,\rho}} = (\varphi,\varphi)^{1/2}_{\chi^{\kappa,\eta,\rho}}. \tag{6}$$

One of the most important properties of the *log J* polynomials is that they are mutually orthogonal in $L^2_{\chi^{\kappa,\eta,\rho}}(I)$, that is,

$$\begin{aligned}&\Big(\mathscr{R}_m^{\kappa,\eta,\rho}(z), \mathscr{R}_j^{\kappa,\eta,\rho}(z)\Big)_{\chi^{\kappa,\eta,\rho}} = 0 \quad \forall j \neq m, \\ &\Big\|\mathscr{R}_j^{\kappa,\eta,\rho}(z)\Big\|_{\chi^{\kappa,\eta,\rho}} = \widehat{\theta}_j^{\kappa,\eta} = \frac{\Gamma(j+\kappa+1)\Gamma(j+\eta+1)}{(2j+\kappa+\eta+1)j!\Gamma(j+\kappa+\eta+1)}.\end{aligned} \tag{7}$$

We define the following first-order differential operator:

$$D^1_{\log}\phi(z) = \frac{d}{d\beta(z,\rho)}\phi(z) = z\phi'(z), \tag{8}$$

and an induction leads to

$$D^k_{\log}\phi(z) = \overbrace{D^1_{\log}\cdot D^1_{\log}\cdots D^1_{\log}}^{k}\phi(z). \tag{9}$$

We also define the non-uniformly weighted *log J* Sobolev space as

$$B^{i,\rho}_{\kappa,\eta}(I) := \{\phi : D^j_{\log}\phi \in L^2_{\chi^{\kappa+j,\eta+j,\rho}}(I),\ 0 \le j \le i\}, \quad i \in \mathbb{N},$$

with

$$\begin{aligned}(\psi,\phi)_{B^{i,\rho}_{\kappa,\eta}} &= \sum_{k=0}^{i}(D^k_{\log}\psi, D^k_{\log}\phi)_{\chi^{\kappa+k,\eta+k,\rho}}, \quad \|\phi\|_{B^{i,\rho}_{\kappa,\eta}} = (\phi,\phi)^{1/2}_{B^{i,\rho}_{\kappa,\eta}}, \\ |\phi|_{B^{i,\rho}_{\kappa,\eta}} &= \|D^i_{\log}\phi\|_{\chi^{\kappa+i,\eta+i,\rho}}.\end{aligned}$$

For the usual shifted–weighted Jacobi Sobolev space, we define

$$B^{i}_{\kappa,\eta}(\Lambda) := \{\phi : \partial^{j}_{z}\phi \in L^{2}_{\chi^{\kappa+j,\eta+j}}(\Lambda), 0 \le j \le i\}, \quad i \in \mathbb{N},$$

where $\chi^{\kappa,\eta} = (-z+1)^{\kappa} z^{\eta}$ with $z \in \Lambda = [0,1]$ is the classical Jacobi weight function.

Assume that $x_0 < x_1 > \cdots < x_{M-1} < x_M$ in I are the roots of $\mathscr{R}^{\kappa,\eta,\rho}_{M+1}(x)$. Let $z(x) = \log\frac{x}{\rho}$. Then, $z_j := z(x_j) = \log\frac{x_j}{\rho}$, $0 \le j \le M$ are zeros of $\mathscr{R}^{\kappa,\eta}_{M+1}(x)$, and $\{\chi_i\}_{i=0}^{M}$ are the corresponding weights.

The *log J*-Gauss quadrature enjoys the exactness

$$\int_I \varphi(z)\chi^{\kappa,\eta,\rho}(z)dz = \sum_{i=0}^{M} \varphi(z_i)\chi_i \quad \forall\varphi(z) \in P^{\log}_{2M+1}. \tag{10}$$

Hence,

$$\sum_{k=0}^{M} \mathscr{R}^{\kappa,\eta,\rho}_{q}(z_k)\mathscr{R}^{\kappa,\eta,\rho}_{j}(z_k)\chi_k = \widehat{\theta}^{\kappa,\eta}_{q}\delta_{q,j}, \qquad \forall\, 0 \le q+j \le 2M+1. \tag{11}$$

For any $\varphi(\rho e^{z}) \in C(I)$, the *log J*-Gauss interpolation operator $I^{\kappa,\eta,\rho}_{z,M} : C(I) \longrightarrow P^{\log}_{M}$ is determined uniquely by

$$I^{\kappa,\eta,\rho}_{z,M}\varphi(z_q) = \varphi(z_q) \qquad 0 \le q \le M. \tag{12}$$

From the above condition, we have $I^{\kappa,\eta,\rho}_{z,M}\varphi = \varphi$ for all $\varphi \in P^{\log}_{M}$. On the other hand, since $I^{\kappa,\eta,\rho}_{z,M}\varphi \in P^{\log}_{M}$, we can write

$$I^{\kappa,\eta,\rho}_{z,M}\varphi(x) = \sum_{i=0}^{M} \widehat{\varphi}^{\kappa,\eta,\rho}_{i}\mathscr{R}^{\kappa,\eta,\rho}_{i}(x), \quad \widehat{\varphi}^{\kappa,\eta,\rho}_{i} = \frac{1}{\widehat{\theta}^{\kappa,\eta}_{i}}\sum_{j=0}^{M} \varphi(x_j)\mathscr{R}^{\kappa,\eta,\rho}_{i}(x_j)\chi_j,\ \forall\varphi \in P^{\log}_{M}(I). \tag{13}$$

The $L^{\infty}(I)$ space is the set of all measurable functions that are essentially bounded. That is, functions g that are bounded almost everywhere on a set of finite measures. The essential supremum norm is used to define the norm of this space and is given as

$$\|g\|_{\infty} = ess\,\underset{x\in I}{sup}|g(x)|.$$

Definition 4. *Let $A(z) = \big(a_{ij}(z)\big)_{m\times n}$ be an $(m \times n)$ matrix function with $z \in I$, we consider the non-negative real-valued function*

$$|\mathbf{A}(z)| = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}(z)|, \tag{14}$$

and the norms

$$\begin{aligned} \|A\|_{\chi^{\kappa,\eta,\rho}} &:= \left(\int_I |A(z)|^2 \chi^{\kappa,\eta,\rho}dz\right)^{1/2}, \\ \|A\|_{\infty} &:= ess\sup_{z\in I}|A(z)|. \end{aligned} \tag{15}$$

