physics

Supplementary Information

Generalized one-dimensional periodic potential wells tending to the Dirac delta potential

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Section S1 Periodic potential of a potential $V = V(x)$ and a zero potential

Bloch functions must satisfy the following boundary and continuity conditions.

$$\Psi_I(b) = \Psi_0(b)$$
$$\Psi_I(-b) = e^{ikT}\Psi_0(b+a)$$
$$\Psi_I^{'}(b) = \Psi_0^{'}(b)$$
$$\Psi_I^{'}(-b) = e^{ikT}\Psi_0^{'}(b+a)$$

The next matrix was obtained:

$$\begin{bmatrix} y_1(b) & y_2(b) & -e^{i\alpha b} & -e^{-i\alpha b} \\ y_1(-b) & y_2(-b) & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)} \\ y_1^{'}(b) & y_2^{'}(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b} \\ y_1^{'}(-b) & y_2^{'}(-b) & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ D_1 \\ C_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the secular determinant.

$$0 = \begin{vmatrix} y_1(b) & y_2(b) & -e^{i\alpha b} & -e^{-i\alpha b} \\ y_1(-b) & y_2(-b) & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)} \\ y_1^{'}(b) & y_2^{'}(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b} \\ y_1^{'}(-b) & y_2^{'}(-b) & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{vmatrix}$$

$$0 = y_1(b)\begin{vmatrix} y_2(-b) & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)} \\ y_2^{'}(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b} \\ y_2^{'}(-b) & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{vmatrix} - y_1(-b)\begin{vmatrix} y_2(b) & -e^{i\alpha b} & -e^{-i\alpha b} \\ y_2^{'}(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b} \\ y_2^{'}(-b) & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{vmatrix}$$
$$+ y_1^{'}(b)\begin{vmatrix} y_2(b) & -e^{i\alpha b} & -e^{-i\alpha b} \\ y_2(-b) & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)} \\ y_2^{'}(-b) & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{vmatrix} - y_1^{'}(-b)\begin{vmatrix} y_2(b) & -e^{i\alpha b} & -e^{-i\alpha b} \\ y_2(-b) & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)} \\ y_2^{'}(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b} \end{vmatrix}$$

Solving each determinant.

$$D1 = y_2(-b)[\alpha^2 e^{ikT}e^{-i\alpha a} - \alpha^2 e^{ikT}e^{i\alpha a}] - y_2^{'}(b)[-i\alpha e^{2ikT} - i\alpha e^{2ikT}] + y_2^{'}(-b)[-i\alpha e^{ikT}e^{i\alpha a} - i\alpha e^{ikT}e^{-i\alpha a}]$$
$$D2 = y_2(b)[\alpha^2 e^{ikT}e^{-i\alpha a} - \alpha^2 e^{ikT}e^{i\alpha a}] - y_2^{'}(b)[-i\alpha e^{ikT}e^{-i\alpha a} - i\alpha e^{ikT}e^{i\alpha a}] + y_2^{'}(-b)[-i\alpha - i\alpha]$$
$$D3 = y_2(b)[-i\alpha e^{2ikT} - i\alpha e^{2ikT}] - y_2(-b)[-i\alpha e^{ikT}e^{-i\alpha a} - i\alpha e^{ikT}e^{i\alpha a}] + y_2^{'}(-b)[e^{ikT}e^{-i\alpha a} - e^{ikT}e^{i\alpha a}]$$
$$D4 = y_2(b)[-i\alpha e^{ikT}e^{i\alpha a} - i\alpha e^{ikT}e^{-i\alpha a}] - y_2(-b)[-i\alpha - i\alpha] + y_2^{'}(b)[e^{ikT}e^{-i\alpha a} - e^{ikT}e^{i\alpha a}]$$

$$D1 = -2i\alpha^2 e^{ikT}\sin(\alpha a)y_2(-b) + 2i\alpha e^{2ikT}y_2^{'}(b) - 2i\alpha e^{ikT}\cos(\alpha a)y_2^{'}(-b)$$
$$D2 = -2i\alpha^2 e^{ikT}\sin(\alpha a)y_2(b) + 2i\alpha e^{ikT}\cos(\alpha a)y_2^{'}(b) - 2i\alpha y_2^{'}(-b)$$
$$D3 = -2i\alpha e^{2ikT}y_2(b) + 2i\alpha e^{ikT}\cos(\alpha a)y_2(-b) - 2ie^{ikT}\sin(\alpha a)y_2^{'}(-b)$$
$$D4 = -2i\alpha e^{ikT}\cos(\alpha a)y_2(b) + 2i\alpha y_2(-b) - 2ie^{ikT}\sin(\alpha a)y_2^{'}(b)$$

The energy equation is given by the following equation.

$$\begin{aligned}
0 &= y_1(b)[-2i\alpha^2 e^{ikT}\sin(\alpha a)y_2(-b) + 2i\alpha e^{2ikT}y_2^{'}(b) - 2i\alpha e^{ikT}\cos(\alpha a)y_2^{'}(-b)] \\
&- y_1(-b)[-2i\alpha^2 e^{ikT}\sin(\alpha a)y_2(b) + 2i\alpha e^{ikT}\cos(\alpha a)y_2^{'}(b) - 2i\alpha y_2^{'}(-b)] \\
&+ y_1^{'}(b)[-2i\alpha e^{2ikT}y_2(b) + 2i\alpha e^{ikT}\cos(\alpha a)y_2(-b) - 2ie^{ikT}\sin(\alpha a)y_2^{'}(-b)] \\
&- y_1^{'}(-b)[-2i\alpha e^{ikT}\cos(\alpha a)y_2(b) + 2i\alpha y_2(-b) - 2ie^{ikT}\sin(\alpha a)y_2^{'}(b)]
\end{aligned}$$

Grouping like terms.

$$\begin{aligned}
0 &= 2i\alpha^2 e^{ikT}\sin(\alpha a)[y_2(b)y_1(-b) - y_2(-b)y_1(b)] \\
&- 2i\alpha e^{ikT}\cos(\alpha a)[y_2^{'}(-b)y_1(b) + y_1(-b)y_2^{'}(b)] \\
&+ 2i\alpha e^{ikT}\cos(\alpha a)[y_1^{'}(b)y_2(-b) + y_1^{'}(-b)y_2(b)] \\
&+ 2ie^{ikT}\sin(\alpha a)[y_2^{'}(b)y_1^{'}(-b) - y_2^{'}(-b)y_1^{'}(b)] \\
&+ 2i\alpha e^{2ikT}[y_2^{'}(b)y_1(b) - y_1^{'}(b)y_2(b)] \\
&+ 2i\alpha[y_2^{'}(-b)y_1(-b) - y_1^{'}(-b)y_2(-b)] \\
0 &= \alpha^2\sin(\alpha a)[y_2(b)y_1(-b) - y_2(-b)y_1(b)] \\
&- \alpha\cos(\alpha a)[y_2^{'}(-b)y_1(b) + y_1(-b)y_2^{'}(b)] \\
&+ \alpha\cos(\alpha a)[y_1^{'}(b)y_2(-b) + y_1^{'}(-b)y_2(b)] \\
&+ \sin(\alpha a)[y_2^{'}(b)y_1^{'}(-b) - y_2^{'}(-b)y_1^{'}(b)] \\
&+ e^{ikT}[y_2^{'}(b)y_1(b) - y_1^{'}(b)y_2(b)] \\
&+ e^{-ikT}[y_2^{'}(-b)y_1(-b) - y_1^{'}(-b)y_2(-b)] \\
0 &= \alpha^2\sin(\alpha a)[y_2(b)y_1(-b) - y_2(-b)y_1(b)] \\
&- \alpha\cos(\alpha a)[y_2^{'}(-b)y_1(b) + y_1(-b)y_2^{'}(b)] \\
&+ \alpha\cos(\alpha a)[y_1^{'}(b)y_2(-b) + y_1^{'}(-b)y_2(b)] \\
&+ \sin(\alpha a)[y_2^{'}(b)y_1^{'}(-b) - y_2^{'}(-b)y_1^{'}(b)] \\
&+ 2W\{y_1(x), y_2(x)\}\,\alpha\cos(kT)
\end{aligned}$$

