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A Model of Competing Gangs in Networks

Alexis Poindron [1,*] **and Nizar Allouch** [2]

[1] Unité d'Économie Appliquée, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France
[2] School of Economics, University of Kent, Canterbury CT2 7NZ, UK; n.allouch@kent.ac.uk
* Correspondence: alexis.poindron.1991@gmail.com

Abstract: Two groups produce a network good perceived by a third party, such as a police or military institution, as a 'public bad', referred to as 'crime' for simplicity. These two groups, considered mafias, are assumed to be antagonists, whether they are enemies or competitors in the same market, causing harm to each other's activities. This paper provides guidelines for the policymaker, typically the police, seeking to minimize overall crime levels by internalizing these negative externalities. One specific question is investigated: the allocation of resources for the police. In general, we recommend a balanced crackdown on both antagonists, but an imbalance in group sizes may lead the police to focus on the more criminal group.

Keywords: networks; crime; strategic complementarity; strategic substitutability

JEL Classification: D19; D74; D79

1. Introduction

The impact of networks on agents' decisions has been a topic of ongoing interest across various fields, ranging from sociology to economics and game theory. The pioneering work by Ballester et al. [1], presenting a model incorporating positive and negative externalities, has sparked considerable attention regarding the role of network influences in amplifying or inhibiting agents' efforts. Along with the properties of the utility functions chosen by the authors, which make the model particularly relevant for crime settings, this combination of externalities of both types invites us to consider groups of criminals. Other noteworthy contributions, including perspectives on delinquent behavior or social norms, are found in works by Calvó-Armengol and Zenou [2], Cao et al. [3], Calvó-Armengol et al. [4], Ballester et al. [5], Ushchev and Zenou [6]. The current paper explores a specific issue related to the coexistence of positive and negative externalities within a network, specifically when society is polarized. We apply the classical structure of Harary [7] to capture society's separation into two rival mafias, with externality being positive within groups and negative across groups.

1.1. Motivation

The initial model of Ballester et al. [1] allowed for perfectly heterogeneous network influences. In the context of a balanced network (the directed graphs version of Harary [7]), this heterogeneity collapses into two groups, within which externalities are positive, and across which externalities are negative.

In the context of delinquent networks, the issue indeed revolves around competition between gangs. While both gangs contribute to crime within their groups, they may also engage in inter-gang conflicts, such as competing for the control of the drug market and impeding on each other's activities, thereby diminishing their overall criminal activities. From society's perspective, only aggregate crime level matters, and one may wonder how civil society, as proxied by the police, could instrumentalize the conflict, possibly by brokering deals with gangs for social peace.

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Throughout the paper, we use the term 'crime' for simplicity and because the setting is relevant enough to deserve particular interest. However, our results could accommodate a broader range of similar strategic interactions: a military opposition between two republics within a federation, cyberattacks between rival hackers, or any strategic interaction that hampers the rivals' activities. We may consider the collaboration between two countries in military, scientific, energy, or diplomatic ventures perceived as a threat to a third country. This third country might contemplate opposing both countries to thwart their activities—whether on a diplomatic or military basis, adopting a 'realpolitik' stance, reaching an agreement with one to oppose the other, or sparking a conflict against only one.

Returning to criminal activity, and by investigating the allocation of resources in the government's fight against gangs, our goal is to elucidate the contradicting effects at play to reach a society's crime level as low as possible. The particular setting of a three-player game—two mafias and the police—establishes the context at a high level of aggregation and allows us to address, in place of the police, the problem of internalizing negative externalities, thereby reducing the overall level of crime. This question will be examined from the perspective of resource allocations, with the police choosing which group to direct its action against.

As compared with Ballester et al. [1], whose main contribution is to identify and remove the 'key player', i.e., the one with the highest inter-centrality (which is not necessarily the one exerting the highest effort), our approach differs and operates at a lower level of granularity. More specifically, in Ballester et al. [1], there is no cost analysis for the police intervention, even though targeting the key player may prove costly since, for example, reconstituting a delinquent network is a real possibility, especially in prisons. Our approach aggregates the activities of agents to encompass a group decision-making aspect and proposing guidelines based on the group's autarkic activities for a third party with the power to impact the game. When gangs are of equal size, we demonstrate that the optimal strategy for the third party is to equally share the resources invested in cracking down on crime against the two groups. However, under the hypothesis of our model, we also show that these results are sensitive to the inequality in the activity/size of the groups and that unequal sizes may turn into exclusive attention against the group of the highest activity, referred to as "the strongest group" (or the "most criminal"), as measured by its autarkic activity (the strongest group is therefore not necessarily the one with the highest cardinality).

Finding the most adequate intervention will surely depend on the type of network and the groups under investigation, but our aggregate approach provides more flexibility by favoring a more diffuse intervention. Even though the removal of the key player could be extended to key groups, as done in Temurshoev [8], and even when computational issues can be successfully addressed in practice, mistakes could have severe implications in terms of resource utilization. Also, bounded information and rationality issues exert severe limitations on the key player approach. On the contrary, since our analysis only needs to know the aggregate crime levels of gangs, it is situated at a coarser granularity: in the presence of limited information about the exact architecture of the network, the police only needs to know the aggregate level of crime, not the exact interactions of criminals.

1.2. Related Work

Generally speaking, the broad notion of network influence designates structured interactions. In some circumstances, the word 'influence' may refer to information considerations, as seen in the context of rumors [9], votes [10–13], diffusion [14–16], opinion formation [17–23], status [24], homophily [25], learning [26], cultural traits [27], epidemiological tensions [28], collective games [29], or information extraction [30–32], not forgetting the anthropological studies on mimikry behavior by Girard [33,34,35,36], where imitation departs from rational decisions but now emerges either from the attribution of prestige perceived by the imitator in a model or from contagion in a society under crisis searching for a scapegoat.

In Ballester et al. [1], where effort equates with crime and externalities with influences, positive and negative influences receive different interpretations. A positive influence exacerbates crime, while a negative one, a notion related to but distinct from 'anti-conformism' [37–39] or anti-coordination [40–42], plays down on agents' level of action. We can think of a 'first-mover advantage' over meeting a demand, e.g., in a drug market. This flexible model, which we will rely on to investigate the particular problem of competing gangs, encapsulates various situations under standard assumptions.