Proposition 1. *It holds for any $\psi(\rho\, e^{x}) \in B^{m}_{\kappa,\eta}(\Lambda)$, $m \ge 1$ and $M+1 \ge m \ge q \ge 0$*

$$\|D^{q}_{\log}\big(\psi - I^{\kappa,\eta,\rho}_{M}\psi\big)\|_{\chi^{\kappa+q,\eta+q,\rho}} \le c\sqrt{\frac{(1+M-m)!}{M!}}M^{q-(1+m)/2}\|\partial^{m}_{x}\{\psi(\rho\, e^{x})\}\|_{\chi^{\kappa+m,\eta+m}}, \tag{16}$$

and it takes the form

$$\|D^q_{\log}(\psi - I^{\kappa,\eta,\rho}_M \psi)\|_{\chi^{\kappa+q,\eta+q,\rho}} \leq cM^{q-m}\|\partial^m_x\{\psi(\rho\, e^x)\}\|_{\chi^{\kappa+m,\eta+m}},\ c \approx 1,\ \textit{for fixed}\quad m\ M \gg 1. \tag{17}$$

In case of $q = 0, 1$, we can write

$$\|\psi - I^{\kappa,\eta,\rho}_M \psi\|_{\chi^{\kappa,\eta,\rho}} \leq cM^{-m}\|\partial^m_x\{\psi(\rho\, e^x)\}\|_{\chi^{\kappa+m,\eta+m}}, \tag{18}$$

$$\|\partial_x(\psi - I^{\kappa,\eta,\rho}_M \psi)\|_{\tilde{\chi}^{\kappa,\eta,\rho}} \leq cM^{1-m}\|\partial^m_x\{\psi(\rho\, e^x)\}\|_{\chi^{\kappa+m,\eta+m}}, \tag{19}$$

where $\tilde{\chi}^{\kappa,\eta,\rho} = x\left(1 - \log\left(\frac{x}{\rho}\right)\right)^{\kappa+1}\left(\log\left(\frac{x}{\rho}\right)\right)^{\eta+1}$.

Lemma 1. *For any $\kappa, \eta \in (-1, -\frac{1}{2})$ and for all $\psi(x) \in B^{1,\rho}_{\kappa,\eta}(I)$, $\psi(\xi) = 0$ for some $\xi \in I$, it holds that*

$$\|\psi\|_\infty \leq \sqrt{2}\|\partial_x \psi\|^{1/2}_{\tilde{\chi}^{\kappa,\eta,\rho}}\|\psi\|^{1/2}_{\chi^{\kappa,\eta,\rho}}. \tag{20}$$

Proposition 2. *For $\kappa, \eta \in (-1, -\dfrac{1}{2}]$,*

$$\|\psi - I^{\kappa,\eta,\rho}_M \psi\|_\infty \leq cM^{1/2-m}\|\partial^m_x \psi(\rho\, e^x)\|_{\chi^{\kappa+m,\eta+m}},\quad \forall \psi(\rho\, e^x) \in B^m_{\kappa,\eta}(\Lambda),\ m \geq 1. \tag{21}$$

Lemma 2.

$$\|I^{\kappa,\eta,\rho}_M\|_\infty := \max_{x \in I}\sum_{j=0}^{M}|h^{\kappa,\eta,\rho}_j(x)| = \begin{cases} O(\log M), & -1 < \kappa, \eta \leq -\frac{1}{2}, \\ O(M^{\mu+\frac{1}{2}}), & \mu = \max(\kappa, \eta),\ \textit{otherwise}, \end{cases} \tag{22}$$

where $\{h^{\kappa,\eta,\rho}_j(x)\}_{j=0}^M$ are the logarithmic Lagrange interpolation functions that related to $\mathscr{R}^{\kappa,\eta,\rho}_{M+1}(x)$.

3. Non-Polynomial Spectral Collocation Scheme

To begin with, we rewrite the differential Equation (1) in the following equivalent compact integral form

$$\Phi(t) = \Phi_\rho + \frac{1}{\Gamma(\kappa)}\int_\rho^t (\beta(t,s))^{\kappa-1}\mathbf{G}(s, \Phi(s))\frac{ds}{s},\ t \in (\rho, L], \tag{23}$$

where

$$\Phi(t) = [\varphi_1, \varphi_2, \ldots, \varphi_M]^T,$$
$$\mathbf{G}(t) = [g_1, g_2, \ldots, g_M]^T.$$

In the following, we will make some useful transformations, which in turn are the basis for the numerical solution scheme and its numerical analysis. In order to convert the integral interval (ρ, t) to I, we consider

$$\beta(s, \rho) = \beta(t, \rho)\beta(r, \rho),$$

or

$$s = s(t, r) = \rho\left(\frac{r}{\rho}\right)^{\beta(t,\rho)}.$$

Hence, the system (23) becomes

$$\Phi(t) = \Phi_\rho + \frac{(\beta(t,\rho))^\kappa}{\Gamma(\kappa)}\int_I (1 - \beta(r,\rho))^{\kappa-1}\mathbf{G}(s(t,r), \Phi(s(t,r)))\frac{dr}{r}. \tag{24}$$

The non-polynomial spectral collocation scheme for (24) consists of finding $\varphi_{m,N}(t) \in P^{\log}_N(I)$, $m = 1, 2, \ldots, M$, such that

$$\Phi_N(t) = \Phi_\rho + \frac{1}{\Gamma(\kappa)} I_{t,N}^{0,0,\rho} (\beta(t,\rho))^\kappa \int_I r^{-1}(1-\beta(r,\rho))^{\kappa-1} I_{r,N}^{\kappa-1,0,\rho} \mathbf{G}(s(t,r), \Phi_N(s(t,r)))dr, \tag{25}$$

where

$$\Phi_N(t) = [\varphi_{1,N}, \varphi_{2,N}, \dots, \varphi_{M,N}]^T,$$

and $I_{z,N}^{\kappa,\eta,\rho}$ is the *log J*-Gauss interpolation operator in the z-direction. For simplicity, we will consider the trial functions as

$$\varphi_{m,N}(t) = \sum_{i-0}^{N} \varphi_{m,i} \mathscr{R}_i^{0,0,\rho}(t), \quad m = 1, \dots, M. \tag{26}$$

Accordingly,

$$I_{t,N}^{0,0,\rho} I_{r,N}^{\kappa-1,0,\rho} (\beta(t,\rho))^\kappa g_m(s(t,r), \Phi_N(s(t,r))) = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{i,j} \mathscr{R}_i^{0,0,\rho}(t) \mathscr{R}_j^{\kappa-1,0,\rho}(r),\ m = 1, \dots, M. \tag{27}$$