Clearing.

$$\begin{aligned}
\cos(kT) = &-\sin(\alpha a)\left[\frac{y_2^{'}(b)y_1^{'}(-b) - y_2^{'}(-b)y_1^{'}(b)}{2W\{y_1(x), y_2(x)\}\,\alpha}\right] - \alpha^2\sin(\alpha a)\left[\frac{y_2(b)y_1(-b) - y_2(-b)y_1(b)}{2W\{y_1(x), y_2(x)\}\,\alpha}\right] \\
&+ \cos(\alpha a)\left[\frac{y_2^{'}(-b)y_1(b) + y_1(-b)y_2^{'}(b) - y_1^{'}(b)y_2(-b) - y_1^{'}(-b)y_2(b)}{2W\{y_1(x), y_2(x)\}}\right]
\end{aligned}$$

$$
\begin{aligned}
\cos(kT) &= \sin(\alpha a)\left[\frac{y_2^{'}(-b)y_1^{'}(b) - y_2^{'}(b)y_1^{'}(-b)}{2W\left\{y_1(x), y_2(x)\right\}\alpha}\right] + \alpha^2 \sin(\alpha a)\left[\frac{y_2(-b)y_1(b) - y_2(b)y_1(-b)}{2W\left\{y_1(x), y_2(x)\right\}\alpha}\right] \\
&+ \cos(\alpha a)\left[\frac{y_2^{'}(-b)y_1(b) + y_1(-b)y_2^{'}(b) - y_1^{'}(b)y_2(-b) - y_1^{'}(-b)y_2(b)}{2W\left\{y_1(x), y_2(x)\right\}}\right] \\
\cos(k(2b+a)) &= \frac{\sin(\alpha a)}{\alpha}\left[\frac{M_1(E) + \alpha^2 M_2(E)}{M_4(E)}\right] + \cos(\alpha a)\left[\frac{M_3(E)}{M_4(E)}\right]
\end{aligned}
$$

Where

$$
\begin{aligned}
M_1(E) &= y_2^{'}(-b)y_1^{'}(b) - y_2^{'}(b)y_1^{'}(-b) \\
M_2(E) &= y_2(-b)y_1(b) - y_2(b)y_1(-b) \\
M_3(E) &= y_1(b)y_2^{'}(-b) - y_1^{'}(-b)y_2(b) + y_1(-b)y_2^{'}(b) - y_1^{'}(b)y_2(-b) \\
M_4(E) &= 2W\left\{y_1(x), y_2(x)\right\}
\end{aligned}
$$

Section S2 Periodic potential of a two potentials $V = V(x)$ and a zero potential

Bloch functions must satisfy the following boundary and continuity conditions.

$$\begin{gathered}\Psi_I(0) = \Psi_{II}(0)\\ \Psi_I'(0) = \Psi_{II}'(0)\\ \Psi_{II}(b) = \Psi_0(b)\\ \Psi_{II}'(b) = \Psi_0'(b)\\ \Psi_I(-b) = e^{ik(2b+a)}\Psi_0(b+a)\\ \Psi_I'(-b) = e^{ik(2b+a)}\Psi_0'(b+a)\end{gathered}$$

The following matrix was obtained:

$$\begin{bmatrix} y_1(0) & y_2(0) & -z_1(0) & -z_2(0) & 0 & 0\\ 0 & 0 & z_1(b) & z_2(b) & -e^{i\alpha b} & -e^{-i\alpha b}\\ y_1(-b) & y_2(-b) & 0 & 0 & -e^{ikT}e^{i\alpha(b+a)} & -e^{ikT}e^{-i\alpha(b+a)}\\ y_1'(0) & y_2'(0) & -z_1'(0) & -z_2'(0) & 0 & 0\\ 0 & 0 & z_1'(b) & z_2'(b) & -i\alpha e^{i\alpha b} & i\alpha e^{-i\alpha b}\\ y_1'(-b) & y_2'(-b) & 0 & 0 & -i\alpha e^{ikT}e^{i\alpha(b+a)} & i\alpha e^{ikT}e^{-i\alpha(b+a)} \end{bmatrix} \cdot \begin{bmatrix} C_1\\ D_1\\ C_2\\ D_2\\ C_3\\ D_3 \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0 \end{bmatrix}$$

As it is very difficult to solve the secular determinant, the elements of the matrix were transformed into easy expressions, as follow:

$$0 = \begin{vmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & 0 & 0\\ 0 & 0 & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H}\\ \mathrm{I} & \mathrm{J} & 0 & 0 & \mathrm{K} & \mathrm{L}\\ \mathrm{M} & \mathrm{N} & \mathrm{P} & \mathrm{Q} & 0 & 0\\ 0 & 0 & \mathrm{R} & \mathrm{S} & \mathrm{T} & \mathrm{U}\\ \mathrm{V} & \mathrm{W} & 0 & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix}$$

$$0 = \mathrm{A}\cdot\begin{vmatrix} 0 & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H}\\ \mathrm{J} & 0 & 0 & \mathrm{K} & \mathrm{L}\\ \mathrm{N} & \mathrm{P} & \mathrm{Q} & 0 & 0\\ 0 & \mathrm{R} & \mathrm{S} & \mathrm{T} & \mathrm{U}\\ \mathrm{W} & 0 & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix} - \mathrm{B}\cdot\begin{vmatrix} 0 & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H}\\ \mathrm{I} & 0 & 0 & \mathrm{K} & \mathrm{L}\\ \mathrm{M} & \mathrm{P} & \mathrm{Q} & 0 & 0\\ 0 & \mathrm{R} & \mathrm{S} & \mathrm{T} & \mathrm{U}\\ \mathrm{V} & 0 & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix} + \mathrm{C}\cdot\begin{vmatrix} 0 & 0 & \mathrm{F} & \mathrm{G} & H\\ \mathrm{I} & \mathrm{J} & 0 & \mathrm{K} & \mathrm{L}\\ \mathrm{M} & \mathrm{N} & \mathrm{Q} & 0 & 0\\ 0 & 0 & \mathrm{S} & \mathrm{T} & \mathrm{U}\\ \mathrm{V} & \mathrm{W} & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix} - \mathrm{D}\cdot\begin{vmatrix} 0 & 0 & \mathrm{E} & \mathrm{G} & \mathrm{H}\\ \mathrm{I} & \mathrm{J} & 0 & \mathrm{K} & \mathrm{L}\\ \mathrm{M} & \mathrm{N} & \mathrm{P} & 0 & 0\\ 0 & 0 & \mathrm{R} & \mathrm{T} & \mathrm{U}\\ \mathrm{V} & \mathrm{W} & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix}$$

