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Review

Unsteady and Inhomogeneous Turbulent Fluctuations around Isotropic Equilibrium

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Abstract: Extracting statistics for turbulent flows directly from the Navier–Stokes equations poses a formidable challenge, particularly when dealing with unsteady or inhomogeneous flows. However, embracing Kolmogorov's inertial range spectrum for isotropic turbulence as a dynamic equilibrium provides a conceptual starting point for perturbation theory. We review theoretical results, combining perturbation approaches, and phenomenological turbulence closures, which allow us to gain valuable insights into the statistics of unsteady and inhomogeneous turbulence. Additionally, we extend the ideas to the case of the mixing of a passive scalar.

Keywords: isotropic turbulence; inhomogeneous turbulence; unsteady turbulence; passive scalar mixing; perturbation approach

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1. Introduction

Perturbation techniques play an important role in a wide range of applications. In general, such methods allow us to obtain insights into complex dynamics for which no analytical solution exists by investigating how the system behaves if it is driven away from a known equilibrium solution. Different techniques fall under the denominator of perturbation theories [1] and in the domain of fluid mechanics only, a large number of approaches exist [2]. In the present article, we will consider one such method to evaluate the temporal and spatial perturbations of an equilibrium turbulent flow. As an introduction, before focusing on turbulent flow, we will outline the general method.

1.1. Perturbation Approaches

We consider an observable, or statistic $\mathcal{X}(p)$, where p can be any set of arguments, such as position $\boldsymbol{x}$, time t, or wavevector $\boldsymbol{k}$. The evolution of $\mathcal{X}$ is described by the equation

$$\mathcal{L}(\mathcal{X}) + F(\mathcal{X}) = 0. \tag{1}$$

The function $\mathcal{L}(\mathcal{X})$ is a known (linear) functional of $\mathcal{X}$. The function F can be any integro-differential nonlinear expression. Let us assume that we know a solution of F for a given value of $\mathcal{X}$

$$F(\mathcal{X}_0) = F_0. \tag{2}$$

The perturbation approach is now used to obtain approximate solutions in the vicinity of $\mathcal{X}_0$. For this, we consider a value of $\mathcal{X}$ not far from $\mathcal{X}_0$,

$$\mathcal{X} = \mathcal{X}_0 + \mathcal{X}_1 \tag{3}$$

where $|\mathcal{X}_1|/|\mathcal{X}_0| \ll 1$. An approximate solution is then obtained by

$$0 = \mathcal{L}(\mathcal{X}) + F_0 + \mathcal{X}_1 \left.\frac{\delta F}{\delta \mathcal{X}}\right|_{\mathcal{X}=\mathcal{X}_0} \tag{4}$$

This allows us to obtain a relation between the shape of the perturbation $\mathcal{X}_1$, $\mathcal{X}_0$ and the other terms in the evolution equation.

In the problems we will consider here, it is the shape of $\mathcal{X}_1$ which is unknown, and characterizing the spatio-temporal behavior of $\mathcal{X}_1$ will be the goal of the present investigation. We are interested in turbulent flow and therefore we consider statistics. The quantity $\mathcal{X}$ can be, for instance, the average kinetic energy, or its wavenumber spectrum. We then need to define, for this case, F and $\mathcal{X}_0$, corresponding to terms in the evolution equation for $\mathcal{X}$, and its equilibrium value, respectively. Determining closed expressions for the evolution of the kinetic energy density and its equilibrium value directly from the Navier–Stokes equations is not an easy task for turbulent flows and we will discuss this in the following sections.

1.2. This Review

The most significant prediction associated with the present work was obtained 30 years ago, in 1994, when Yoshizawa suggested how unsteadiness influences the spectral kinetic energy density of a turbulent flow [3]. However, it was not until 2011 that these ideas found empirical validation in numerical simulations [4], and only recently, in 2024, has there been a proposal to extend these findings to inhomogeneous flows [5].

This review aims to synthesize both theoretical and numerical advancements over the past thirty years, contextualizing them within the broad landscape of isotropic turbulence modeling. By maintaining a close connection with the theory of isotropic turbulence, this approach distinguishes itself from reviews on conventional modeling strategies for complex flows. It is hoped that this closer alignment with theory will establish a more robust theoretical foundation for models pertaining to unsteady and inhomogeneous turbulence. The ultimate objective of this review is to pinpoint the most promising avenues within this still active research domain.

In Section 2, we will discuss the equilibrium $\mathcal{X}_0$ that we work with. In Section 3, the theoretical attempts to obtain a closed expression for F as a function of $\mathcal{X}$ are reviewed. In Section 4, we discuss the temporal perturbation approach to determine the shape of the perturbation to the kinetic energy spectrum. A recent application to inhomogeneous flows is discussed in Section 5, and an extension to mixing is proposed in Section 6. An important question is whether these spectral corrections can also describe large-scale behavior, an issue discussed in Section 7. We conclude by briefly reviewing similar approaches applied to anisotropy and wall bounded flows in Section 8.

2. Equilibrium in Turbulence: Finding $\mathcal{X}_0$

In statistical physics, the notion of (thermodynamic) equilibrium denotes a state which does not evolve and where macroscopic fluctuations are absent. The Navier–Stokes equations are compatible with such an equilibrium, when the range of modes is constrained by Galerkin truncation. Indeed, considering the statistics of an isolated number of Fourier modes [6] it was theoretically shown that the Navier–Stokes equations admit an equipartition solution, where all Fourier modes are in equilibrium. The definition of the three-dimensional energy spectrum $E(k)$, with k the wavenumber, then implies that the energy density is given by

$$E(k) \sim k^2. \tag{5}$$

This equilibrium is modified at the largest wavenumbers if the flow contains helicity [7]. In systems governed by the Euler-equations, numerically integrated by a conservative Galerkin method, this k^2 dependence can be assessed conveniently [8,9].