A straightforward calculation by using (27) and (7) gives

$$\begin{aligned}
&\frac{1}{\Gamma(\kappa)} I_{t,N}^{0,0,\rho}\left[(\beta(t,\rho))^\kappa \int_I r^{-1}(1-\beta(r,\rho))^{\kappa-1} I_{r,N}^{\kappa-1,0,\rho} g_m(s(t,r), \Phi_N(s(t,r)))dr\right] \\
&= \frac{1}{\Gamma(\kappa)} \sum_{i=0}^{N}\sum_{j=0}^{N} v_{m,i,j} \mathscr{R}_i^{0,0,\rho}(t) \int_I r^{-1}(1-\beta(r,\rho))^{\kappa-1} \mathscr{R}_j^{\kappa-1,0,\rho}(r)dr \\
&= \frac{1}{\Gamma(\kappa+1)} \sum_{i=0}^{N} v_{m,i,0} \mathscr{R}_i^{0,0,\rho}(t), \quad m = 1, \dots, M.
\end{aligned} \tag{28}$$

Let $\left\{\chi_p^{\kappa,\eta,\rho}, x_p^{\kappa,\eta,\rho}\right\}_{p=0}^{N}$ be the weights and the nodes of Gauss type logarithmic Jacobi interpolation. A direct application of (27) and (13) yields

$$\begin{aligned}
&v_{i,0} = \kappa\,(2i+1) \times \\
&\sum_{p=0}^{N}\sum_{q=0}^{N} \left(\beta(t_p^{0,0,\rho}, \rho)\right)^\kappa g_m\left(s\left(t_p^{0,0,\rho}, r_q^{\kappa-1,0,\rho}\right), \Phi_N\left(s\left(t_p^{0,0,\rho}, r_q^{\kappa-1,0,\rho}\right)\right)\right) \mathscr{R}_i^{0,0,\rho}\left(t_p^{0,0,\rho}\right) \chi_p^{0,0,\rho} \chi_q^{\kappa-1,0,\rho}.
\end{aligned} \tag{29}$$

Hence, we deduce that

$$\sum_{i-0}^{N} \varphi_{m,i} \mathscr{R}_i^{0,0,\rho}(t) = \varphi_\rho \mathscr{R}_0^{0,0,\rho}(t) + \frac{1}{\Gamma(\kappa+1)} \sum_{i=0}^{N} v_{m,i,0} \mathscr{R}_i^{0,0,\rho}(t). \tag{30}$$

Comparing the coefficients of (30), we obtain

$$\begin{aligned}
\varphi_{m,0} &= \varphi_\rho + \frac{v_{m,0,0}}{\Gamma(\kappa+1)}, \\
\varphi_{m,i} &= \frac{v_{m,i,0}}{\Gamma(\kappa+1)}, \quad 1 \le i \le N, \quad m = 1, \dots, M.
\end{aligned} \tag{31}$$

4. Auxiliary Lemmas

Herein, we derive the rate of convergence of the scheme (25) in the $L^2_{\chi^{0,0,\rho}}$-norm. Accordingly, we introduce some lemmas.

Let $r_i^{\kappa,\eta,\rho}$ be the *log J*-Gauss nodes in I and $s_i^{\kappa,\eta,\rho} = s\left(x, r_i^{\kappa,\eta,\rho}\right)$. The mapped *log J*-Gauss interpolation operator ${}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa,\eta,\rho} : C(\rho, x) \longrightarrow P_N^{\log}(\rho, x)$ is defined by

$${}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa,\eta,\rho} u\left(s_i^{\kappa,\eta,\rho}\right) = u\left(s_i^{\kappa,\eta,\rho}\right), \quad 0 \le i \le N. \tag{32}$$

Hence,

$$ {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa,\eta,\rho} u\left(s_i^{\kappa,\eta,\rho}\right) = u\left(s_i^{\kappa,\eta,\rho}\right) = u\left(s(x, r_i^{\kappa,\eta,\rho})\right) = \mathcal{I}_{r,N}^{\kappa,\eta,\rho} u\left(s(x, r_i^{\kappa,\eta,\rho})\right), \tag{33} $$

and

$$ {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa,\eta,\rho} u(s) = \mathcal{I}_{r,N}^{\kappa,\eta,\rho} u(s(x,r))\Big|_{\beta(r,\rho)=\frac{\beta(s,\rho)}{\beta(x,\rho)}}. \tag{34} $$

Moreover, the following results can be easily derived:

$$ \begin{aligned} \int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\, {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho} \varphi(s) ds &= (\beta(x,\rho))^\kappa \int_I r^{-1}(1-\beta(r,\rho))^{\kappa-1} \mathcal{I}_{r,N}^{\kappa-1,0,\rho} \varphi(s(x,r)) dr \\ &= (\beta(x,\rho))^\kappa \sum_{j=0}^N \varphi\left(s(x, r_j^{\kappa-1,0,\rho})\right) \chi_j^{\kappa-1,0,\rho} \\ &= (\beta(x,\rho))^\kappa \sum_{j=0}^N \varphi\left(s_j^{\kappa-1,0,\rho}\right) \chi_j^{\kappa-1,0,\rho}. \end{aligned} \tag{35} $$

Similarly,

$$ \int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1} \left({}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho} \varphi(s)\right)^2 ds = (\beta(x,\rho))^\kappa \sum_{j=0}^N \varphi^2\left(s_j^{\kappa-1,0,\rho}\right) \chi_j^{\kappa-1,0,\rho}. \tag{36} $$

Then, for any $1 \le s \le N+1$, we have

$$ \begin{aligned} &\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1} \left| \left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right) \varphi(s) \right|^2 ds \\ &= (\beta(x,\rho))^\kappa \int_I r^{-1}(1-\beta(r,\rho))^{\kappa-1} \left| \left(\mathcal{I} - \mathcal{I}_{r,N}^{\kappa-1,0,\rho}\right) \varphi(s(x,r)) \right|^2 dr \\ &\le cN^{-2m}(\beta(x,\rho))^\kappa \int_I r^{-1}(1-\beta(r,\rho))^{\kappa+m-1}(\beta(r,\rho))^m \left| D_{\log,r}^m \varphi(s(x,r)) \right|^2 dr \\ &= cN^{-2m} \int_\rho^x s^{-1}(\beta(x,s))^{\kappa+m-1}(\beta(s,\rho))^m \left| D_{\log,s}^m \varphi(s) \right|^2 ds, \end{aligned} \tag{37} $$

where $\mathcal{I}$ is the identity operator.