The Ballester et al. [1] model belongs to the family of models with continuous actions/opinions and static decisions: there is no 'repeated game' dimension, as we could find, e.g., in opinion formation models [17,22,43–45], mainly focusing their interest on opinion reversal and diffusion [25,46,47], including in development economics [16]. Its main technical difficulty is the nonnegativity of actions, raising the delicate question of the interiority of equilibrium and the actual network of active players (see, for example, Bramoullé and Kranton [48] for a standard model on a similar framework).

Among the other workhorse models of this literature, the key paper of Bramoullé et al. [49] also investigates public goods in networks in the vein of Ballester et al. [1] and Ballester and Zenou [50], where the network good is actually a kind of 'public bad'. The problem had already been investigated in the context of groups by Buchholz et al. [51]. In an economy, one group perceives the action as good and the other as bad. Other frameworks have also been proposed, e.g., Cabrales et al. [52], encompassing network formation and the productive side in a single spillover model.

Though the key player does not play a role in our model, the aggregation of individual activities remains dependent on the network's topology, and this topology is typically a sociological dimension of the problem. Therefore, we have to mention the theory of social power, where the position in the network is central; see, for example, Friedkin [53], de Swart and Rusinowska [54], van den Brink and Steffen [55]. In contrast with models like that of Acemoglu et al. [56], where leaders are specifically designed, leadership may also emerge from centrality, which ties the notion of power to topological considerations [53,57–60]. In the literature on social and economic networks, the role of an agent's position in the network in the outcome of a game is a standard field of investigation; we refer to Jackson [61], Bramoullé et al. [62] for two authoritative reviews of theoretical models on social and economic networks.

1.3. Organization of the Paper

The paper is structured as follows. The outlines of the Ballester et al. [1] model and the specific setting we will be studying are exposed in Section 2. Policy recommendations for the police are discussed in Section 3. Section 4 concludes with the limitations of our model and some perspectives.

2. The Model

Our model adapts Ballester et al. [1,5] in a specific setting involving two rival groups. We consider crime networks as a practical application of the model, given that the properties of the utility functions align well with the impact of agents' actions (not only utilities) based on whether these actions originate from allies or enemies, as we will see below. A notable departure from the concerns addressed by Ballester et al. [1] is that we focus on examining aggregate levels within groups rather than targeting a key player. In particular, the significance of the network's architecture is only relevant in terms of its consequences on an aggregate scale.

We denote by N the set of n agents (criminals) in a network. Agents are located on an undirected and unweighted network described by the adjacency matrix G. If agent $j \neq i$ is a *neighbor* of i then $g_{ij} = 1$, otherwise $g_{ij} = 0$. The set of neighbors of i is denoted by $\mathcal{N}_i$, i.e., $\mathcal{N}_i := \{j \mid g_{ij} \neq 0\}$. Each agent i exerts a level of effort $x_i \geq 0$ and obtains utility $u_i(x_i, \mathbf{x}_{-i})$, where $\mathbf{x}_{-i}$ is the vector of efforts exerted by other agents (we use bold lowercase letters for vectors).

The utility functions are taken quadratic:

$$u_i(x_i, \mathbf{x}_{-i}) = x_i - \frac{1}{2}x_i^2 + \sum_{j \in \mathcal{N}_i} \beta_{ij} x_i x_j, \tag{1}$$

where β_{ij} denotes the nature and intensity of the externality between agents i and j.

In interpretation, when $\beta_{ij} > 0$ efforts are strategic complements and when $\beta_{ij} < 0$ efforts are strategic substitutes. This observation, specific to the family of bilinear utility functions, is the one that justifies their use to model delinquent networks. Individual effort is then identified with crime. A positive network influence $\beta_{ij} > 0$ can be thought of as an incentive to commit crime (commonly known as 'bad company') or, e.g., as an exacerbation of violence between criminals. On the contrary, $\beta_{ij} < 0$ can be thought of as the control of a drug market, where some criminals' actions impede others' actions.

An agent i seeks to maximize their payoffs and has a best-response function:

$$x_i = f_i(\mathbf{x}_{-i}) \stackrel{\text{def}}{=} \max\left(0,\ 1 + \sum_{j \in \mathcal{N}_i} \beta_{ij} x_j\right). \tag{2}$$

At a Nash equilibrium $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ of the game, each agent's action is a best-response to their neighbors' actions, that is, $x_i^* = f_i(\mathbf{x}_{-i}^*)$ for each agent $i \in N$.

We now apply the Ballester et al. [1] model to our two competing groups model. The society is partitioned into two groups, A and B, of sizes n_A and n_B. To simplify the computations:

- We restrict our analysis to two concurrent groups, i.e., that exert negative externalities on each other (communitarian model with two groups).
- All externalities are of the same intensity within group $\delta \geq 0$ and inter-group $-\mu \leq 0$.
- We consider the 'full inter-connection case', where any two agents of different groups are linked. One interpretation of the full interconnection case is that inter-group confrontations are uniform in the sense that the aggregate crime production of the opposite group hurts each agent. Another interpretation is probabilistic: μ represents the probability of facing each agent of the opposite group.