Nevertheless, realistic turbulent flows do not conserve energy since the energy dissipation acts as a sink. To conserve energy in the system, a continuous energy input is necessary. If a statistical steady state is attained where the input and dissipation of energy are in equilibrium, one obtains a dynamical equilibrium, different from the thermodynamic equipartition equilibrium. It is this dynamic equilibrium for scales smaller than the injection which we will focus on in the present case, and it is this equilibrium which represents our $\mathcal{X}_0$.

The dynamical equilibrium range, or inertial range, was predicted by Kolmogorov [10] and Obukhov to be proportional to $k^{-5/3}$,

$$E(k,t) = C_K \epsilon(t)^{2/3} k^{-5/3}, \tag{6}$$

with ϵ denoting the energy dissipation rate and C_K denoting a constant. The first convincing evidence of expression (6), which we will refer to as K41, in a real turbulent flow, was obtained in tidal channel measurements [11]. Ever since, Kolmogorov's theory has gained overwhelming evidence in experiments [12] or in numerical simulations [13].

Interestingly, both equilibria, Equations (5) and (6), can coexist in different wavenumber ranges of a flow. Indeed, for flow-scales larger than the energy-injection scale, equipartition (Equation (5)) is observed in steady state numerical simulations [14]. Furthermore, at scales small enough so that the kinetic energy becomes comparable to, or smaller than, the energy associated with thermal noise, the equipartition of energy is also observed [15]. Most of the triadic turbulence closures and some of the phenomenological turbulence closures (approaches discussed in the next section) do reproduce this equilibrium solution [16–19].

K41 is an asymptotic theory, valid exactly only in the limit of infinite Reynolds number and, in general, corrections due to the finiteness of the Reynolds number are present [20–25]. Furthermore, as suggested by Kolmogorov himself [26], intrinsic corrections could possibly be added to the K41 theory due to the fact that the dissipation rate is not a uniformly distributed quantity in space. Even though, for higher order statistics, these corrections seem indisputable [27,28] for second-order quantities such as the kinetic energy spectrum, these corrections are small and within error-bars not easily distinguishable from finite Reynolds number effects [22]. We will not further take them into account.

In the following, we will thus consider that the equilibrium we are interested in is a flow statistically characterized by a K41 inertial range. This latter spectrum is thus the $\mathcal{X}_0$ in expression (4). We need, at this point, in Equation (4), an expression for F, which is simple enough to allow us to determine the functional (or Fréchet) derivative. This will be discussed in the following section.

3. Isotropic Turbulence: Deriving $F(\mathcal{X})$

In this section, we focus on the necessary requirement to apply the method represented by Equation (4) to the energy spectrum: we need the governing equation for the kinetic energy spectrum. We focus in this section on (statistically) isotropic turbulence, a concept introduced by Taylor [29] and popularized by Batchelor [30].

For the case of isotropic turbulence, the equation for the evolution of the kinetic energy spectrum writes

$$\frac{\partial E}{\partial t} = P - 2\nu k^2 E + T, \tag{7}$$

where we omitted time and wavenumber dependence for brevity. The first two terms on the right-hand side of this equation are, respectively, production P and dissipation $2\nu k^2 E$. The last term is the nonlinear transfer T, which can be written as the wavenumber divergence of the energy flux Π at scale k,

$$T = -\frac{\partial \Pi}{\partial k}. \tag{8}$$

This quantity T is the term associated with F in Equation (1). To carry out our perturbation analysis, we need F to be a functional of $\mathcal{X}$, in our case, the kinetic energy spectrum. The Navier–Stokes equations are a (quadratic) nonlinear function of the velocity. Therefore, the equation of second-order moments, such as the kinetic energy spectrum, will contain terms containing triple-velocity correlations. This is the case for the transfer spectrum, which is thereby not a closed function of the kinetic energy spectrum, but is determined by the interactions of a set of three Fourier modes (or triadic interactions). We can at this time not directly apply the perturbation theory sketched in Section 1.1 to Equation (7), since we cannot simply determine the differential $\delta F/\delta E$. We now summarize the efforts to obtain such a closed equation.

A first attempt could be to simply ignore T, as would be the case in a Gaussian field, but this does not allow turbulent-like solutions, since different flow-scales are not able to interact in the absence of nonlinearity. Perhaps the first encouraging attempt to obtain a closed expression for the transfer of the kinetic energy from the Navier–Stokes equations was based on an idea from Millionschikov [31,32]: if the cumulants (i.e., the deviations from Gaussianity) to the fourth-order velocity moments are ignored, the statistical moment hierarchy obtained from the Navier–Stokes equations can be closed. This retains mode coupling leading to finite transfer. The algebra to obtain an expression for T was carried out by Proudman and Reid [33] and Tatsumi [34]. The resulting expression was numerically integrated by Ogura [35] and shown to lead to negative values of kinetic energy, in mathematical disagreement with reality. The reason for this violation of realizability is that ignoring the cumulants to the fourth order moments leads to insufficient damping of the third-order moments, so that these can become too large [8].