Lemma 3. *The following estimate holds for the error function $e_N(x) = \Phi(x) - \Phi_N(x)$*

$$ \|e_N\|_{\chi^{0,0,\rho}} \le \sum_{j=1}^3 \left\|\boldsymbol{E}_j\right\|_{\chi^{0,0,\rho}}, \tag{38} $$

where

$$ \begin{aligned} \boldsymbol{E}_1 &= \Phi(x) - \mathcal{I}_{x,N}^{0,0,\rho} \Phi(x), \\ \boldsymbol{E}_2 &= \mathcal{I}_{x,N}^{0,0,\rho} \int_\rho^x \boldsymbol{R}(x,s)(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}) \quad \boldsymbol{G}(s,\Phi(s)) ds, \\ \boldsymbol{E}_3 &= \mathcal{I}_{x,N}^{0,0,\rho} \int_\rho^x \boldsymbol{R}(x,s) \quad {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho} (\boldsymbol{G}(s,\Phi(s)) - \boldsymbol{G}(s,\Phi_N(s))) ds, \end{aligned} $$

and $\boldsymbol{R}(x,s) = (R_{ij})$ with $R_{ij} = \frac{s^{-1}(\beta(x,s))^{\kappa-1}}{\Gamma(\kappa)} \delta_{ij}$, $i,\ j = 1, \ldots, M$.

Proof.

$$ \|e_N\|_{\chi^{0,0,\rho}} \le \left\|\Phi - \mathcal{I}_{x,N}^{0,0,\rho}\Phi\right\|_{\chi^{0,0,\rho}} + \left\|\mathcal{I}_{x,N}^{0,0,\rho}\Phi - \Phi_N\right\|_{\chi^{0,0,\rho}}. \tag{39} $$

It is clear from (23) that

$$\mathcal{I}_{x,N}^{0,0,\rho}\Phi(x) = \Phi_\rho + \frac{1}{\Gamma(\kappa)}\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\mathbf{G}(s,\Phi(s))ds, \tag{40}$$

and

$$\Phi_N(x) = \Phi_\rho + \frac{1}{\Gamma(\kappa)}\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\,{}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\mathbf{G}(s,\Phi_N(s))ds. \tag{41}$$

Subtracting (40) from (42) yields

$$\begin{aligned}&\mathcal{I}_{x,N}^{0,0,\rho}\Phi(x) - \Phi_N(x)\\ &= \frac{1}{\Gamma(\kappa)}\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\Big(\mathbf{G}(s,\Phi(s)) - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\mathbf{G}(s,\Phi_N(s))\Big)ds,\end{aligned} \tag{42}$$

which has the form

$$\begin{aligned}\mathcal{I}_{x,N}^{0,0,\rho}\Phi(x) - \Phi_N(x) &= \frac{1}{\Gamma(\kappa)}\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\Big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)\mathbf{G}(s,\Phi(s))ds\\ &+ \frac{1}{\Gamma(\kappa)}\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\,{}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\mathbf{G}(s,\Phi(s)) - \mathbf{G}(s,\Phi_N(s)))ds.\end{aligned} \tag{43}$$

□

Assume that the Nemytskii operator $\mathbb{F}$ for f is given as

$$\mathbb{F}(\varphi)(x) := f(x,\varphi(x)).$$

5. Convergence Analysis

5.1. Convergence Analysis in $L^2_{\chi^{0,0,\rho}}$-Norm

Theorem 1. *Let $\Phi(t)$ and $\Phi_N(t)$ be the solutions of systems (23) and (25), respectively. Let $\Phi(x) \in B_{0,0}^{m,\rho}(I)$, $\mathbb{F} : B_{0,0}^{m,\rho}(I) \longrightarrow B_{\kappa-1,0}^{m,\rho}(I)$ with $1 \le m \le N+1$ and $N \ge 1$. Then, we have the following estimate*

$$\|\Phi - \Phi_N\|_{\chi^{0,0,\rho}} \le cN^{-m}\Big(\|D_{\log}^m\Phi\|^2_{\chi^{m,m,\rho}} + \|D_{\log}^m\mathbf{G}(\cdot,\Phi(\cdot))\|^2_{\chi^{\kappa+m-1,m,\rho}}\Big). \tag{44}$$

Proof. Using Proposition 1, we obtain

$$\|\mathbf{E}_1\|_{\chi^{0,0,\rho}} = \Big\|\Phi - \mathcal{I}_{x,N}^{0,0,\rho}\Phi\Big\|_{\chi^{0,0,\rho}} \le cN^{-m}\|D_{\log}^m\Phi\|^2_{\chi^{m,m,\rho}} \le cN^{-m}\|\partial_x^m\Phi(\rho\, e^x)\|_{\chi^{m,m}}. \tag{45}$$

Using the *log J*–Gauss integration formula gives

$$\begin{aligned}\|\mathbf{E}_2\|_{\chi^{0,0,\rho}} &= \left\|\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x \mathbf{R}(x,s)\Big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)\mathbf{G}(s,\Phi(s))ds\right\|_{\chi^{0,0,\rho}}\\ &= \left\|\sum_{k=1}^M \mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x R_{kk}(x,s)\Big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)g_k(s,\Phi(s))ds\right\|_{\chi^{0,0,\rho}}\\ &= \left[\int_I \chi^{0,0,\rho}\left(\sum_{k=1}^M \mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x R_{kk}(x,s)\Big(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)g_k(s,\Phi(s))ds\right)dx\right]^{1/2}\\ &= \left[\sum_{j=0}^N \chi_j^{0,0,\rho}\left(\sum_{k=1}^M \int_\rho^{x_j^{0,0,\rho}} R_{kk}(x_j^{0,0,\rho},s)\Big(\mathcal{I} - {}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)g_k(s,\Phi(s))ds\right)^2\right]^{1/2}\\ &\le \left[\sum_{j=0}^N \chi_j^{0,0,\rho}\sum_{k=1}^M\left(\int_\rho^{x_j^{0,0,\rho}} R_{kk}(x_j^{0,0,\rho},s)\Big(\mathcal{I} - {}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\Big)g_k(s,\Phi(s))ds\right)^2\sum_{k=1}^M(1)^2\right]^{1/2}\end{aligned}$$