$$0 = \begin{vmatrix} 0 & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{H}\\ \text{AJ - BI} & 0 & 0 & \mathrm{K} & \mathrm{L}\\ \text{AN - BM} & \mathrm{P} & \mathrm{Q} & 0 & 0\\ 0 & \mathrm{R} & \mathrm{S} & \mathrm{T} & \mathrm{U}\\ \text{AW - BV} & 0 & 0 & \mathrm{X} & \mathrm{Y} \end{vmatrix} + \begin{vmatrix} \text{CF - DE} & 0 & 0 & \mathrm{G} & \mathrm{H}\\ 0 & \mathrm{I} & \mathrm{J} & \mathrm{K} & \mathrm{L}\\ \text{CQ - DP} & \mathrm{M} & \mathrm{N} & 0 & 0\\ \text{CS - DR} & 0 & 0 & \mathrm{T} & \mathrm{U}\\ 0 & \mathrm{V} & \mathrm{W} & \mathrm{X} & \mathrm{Y} \end{vmatrix}$$

Solving the resulting determinants

$$\begin{aligned} 0 = {} & \text{- (AJ - BI)(EQ - PF)(TY - XU) - (AJ-BI)(PS-RQ)(GY-XH)}\\ & \text{+ (AW - BV)(PF - EQ)(KU - TL) + (AW - BV)(PS - RQ)(GL - KH)}\\ & \text{- (CF - DE)(JM - NI)(TY - XU) - (CF - DE)(WM - NV)(KU - TL)}\\ & \text{+ (CS - DR)(JM - NI)(GY - XH) + (CS - DR)(NV - WM)(GL - KH)}\\ & \text{+ (AN - BM)(RF - ES)(KY - XL) - (CQ - DP)(WI - JV)(GU - TH)} \end{aligned}$$

Grouping the result of the determinant in terms of complex exponentials

$$
\begin{aligned}
0 = \ & (\text{XU - TY})[\ (\text{AJ - BI})(\text{EQ - PF}) + (\text{CF - DE})(\text{JM - NI})\ ] \\
& + (\text{GY - XH})[\ (\text{CS - DR})(\text{JM - NI}) + (\text{BI - AJ})(\text{PS - RQ})\ ] \\
& + (\text{KU - TL})[\ (\text{AW - BV})(\text{PF - EQ}) + (\text{DE - CF})(\text{WM - NV})\ ] \\
& + (\text{GL - KH})[\ (\text{AW - BV})(\text{PS - RQ}) + (\text{CS - DR})(\text{NV - WM})\ ] \\
& + (\text{KY - XL})[\ (\text{AN - BM})(\ \text{RF - ES})\ ] - (\text{GU - TH})[\ (\text{CQ - DP})(\text{WI - JV})\ ]
\end{aligned}
$$

Where

$$
\begin{aligned}
\gamma &= e^{ikT} \\
\text{GY - XH} &= -2i\alpha\gamma\cos(\alpha a) \\
\text{KU - TL} &= -2i\alpha\gamma\cos(\alpha a) \\
\text{GL - KH} &= -2i\gamma\sin(\alpha a) \\
\text{XU - TY} &= -2i\alpha^2\gamma\sin(\alpha a) \\
\text{KY - XL} &= -2i\alpha\gamma^2 \\
\text{GU - TH} &= -2i\alpha\alpha \\
\text{AN - BM} &= W\{y_1(x), y_2(x)\} \\
\text{WI - JV} &= W\{y_1(x), y_2(x)\} \\
\text{CQ - DP} &= W\{z_1(x), z_2(x)\} \\
\text{ES - RF} &= W\{z_1(x), z_2(x)\} \\
\gamma + \gamma^{-1} &= 2\cos(kT)
\end{aligned}
$$

Therefore.

$$
\begin{aligned}
0 = \ & \alpha\sin(\alpha a)[\ (\text{AJ - BI})(\text{EQ - PF}) + (\text{CF - DE})(\text{JM - NI})\ ] \\
& + \cos(\alpha a)[\ (\text{CS - DR})(\text{JM - NI}) + (\text{BI - AJ})(\text{PS - RQ})\ ] \\
& + \cos(\alpha a)[\ (\text{AW - BV})(\text{PF - EQ}) + (\text{DE - CF})(\text{WM - NV})\ ] \\
& + \frac{\sin(\alpha a)}{\alpha}[\ (\text{AW - BV})(\text{PS - RQ}) + (\text{CS - DR})(\text{NV - WM})\ ] \\
& - 2\cos(kT) W\{y_1(x), y_2(x)\} W\{z_1(x), z_2(x)\}
\end{aligned}
$$

The transcendental energy equation for the two potentials $V = V(x)$ and a zero potential is the following.

$$
\begin{aligned}
2\cos(kT) = \ & \alpha\sin(\alpha a)\frac{[\ (\text{AJ - BI})(\text{EQ - PF}) + (\text{CF - DE})(\text{JM - NI})\ ]}{2W\{y_1(x), y_2(x)\} W\{z_1(x), z_2(x)\}} \\
& + \cos(\alpha a)\frac{[\ (\text{CS - DR})(\text{JM - NI}) + (\text{BI - AJ})(\text{PS - RQ})\ ]}{2W\{y_1(x), y_2(x)\} W\{z_1(x), z_2(x)\}} \\
& + \cos(\alpha a)\frac{[\ (\text{AW - BV})(\text{PF - EQ}) + (\text{DE - CF})(\text{WM - NV})\ ]}{2W\{y_1(x), y_2(x)\} W\{z_1(x), z_2(x)\}} \\
& + \frac{\sin(\alpha a)}{\alpha}\frac{[\ (\text{AW - BV})(\text{PS - RQ}) + (\text{CS - DR})(\text{NV - WM})\ ]}{2W\{y_1(x), y_2(x)\} W\{z_1(x), z_2(x)\}} \\
\cos(k(2b+a)) = \ & \frac{\sin(\alpha a)}{\alpha}\left[\frac{N_1(E) + \alpha^2 N_2(E)}{N_4(E)}\right] + \cos(\alpha a)\left[\frac{N_3(E)}{N_4(E)}\right]
\end{aligned}
$$