Kraichnan contributed enormously to the theory of isotropic turbulence by introducing a renormalized perturbation technique, where the small parameter is associated with the contribution of a single triadic interaction to the transfer. The resulting theory is called the direct-interaction approximation (DIA) [36,37]. Despite its mathematical elegancy, and the realizability of its results (no negative kinetic energy), the asymptotic energy spectrum associated with the DIA is in disagreement with Kolmogorov's inertial range theory. The origin of this discrepancy was pinpointed by Kraichnan to be the Eulerian character of the theory [38]. Indeed, the DIA is a two-time theory, involving memory-times. In a Eulerian reference frame, time-correlations between two points are importantly affected if one adds a sweeping velocity, an effect associated with Galilean invariance (adding a uniform velocity will affect the Eulerian time-correlation but should not affect the energy transfer). This motivated the formulation of Lagrangian versions of the DIA [39,40]. These theories are of a formidable mathematical complexity. An exception is the Lagrangian renormalized approximation [41–43], which, through the precise combination of the choice of the points in the reference frame, is of lesser complexity.

An alternative approach used the insights obtained from Kraichnan's theories to replace the Lagrangian character by more ad hoc damping, either by directly modeling the missing cumulants by a spectral eddy-viscosity in the balance for the third-order moments, yielding the Eddy-damped quasi-normal Markovian (EDQNM) approach [44], or by evaluating the damping using the help of a test-field which measures the Lagrangian decorrelation [45], or by using a scalar field to measure the correlation-time directly, yielding the Lagrangian Markovianized field approximation (LMFA) [46,47]. An Eulerian two-time theory, allowing agreement with K41 due to the introduction of a well-chosen velocity propagator function, is the local energy transfer theory [48,49]. All these theories can be integrated numerically, and their solution is compatible with K41. Nevertheless, since the final closed equations are based on convolution integrals, the Fréchet derivative used in Section 1.1 is not easily determined from these closures.

Another line of research, leading to simpler expressions, is more phenomenological and aimed at directly reproducing the features of turbulence, but not starting from first principles (i.e., not starting from the Navier–Stokes equations). This approach yielded closure models for Π involving simple integrals and derivatives. Notable examples are the model by Heisenberg [50] taking into account scale-nonlocality, and Leith's [17] diffusion approach, allowing both the thermal-equilibrium solution and K41. An overview of these approaches is given in a dedicated textbook [51]. Certain models can be obtained from the above-discussed closure theories like the DIA by taking the appropriate limits of the triadic wavenumber interactions [52,53]. Since these approaches aim for simplicity and do not simply relate to the Navier–Stokes equations, we discuss the simplest of all, compatible with K41, proposed by Kovaznay [54]

$$\Pi = \left(C_K^{-1} k^{5/3} E\right)^{3/2}. \tag{9}$$

This form of the spectral flux is dimensionally correct, yields a conserved energy flux, and allows a steady state energy spectrum of the form Equation (6). One of the features it does not reproduce is the thermal equilibrium solution. If this was desired, the Leith model would be better adapted. For our purposes Expression (9) will be sufficient and we will retain it for the remainder of our discussion.

We note that the development of an analytical theory of turbulence has been a long-standing objective in fluid mechanics research. The present review does not address all attempts and more complete reviews can be found, for instance, in [55–57]. In the following, we will focus on the application of perturbation methods to inertial range scaling, without worrying too much about the exact formulation of the closure, as long as it is compatible with the physics that we are interested in.

4. Temporal Perturbations to K41

In this section, we focus on the inertial range given by Equation (6). We take Equation (7) and integrate from k to infinity, where k is chosen to be larger than the scales where the forcing is significant, and smaller than those where the dissipation is important. We have then

$$\int_k^\infty \frac{\partial E}{\partial t} dk = \Pi - \epsilon. \tag{10}$$

To this expression, we apply our perturbation approach. Identifying

$$\mathcal{X}_0 \to C_K \epsilon^{2/3} k^{-5/3} \tag{11}$$

$$F(\mathcal{X}) \to \Pi = \left(C_K^{-1} k^{5/3} E\right)^{3/2} \tag{12}$$

$$\mathcal{L}(\mathcal{X}) \to -\int_k^\infty \frac{\partial E}{\partial t} dk - \epsilon \tag{13}$$

we find $F_0 = \epsilon$ and $\delta F/\delta E = \frac{3}{2}\Pi/E$. We then obtain to the leading order (i.e., ignoring $\partial E_1/\partial t$)

$$E_1 = \frac{2}{3}\frac{E_0}{\epsilon}\int_k^\infty \frac{\partial E_0}{\partial t} dk. \tag{14}$$

This yields, using the K41 expression for E_0, and for a wide inertial range,

$$E_1 = \frac{2}{3} C_K^2 \dot{\epsilon} \epsilon^{-2/3} k^{-7/3}. \tag{15}$$

This expression was first proposed by Yoshizawa [3] in a study using two-scale DIA. Further theoretical investigations discussing this scaling are [58–61].

An important implication for the inertial range scaling of Equation (15) is that the exponent $-7/3$ indicates a faster decay as a function of wavenumber than the Kolmogorov spectrum. For large values of k, the correction is therefore sub-dominant, and Kolmogorov-scaling will prevail. The ratio in the inertial range is

$$\frac{E_1}{E_0} \sim \frac{\dot{\epsilon}}{\epsilon}\tau_k, \tag{16}$$

where $\tau_k \sim \epsilon^{-1/3} k^{-2/3}$, which allows us to estimate for a given perturbation of ϵ, the scale or wavenumber beyond which the corrections become insignificant. For large values of the wavenumber, the approximation should thus be better since the precision of linear perturbation theory improves for small values of the perturbation, but at the same time, the corrections become very small compared to the equilibrium spectrum and are therefore harder to detect. However, by subtracting the equilibrium spectrum Equation (6) from the instantaneous spectrum, one is able to identify the non-equilibrium contribution [62].