Using the Cauchy–Schwarz inequality leads to the following estimate:

$$
\begin{aligned}
&\|\mathbf{E}_2\|_{\chi^{0,0,\rho}} \\
&\le C\left[\sum_{j=0}^{N}\sum_{k=1}^{M}\chi_j^{0,0,\rho}\int_{\rho}^{x_j^{0,0,\rho}} R_{kk}(x_j^{0,0,\rho},s)\,ds\int_{\rho}^{x_j^{0,0,\rho}} R_{kk}(x_j^{0,0,\rho},s)\left|\left(\mathcal{I}-{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)g_k(s,\Phi(s))\right|^2 ds\right]^{1/2} \\
&\le C\left[\sum_{j=0}^{N}\sum_{k=1}^{M}\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)g_k(s,\Phi(s))\right|^2 ds\right]^{1/2} \\
&\le C\left(\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\right)^{1/2}\left(\sum_{k=1}^{M}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)g_k(s,\Phi(s))\right|^2 ds\right)^{1/2} \qquad (46)\\
&\le C\left(\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\right)^{1/2}\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left|\left(\mathcal{I}-{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)\mathbf{G}(s,\Phi(s))\right|^2 ds\right)^{1/2} \\
&\le cN^{-m}\left[\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa+m-1}(\beta(s,\rho))^m\left|D_{\log,s}^{m}\mathbf{G}(s,\Phi(s))\right|^2 ds\right]^{1/2} \\
&\le cN^{-m}\|D_{\log}^{m}\mathbf{G}(\cdot,\Phi(\cdot))\|_{\chi^{\kappa+m-1,m,\rho}}^{2}.
\end{aligned}
$$

An estimate for the term $\|E_3\|_{\chi^{0,0,\rho}}$ can be obtained by using the *log J*-Gauss integration formula to give

$$
\begin{aligned}
\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} &= \left\|\mathbf{R}(x,s)\mathcal{I}_{x,N}^{0,0,\rho}\int_{\rho}^{x} {}_{x}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\mathbf{G}(s,\Phi(s))-\mathbf{G}(s,\Phi_N(s)))ds\right\|_{\chi^{0,0,\rho}} \\
&= \frac{1}{\Gamma(\kappa)}\left[\int_I \chi^{0,0,\rho}\left(\sum_{k=1}^{M}\mathcal{I}_{x,N}^{0,0,\rho}\int_{\rho}^{x} s^{-1}(\beta(x,s))^{\kappa-1}{}_{x}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s))-g_k(s,\Phi_N(s)))ds\right)^2 dx\right]^{1/2} \\
&= \frac{1}{\Gamma(\kappa)}\left[\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M}{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s))-g_k(s,\Phi_N(s)))ds\right)^2\right]^{1/2}
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, we obtain

$$
\begin{aligned}
&\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} \\
&\le \frac{1}{\Gamma(\kappa)}\left[\sum_{j=0}^{N}\chi_j^{0,0,\rho}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}(\beta(x,s))^{\kappa-1}ds\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}(\beta(x,s))^{\kappa-1}\left(\left|\sum_{k=1}^{M}{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s))-g_k(s,\Phi_N(s)))\right|\right)^2 ds\right]^{1/2} \qquad (47)\\
&\le \frac{1}{\Gamma(\kappa)}\left[\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\log\frac{x}{\rho}\right)^{\kappa}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{k=1}^{M}\left|{}_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s))-g_k(s,\Phi_N(s)))\right|\right)^2 ds\right]^{1/2}
\end{aligned}
$$

and using the logarithmic Jacobi–Gauss quadrature Formula (35), we obtain

$$
\begin{aligned}
&\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} \\
&\le \frac{1}{\Gamma(\kappa+1)}\left[\sum_{j=0}^{N}\kappa\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{2\kappa}\right. \qquad (48)\\
&\left.\times\sum_{q=0}^{N}\chi_q^{\kappa-1,0,\rho}\left(\sum_{k=1}^{M}\left|g_k\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right),\Phi\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right)\right)\right)-g_k\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right),\Phi_N\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right)\right)\right)\right|\right)^2\right]^{1/2}.
\end{aligned}
$$

Using the Lipschitz condition, we obtain

$$\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} \leq \frac{L}{\Gamma(\kappa+1)}\left[\sum_{j=0}^{N}\kappa\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{2\kappa}\sum_{q=0}^{N}\left(\sum_{i=1}^{M}\chi_q^{\kappa-1,0,\rho}\left|\varphi_i\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right)\right)-\varphi_{N,i}\left(s\left(x_j^{0,0,\rho},r_q^{\kappa-1,0,\rho}\right)\right)\right|\right)^2\right]^{1/2}, \tag{49}$$

and using (36), we obtain

$$\begin{aligned}
&\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} \\
&\leq \frac{L}{\Gamma(\kappa+1)}\left[\sum_{j=0}^{N}\kappa\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\varphi_i(s)-\varphi_{N,i}(s))\right|\right)^2 ds\right]^{1/2} \\
&\|E_3\|_{\chi^{0,0,\rho}} \\
&\leq \frac{L}{\Gamma(\kappa+1)}\left(\sum_{j=0}^{N}\kappa\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}\right)^{1/2}\max_{0\leq j\leq N}\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\varphi_i(s)-\varphi_{i,N}(s))\right|\right)^2 ds\right)^{1/2}.
\end{aligned} \tag{50}$$

For any $x_j^{0,0,\rho}\in I$, let $f(\kappa)=\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa}$. We note that $f(\kappa)$ is a convex function of κ. Hence, by Jensen's inequality for all $\kappa\in(0,1)$

$$f(\kappa)=(1-\kappa)f(0)+\kappa f(1).$$

The above inequality yields

$$\begin{aligned}
\kappa\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left(\beta(x_j^{0,0,\rho},\rho)\right)^{\kappa} &\leq \kappa\sum_{j=0}^{N}\chi_j^{0,0,\rho}\left[1-\kappa+\kappa\left(\beta(x_j^{0,0,\rho},\rho)\right)\right] \\
&\leq \kappa\left[1-\kappa+\kappa\int_I s^{-1}\left(\log\frac{x}{a}\right)dx\right]\leq\kappa\left(1-\frac{\kappa}{2}\right)\leq\frac{1}{2}.
\end{aligned} \tag{51}$$