From the assumptions above (in particular, since for $i \in A$, $B \cap \mathcal{N}_i = B$, by the full inter-connection case), we can rewrite (1) as follows. If $i \in A$, then:

$$u_i(x_i, \mathbf{x}_{-i}) = \left[x_i - \frac{1}{2}x_i^2 + \delta x_i \sum_{j \in A \cap \mathcal{N}_i} x_j - \mu x_i \sum_{j \in B} x_j \right]. \tag{3}$$

Similarly, if $i \in B$, then:

$$u_i(x_i, \mathbf{x}_{-i}) = \left[x_i - \frac{1}{2}x_i^2 + \delta x_i \sum_{j \in B \cap \mathcal{N}_i} x_j - \mu x_i \sum_{j \in A} x_j \right]. \tag{4}$$

Let $\mathbf{\Gamma}_{[AA]} = \mathbf{I} - \delta \mathbf{G}_A$ where $\mathbf{G}_A$ denotes the adjacency matrix of the network of interactions within group A and lets $\mathbf{\Gamma}_{[BB]} = \mathbf{I} - \delta \mathbf{G}_B$ where $\mathbf{G}_B$ denotes the adjacency matrix of the network of interactions within group B. Let us write

$$\mathbf{\Gamma} = \left[\begin{array}{c|c} \mathbf{\Gamma}_{[AA]} & \mathbf{\Gamma}_{[AB]} \\ \hline \mathbf{\Gamma}_{[BA]} & \mathbf{\Gamma}_{[BB]} \end{array} \right],$$

where $\mathbf{\Gamma}_{[AB]}$ and $\mathbf{\Gamma}_{[BA]}$ denote the links connecting A to B and B to A. Since we assume the 'full inter-connection case', it holds that all the entries of $\mathbf{\Gamma}_{[AB]}$ and $\mathbf{\Gamma}_{[BA]}$ are μ. In our model, a Nash equilibrium is interior when all agents in the network exert a strictly positive level of effort (crime).

Property 1. *When it exists, the interior Nash equilibrium* $\mathbf{x}^*$ *verifies:*

$$\mathbf{\Gamma x}^* = \mathbf{1},$$

where $\mathbf{1}$ *is the vector whose all coordinates are* 1.

Proof. Indeed, at an interior Nash equilibrium $\mathbf{x}^*$, it follows from the best response functions that for an agent $i \in A$ it holds that :

$$x_i^* = 1 + \delta \sum_{j \in A \cap \mathcal{N}_i} x_j^* - \mu \sum_{j \in B} x^*{}_j. \tag{5}$$

and for an agent $i \in B$, it holds that:

$$x_i^* = 1 + \delta \sum_{j \in B \cap \mathcal{N}_i} x_j^* - \mu \sum_{j \in A} x_j^*. \tag{6}$$

By re-arranging (5) and (6) we obtain

$$\mathbf{\Gamma x}^* = \mathbf{1}.$$

□

Let us define $\mathbf{x}_A := \mathbf{\Gamma}_{[AA]}^{-1}\mathbf{1}$ and $\mathbf{x}_B := \mathbf{\Gamma}_{[BB]}^{-1}\mathbf{1}$. We also consider $x_A := \mathbf{1}^T \cdot \mathbf{x}_A$ and $x_B := \mathbf{1}^T \cdot \mathbf{x}_B$, which represent the autarkic crime levels in groups A and B, respectively (note that the expression $\mathbf{1}^T \cdot \mathbf{y}$ denotes the sum of the entries in the vector $\mathbf{y}$). The next result provides a closed-form expression of the production of crime $\mathbf{x}^*$, as a function of the autarkic productions of crime $\mathbf{x}_A$ and $\mathbf{x}_B$, namely, the crime levels produced if each of the two groups was alone, or if they were not impeding on each other's actions ($\mu = 0$). In the sequel, we will also write $x^* := \mathbf{1}^T \cdot \mathbf{x}^*$ the total quantity of crime, or *(total) crime level*, defined as the sum of all individual contributions (both groups together). x^* is arguably the only relevant crime index from the police perspective, aiming to achieve the highest possible level of societal safety.

Theorem 1 is based on two assumptions: first, that negative externalities are not excessively high, and second, the positive-definiteness of the matrix of interactions $\mathbf{\Gamma}$, that ensures a unique Nash equilibrium.

Theorem 1. *Assume that* $\mu \leq \frac{1}{\max(x_A, x_B)}$ *and* $\mathbf{\Gamma}$ *is positive-definite. Then, there exists a unique Nash equilibrium, which is interior, and which has an expression as follows:*

$$\mathbf{x}^* = \frac{1}{1 - \mu^2 x_A x_B} \cdot \begin{pmatrix} (1 - \mu x_B) \cdot \mathbf{x}_A \\ (1 - \mu x_A) \cdot \mathbf{x}_B \end{pmatrix} =: \begin{pmatrix} \mathbf{x}_A^* \\ \mathbf{x}_B^* \end{pmatrix}.$$

Proof. The proof of Theorem 1, together with all the subsequent proofs, appears in the Appendix A. □

We will now provide a sufficient condition on the interaction network that ensures that $\mathbf{\Gamma}$ is positive definite, as stipulated in Theorem 1. Given a matrix $\mathbf{G}$, let $\lambda_{\max}(\mathbf{G})$ denote its largest eigenvalue and $\lambda_{\min}(\mathbf{G})$ denote its lowest eigenvalue.

Proposition 1. *Assume that* $1 - \delta \max(\lambda_{\max}(\mathbf{G}_A), \lambda_{\max}(\mathbf{G}_B)) - \mu\sqrt{n_A n_B} > 0$. *Then,* $\mathbf{\Gamma}$ *is positive definite.*

It follows from Theorem 1 that when, for example, $x_A < x_B$, Group A (the group with the smaller autarkic crime level) is impacted by a higher reduction factor than Group B.

We are now in a position to express, in Proposition 2, the crime level x^* as a function of each group's autarkic activities, along with a few of its properties.

Proposition 2. *Under the assumptions and with the notations of Theorem 1:*

(i) *The crime level x^* is:*

$$x^* = \frac{(1-\mu x_B)x_A + (1-\mu x_A)x_B}{1-\mu^2 x_A x_B} = \frac{x_A + x_B - \frac{2}{\mu}}{1-\mu^2 x_A x_B} + \frac{2}{\mu}.$$

(For $\mu \neq 0$ in the second equality.)

(ii) $x^* \leq x_A + x_B$.

(iii) *Suppose that $x_A + x_B$ is fixed. Then, $x^* - (x_A + x_B)$ attains its maximum when $x_A = x_B$.*

Proposition 2 shows how much the links between the two groups reduces the total crime level. We also proved that, given a fixed sum of the autarkic levels of crime, the maximum total crime reduction is achieved when the two groups have identical autarkic crime levels (Proposition 2iii).