An example of a clear $k^{-7/3}$ correction to the kinetic energy spectrum is given in Figure 1, where we show EDQNM results for the equilibrium and perturbation spectra, for the case of periodically forced turbulence [62], i.e., for a flow where the large scales are modulated by a periodic forcing [63]. In this Figure, E_0 is obtained by time-averaging the spectrum, after normalizing it using Kolmogorov variables (length and timescales based on the viscosity and dissipation). Subtracting this estimate of E_0 from the instantaneous spectra and averaging the norm of the resulting spectra yields the spectrum $\langle |E_1| \rangle$.

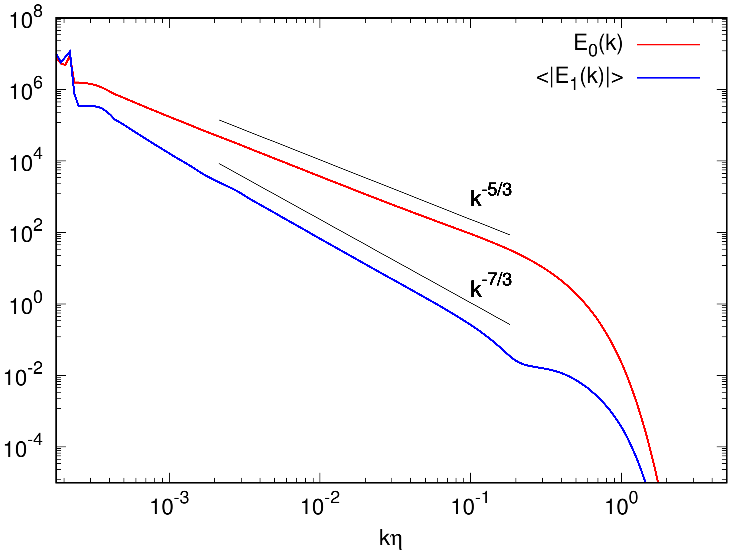

Figure 1. Equilibrium spectrum E_0 and the average of the norm of the perturbation spectrum E_1 for the case of periodically forced turbulence. Details and parameters of the set-up are found in [62]. The wavenumber is normalized by the Kolmogorov-scale η.

To derive Equation (15), there has been no assumption concerning the nature of the temporal perturbations, i.e., the time-signal of the perturbation, so that the perturbations should be observed universally. Evidence of this scaling was indeed observed in a range of different flows. The first confirmation of the non-equilibrium scaling was reported in the numerical simulations of Horiuti and Ozawa in shear flow [4], followed by observations in isotropic turbulence in Horiuti and Tamaki [64]. Recent further evidence was obtained by Berti et al. in three-dimensional turbulence forced by Taylor-Green forcing [65] and assessing the isotropic EDQNM equations [62] (see Figure 1).

5. Inhomogeneous Perturbations to K41

Given the success in describing temporal fluctuations in turbulence using perturbation theory on the Kolmogorov spectrum, one might want to extend the approach to spatial inhomogeneity. The progress to derive statistical theories for inhomogeneous turbulence can be summarized by a small number of attempts, including the inhomogeneous test-field model [66] and inhomogeneous extensions of the EDQNM approach [67–69]. A particular difficulty is that, in inhomogeneous turbulence, the fluxes in scale and space are intertwined and it seems that approximations are needed to separately consider them [70–72].

Indeed, this coupling of scale and space renders the description of statistically inhomogeneous turbulence tremendously complex. Furthermore, whereas temporal perturbations can be assessed in an isotropic setting, inhomogeneous flows are intrinsically statistically anisotropic. The general statistical description of inhomogeneous turbulence, taking into account both space and scale variations, can be covered by the Karman–Howarth–Hill equations [73,74]. A comprehensible analytical treatment of these equations to obtain the statistics of inhomogeneous turbulence does not yet exist. To study the corrections due to inhomogeneity, as in the previous section for temporal fluctuations, we abandon our attempts to rigorously derive closures from the Navier–Stokes equations and, again, we use the simplest possible phenomenological closure models.

We consider the case where the flow is statistically stationary and inhomogeneous in a single direction z only. Assuming that we can separately consider space and scale fluxes in the Lin-equation, we write for the inhomogeneous spectrum $E(k,z)$

$$0 = P - 2\nu k^2 E - \frac{\partial \Pi}{\partial k} - \frac{\partial \Phi_z}{\partial z}. \tag{17}$$

One should thus realize that, since we separate scale and space fluxes, this is already a modeled equation. However, the important physical mechanisms are present. In Section 4, the wavenumber flux Π was modeled by the simplest possible model Equation (9). If we extend this approach here and keep the model for Π and model the spatial energy flux by a simple gradient diffusion approach,

$$\Phi_z = -D_T \frac{\partial E}{\partial z}, \tag{18}$$

we obtain a closed equation for $E(k,z)$. The inhomogeneous equivalent of Equation (14) then becomes, for uniform D_T,

$$E_1 = -\frac{2}{3}\frac{E_0}{\epsilon}\int_k^\infty D_T \frac{\partial^2 E}{\partial z^2} dk, \tag{19}$$

which yields

$$E_1 = -\frac{2C_K^2}{3} D_T \left(\frac{\partial^2 \epsilon}{\partial z^2} - \frac{\epsilon^{-1}}{3}\left(\frac{\partial \epsilon}{\partial z}\right)^2 \right) \epsilon^{-2/3} k^{-7/3}. \tag{20}$$

The assumption of uniform turbulent diffusivity D_T can be refined. Indeed, a spectral equivalent was considered in [70,75], yielding a k-independent value for large k. Furthermore, D_T can be a function of space. This effect only leads, however, to second-order corrections [5]. Also, the second term in brackets is for small fluctuations of ϵ around equilibrium negligible compared to the first.