Hence, by using the above inequality and the triangle inequality, we deduce that

$$\begin{aligned}
\|\mathbf{E}_3\|_{\chi^{0,0,\rho}} &\leq \frac{L}{\sqrt{2}\,\Gamma(\kappa+1)}\max_{0\leq j\leq N}\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\varphi_i(s)-\varphi_{N,i}(s))\right|\right)^2 ds\right)^{1/2} \\
&\leq \frac{L}{\sqrt{2}\,\Gamma(\kappa+1)}\times\max_{0\leq j\leq N}\left[\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}\left|_{x_j^{0,0,\rho}}\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\varphi_i(s)-\varphi_i(s)\right|\right)^2 ds\right)^{1/2}\right. \\
&\quad\left.+\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}|\varphi_i(s)-\varphi_{N,i}(s)|\right)^2 ds\right)^{1/2}\right] \\
&\leq cN^{-m}\max_{0\leq j\leq N}\left(\int_{\rho}^{x_j}(\beta(s,\rho))^m\left(\sum_{i=1}^{M}\left|D_{\log,s}^{m}\varphi_i(s)\right|\right)^2 ds\right)^{1/2} \\
&\quad+\frac{L}{\sqrt{2}\,\Gamma(\kappa+1)}\times\max_{0\leq j\leq N}\left(\int_{\rho}^{x_j^{0,0,\rho}} s^{-1}\left(\beta(x_j^{0,0,\rho},s)\right)^{\kappa-1}\left(\sum_{i=1}^{M}|\varphi_i(s)-\varphi_{N,i}(s)|\right)^2 ds\right)^{1/2} \\
&\leq cN^{-m}\|D_{\log}^{m}\Phi\|^2_{\chi^{m,m,\rho}}+\frac{L}{\sqrt{2}\Gamma(\kappa+1)}\|e_N\|^2_{\chi^{m,m,\rho}}.
\end{aligned} \tag{52}$$

Hence, a combination of (45), (46), (52) and the Lipschitz constant $L<\Gamma(\kappa+1)$ leads to the desired result. □

5.2. Convergence Analysis in L^{∞}–Norm

Theorem 2. *Let $\Phi(x)$ be the exact solution of Equation (23) and $\Phi_N(x)$ be its approximate solution. Assume that $\kappa \in (0,1)$, $\Phi \in B^{m,\rho}_{0,0}(I)$. Then, we have the following estimate:*

$$\begin{aligned}\|\Phi-\Phi_N\|_{\chi^{0,0,\rho}} \le cN^{1-m}\|\partial_x^m\Phi(\rho e^x)\|_{\chi^{-1/2,-1/2}} + cN^{\frac{1}{2}-m}\|D^m_{\log}\boldsymbol{G}(\cdot,\Phi(\cdot))\|^2_{\chi^{\kappa+m-1,m,\rho}} \\ + cN^{\frac{1}{2}-m}\|D^m_{\log}\Phi\|^2_{\chi^{m,m,\rho}} + cN^m\|e_N\|^2_{\chi^{m,m,\rho}}.\end{aligned} \tag{53}$$

Proof. It follows from (38) that

$$\|\mathbf{E}_N\|_\infty \le \left\|\Phi - \mathcal{I}^{0,0,\rho}_{x,N}\Phi\right\|_\infty + \left\|\mathcal{I}^{0,0,\rho}_{x,N}\Phi - \Phi_N\right\|_\infty \le \sum_{j=1}^{3}\|\mathbf{E}_j\|_\infty. \tag{54}$$

Then, we have

$$\begin{aligned}|\mathbf{E}_1| &= \left\|(I-\mathcal{I}^{0,0,\rho}_{x,N})\Phi\right\|_\infty \\ &= \left\|\sum_{k=1}^{M}\left|(I-\mathcal{I}^{0,0,\rho}_{x,N})g_k\right|\right\|_\infty \\ &\le \sum_{k=1}^{M}\left\|(I-\mathcal{I}^{0,0,\rho}_{x,N})g_k\right\|_\infty \\ &= \sum_{k=1}^{M}\left\|g_k - \mathcal{I}^{-1/2}_{x,N}\phi_{N,k} + \mathcal{I}^{0,0,\rho}_{x,N}\mathcal{I}^{-1/2}_{x,N}\phi_{N,k} - \mathcal{I}^{0,0,\rho}_{x,N}g_k\right\|_\infty.\end{aligned} \tag{55}$$

Using the Jacobi–Gauss interpolation error estimate (see [20] page 133), for any $g_k \in H^m_{\chi^{-1/2,-1/2}}$ with $1 \le m \le N+1$,

$$\left\|g_k - \mathcal{I}^{-1/2,-1/2}_{x,N}g_k\right\|_\infty \le cN^{\frac{1}{2}-m}\|\partial_x^m\Phi(\rho e^x)\|_{\chi^{-1/2,-1/2}}, \tag{56}$$

where c is a positive constant independent of $m,\ N$ and g_k. Then, we obtain

$$\begin{aligned}\left\|(I-\mathcal{I}^{0,0,\rho}_{x,N})\Phi\right\|_\infty &\le \sum_{k=1}^{M}\left(1+\left\|\mathcal{I}^{0,0,\rho}_{x,N}\right\|_\infty\right)\left\|g_k - \mathcal{I}^{-1/2}_{x,N}g_k\right\|_\infty \\ &\le cN^{1-m}\sum_{k=1}^{M}\|\partial_x^m g_k\|_{\chi^{-1/2,-1/2}} \\ &= cN^{1-m}\|\partial_x^m\Phi(\rho e^x)\|_{\chi^{-1/2,-1/2}}.\end{aligned} \tag{57}$$

Next, by Lemma 2, we obtain

$$\begin{aligned}|\mathbf{E}_2| &= \left|\mathcal{I}^{0,0,\rho}_{x,N}\int_\rho^x \boldsymbol{R}(x,s)(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\kappa-1,0,\rho}_{s,N})\quad \mathbf{G}(s,\Phi(s))ds\right| \\ &= \frac{1}{\Gamma(\kappa)}\left|\mathcal{I}^{0,0,\rho}_{x,N}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M}\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\kappa-1,0,\rho}_{s,N}\right)g_k(s,\Phi(s))ds\right| \\ &\le \frac{1}{\Gamma(\kappa)}\left\|\mathcal{I}^{0,0,\rho}_{x,N}\right\|_\infty \max_{x\in I}\left|\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M}\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\kappa-1,0,\rho}_{s,N}\right)g_k(s,\Phi(s))ds\right| \\ &\le c\,N^{\frac{1}{2}}\max_{x\in I}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\left|\sum_{k=1}^{M}\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\kappa-1,0,\rho}_{s,N}\right)g_k(s,\Phi(s))ds\right| \\ &\le c\,N^{\frac{1}{2}}\max_{x\in I}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M}\left|\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}^{\kappa-1,0,\rho}_{s,N}\right)g_k(s,\Phi(s))ds\right|.\end{aligned}$$