Where

$$
\begin{aligned}
N_1(E) &= [y_1(0)y_2'(-b) - y_2(0)y_1'(-b)] \cdot [z_2'(0)z_1'(b) - z_1'(0)z_2'(b)] \\
&\quad + [z_2(0)z_1'(b) - z_1(0)z_2'(b)] \cdot [y_1'(-b)y_2'(0) - y_2'(-b)y_1'(0)] \\
N_2(E) &= [y_1(0)y_2(-b) - y_2(0)y_1(-b)] \cdot [z_2(b)z_1'(0) - z_1(b)z_2'(0)] \\
&\quad + [z_2(0)z_1(b) - z_1(0)z_2(b)] \cdot [y_2(-b)y_1'(0) - y_1(-b)y_2'(0)] \\
N_3(E) &= [y_2(0)y_1(-b) - y_1(0)y_2(-b)] \cdot [z_2'(0)z_1'(b) - z_1'(0)z_2'(b)] \\
&\quad + [z_2(0)z_1'(b) - z_1(0)z_2'(b)] \cdot [y_2(-b)y_1'(0) - y_1(-b)y_2'(0)] \\
&\quad + [y_1(0)y_2'(-b) - y_2(0)y_1'(-b)] \cdot [z_1(b)z_2'(0) - z_2(b)z_1'(0)] \\
&\quad + [z_1(0)z_2(b) - z_2(0)z_1(b)] \cdot [y_2'(-b)y_1'(0) - y_1'(-b)y_2'(0)] \\
N_4(E) &= 2W\{y_1(x), y_2(x)\} \cdot W\{z_1(x), z_2(x)\}
\end{aligned}
$$

Where the solutions of the SE for the triangular potential $y_1(x), y_2(x)$ and $z_1(x), z_2(x)$ are given by following equations.

$$
\begin{aligned}
y_1(x) &= Ai\left(\left(\frac{\sqrt{2m}}{\hbar}\left(\frac{b}{w}\right)\right)^{2/3}\left(\frac{w(x+b)}{b} - E\right)\right) \\
y_2(x) &= Bi\left(\left(\frac{\sqrt{2m}}{\hbar}\left(\frac{b}{w}\right)\right)^{2/3}\left(\frac{w(x+b)}{b} - E\right)\right) \\
z_1(x) &= Ai\left(\left(\frac{\sqrt{2m}}{\hbar}\left(-\frac{b}{w}\right)\right)^{2/3}\left(-\frac{w(x-b)}{b} - E\right)\right) \\
z_2(x) &= Bi\left(\left(\frac{\sqrt{2m}}{\hbar}\left(-\frac{b}{w}\right)\right)^{2/3}\left(-\frac{w(x-b)}{b} - E\right)\right)
\end{aligned}
$$

The transcendental energy equation found is also useful for dealing with asymmetric periodic potentials.

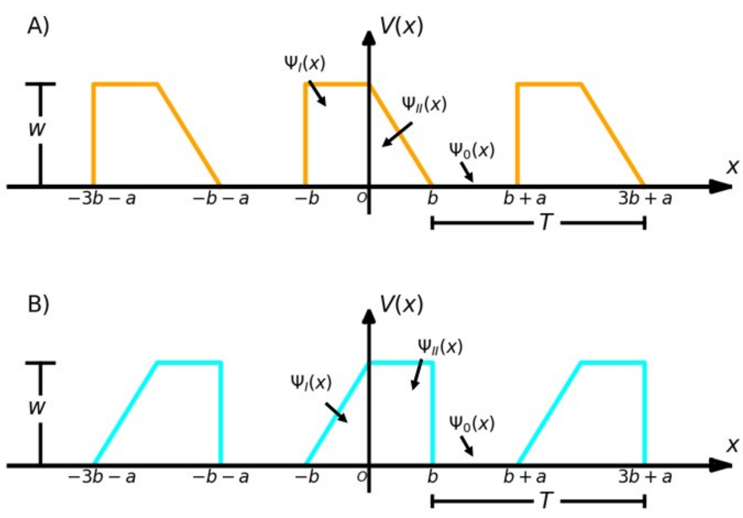

Figure S1: Representation of asymmetric periodic potentials. A) Rectangular - triangular periodic potential, B) Triangular - rectangular periodic potential.

Section S3 The transcendental energy equation

The transcendental energy equation for periodic potentials has the following form:

$$\cos(kT) = f(E)\sin(\alpha(E)a) + g(E)\cos(\alpha(E)a)$$

Where:

$$f(E) = \frac{M_1(E) + \alpha^2 M_2(E)}{\alpha M_4(E)} \qquad f(E) = \frac{N_1(E) + \alpha^2 N_2(E)}{\alpha N_4(E)}$$
$$g(E) = \frac{M_3(E)}{M_4(E)} \qquad g(E) = \frac{N_3(E)}{N_4(E)}$$

The variables $M_1(E), M_2(E), M_3(E)$, and $M_4(E)$ for a one potential of the form $V = V(x)$ coupled to a zero potential are given by:

$$\begin{aligned}
M_1(E) &= y_2'(-b)y_1'(b) - y_2'(b)y_1'(-b)\\
M_2(E) &= y_2(-b)y_1(b) - y_2(b)y_1(-b)\\
M_3(E) &= y_1(b)y_2'(-b) - y_1'(-b)y_2(b) + y_1(-b)y_2'(b) - y_1'(b)y_2(-b)\\
M_4(E) &= 2W\{y_1(x), y_2(x)\}
\end{aligned}$$

The variables $N_1(E), N_2(E), N_3(E)$ and $N_4(E)$ for two potentials of the form $V = V(x)$ coupled to a zero potential are given by:

$$\begin{aligned}
N_1(E) = {} & [y_1(0)y_2'(-b) - y_2(0)y_1'(-b)] \cdot [z_2'(0)z_1'(b) - z_1'(0)z_2'(b)]\\
& + [z_2(0)z_1'(b) - z_1(0)z_2'(b)] \cdot [y_1'(-b)y_2'(0) - y_2'(-b)y_1'(0)]\\
N_2(E) = {} & [y_1(0)y_2(-b) - y_2(0)y_1(-b)] \cdot [z_2(b)z_1'(0) - z_1(b)z_2'(0)]\\
& + [z_2(0)z_1(b) - z_1(0)z_2(b)] \cdot [y_2(-b)y_1'(0) - y_1(-b)y_2'(0)]\\
N_3(E) = {} & [y_2(0)y_1(-b) - y_1(0)y_2(-b)] \cdot [z_2'(0)z_1'(b) - z_1'(0)z_2'(b)]\\
& + [z_2(0)z_1'(b) - z_1(0)z_2'(b)] \cdot [y_2(-b)y_1'(0) - y_1(-b)y_2'(0)]\\
& + [y_1(0)y_2'(-b) - y_2(0)y_1'(-b)] \cdot [z_1(b)z_2'(0) - z_2(b)z_1'(0)]\\
& + [z_1(0)z_2(b) - z_2(0)z_1(b)] \cdot [y_2'(-b)y_1'(0) - y_1'(-b)y_2'(0)]\\
N_4(E) = {} & 2W\{y_1(x), y_2(x)\} \cdot W\{z_1(x), z_2(x)\}
\end{aligned}$$

The variable $\alpha = \alpha(E)$ is an energy function and its energy derivatives are very important for the calculation of group speed and effective mass.

$$\begin{aligned}
\alpha(E) &= \frac{\sqrt{2mE}}{\hbar}\\
\frac{d\alpha(E)}{dE} &= \frac{m}{\hbar\sqrt{2mE}}\\
\frac{d^2\alpha(E)}{dE^2} &= -\frac{m^2}{\hbar(2mE)^{3/2}}
\end{aligned}$$

The values of $N_1(E)$, $N_2(E)$, $N_3(E)$, and $N_4(E)$ are shown below for a combination of periodic potentials with an asymmetric pattern, potentials for which a solution is known that can be numerically simulated using the high-level programming language Python.