Example 1. *Let us investigate the case of regular networks within each group to push the computations forward. We assume that internal networks for the groups are regular of degrees r_A and r_B, respectively. Then, $\mathbf{x}_A$ and $\mathbf{x}_B$ can be expressed in terms of the degree of the regular network and the discount factor (see Allouch [63], Proposition 7): $\mathbf{x}_A = \frac{1}{1-\delta r_A}\mathbf{1}$, $\mathbf{x}_B = \frac{1}{1-\delta r_B}\mathbf{1}$, and therefore $x_A = \frac{n_A}{1-\delta r_A}$ and $x_B = \frac{n_B}{1-\delta r_B}$. We obtain:*

$$x^* = \frac{(1-\delta r_A - \mu n_A)n_B + (1-\delta r_B - \mu n_B)n_A}{(1-\delta r_A)(1-\delta r_B) - \mu^2 n_A n_B}.$$

If, for example, we have $n_A = n_B = 4$ and $r_A = r_B = 2$, then we obtain:

$$x_A = x_B = \frac{4}{1-2\delta} \quad \text{and} \quad x^* = \frac{8}{1-2\delta+4\mu}. \tag{7}$$

From (7), we see that policies decreasing the emulation factor δ (we can think, for example, of an "education approach" aimed at increasing the opportunity cost of engaging in terrorist activities[1]) unambiguously decreases the crime level. However, very interestingly, if the police had a choice between decreasing the emulation factor δ by ϵ or increasing the inter-group fight intensity μ by ϵ, the latter policy will outperform the former policy.

Let us end this example with one question: What is the effect of degrees of unbalance on the aggregate level of crime? Let us redo the computations with $n_A = n_B = n$ but with potentially different degrees r_A andr_B, such that $r_A + r_B = K$, where K is a constant (also even):

$$x^* = \frac{2n - \delta K - 2\mu n^2}{(1-\delta r_A)(1-\delta K + \delta r_A) - \mu^2 n^2}.$$

$$\frac{dx^*}{dr_A} = \frac{-(2n - \delta K - 2\mu n^2)(\delta^2 K - 2\delta^2 r_A)}{((1-\delta r_A)(1-\delta K + \delta r_A) - \mu^2 n^2)^2},$$

which is null if and only if $r_A = \frac{K}{2}$, that is, $r_A = r_B$. This extremum represents a minimum. The total quantity of crime is minimized when groups possess identical internal degrees. In interpretation, the equality of internal degrees results in similar crime levels for the two groups, exacerbating losses on both sides and ultimately leading to a reduced total quantity of crime.

Scenario 1. *Given our focus on the internalization of negative externalities, there is a compelling case for concern regarding the unification of the groups. It is crucial to assess whether these groups may share common interests and, specifically, to gauge the strength of the threat that they might abruptly merge into a single mafia. The more similar the groups, the more pronounced the threat becomes. From the perspective of law enforcement, what would be the consequences if, instead of opposing each other, the two groups were to unite? (In a sense, what is the incentive for 'making* *mischief'?) We calculate the total quantity of crime by replacing μ with $-\mu$. Consequently, the total quantity of crime in the unified case is as follows:*

$$y^* = \frac{(1+\mu x_B)x_A + (1+\mu x_A)x_B}{1-\mu^2 x_A x_B} = \frac{x_A + x_B + \frac{2}{\mu}}{1-\mu^2 x_A x_B} - \frac{2}{\mu}.$$

As a consequence, the premium of causing mischief is given by:

$$y^* - x^* = \frac{4\mu x_A x_B}{1-\mu^2 x_A x_B}.$$

It is noteworthy that, at a given total autarkic production of crime (for a given $x_A + x_B$), the more equal the groups are ($x_A \approx x_B$), the higher the fighting premium is from the perspective of the police. This observation aligns with the findings of Example 1.

3. Consequence for the Police: Focusing or Splitting the Resources?

In this section, our goal is to answer the question: Where should the police allocate its resources to minimize the quantity of crime?

We will explore two extreme cases. In Scenario 2, the effect on crime in both groups is assumed to be proportional to the effort of the police. This assumption reflects a situation where the police conducts random controls against drugs, implements reinforced patrols, or takes any action characterized by the absence of scale. In Scenario 3, the effect depends on the mafia size. This multiplicative assumption involves attributing an effect on crime proportional to the group's size being fought against. This aims to represent a situation where there are 'economies of scale' concerning the effort of the police, such as in scenarios involving infiltration or cybersecurity measures against a specific threat type, where the difficulty of dismantling a network does not depend much on its size. In reality, the impact of spending is likely to be a combination of these two extreme scenarios.

In both scenarios, the police is assumed to have a budget that it divides into two parts. Specifically, an amount $\epsilon \in [0,1]$ is allocated to combat Group A, and an amount $1-\epsilon \in [0,1]$ is allocated to combat Group B. It is essential to note that the quantitative effect, i.e., how efficient this crackdown on crime actually is, is not considered at a higher level of policy-making. No parliament responsible for budget approval contemplates the expected impact of cracking down on crime, which would need to be balanced with other types of spending for the public sector. Nevertheless, we present, in our simulations, the actual impact on crime levels of an optimal policy.

Scenario 2 (An impact on crime with proportional effects)**.** *As a first extreme assumption, the present scenario assumes the decrease in crime to be directly proportional to the resources, meaning that spending ϵ results in a decrease in crime of $L \cdot \epsilon$, for some constant L. For $x_A \neq x_B$, let us first treat the case $\mu = 0$. In the absence of police, the crime level is simply the sum of autarkic levels: $x^* = x_A + x_B$. Under police action, we have $x^* = (x_A - L\epsilon) + (x_B - L(1-\epsilon)) = x_A + x_B - L$, which does not depend on ϵ: all repartitions of resources are equivalent for the police in terms of decision making. However, the crime level depends linearly on L; as we will see in Figure 1b,c, the effect is more sophisticated when $\mu \neq 0$, which is the case we examine now. Applying the first equality of Proposition 2, if x^* is interior, then:*

$$x^* = \frac{(1-\mu(x_B - L(1-\epsilon)))(x_A - L\epsilon) + (1-\mu(x_A - L\epsilon))(x_B - L(1-\epsilon))}{1-\mu^2(x_A - L\epsilon)(x_B - L(1-\epsilon))}.$$