By the Cauchy–Schwarz inequality and (36), we have

$$\begin{aligned}
|\mathbf{E}_2| &\leq c\,N^{\frac{1}{2}} \max_{x\in I}\Bigg[\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}ds \\
&\quad \times \int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{k=1}^{M}\left|\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)g_k(s,\Phi(s))\right|\right)^2 ds\Bigg]^{\frac{1}{2}} \\
&\leq c\,N^{\frac{1}{2}} \max_{x\in I}\left[\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{k=1}^{M}\left|\left(\mathcal{I} - {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\right)g_k(s,\Phi(s))\right|\right)^2 ds\right]^{\frac{1}{2}} \\
&\leq c\,N^{\frac{1}{2}-m} \max_{x\in I}\left[\int_\rho^x s^{-1}(\beta(x,s))^{\kappa+m-1}\left(\log\frac{s}{a}\right)^m\left|D^m_{\log,s}\mathbf{G}(s,\Phi(s))\right|^2 ds\right]^{\frac{1}{2}} \\
&\leq cN^{\frac{1}{2}-m}\|D^m_{\log}\mathbf{G}(\cdot,\Phi(\cdot))\|^2_{\chi^{\kappa+m-1,m,\rho}}.
\end{aligned} \tag{58}$$

Similarly, using Lemma 2 leads to

$$\begin{aligned}
|\mathbf{E}_3| &= \left|\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x \boldsymbol{R}(x,s) \quad {}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(\mathbf{G}(s,\Phi(s)) - \mathbf{G}(s,\Phi_N(s)))ds\right| \\
&= \frac{1}{\Gamma(\kappa)}\left|\mathcal{I}_{x,N}^{0,0,\rho}\int_\rho^x \left(s^{-1}\beta(x,s)\right)^{\kappa-1}\sum_{k=1}^{M} {}_z\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s)) - g_k(s,\Phi_N(s)))ds\right| \\
&\leq c\left\|\mathcal{I}_{x,N}^{0,0,\rho}\right\|_\infty \max_{x\in I}\left|\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M} {}_z\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s)) - g_k(s,\Phi_N(s)))ds\right| \\
&\leq cN^{\frac{1}{2}} \max_{x\in I}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\left|\sum_{k=1}^{M} {}_z\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s)) - g_k(s,\Phi_N(s)))ds\right| \\
&\leq cN^{\frac{1}{2}} \max_{x\in I}\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\sum_{k=1}^{M}\left|{}_z\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s)) - g_k(s,\Phi_N(s)))ds\right|.
\end{aligned}$$

Applying the Cauchy–Schwarz inequality and (36), we obtain

$$\begin{aligned}
|\mathbf{E}_3| &\leq cN^{\frac{1}{2}} \max_{x\in I}\Bigg[\int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}ds \\
&\quad \times \int_\rho^x s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{k=1}^{M}\left|{}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}(g_k(s,\Phi(s)) - g_k(s,\Phi_N(s)))\right|\right)^2 ds\Bigg]^{\frac{1}{2}} \\
&\leq cN^{\frac{1}{2}} \max_{x\in I}\Bigg[(\beta(x,\rho))^\kappa \\
&\quad \times \sum_{q=0}^{N}\left(\sum_{k=1}^{M}\left|g_k(s_q^{\kappa-1,0,\rho},\Phi(s_q^{\kappa-1,0,\rho})) - g_k(s_q^{\kappa-1,0,\rho},\Phi_N(s_q^{\kappa-1,0,\rho}))\right|\right)^2 \chi_q^{\kappa-1,0,\rho}\Bigg]^{\frac{1}{2}}.
\end{aligned}$$

Further, by the triangle inequality, (35) and (36), we deduce that

$$
\begin{aligned}
|\mathbf{E}_3| &\le cN^{\frac{1}{2}} \max_{x\in I}\left[(\beta(x,\rho))^{\kappa}\sum_{q=0}^{N}\left(\sum_{i=1}^{M}\left|\phi_i(s_q^{\kappa-1,0,\rho})-\phi_{N,i}(s_q^{\kappa-1,0,\rho})\right|\right)^2\chi_q^{\kappa-1,0,\rho}\right]^{\frac{1}{2}}\\
&\le cN^{\frac{1}{2}} \max_{x\in I}\left[\int_{\rho}^{x}s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{i=1}^{M}\left|{}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\phi_i(s)-\phi_{N,i}(s)\right|\right)^2 ds\right]^{\frac{1}{2}}\\
&\le cN^{\frac{1}{2}} \max_{x\in I}\left[\int_{\rho}^{x}s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{i=1}^{M}\left|{}_x\widetilde{\mathcal{I}}_{s,N}^{\kappa-1,0,\rho}\phi_i(s)-\phi_i(s)\right|\right)^2 ds\right.\\
&\qquad\left.+\int_{\rho}^{x}s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{i=1}^{M}|\phi_i(s)-\phi_{N,i}(s)|\right)^2 ds\right]^{\frac{1}{2}}\\
&\le cN^{\frac{1}{2}-m}\max_{x\in I}\left(\int_{\rho}^{x}(\beta(s,\rho))^{m}\left(\sum_{i=1}^{M}\left|D_{\log,s}^{m}\varphi_i(s)\right|\right)^2 ds\right)^{1/2}\\
&\qquad+cN^{\frac{1}{2}}\max_{x\in I}\left(\int_{\rho}^{x}s^{-1}(\beta(x,s))^{\kappa-1}\left(\sum_{i=1}^{M}|\varphi_i(s)-\varphi_{N,i}(s)|\right)^2 ds\right)^{1/2}\\
&\le cN^{\frac{1}{2}-m}\|D_{\log}^{m}\Phi\|_{\chi^{m,m,\rho}}^2+cN^{m}\|e_N\|_{\chi^{m,m,\rho}}^2.
\end{aligned}
\tag{59}
$$

Hence, a combination of (54), (57), (58) and (59) yields (53). □

6. Numerical Results

In this section, we introduce two numerical examples to illustrate the effectiveness of the proposed method.