The inhomogeneous spectral corrections have, for the moment, received little attention. The recent numerical investigation [5], however, shows evidence that the expression (20) is consistent with the statistics of the three-dimensional Kolmogorov flow. Further research could consider the refinement of the modeling of the turbulent diffusion and the extension and assessment of these ideas to integral quantities.

6. Mixing

The foregoing ideas can also be applied to the mixing of a passive scalar by turbulent flow. The evolution-equation of the spectrum $E_\theta(k,t)$ of the scalar variance in an isotropic turbulent flow is given by

$$\frac{\partial E_\theta}{\partial t} = P_\theta - 2\kappa k^2 E_\theta - \frac{\partial \Pi_\theta}{\partial k}. \tag{21}$$

This expression is the equivalent of Equation (7). We similarly find a production term P_θ and a dissipation term $2\kappa k^2 E_\theta$, with κ, the molecular diffusivity of the scalar. We will focus on the case of unity Prandtl number, where $\kappa = \nu$. The last term is again an unclosed term, associated with the flux Π_θ of θ^2-stuff through scale-space (the term θ^2-stuff was introduced by Batchelor [76], and we could also call it scalar variance).

To apply the temporal perturbation approach to the case of scalar mixing one needs again to define $\mathcal{X}_0$ and $\mathcal{L}$ and F. The study of turbulent mixing closely followed the development of the theory for isotropic turbulence and we will briefly review it here.

The equivalent of Kolmogorov's inertial range theory for mixing is due to Obukhov and Corrsin [77,78] and reads

$$E_\theta = C_\theta \epsilon_\theta \epsilon^{-1/3} k^{-5/3}, \tag{22}$$

where ϵ_θ is the dissipation of scalar variance, obtained by integrating the dissipation term in Equation (21) over wavenumbers. Expression (22) will thus be the $\mathcal{X}_0$ for the case of turbulent scalar mixing. The next step consists in obtaining a closed form for the scalar transfer. The approaches described in Section 3 were also applied to mixing. For instance, the DIA for a passive scalar was developed by Roberts [79], and the Lagrangian version was developed by Kraichnan in the same article where he treated the velocity [39]. The test-field model for scalar mixing is discussed in [80] and the EDQNM approach to the scalar was first discussed in [81,82]. Again, these models are elegant, but not easily treatable in a perturbation approach. We will therefore refer to phenomenological closure [83], as discussed in Section 3 for the energy spectrum.

For mixing, the equivalent of the Kovaznay model reads

$$\Pi_\theta = C_K^{-1/2} C_\theta^{-1} k^{5/2} E_\theta E^{1/2}, \tag{23}$$

which will constitute our $F(\mathcal{X})$ for scalar mixing. Carrying out the same procedure as in Section 1.1, we obtain for the perturbations to the equilibrium spectrum

$$E_{\theta 1} = \frac{E_{\theta 0}}{\epsilon_\theta} \int_k^\infty \frac{\partial E_{\theta 0}}{\partial t} dk. \tag{24}$$

This yields

$$E_{\theta 1} = C_\theta^2 \left(\frac{\dot{\epsilon}_\theta}{\epsilon_\theta} - \frac{\dot{\epsilon}}{3\epsilon} \right) \epsilon_\theta \epsilon^{-2/3} k^{-7/3}. \tag{25}$$

The validity of the present approach to the case of the passive scalar has not been assessed and verifying the expression (25) constitutes a logical perspective for further research.

7. Implications for the Large-Scale Statistics

An intriguing question is that of whether the ideas of the last three sections can be extended to integral scales. For the temporal fluctuations, in principle, if the fluctuations are slow, so that the value of $\dot{\epsilon}/\epsilon$ is not too large compared to the typical timescale of the large eddies, the approximation should hold at the large scales. Indeed, this was explicitly assumed in the theoretical study by Bos and Rubinstein [84].

In that study, an attempt was made to explain fluctuations in the normalized dissipation rate, defined as

$$C_\epsilon = \frac{\epsilon L}{U^3} \tag{26}$$

where $U = \sqrt{2 \int E(k) dk / 3}$ and L is the longitudinal correlation lengthscale. Indeed, C_ϵ should at first order be constant, as was suggested by Taylor [29]. However, the fluctuations around this equilibrium were shown to be close to a functional relation of the Reynolds number R_λ [85,86]. Integrating Equations (6) and (15), it was obtained in [84] that this variation should be given by

$$C_\epsilon(t) \sim R_\lambda(t)^{-15/14}. \tag{27}$$

The result (27) was compared in [84] to the numerical results of [86] and the wind-tunnel results of Hearst and Lavoie [87]. Subsequently, confirmation was obtained in windtunnel measurements [88,89] and in atmospheric measurements [90]. These observations suggest that the temporal corrections to the large scales are following the same trends as the spectral corrections, an observation which needs further research.

Other preliminary investigations have focused on the integral quantities for the case of modulated scalar mixing, discussed in Section 6 [91,92]. Again, these approaches extend the results for inertial range quantities to integral lengths and timescales. A recent extension where the small scales are perturbed instead of the large ones is proposed [93] and numerically verified [94], yielding an expression similar to Equation (27), but with an exponent close to -2 instead of $-15/14$.

There seems, therefore, to be evidence that the Yoshizawa scaling and its extensions can be used for the large-scale descriptions. This might seem surprising given the non-universality of large-scale turbulence and further research is needed to assess why this approach is justified.