- **Asymmetric rectangular - triangular periodic potential**

$$N_1(E) = -2\beta\cosh(\beta b)\cdot[z_2^{'}(0)z_1^{'}(b) - z_1^{'}(0)z_2^{'}(b)] + 2\beta^2\sinh\beta b\cdot[z_2(0)z_1^{'}(b) - z_1(0)z_2^{'}(b)]$$

$$N_2(E) = -2\sinh(\beta b)\cdot[z_2(b)z_1^{'}(0) - z_1(b)z_2^{'}(0)] + 2\beta\cosh(\beta b)\cdot[z_1(0)z_2(b) - z_2(0)z_1(b)]$$

$$\begin{aligned} N_3(E) = &-2\sinh(\beta b)\cdot[z_2^{'}(0)z_1^{'}(b) - z_1^{'}(0)z_2^{'}(b)] + 2\beta\cosh(\beta b)\cdot[z_2(0)z_1^{'}(b) - z_1(0)z_2^{'}(b)] \\ &- 2\beta\cosh(\beta b)\cdot[z_1(b)z_2^{'}(0) - z_2(b)z_1^{'}(0)] - 2\beta^2\sinh(\beta b)\cdot[z_1(0)z_2(b) - z_2(0)z_1(b)] \end{aligned}$$

$$N_4(E) = -2\beta W\{z_1(x), z_2(x)\}$$

where the set of linearly independent functions $z_1(x)$ and $z_2(x)$ are the special Airy functions.

$$z_1(x) = Ai\left(\left(\frac{\sqrt{2m}}{\hbar}\left(-\frac{b}{w}\right)\right)^{2/3}\left(-\frac{w(x-b)}{b} - E\right)\right)$$
$$z_2(x) = Bi\left(\left(\frac{\sqrt{2m}}{\hbar}\left(-\frac{b}{w}\right)\right)^{2/3}\left(-\frac{w(x-b)}{b} - E\right)\right)$$

- **Asymmetric triangular - rectangular periodic potential**

$$\begin{aligned} N_1(E) = &-2\beta^2\sinh(\beta b)\cdot[y_1(0)y_2^{'}(-b) - y_2(0)y_1^{'}(-b)] \\ &+ 2\beta\cosh(\beta b)\cdot[y_1^{'}(-b)y_2^{'}(0) - y_2^{'}(-b)y_1^{'}(0)] \end{aligned}$$

$$\begin{aligned} N_2(E) = &\ 2\beta\cosh(\beta b)\cdot[y_2(0)y_1(-b) - y_1(0)y_2(-b)] \\ &- 2\sinh(\beta b)\cdot[y_2(-b)y_1^{'}(0) - y_1(-b)y_2^{'}(0)] \end{aligned}$$

$$\begin{aligned} N_3(E) = &-2\beta^2\sinh(\beta b)\cdot[y_2(0)y_1(-b) - y_1(0)y_2(-b)] \\ &+ 2\beta\cosh(\beta b)\cdot[y_2(-b)y_1^{'}(0) - y_1(-b)y_2^{'}(0)] \\ &- 2\beta\cosh(\beta b)\cdot[y_1(0)y_2^{'}(-b) - y_2(0)y_1^{'}(-b)] \\ &- 2\sinh(\beta b)\cdot[y_2^{'}(-b)y_1^{'}(0) - y_1^{'}(-b)y_2^{'}(0)] \end{aligned}$$

$$N_4(E) = -2\beta W\{y_1(x), y_2(x)\}$$

Where the set of linearly independent functions $y_1(x)$ and $y_2(x)$ are the special Airy functions.

$$y_1(x) = Ai\left(\left(\frac{\sqrt{2m}}{\hbar}\left(\frac{b}{w}\right)\right)^{2/3}\left(\frac{w(x+b)}{b} - E\right)\right)$$
$$y_2(x) = Bi\left(\left(\frac{\sqrt{2m}}{\hbar}\left(\frac{b}{w}\right)\right)^{2/3}\left(\frac{w(x+b)}{b} - E\right)\right)$$

In the same way, for the cases of symmetric potentials and asymmetric potentials, they must comply with the following limits, since in extreme cases, that is, when w tends to infinity and b has zero, such asymmetry is negligible or imperceptible.

$$\lim_{(b,w)\to(0,\infty)}\left[\frac{N_1(E)+\alpha^2 N_2(E)}{N_4(E)}\right]=G(b,w)$$
$$\lim_{(b,w)\to(0,\infty)}\left[\frac{N_3(E)}{N_4(E)}\right]=1$$

Where $G(b,w)$ is the value of the limit, which depends on the width of the potential b and the height of the potential w respectively.

Next, the numerical simulation of the asymmetric periodic potentials previously developed in this complementary information is carried out. For the cases in which its transcendental energy equation tends to the transcendental energy equation of the Dirac delta potential.