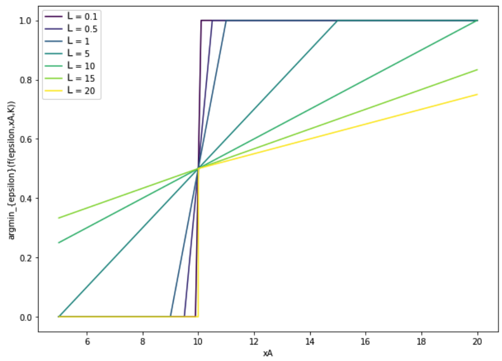
The derivative with respect to ϵ of the total amount of crime is:

$$\frac{dx^*}{d\epsilon} = \frac{L\mu((L - x_B - x_B)\mu + 2)(2L\epsilon - L + x_B - x_A)}{(1-\mu^2(x_A - L\epsilon)(x_B - L(1-\epsilon)))^2},$$

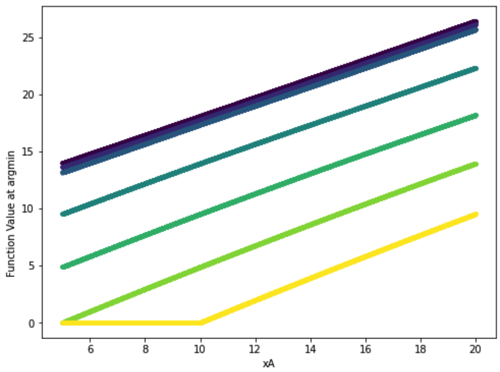
whose only root is:

$$\epsilon^* = \frac{1}{2} - \frac{\Delta}{2L}, \text{ where } \Delta = x_B - x_A.$$

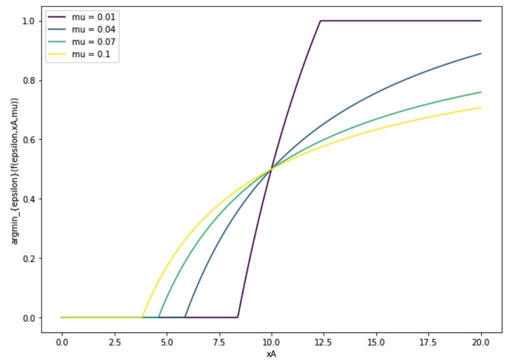
When Δ is sufficiently small, we obtain $\epsilon^* \in]0,1[$ (sharing the resources, i.e., interior solution). More precisely, the optimum is interior if and only if $\Delta \leq L$, that is, when group imbalance is not too high, or the police is sufficiently efficient. Otherwise, we obtain a corner solution: all resources are directed towards the fight against the strongest group, e.g., when $x_B > x_A$, against Group B ($\epsilon^* = 0$). The second-order conditions can be easily checked, as the numerator of the derivative is linear in ϵ. This result implies for the police that, as the groups' unbalance is growing, in order to reach a minimal level of total crime, it should direct relatively more resources against the strongest group. Remarkably, the parameter μ plays no role in the decision making (though it obviously impacts the crime level x^*). The $\arg\min$ *of $x^*(\epsilon)$ for different parameters of μ is displayed in Figure 1b,c. The requirement for the crime level to be positive translates into $x_A + x_B \geq L$, otherwise, the police should not spend all the available resources to bring the total level of crime to zero. This remark, as we will see, also holds in the next scenario (with the scale effects of spending on crime).*



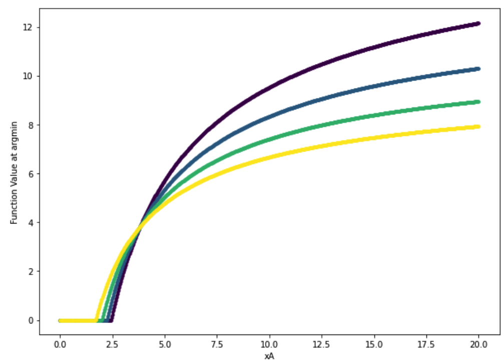
(**a**) Scenario 2. $\arg\min_\epsilon x^*(\epsilon, x_A, L)$



(**b**) Scenario 2. $x^*(\epsilon^*)$ with $\mu = 0.01$ and the no spoiling condition. Color code in (a).

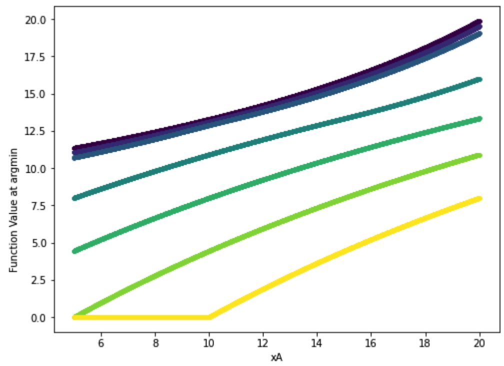


(**d**) Scenario 3 (scale effect). $\arg\min_\epsilon x^*(\epsilon, x_A, \mu)$



(**e**) Scenario 3 (scale effect). $x^*(\epsilon^*)$. Color code in (c).

Figure 1. *Cont.*



(**c**) Scenario 2. $x^*(\epsilon^*)$ with $\mu = 0.05$ and the no spoiling condition. Color code in (a).

Figure 1. $\arg\min_\epsilon x^*$ and x^* in Scenarios 2 and 3. In these simulations, we fix $x_B = 10$.