8. Conclusions and Perspectives

To summarize: it was highlighted in this review how isotropic turbulence can serve as a basis for linear perturbation theory. For the inertial-range in three dimensional turbulence, the spectral corrections are, for the cases discussed in this review, proportional to

$$E_1(k,t) \sim \frac{\dot{\epsilon}}{\epsilon}\epsilon^{1/3}k^{-7/3} \tag{28}$$

$$E_1(k,z) \sim -\frac{\partial^2 \epsilon}{\partial z^2}\frac{D_T}{\epsilon}\epsilon^{1/3}k^{-7/3} \tag{29}$$

$$E_{\theta 1}(k,t) \sim \left(\frac{\dot{\epsilon}_\theta}{\epsilon_\theta} - \frac{\dot{\epsilon}}{3\epsilon}\right)\epsilon_\theta \epsilon^{-2/3}k^{-7/3}. \tag{30}$$

In particular, for the moment, this last expression remains a prediction, and should be verified by simulation or experiment.

We have not reviewed a great part of the literature, focusing on the anisotropic properties of turbulence [95,96]. Perturbation approaches have been successfully applied to investigate the anisotropy of flows in the presence of certain external forcing mechanisms. For instance, following dimensional arguments by Lumley [97], homogeneous shear-flow was investigated both theoretically [98] and numerically [99]. Scaling corrections for different anisotropic contributions were shown to be proportional to either $k^{-7/3}$ or k^{-3}. Scaling obtained by perturbation analysis in stratified turbulence was considered in [100], and in quasi-static magnetohydrodynamics in [101].

All these developments have considered flows in homogeneous or periodic flow-settings. Clearly things, will change in the vicinity of solid walls. An interesting first approach to extend these ideas to wall bounded flow was recently proposed [102,103], opening a whole new direction for the investigation of turbulent flows using perturbation approaches.

We do want to insist that most of the methods used in the present review are fairly general. An extension to other turbulent flows is thereby an obvious direction for further research. Indeed, a large number of turbulent flows exhibit scaling ranges [104] and can be described by empirical transfer-flux models. For instance, we mention magnetohydrodynamic turbulence, even though the precise scaling of the equilibrium spectrum is not uniquely determined (see, e.g., [105] for a recent review).

Yet another field of research which can take advantage of the present results is the field of atmospheric research. Indeed, in atmospheric turbulence, the interplay between the turbulent fluxes of heat and momentum yields dynamics which are not only governed by the turbulent kinetic energy dynamics but also by the influence of buoyancy. In such flows, it might be reasonable to consider another equilibrium state that is different from K41 and to choose an associated spectral energy transfer closure. This would allow one to measure the influence of unsteadiness and inhomogeneity on these non-Kolmogorov flows. An interesting perspective then consists in linking closure approaches for this system [106,107] to the present approach. It is encouraging that, as mentioned in Section 7, measurements have shown evidence of the validity of the current approach to describe the fluctuations of integral quantities in the atmospheric boundary layer [90]. Further research is needed.

Altogether, advancing the present concepts to encompass the (simultaneous) effects of magnetic fields, rotation, buoyancy, and shear, while validating the extrapolation to large scales, will allow us to obtain a refined understanding of out-of-equilibrium statistics. Such insights hold significant promise for a wide range of applications across engineering, geophysical, and astrophysical flows.