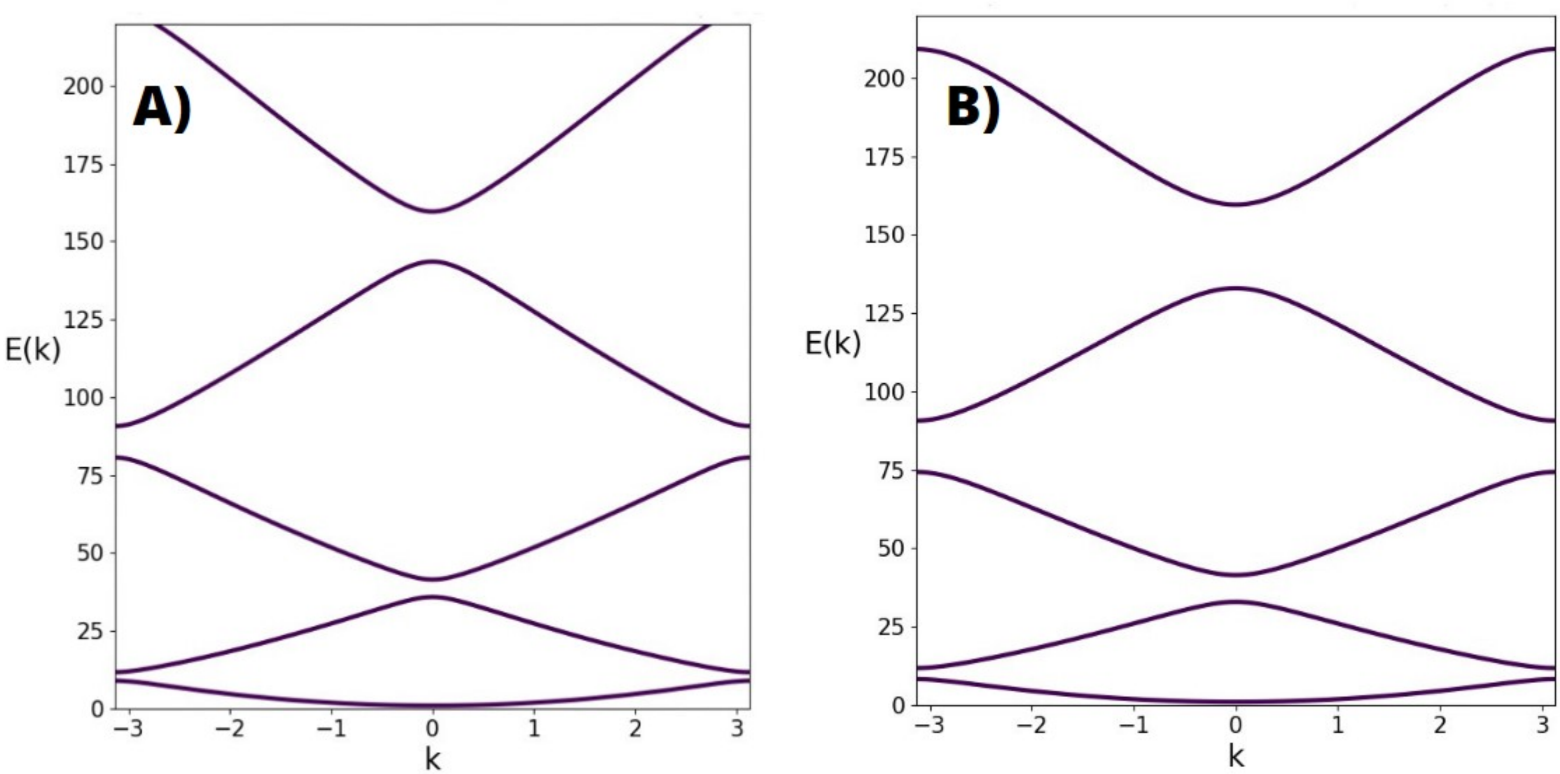

Figure S2: Periodic asymmetric potential with separation between the asymmetric potentials $a = 1$, width of the asymmetric potential $b = 0.001$ and height of the asymmetric potential $w = 1000$. A) Periodic asymmetric rectangular-triangular potential. B) Periodic asymmetric triangular-rectangular potential.

Section S4 The group speed

$$\begin{aligned}
\cos(kT) &= f(E)\sin(\alpha(E)a) + g(E)\cos(\alpha(E)a) \\
-T\sin(kT) &= \frac{df(E)}{dE}\frac{dE}{dk}\sin(\alpha(E)a) + af(E)\frac{d\alpha(E)}{dE}\frac{dE}{dk}\cos(\alpha(E)a) \\
&\quad + \frac{dg(E)}{dE}\frac{dE}{dk}\cos(\alpha(E)a) - ag(E)\frac{d\alpha(E)}{dE}\frac{dE}{dk}\sin(\alpha(E)a) \\
-T\sin(kT) &= \left(\frac{df(E)}{dE} - ag(E)\frac{d\alpha(E)}{dE}\right)\frac{dE}{dk}\sin(\alpha(E)a) + \left(\frac{dg(E)}{dE} + af(E)\frac{d\alpha(E)}{dE}\right)\frac{dE}{dk}\cos(\alpha(E)a) \\
T\sin(kT) &= \left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\frac{dE}{dk}\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\frac{dE}{dk}\cos(\alpha(E)a) \\
T\sin(kT) &= \left[\left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\cos(\alpha(E)a)\right]\frac{dE}{dk} \\
\frac{dE}{dk} &= T\sin(kT)\left[\left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\cos(\alpha(E)a)\right]^{-1} \\
v_G &= \frac{T\sin(kT)}{\hbar}\left[\left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\cos(\alpha(E)a)\right]^{-1} \\
v_G &= \frac{T\sin(kT)}{\hbar H_0(E)}
\end{aligned}$$

Where

$$H_0(E) = \left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\cos(\alpha(E)a)$$

Numerical simulation of the group speed of the asymmetric periodic potentials and the triangular periodic potential.

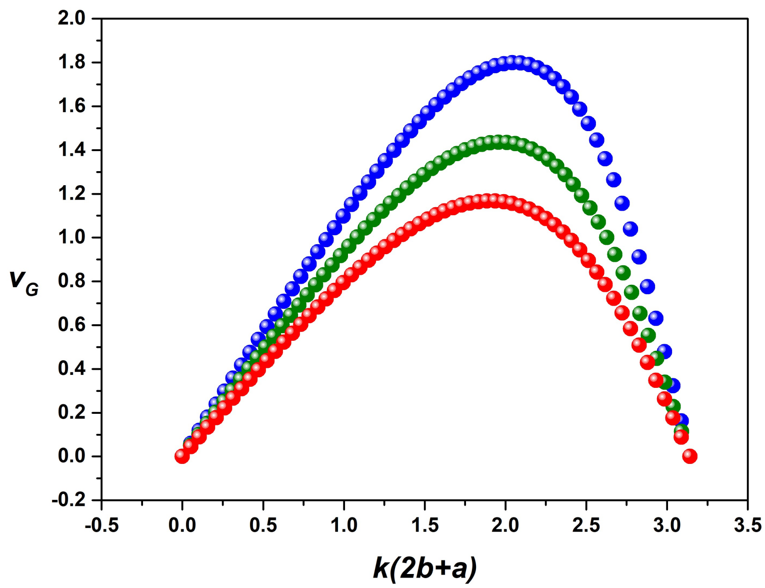

Figure S3: Group speed with width $b = 0.25$, spacing between potentials $a = 1$ and height $w = 10$. Blue color for the periodic potential asymmetrical rectangular - triangular, green color for the periodic triangular potential and red color for the periodic potential asymetrical triangular-rectangular.