Scenario 3 (An impact on crime with economies of scale)**.** *The other extreme assumption is to set the decrease in crime as multiplicative, i.e., to be proportional to the resources spent against crime. We assume that by spending a proportion ϵ of its resources on Group A, the police brings the autarkic crime level x_A to $x_A - \epsilon x_A = (1-\epsilon)x_A$. Similarly, the autarkic level x_B becomes ϵx_B. The idea is to endow the police with a lump sum that exactly suffices to suppress each group separately, independently of their size, allocating the full resources. In a first approach, we may assume that $x_A = x_B$, a special case which has the advantage of being tractable. We consequently denote $x_A = x_B = x$. Applying Proposition 2(i):*

$$x^* = \frac{[1-\mu\epsilon x_B](1-\epsilon)x_A + [1-\mu(1-\epsilon)x_A]\epsilon x_B}{1-\mu^2\epsilon(1-\epsilon)x_A x_B} = \frac{x - \frac{2}{\mu}}{1-\mu^2\epsilon(1-\epsilon)x^2} + \frac{2}{\mu}.$$

(With $\mu \neq 0$ for the second equality.)

The minimum total level of crime x^* is obtained at $\epsilon = \frac{1}{2}$. The police should spread its resources equally between the two groups when they are of similar size, implying that there is no reason for the police to break the symmetry of the problem by favoring one group over the other (which would have been the case with a concave symmetrical function).

For $x_A \neq x_B$, let us first treat the case $\mu = 0$. In the absence of police, the total level of crime is the sum of the autarkic levels: $x^* = x_A + x_B$. Let us now investigate the case $x_A \neq x_B$. Also assume that $\mu \leq \frac{1}{\max(x_A, x_B)}$ (hypothesis of Theorem 1). One first remark is that, whatever the allocation that the police is choosing, even if not optimal, it should not reach a total level of crime higher than the autarkic level of the largest group. Indeed, assuming $x_A > x_B$:

$$\begin{aligned}
x_A - x^* &= x_A - \frac{[1-\mu\epsilon x_B](1-\epsilon)x_A + [1-\mu(1-\epsilon)x_A]\epsilon x_B}{1-\mu^2\epsilon(1-\epsilon)x_A x_B} \\
&= \frac{x_A(1-\mu^2\epsilon(1-\epsilon)x_A x_B) - [1-\mu\epsilon x_B](1-\epsilon)x_A - [1-\mu(1-\epsilon)x_A]\epsilon x_B}{1-\mu^2\epsilon(1-\epsilon)x_A x_B} \\
&= \frac{\epsilon(x_A - x_B) - \mu^2\epsilon(1-\epsilon)x_A^2 x_B + 2\mu\epsilon(1-\epsilon)x_A x_B}{1-\mu^2\epsilon(1-\epsilon)x_A x_B} \\
&= \frac{\epsilon[(x_A - x_B) + \mu\epsilon x_A x_B(2-\mu x_A)]}{1-\mu^2\epsilon(1-\epsilon)x_A x_B} \\
&\geq 0.
\end{aligned}$$

A fortiori, this inequality must hold at optimum. We can check this fact in Figure 1e (all curves below $x_B = 10$ for $x_A \leq 10$, and below the 45° line for $x_A \geq 10$). Clearly, this inequality did not hold in the last scenario, where the total level of crime, even at optimal resources allocation, did *not display a similar concave pattern. We are now searching for the optimal allocation of resources. We calculate:*

$$\frac{dx^*}{d\epsilon} = -\frac{(x_A x_B^2 - x_A^2 x_B)\mu^2\epsilon^2 + (2x_A^2 x_B \mu^2 - 4x_A x_B \mu)\epsilon - x_A^2 x_B \mu^2 + 2x_A x_B \mu - x_B + x_A}{(1 - x_A x_B \mu^2 (1-\epsilon)\epsilon)^2}. \tag{8}$$

The numerator is a binomial in ϵ, whose roots in $\mathbb{C}$ are:

$$\epsilon_{\pm} = \frac{x_A^2 x_B \mu - 2x_A x_B \pm \sqrt{x_A x_B}(x_A x_B \mu - x_A - x_B)}{(x_A x_B^2 - x_A^2 x_B)\mu}.$$

Imposing $x_B \geq x_A$, only ϵ_+ can possibly be a positive root, and then, the numerator being a binomial in ϵ, the second-order conditions can be easily checked: this value of ϵ would correspond indeed to a minimum. The case $\epsilon_+ > 1$ (and when therefore the police sets $\epsilon^* = 1$) is reached when μ is sufficiently small, more precisely, if and only if $\mu \leq \frac{2x_A x_B + x_A + x_B}{x_A^2 x_B + (x_A x_B)^{3/2}} =: M(x_A, x_B)$, for which the bound is found to be decreasing in x_B, meaning that for x_A fixed, the corner solution is reached for lower values of μ as the sizes of the groups become more unbalanced. Let us investigate when $\epsilon_+ \leq 1$. The $\arg\min$ of $x^*(\epsilon)$ for different parameters of μ is displayed in Figure 1d and the value of $f(\epsilon^*)$) is displayed in Figure 1e. Notice that the value x_A at which $f(\epsilon^*)$ reaches zero does not coincide with the reaching of a corner solution for $\arg\min_\epsilon x^*$ (it is smaller). Typically, even by directing all its resources against one of the two gangs (the strongest one), the other gang is not annihilated, except if it is weak enough.

One noticeable qualitative difference between the two scenarios is that the scale-free one (Scenario 2) displays symmetry around $x_A = x_B$ and not the multiplicative scenario (Scenario 3). Nonetheless, the similarity of optimal behaviors in both scenarios confirms the robustness of the model.

4. Conclusions

Now that the different effects at play have been clarified, the police have more cards in hand to assess the situation. We have demonstrated that the police should spread resources when the group sizes are similar, and direct relatively more resources towards the strongest group otherwise. Furthermore, we have shown that the more alike the groups are, the more the police gains from cracking down on crime. We end this paper with a clarification of our model's limitations and a list of perspectives.