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References

1. Nayfeh, A.H. *Perturbation Methods*; John Wiley & Sons: Hoboken, NJ, USA, 2008.
2. Van Dyke, M. Perturbation methods in fluid mechanics/Annotated edition. *NASA STI/Recon Tech. Rep. A* **1975**, *75*, 46926.
3. Yoshizawa, A. Nonequilibrium effect of the turbulent-energy-production process on the inertial-range energy spectrum. *Phys. Rev. E* **1994**, *49*, 4065. [CrossRef] [PubMed]
4. Horiuti, K.; Ozawa, T. Multimode stretched spiral vortex and nonequilibrium energy spectrum in homogeneous shear flow turbulence. *Phys. Fluids* **2011**, *23*, 035107. [CrossRef]
5. Araki, R.; Bos, W.J.T. Inertial range scaling of inhomogeneous turbulence. *J. Fluid Mech.* **2024**, *978*, A9. [CrossRef]
6. Lee, T. On some statistical properties of hydrodynamical and magnetohydrodynamical fields. *Q. Appl. Math.* **1952**, *10*, 69. [CrossRef]
7. Kraichnan, R.H. Helical turbulence and absolute equilibrium. *J. Fluid Mech.* **1973**, *59*, 745. [CrossRef]
8. Orszag, S. Statistical theory of turbulence. In *Fluid Dynamics, Lecture Notes Les Houches*; Oxford Academic: Oxford, UK, 1973; pp. 237–374.
9. Cichowlas, C.; Bonaïti, P.; Debbasch, F.; Brachet, M. Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows. *Phys. Rev. Lett.* **2005**, *95*, 264502. [CrossRef] [PubMed]
10. Kolmogorov, A.N. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk. SSSR* **1941**, *30*, 301.
11. Grant, H.L.; Stewart, R.W.; Moilliet, A. Turbulence spectra from a tidal channel. *J. Fluid Mech.* **1962**, *12*, 241. [CrossRef]
12. Sreenivasan, K.R. On the universality of the Kolmogorov constant. *Phys. Fluids* **1995**, *7*, 2778–2784. [CrossRef]
13. Ishihara, T.; Gotoh, T.; Kaneda, Y. Study of high Reynolds number isotropic turbulence by Direct Numerical Simulation. *Annu. Rev. Fluid Mech.* **2009**, *41*, 65. [CrossRef]
14. Alexakis, A.; Brachet, M.E. On the thermal equilibrium state of large-scale flows. *J. Fluid Mech.* **2019**, *872*, 594–625. [CrossRef]
15. Bandak, D.; Goldenfeld, N.; Mailybaev, A.A.; Eyink, G. Dissipation-range fluid turbulence and thermal noise. *Phys. Rev. E* **2022**, *105*, 065113. [CrossRef] [PubMed]
16. Lesieur, M. *Turbulence in Fluids*; Kluwer: Dordrecht, The Netherlands 1990.
17. Leith, C. Diffusion approximation to inertial energy transfer in isotropic turbulence. *Phys. Fluids* **1967**, *10*, 435–472. [CrossRef]
18. Connaughton, C.; Nazarenko, S. Warm Cascades and Anomalous Scaling in a Diffusion Model of Turbulence. *Phys. Rev. Lett.* **2004**, *92*, 044501. [CrossRef] [PubMed]
19. Bos, W.J.T.; Bertoglio, J.P. Dynamics of spectrally truncated inviscid turbulence. *Phys. Fluids* **2006**, *18*, 071701. [CrossRef]
20. Qian, J. Inertial range and the finite Reynolds number effect of turbulence. *Phys. Rev. E* **1997**, *55*, 337. [CrossRef]
21. Qian, J. Slow decay of the finite Reynolds number effect of turbulence. *Phys. Rev. E* **1999**, *60*, 3409. [CrossRef] [PubMed]
22. Bos, W.J.T.; Chevillard, L.; Scott, J.F.; Rubinstein, R. Reynolds number effect on the velocity increment skewness in isotropic turbulence. *Phys. Fluids* **2012**, *24*, 015108. [CrossRef]
23. Antonia, R.; Djenidi, L.; Danaila, L.; Tang, S. Small scale turbulence and the finite Reynolds number effect. *Phys. Fluids* **2017**, *29*, 020715. [CrossRef]
24. McComb, W.D.; Berera, A.; Yoffe, S.R.; Linkmann, M.F. Energy transfer and dissipation in forced isotropic turbulence. *Phys. Rev. E* **2015**, *91*, 043013. [CrossRef] [PubMed]
25. Gagne, Y.; Castaing, B.; Baudet, C.; Malécot, Y. Reynolds dependence of third-order velocity structure functions. *Phys. Fluids* **2004**, *16*, 482–485. [CrossRef]
26. Kolmogorov, A. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **1962**, *13*, 82. [CrossRef]
27. Anselmet, F.; Gagne, Y.; Hopfinger, E.J.; Antonia, R.A. High-order velocity structure functions in turbulent shear flows. *J. Fluid Mech.* **1984**, *140*, 63. [CrossRef]
28. Frisch, U. *Turbulence, the Legacy of A.N. Kolmogorov*; Cambridge University Press: Cambridge, UK, 1995.
29. Taylor, G.I. Statistical theory of turbulence. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1935**, *151*, 421–444. [CrossRef]
30. Batchelor, G.K. *The Theory of Homogeneous Turbulence*; Cambridge University Press: Cambridge, UK, 1953.
31. Millionschikov, M. On theory of homogeneous isotropic turbulence. *Dokl. Akad. Nauk SSSR* **1941**, *32*, 615.
32. Monin, A.S.; Yaglom, A.M. *Statistical Fluid Mechanics II*; MIT Press: Cambridge, MA, USA, 1975.