Section S5 The effective mass

$$
\begin{aligned}
T\sin(kT) &= \left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\frac{dE}{dk}\sin(\alpha(E)a)-\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\frac{dE}{dk}\cos(\alpha(E)a)\\
T^2\cos(kT) &= \left[\frac{dE}{dk}\left(ag(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{dg(E)}{dE}\frac{d\alpha(E)}{dE}-\frac{d^2f(E)}{dE^2}\right)\right]\frac{dE}{dk}\sin(\alpha(E)a)\\
&+\left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\left[a\frac{d\alpha(E)}{dE}\left(\frac{dE}{dk}\right)^2\cos(\alpha(E)a)+\frac{d^2E}{dk^2}\sin(\alpha(E)a)\right]\\
&-\left[\frac{dE}{dk}\left(af(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{df(E)}{dE}\frac{d\alpha(E)}{dE}+\frac{d^2g(E)}{dE^2}\right)\right]\frac{dE}{dk}\cos(\alpha(E)a)\\
&-\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\left[-a\frac{d\alpha(E)}{dE}\left(\frac{dE}{dk}\right)^2\sin(\alpha(E)a)+\frac{d^2E}{dk^2}\cos(\alpha(E)a)\right]\\
T^2\cos(kT) &= \left(\frac{dE}{dk}\right)^2\left(ag(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{dg(E)}{dE}\frac{d\alpha(E)}{dE}-\frac{d^2f(E)}{dE^2}\right)\sin(\alpha(E)a)\\
&+\left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\left[a\frac{d\alpha(E)}{dE}\left(\frac{dE}{dk}\right)^2\cos(\alpha(E)a)+\frac{d^2E}{dk^2}\sin(\alpha(E)a)\right]\\
&-\left(\frac{dE}{dk}\right)^2\left(af(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{df(E)}{dE}\frac{d\alpha(E)}{dE}+\frac{d^2g(E)}{dE^2}\right)\cos(\alpha(E)a)\\
&-\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\left[-a\frac{d\alpha(E)}{dE}\left(\frac{dE}{dk}\right)^2\sin(\alpha(E)a)+\frac{d^2E}{dk^2}\cos(\alpha(E)a)\right]\\
T^2\cos(kT) &= \left(\frac{dE}{dk}\right)^2\left[ag(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{dg(E)}{dE}\frac{d\alpha(E)}{dE}-\frac{d^2f(E)}{dE^2}\right]\sin(\alpha(E)a)\\
&-\left(\frac{dE}{dk}\right)^2\left[af(E)\frac{d^2\alpha(E)}{dE^2}+a\frac{df(E)}{dE}\frac{d\alpha(E)}{dE}+\frac{d^2g(E)}{dE^2}\right]\cos(\alpha(E)a)\\
&+a\frac{d\alpha(E)}{dE}\left(\frac{dE}{dk}\right)^2\left[\left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\cos(\alpha(E)a)+\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\sin(\alpha(E)a)\right]\\
&+\frac{d^2E}{dk^2}\left[\left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\sin(\alpha(E)a)-\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\cos(\alpha(E)a)\right]\\
T^2\cos(kT) &= \left(\frac{dE}{dk}\right)^2\left[ag(E)\frac{d^2\alpha(E)}{dE^2}+2a\frac{dg(E)}{dE}\frac{d\alpha(E)}{dE}-\frac{d^2f(E)}{dE^2}+a^2f(E)\left(\frac{d\alpha(E)}{dE}\right)^2\right]\sin(\alpha(E)a)\\
&-\left(\frac{dE}{dk}\right)^2\left[af(E)\frac{d^2\alpha(E)}{dE^2}+2a\frac{df(E)}{dE}\frac{d\alpha(E)}{dE}+\frac{d^2g(E)}{dE^2}-a^2g(E)\left(\frac{d\alpha(E)}{dE}\right)^2\right]\cos(\alpha(E)a)\\
&+\frac{d^2E}{dk^2}\left[\left(ag(E)\frac{d\alpha(E)}{dE}-\frac{df(E)}{dE}\right)\sin(\alpha(E)a)-\left(af(E)\frac{d\alpha(E)}{dE}+\frac{dg(E)}{dE}\right)\cos(\alpha(E)a)\right]\\
\frac{d^2E}{dk^2} &= \frac{T^2\cos(kT)-\hbar^2v_G^2H_1(E)}{H_2(E)}
\end{aligned}
$$

The effective mass is given by the following equation.

$$
\begin{aligned}
m^* &= \hbar^2 \left(\frac{d^2E}{dk^2}\right)^{-1} \\
m^* &= \hbar^2 \left(\frac{T^2\cos(kT) - \hbar^2 v_G^2 H_1(E)}{H_2(E)}\right)^{-1} \\
m^* &= \frac{\hbar^2 H_2(E)}{T^2\cos(kT) - \hbar^2 v_G^2 H_1(E)}
\end{aligned}
$$

Where:

$$
\begin{aligned}
H_1(E) = &\left[ag(E)\frac{d^2\alpha(E)}{dE^2} + 2a\frac{dg(E)}{dE}\frac{d\alpha(E)}{dE} - \frac{d^2f(E)}{dE^2} + a^2f(E)\left(\frac{d\alpha(E)}{dE}\right)^2\right]\sin(\alpha(E)a) \\
&- \left[af(E)\frac{d^2\alpha(E)}{dE^2} + 2a\frac{df(E)}{dE}\frac{d\alpha(E)}{dE} + \frac{d^2g(E)}{dE^2} - a^2g(E)\left(\frac{d\alpha(E)}{dE}\right)^2\right]\cos(\alpha(E)a)
\end{aligned}
$$

$$
H_2(E) = \left(ag(E)\frac{d\alpha(E)}{dE} - \frac{df(E)}{dE}\right)\sin(\alpha(E)a) - \left(af(E)\frac{d\alpha(E)}{dE} + \frac{dg(E)}{dE}\right)\cos(\alpha(E)a)
$$

Numerical simulation of the effective mass of the asymmetric periodic potentials and the triangular periodic potential.

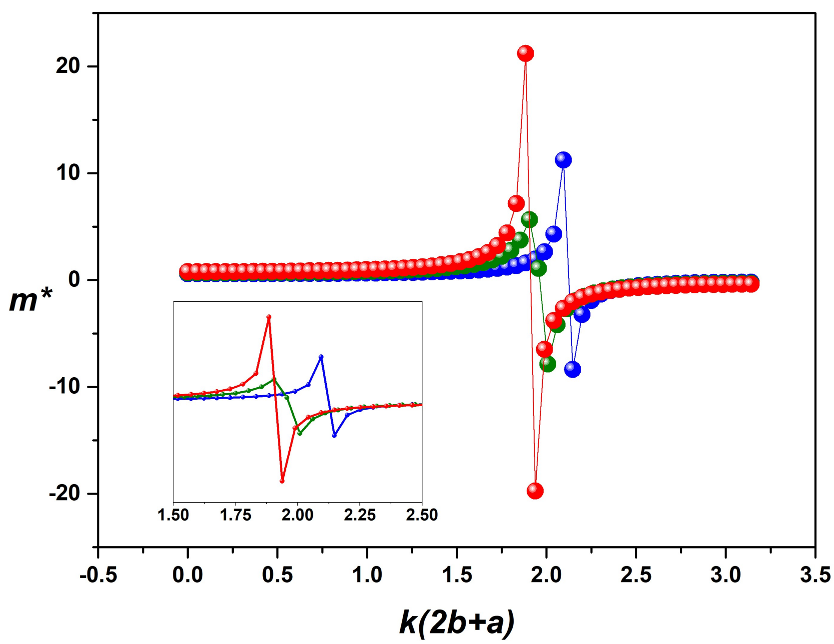

Figure S4: Effective mass with width $b = 0.25$, spacing between potentials $a = 1$ and height $w = 10$. Blue color blue for the periodic potential asymmetrical rectangular - triangular, green color for the periodic triangular potential and red color for the periodic potential asymetrical triangular-rectangular.

Calculations of the energy functions $H_1(E)$ and $H_2(E)$ for the effective mass.