4.1. Limitations of the Model

We identify at least three primary limitations of the model:

- While μ can be proxied and arguably estimated, the question arises: how would the police observe autarkic activities?
- Agents exerting complementary efforts, such as those belonging to the same gang, typically communicate closely. Thus, a more nuanced approach to the problem exists beyond simply targeting groups or key players. We would assume that connected criminals face correlated probabilities of being captured, not only because they could betray each other but simply because they are involved in the same criminal activities. Mathematically, if $\beta_{ij} > 0$ (indicating that agent j increases agent i criminal activity), then if j is caught, and i should face a higher probability of being caught as well.
- The model can be argued to be excessively quantitative. In contrast to Calvó-Armengol and Zenou [2], which proposes a model where criminals perceive more expected benefits in crime than in the job market, and where agents first decide whether to participate in the job market or the crime market and then determine the level of crime to engage in, our model lacks a qualitative dimension of delinquency. Exerting a strictly positive level of crime, even a small one, is still being involved in crime. Being active should significantly differ from not being active. The absence of a binary decision on criminal activity participation somehow bypasses the decision-making aspect of criminal behavior, the moral dimension of criminality and its sociological

implications, along with the intricate dynamics of crime and activity within mafias (including the snowball effect of illegal activities).

4.2. Perspectives

- A more in-depth exploration of the nature of crime is essential. For instance, consider focusing on the quantity of drugs purchased on the drug market. While the negative externality effectively portrays the substitution effect (where a drug consumer shifting from one gang diminishes the sales of another), it fails to adequately represent the heightened competition that should result in a more intense conflict between gangs. This negative externality lacks the depth to capture the potential escalation into a more ferocious fight. It prompts us to question the specific crime under investigation and whether we intend to disregard inter-gang murders in our model.
- Numerous traditional games on networks are static, including this one. Dynamic and endogenous games on agents' behavior and link formation align more closely with the actual challenges faced by the police in their fight against criminality.
- Finally, it is crucial to recall that, in real life, the network is not fixed. For instance, the war against terrorism is not only focused on eliminating terrorists but also involves shaping the perception of conflicts in the eyes of social groups with access to the media.

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Appendix A

Proof of Theorem 1. Since $\mathbf{\Gamma}$ is positive definite, it follows from standard results in the literature [49,67] that there exists a unique Nash equilibrium. One approach for the proof is to state the problem as a Linear Complementarity Problem (**LCP**) $L(-\mathbf{1}, \mathbf{\Gamma})$:

"Determine $\mathbf{x} \geq \mathbf{0}$ such that

$$\mathbf{\Gamma x} - \mathbf{1} \geq \mathbf{0} \quad \text{and} \quad (\mathbf{\Gamma x} - \mathbf{1})^T \mathbf{x} = 0."$$

This problem has a unique solution, which is the unique Nash equilibrium.

Now, we will show that the Nash equilibrium is interior if $\mu \leq \min(\frac{1}{x_A}, \frac{1}{x_B})$. Indeed, an interior Nash equilibrium needs to obey the first-order conditions:

For $i \in A$: $\dfrac{du_i(x_i)}{dx_i} = 0 \Leftrightarrow 1 - x_i + \delta \displaystyle\sum_{j \in \mathcal{N}_i \cap A} x_j - \mu \sum_{j \in B} x_j = 0.$

For $i \in B$: $\dfrac{du_i(x_i)}{dx_i} = 0 \Leftrightarrow 1 - x_i + \delta \displaystyle\sum_{j \in \mathcal{N}_i \cap B} x_j - \mu \sum_{j \in A} x_j = 0.$

Equivalently,

$$\mathbf{x}^* = \mathbf{\Gamma}^{-1} \cdot \mathbf{1} = \left[\begin{array}{c|c} \mathbf{\Gamma}_{[AA]} & \mathbf{\Gamma}_{[AB]} \\ \hline \mathbf{\Gamma}_{[BA]} & \mathbf{\Gamma}[BB] \end{array}\right]^{-1} \cdot \mathbf{1}.$$

We will apply the formula of the inverse of the partitioned matrix Horn and Johnson [68]. Indeed, since $\mathbf{\Gamma}[AA]$ and $\mathbf{\Gamma}_{[BB]}$ are also invertible, the inverse of $\mathbf{\Gamma}$ writes:

$$(\mathbf{\Gamma})^{-1} = \begin{pmatrix} (\mathbf{\Gamma}_{[AA]} - \mathbf{\Gamma}_{[AB]}\mathbf{\Gamma}_{[BB]}^{-1}\mathbf{\Gamma}_{[BA]})^{-1} & -(\mathbf{\Gamma}_{[AA]} - \mathbf{\Gamma}_{[AB]}\mathbf{\Gamma}_{[BB]}^{-1}\mathbf{\Gamma}_{[BA]})^{-1}\mathbf{\Gamma}_{[AB]}\mathbf{\Gamma}_{[BB]}^{-1} \\ -(\mathbf{\Gamma}_{[BB]} - \mathbf{\Gamma}_{[BA]}\mathbf{\Gamma}_{[AA]}^{-1}\mathbf{\Gamma}_{[AB]})^{-1}\mathbf{\Gamma}_{[BA]}\mathbf{\Gamma}_{[AA]}^{-1} & (\mathbf{\Gamma}_{[BB]} - \mathbf{\Gamma}_{[BA]}\mathbf{\Gamma}_{[AA]}^{-1}\mathbf{\Gamma}_{[AB]})^{-1} \end{pmatrix} \tag{A1}$$

We want to express $\mathbf{x}^*$ as a function of $\mathbf{x}_A$, $\mathbf{x}_B$, $x_A(= \mathbf{1}_A^T \cdot \mathbf{x}_A)$, $x_B(= \mathbf{1}_B^T \cdot \mathbf{x}_B)$. Recall that the non-diagonal terms of $\mathbf{\Gamma}_{[AA]}$ and $\mathbf{\Gamma}_{[BB]}$ are non-negative. We have:

$$\mathbf{\Gamma}_{[AB]} = \mu \mathbf{1}_A \mathbf{1}_B^T \text{ and } \mathbf{\Gamma}_{[BA]} = \mu \mathbf{1}_B \mathbf{1}_A^T$$

Hence, we have:

$$\mathbf{\Gamma}_{[AB]}(\mathbf{\Gamma}_{[BB]}^{-1})\mathbf{\Gamma}_{[BA]} = \mu^2 x_B \mathbf{1}_A \mathbf{1}_A^T,$$

And:

$$\mathbf{\Gamma}_{[BA]}(\mathbf{\Gamma}_{[AA]}^{-1})\mathbf{\Gamma}_{[AB]} = \mu^2 x_A \mathbf{1}_B \mathbf{1}_B^T.$$