Example 1. *We consider the following coupled system:*

$$
\begin{aligned}
&{}_{1}^{CH}D_t^{\kappa}\varphi_1(t)=g_1(t,\varphi_1(t),\varphi_2(t)),\quad \kappa\in(0,1),\\
&{}_{1}^{CH}D_t^{\kappa}\varphi_2(t)=g_2(t,\varphi_1(t),\varphi_2(t)),\quad \kappa\in(0,1),\\
&\varphi_1(1)=\varphi_2(1)=0.
\end{aligned}
\tag{60}
$$

For this problem, the exact solution is given as

$$
\varphi_1(t)=(\log t)^{4+\kappa},
$$

$$
\varphi_2(t)=(\log t)^{3+\kappa},
$$

and

$$
g_1(t,\varphi_1,\varphi_2)=\frac{\Gamma(5+\kappa)}{\Gamma(5)}(\log t)^4-\varphi_2^3+\left((\log t)^{4+\kappa}\right)^3,
$$

$$
g_2(t,\varphi_1,\varphi_1)=\frac{\Gamma(4+\kappa)}{\Gamma(4)}(\log t)^3-\varphi_1^3+\left((\log t)^{3+\kappa}\right)^3.
$$

In Tables 1 and 2, we report the $L^2_{\chi^{0,0,1}}$ and L^∞ errors for different values of κ. It is clear that with increasing N, the errors are decreased.

Table 1. The errors for φ_1 with the fractional orders $\kappa = 0.4$, 0.8 for Example 1.

	$L^2_{\chi^{0,0,1}}$-Errors		L^∞-Errors	
N	$\kappa = 0.4$	$\kappa = 0.8$	$\kappa = 0.4$	$\kappa = 0.8$
5	1.1×10^{-5}	1.5×10^{-5}	5.2×10^{-5}	7.0×10^{-5}
10	1.9×10^{-8}	6.4×10^{-9}	1.8×10^{-7}	6.0×10^{-8}
15	4.4×10^{-10}	1.0×10^{-10}	6.1×10^{-9}	1.4×10^{-9}
20	5.3×10^{-12}	1.1×10^{-10}	3.8×10^{-10}	1.0×10^{-10}

Table 2. The errors for φ_2 with the fractional orders $\kappa = 0.4$, 0.8 for Example 1.

	$L^2_{\chi^{0,0,1}}$-Errors		L^∞-Errors	
N	$\kappa = 0.4$	$\kappa = 0.8$	$\kappa = 0.4$	$\kappa = 0.8$
5	1.3×10^{-5}	1.0×10^{-5}	7.6×10^{-5}	5.4×10^{-5}
10	1.1×10^{-7}	3.0×10^{-8}	1.2×10^{-6}	3.3×10^{-7}
15	5.7×10^{-9}	1.1×10^{-9}	1.5×10^{-8}	1.8×10^{-9}
20	8.3×10^{-11}	1.1×10^{-10}	5.9×10^{-10}	2.9×10^{-10}

Example 2. *We consider the following coupled system:*

$$\begin{aligned} &{}^{CH}_{1}D^{\kappa}_{t}\varphi_1(t) = g_1(t,\varphi_1(t),\varphi_2(t)), \quad \kappa \in (0,1), \\ &{}^{CH}_{1}D^{\kappa}_{t}\varphi_2(t) = g_2(t,\varphi_1(t),\varphi_2(t)), \quad \kappa \in (0,1), \\ &\varphi_1(1) = \varphi_2(1) = 0. \end{aligned} \tag{61}$$

For this problem, the exact solution is given as

$$\varphi_1(t) = (\log t)^5 + 2(\log t)^3,$$

$$\varphi_2(t) = -(\log t)^4 + 2(\log t)^3,$$

and

$$g_1(t,\varphi_1,\varphi_2) = \frac{\Gamma(6)}{\Gamma(6-\kappa)}(\log t)^{5-\kappa} + \frac{2\Gamma(4)}{\Gamma(4-\kappa)}(\log t)^{3-\kappa} - \varphi_2^2 + \left(-(\log t)^4 + 2(\log t)^3\right)^2,$$

$$g_2(t,\varphi_1,\varphi_2) = -\frac{\Gamma(5)}{\Gamma(5-\kappa)}(\log t)^{4-\kappa} + \frac{2\Gamma(4)}{\Gamma(4-\kappa)}(\log t)^{3-\kappa} - \varphi_1^2 + \left((\log t)^5 + 2(\log t)^3\right)^2.$$

In Tables 3 and 4, we report the $L^2_{\chi^{0,0,1}}$ and L^∞ errors for different values of κ. It is clear that with increasing the number of bases functions, the errors are decreased.

Table 3. The errors for φ_1 with the fractional orders $\kappa = 0.4$, 0.8 for Example 2.

	$L^2_{\chi^{0,0,1}}$-Errors		L^∞-Errors	
N	$\kappa = 0.4$	$\kappa = 0.8$	$\kappa = 0.4$	$\kappa = 0.8$
5	4.5×10^{-7}	7.1×10^{-7}	2.2×10^{-6}	1.8×10^{-6}
10	6.1×10^{-9}	1.6×10^{-8}	3.7×10^{-8}	4.4×10^{-8}
15	4.3×10^{-10}	1.6×10^{-9}	2.6×10^{-9}	4.3×10^{-9}
20	6.2×10^{-11}	2.9×10^{-10}	3.8×10^{-10}	7.8×10^{-10}
25	1.3×10^{-11}	7.6×10^{-11}	8.3×10^{-11}	2.0×10^{-10}

Table 4. The errors for φ_2 with the fractional orders $\kappa = 0.4$, 0.8 for Example 2.

	$L^2_{\chi^{0,0,1}}$-Errors		L^∞-Errors	
N	$\kappa = 0.4$	$\kappa = 0.8$	$\kappa = 0.4$	$\kappa = 0.8$
5	5.9×10^{-7}	4.2×10^{-7}	3.2×10^{-6}	7.7×10^{-7}
10	8.2×10^{-9}	9.4×10^{-9}	5.7×10^{-8}	1.7×10^{-8}
15	5.7×10^{-10}	8.9×10^{-10}	4.0×10^{-9}	1.6×10^{-10}
20	8.3×10^{-11}	1.6×10^{-10}	5.9×10^{-10}	2.9×10^{-10}
25	1.8×10^{-11}	4.2×10^{-11}	1.2×10^{-10}	7.6×10^{-11}

7. Conclusions

We derived a collocation spectral scheme for systems of nonlinear C-H differential equations. We constructed a non-polynomial spectral collocation scheme, described its effective implementation, and derived its convergence analysis. In addition, we provided numerical examples to support our theoretical analysis. The numerical results demonstrate the accuracy and effectiveness of the proposed scheme. We also conclude that the described technique produces very accurate results, even when employing a small number of base functions. In further research, we will consider an efficient spectral collocation method for nonlinear systems of fractional pantograph delay differential equations.

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