Section S5.1 Dirac delta potential

$$H_1(E) = \frac{m^2}{a(2mE)^{3/2}}\left(\frac{a^2P}{\hbar} - \frac{a^2}{\hbar} - \frac{3P\hbar}{2mE}\right)\sin\left(\frac{\sqrt{2mE}a}{\hbar}\right) + \left(\frac{3P}{4E^2} + \frac{a^2m}{2\hbar^2E}\right)\cos\left(\frac{\sqrt{2mE}a}{\hbar}\right)$$

$$H_2(E) = \left(\frac{am}{\hbar\sqrt{2mE}} + \frac{P\hbar m}{a(2mE)^{3/2}}\right)\sin\left(\frac{\sqrt{2mE}a}{\hbar}\right) - \left(\frac{P}{2E}\right)\cos\left(\frac{\sqrt{2mE}a}{\hbar}\right)$$

Section S5.2 Rectangular potential

$$\begin{aligned}
H_1(E) = &-\frac{am^2}{\hbar(2mE)^{3/2}}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right) - \frac{2abm}{\hbar^2\sqrt{E(w-E)}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&- \frac{3w^2(w-2E)}{8(E(w-E))^{5/2}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&- \frac{2w^2bm^2}{\hbar(w-E)^2(2mE)^{3/2}}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&- \frac{bm(2E^2-wE+w^2)}{2\hbar E(w-E)\sqrt{2mE}}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&- \frac{b^2m(w-2E)}{\hbar^2\sqrt{E(w-E)}(w-E)}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{a^2m(w-2E)}{4\hbar^2E\sqrt{E(w-E)}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{am^2(w-2E)}{2\hbar(2mE)^{3/2}\sqrt{E(w-E)}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{amw^2}{2\hbar\sqrt{2mE}(E(w-E))^{3/2}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{abm(w-2E)}{\hbar^2E(w-E)}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right) - \frac{2b^2}{\hbar^2(w-E)}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{2bm^2}{\hbar(2m(w-E))^{3/2}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right) - \frac{a^2m}{2\hbar^2E}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)
\end{aligned}$$

$$\begin{aligned}
H_2(E) = &\frac{ma}{\hbar\sqrt{2mE}}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right) + \frac{w^2}{4(E(w-E))^{3/2}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&+ \frac{\sqrt{m}b(w-2E)}{\hbar\sqrt{2E}(w-E)}\cosh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\sin\left(\frac{\sqrt{2mE}}{\hbar}\right) + \frac{a\sqrt{m}(w-2E)}{2E\hbar\sqrt{2(w-E)}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)\\
&- \frac{\sqrt{2m}}{\hbar\sqrt{w-E}}\sinh\left(\frac{2\sqrt{2m(w-E)}b}{\hbar}\right)\cos\left(\frac{\sqrt{2mE}}{\hbar}\right)
\end{aligned}$$

Section S6 Supplementary tables

Section S6.1 Dirac delta potential

Group Speed	P	ka	v_G	Effective Mass	P	ka	E
First Energy Band	1	2.57	4.46	First Energy Band	1	2.55	7.89
	5	2.00	2.37		5	2.03	8.11
	10	1.85	1.49		10	1.83	8.58
	25	1.70	0.70		25	1.67	9.20
	50	1.65	0.37		50	1.62	9.51
	100	1.60	0.19		100	1.62	9.69
	200	1.60	0.10		200	1.57	9.77
	500	1.60	0.34		500	1.57	9.83
Second Band Energy	1	0.52	-10.9	Second Band Energy	1	0.49	35.2
	5	1.13	-7.83		5	1.11	33.3
	10	1.29	-5.50		10	1.31	34.6
	25	1.44	-2.75		25	1.47	36.8
	50	1.49	-1.48		50	1.52	38.0
	100	1.55	-0.77		100	1.52	38.7
	200	1.55	-0.39		200	1.57	39.0
	500	1.55	-0.16		500	1.57	39.3

Table S1: Values of ka, v_G, and E for the group speed and effective mass of the Dirac delta potential. The value of $a = 1$.

Section S6.2 Rectangular potential

Group Speed	w	$k(2b+a)$	v_G	Effective Mass	w	$k(2b+a)$	E
First Energy Band	1	ND	ND	First Energy Band	1	ND	ND
	10	3.09	6.14		10	3.12	9.68
	50	3.04	5.98		50	3.06	9.44
	100	2.99	5.82		100	3.01	9.21
	500	2.73	5.05		500	2.75	8.33
	1000	2.58	4.44		1000	2.55	7.87
	5000	2.01	2.35		5000	2.03	8.08
	10000	1.85	1.48		10000	1.83	8.55
	50000	1.65	0.36		50000	1.62	9.48
Second Band Energy	1	ND	ND	Second Band Energy	1	ND	ND
	10	ND	ND		10	ND	ND
	50	0.10	-12.3		50	0.07	38.4
	100	0.10	-12.2		100	0.12	37.9
	500	0.31	-11.5		500	0.33	36.1
	1000	0.52	-10.9		1000	0.49	35.1
	5000	1.13	-7.78		5000	1.11	33.2
	10000	1.29	-5.44		10000	1.31	34.4
	50000	1.49	-1.43		50000	1.52	37.9

Table S2: Values of $k(2b+a)$, v_G, and E for the group speed and effective mass of the rectangular potential. The values of $a = 1$ and $b = 0.001$. ND = Not Defined

Section S6.3 Triangular potential, triangular-rectangular potential and rectangular-triangular potential

Group Speed	w	$k(2b+a)$	v_G	Effective Mass	w	$k(2b+a)$	E
Triangular	1	2.86	5.61	Triangular	1	2.57	6.76
	5	2.86	5.50		5	2.57	6.64
	10	2.86	5.42		10	2.57	6.57
	50	2.85	5.11		50	2.43	5.73
	100	2.85	4.88		100	2.43	5.63
	500	2.57	4.34		500	2.43	5.32
	1000	2.57	4.11		1000	2.43	5.12
	5000	2.57	3.35		5000	2.43	4.51
	10000	2.28	3.01		10000	2.43	4.22
	50000	2.28	2.18		50000	2.14	2.79
Triangular - Rectangular	1	ND	ND	Triangular - Rectangular	1	ND	ND
	5	ND	ND		5	ND	ND
	10	2.83	5.50		10	2.98	8.63
	50	2.83	5.33		50	2.98	8.44
	100	2.83	5.18		100	2.98	8.33
	500	2.51	4.37		500	2.67	6.88
	1000	2.51	3.95		1000	2.67	6.92
	5000	2.20	2.14		5000	2.04	5.67
	10000	1.88	1.34		10000	1.73	5.56
	50000	1.57	0.26		50000	1.73	5.59
Rectangular - Triangular	1	ND	ND	Rectangular - Triangular	1	ND	ND
	5	ND	ND		5	ND	ND
	10	2.83	5.58		10	1.57	4.83
	50	2.83	5.44		50	2.20	5.55
	100	2.83	5.33		100	2.20	5.19
	500	2.83	4.67		500	2.36	5.93
	1000	2.51	4.37		1000	2.36	6.16
	5000	2.20	2.65		5000	2.04	6.29
	10000	1.88	1.71		10000	2.04	6.85
	50000	1.57	0.37		50000	1.73	6.83

Table S3: Values of $k(2b+a)$, v_G, and E for the group speed and effective mass of the triangular potential, triangular-rectangular potential and rectangular-triangular potential in the first band energy. The values of $a = 1$ and $b = 0.001$. ND = Not Defined.