Applying the Sherman-Morrison [68,69]:

$$(\mathbf{\Gamma}_{[AA]} + \mathbf{\Gamma}_{[AB]}\mathbf{\Gamma}_{[BB]}^{-1}\mathbf{\Gamma}_{[BA]})^{-1} = (\mathbf{\Gamma}_{[AA]} + \mu^2 x_B \mathbf{1}_A \mathbf{1}_A^T)^{-1} = \mathbf{\Gamma}_{[AA]}^{-1} + \frac{\mu^2 x_B}{1 - \mu^2 x_B \mathbf{1}_A^T \mathbf{\Gamma}_{[AA]}^{-1} \mathbf{1}_A} \mathbf{\Gamma}_{[AA]}^{-1} \mathbf{1}_A \mathbf{1}_A^T \mathbf{\Gamma}_{[AA]}^{-1}$$

$$= \mathbf{\Gamma}_{[AA]}^{-1} + \frac{\mu^2 x_B}{1 + \mu^2 x_B \mathbf{1}_A^T \mathbf{\Gamma}_{[AA]}^{-1} \mathbf{1}_A} \cdot \mathbf{x}_A \mathbf{1}_A^T \mathbf{\Gamma}_{[AA]}^{-1} = \mathbf{\Gamma}_{[AA]}^{-1} + \frac{\mu^2 x_B}{1 - \mu^2 x_A x_B} \cdot \mathbf{x}_A \mathbf{1}_A^T \mathbf{\Gamma}_{[AA]}^{-1}$$

and

$$(\mathbf{\Gamma}_{[BB]} - \mathbf{\Gamma}_{[BA]}\mathbf{\Gamma}_{[AA]}^{-1}\mathbf{\Gamma}_{[AB]})^{-1} = \mathbf{\Gamma}_{[BB]}^{-1} + \frac{\mu^2 x_A}{1 - \mu^2 x_A x_B} \cdot \mathbf{x}_B \mathbf{1}_B^T \mathbf{\Gamma}_{[BB]}^{-1}.$$

Therefore,

$$\mathbf{x}^* = \begin{pmatrix} (1 + \frac{\mu^2 x_A x_B}{1 - \mu^2 x_A x_B})(1 - \mu x_B) \cdot \mathbf{x}_A \\ (1 + \frac{\mu^2 x_A x_B}{1 - \mu^2 x_A x_B})(1 - \mu x_A) \cdot \mathbf{x}_B \end{pmatrix} = \frac{1}{1 - \mu^2 x_A x_B} \begin{pmatrix} (1 - \mu x_B) \cdot \mathbf{x}_A \\ (1 - \mu x_A) \cdot \mathbf{x}_B \end{pmatrix}.$$

□

Proof of Proposition 1. Let

$$\mathbf{D} = \left[\begin{array}{c|c} \mathbf{\Gamma}_{[AA]} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Gamma}_{[BB]} \end{array}\right].$$

From Weyl's inequality theorem [68,70], it holds that:

$$\lambda_{\min}(\mathbf{\Gamma}) \geq \lambda_{\min}(\mathbf{D}) + \lambda_{\min}(\mathbf{\Gamma} - \mathbf{D}).$$

Since $\mathbf{\Gamma}_{[AA]} = \mathbf{I} - \delta \mathbf{G}_A$, $\mathbf{\Gamma}_{[BB]} = \mathbf{I} - \delta \mathbf{G}_B$ and $\mathbf{D}$ is diagonal, it holds that:

$$\lambda_{\min}(\mathbf{D}) = 1 - \delta \max(\lambda_{\max}(\mathbf{G}_A), \lambda_{\max}(\mathbf{G}_B)).$$

Since $\frac{1}{\mu}(\mathbf{\Gamma} - \mathbf{D})$ is the adjacency matrix of a complete bipartite network, it holds that

$$\lambda_{\min}(\mathbf{\Gamma} - \mathbf{D}) = -\mu\sqrt{n_A n_B}.$$

Hence, $\mathbf{\Gamma}$ is positive definite since

$$\lambda_{\min}(\mathbf{\Gamma}) \geq 1 - \delta \max(\lambda_{\max}(\mathbf{G}_A), \lambda_{\max}(\mathbf{G}_B))) - \mu\sqrt{n_A n_B} > 0.\square$$

□

Proof of Proposition 2.

(i) Straightforward from Theorem 1.

(ii) $x^* < x_A + x_B \Leftrightarrow \frac{(1-\mu x_B)x_A + (1-\mu x_A)x_B}{1-\mu^2 x_A x_B} < x_A + x_B \Leftrightarrow (x_A + x_B) - 2\mu x_A x_B < (x_A + x_B) - \mu^2 x_A x_B(x_A + x_B) \Leftrightarrow \mu < \frac{2}{x_A + x_B} = \frac{2}{\frac{1}{1/x_A} + \frac{1}{1/x_B}} =: \mathcal{H}(\frac{1}{x_A}, \frac{1}{x_B})$, the harmonic mean of $\frac{1}{x_A}$ and $\frac{1}{x_B}$. But, since $\mu \leq \frac{1}{\max(x_A, x_B)}$ it also holds that $\mu < \mathcal{H}(\frac{1}{x_A}, \frac{1}{x_B})$.

(iii) Let us fix $x_A + x_B = K$. We have:

$$x^* - (x_A + x_B) = \frac{K - \frac{2}{\mu}}{1 - \mu^2 x_A (K - x_A)} + \frac{2}{\mu} - K,$$

which attains its maximum when $x_A = x_B = \frac{K}{2}$.

□

Notes

[1] The classical idea according to which aid (for example, in the form of education) creates reservation utility is tested, e.g., in Azam and Thelen [64]; according to Azam [65], the effect of education on the opportunity cost can be mitigated by the 'revelation' of their type to potential terrorists. One can also mention Collier and Hoeffler [66], who investigate a utility-based model and conditions under which rebels have an interest in sparking a civil war. A version of our model capturing the decision-making of people to engage in gangs would be a significant improvement, as we suggest in the perspectives section.

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