33. Proudman, I.; Reid, W. On the Decay of a Normally Distributed and Homogeneous Turbulent Velocity Field. *Philos. Trans. R. Soc. Lond. A* **1954**, *247*, 163–189.
34. Tatsumi, T. The Theory of Decay Process of Incompressible Isotropic Turbulence. *Proc. R. Soc. Lond. A* **1957**, *239*, 16.
35. Ogura, Y. A consequence of the zero-fourth-cumulant approximation in the decay of isotropic turbulence. *J. Fluid Mech.* **1963**, *16*, 33–40.
36. Kraichnan, R.H. Irreversible Statistical Mechanics of Incompressible Hydromagnetic Turbulence. *Phys. Rev.* **1958**, *109*, 1407. [CrossRef]
37. Kraichnan, R.H. The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **1959**, *5*, 497–543. [CrossRef]
38. Kraichnan, R.H. Kolmogorov's Hypotheses and Eulerian Turbulence Theory. *Phys. Fluids* **1964**, *7*, 1723. [CrossRef]
39. Kraichnan, R.H. Lagrangian-History Closure Approximation for Turbulence. *Phys. Fluids* **1965**, *8*, 575. [CrossRef]
40. Kraichnan, R.H.; Herring, J. A strain-based Lagrangian-History turbulence theory. *J. Fluid Mech.* **1978**, *88*, 355–367. [CrossRef]
41. Kaneda, Y. Renormalized expansions in the theory of turbulence with the use of the Lagrangian position function. *J. Fluid Mech.* **1981**, *107*, 131–145. [CrossRef]
42. Kaneda, Y. Inertial range structure of turbulent velocity and scalar fields in a Lagrangian renormalized approximation. *Phys. Fluids* **1986**, *29*, 701–708. [CrossRef]
43. Kida, S.; Goto, S. A Lagrangian direct-interaction approximation for homogeneous isotropic turbulence. *J. Fluid Mech.* **1997**, *345*, 307. [CrossRef]
44. Orszag, S.A. Analytical theories of Turbulence. *J. Fluid Mech.* **1970**, *41*, 363. [CrossRef]
45. Kraichnan, R.H. An almost-Markovian Galilean-invariant turbulence model. *J. Fluid Mech.* **1971**, *47*, 513. [CrossRef]
46. Bos, W.J.T.; Bertoglio, J.P. A single-time two-point closure based on fluid particle displacements. *Phys. Fluids* **2006**, *18*, 031706. [CrossRef]
47. Bos, W.J.T.; Bertoglio, J.P. Lagrangian Markovianized Field Approximation for turbulence. *J. Turbul.* **2013**, *14*, 99. [CrossRef]
48. McComb, W. A local energy-transfer theory of isotropic turbulence. *J. Phys. Math. Nucl. Gen.* **1974**, *7*, 632. [CrossRef]
49. McComb, W.; Yoffe, S. A formal derivation of the local energy transfer (LET) theory of homogeneous turbulence. *J. Phys. A Math. Theor.* **2017**, *50*, 375501. [CrossRef]
50. Heisenberg, W. Zur statistischen Theorie der Turbulenz. *Z. Phys.* **1948**, *124*, 628. [CrossRef]
51. Rubinstein, R.; Clark, T. *Reassessment of the Classical Turbulence Closures*; Cambridge Scholars Publishing: Newcastle upon Tyne, UK, 2022.
52. Rubinstein, R.; Clark, T. A generalized Heisenberg model for turbulent spectral dynamics. *Theor. Comput. Fluid Dyn.* **2004**, *17*, 249. [CrossRef]
53. Clark, T.; Rubinstein, R.; Weinstock, J. Reassessment of the classical turbulence closures: The Leith diffusion model. *J. Turbul.* **2009**, *10*, N35. [CrossRef]
54. Kovaznay, L. Spectrum of locally isotropic turbulence. *J. Aeronaut. Sci.* **1948**, *15*, 745. [CrossRef]
55. Sagaut, P.; Cambon, C. *Homogeneous Turbulence Dynamics*; Springer: Berlin/Heidelberg, Germany, 2008; Volume 10.
56. McComb, W.D. *Homogeneous, Isotropic Turbulence: Phenomenology, Renormalization and Statistical Closures*; OUP Oxford: Oxford, UK, 2014; Volume 162.
57. Zhou, Y. Turbulence theories and statistical closure approaches. *Phys. Rep.* **2021**, *935*, 1–117.
58. Rubinstein, R.; Clark, T.; Livescu, D.; Luo, L.S. Time-dependent isotropic turbulence. *J. Turbul.* **2004**, *5*, 011. [CrossRef]
59. Woodruff, S.; Rubinstein, R. Multiple-scale perturbation analysis of slowly evolving turbulence. *J. Fluid Mech.* **2006**, *565*, 95. [CrossRef]
60. Goto, S.; Vassilicos, J.C. Local equilibrium hypothesis and Taylor's dissipation law. *Fluid Dyn. Res.* **2016**, *48*, 021402. [CrossRef]
61. Rubinstein, R.; Clark, T.T. "Equilibrium" and "non-equilibrium" turbulence. *Theor. Appl. Mech. Lett.* **2017**, *7*, 301–305. [CrossRef]
62. Fang, L.; Bos, W.J.T. An EDQNM study of the dissipation rate in isotropic non-equilibrium turbulence. *J. Turbul.* **2023**, *24*, 217–234. [CrossRef]
63. Bos, W.J.T.; Clark, T.T.; Rubinstein, R. Small scale response and modeling of periodically forced turbulence. *Phys. Fluids* **2007**, *19*, 055107. [CrossRef]
64. Horiuti, K.; Tamaki, T. Nonequilibrium energy spectrum in the subgrid-scale one-equation model in large-eddy simulation. *Phys. Fluids* **2013**, *25*, 125104. [CrossRef]
65. Berti, S.; Boffetta, G.; Musacchio, S. Mean flow and fluctuations in the three-dimensional turbulent cellular flow. *Phys. Rev. Fluids* **2023**, *8*, 054601. [CrossRef]
66. Kraichnan, R.H. Test-field model for inhomogeneous turbulence. *J. Fluid Mech.* **1972**, *56*, 287–304. [CrossRef]
67. Laporta, A. Etude Spectrale et Modélisation de la Turbulence Inhomogène. Ph.D. Thesis, Ecole Centrale de Lyon, Écully, France, 1995.
68. Laporta, A.; Bertoglio, J.P. A model for inhomogeneous turbulence based on two-point correlations. In *Advances in Turbulence V*; Springer: Berlin/Heidelberg, Germany, 1995; pp. 286–297.
69. Parpais, S. Développement d'un Modèle Spectral Pour la Turbulence Inhomogène. Résolution par une Méthode d'éléments Finis. Ph.D. Thesis, Ecole Centrale de Lyon, Écully, France, 1997.
70. Besnard, D.; Harlow, F.; Rauenzahn, R.; Zemach, C. Spectral transport model for turbulence. *Theor. Comput. Fluid Dyn.* **1996**, *8*, 1–35. [CrossRef]
71. Touil, H.; Bertoglio, J.P.; Shao, L. The decay of turbulence in a bounded domain. *J. Turbul.* **2002**, *3*, N49. [CrossRef]
72. Touil, H.; Parpais, S.; Bertoglio, J.P. A spectral closure applied to anisotropic inhomogeneous turbulence. In Proceedings of the 8th European Turbulence Conference, Barcelona, Spain, 27–30 June 2000.

73. De Karman, T.; Howarth, L. On the statistical theory of isotropic turbulence. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1938**, *164*, 192–215. [CrossRef]
74. Hill, R.J. Exact second-order structure-function relationships. *J. Fluid Mech.* **2002**, *468*, 317–326. [CrossRef]
75. Cadiou, A.; Hanjalić, K.; Stawiarski, K. A two-scale second-moment turbulence closure based on weighted spectrum integration. *Theor. Comput. Fluid Dyn.* **2004**, *18*, 1–26. [CrossRef]
76. Batchelor, G.K. Small-scale variation of convected quantities like temperature in turbulent fluid. Part 1. General discussion and the case of small conductivity. *J. Fluid Mech.* **1959**, *5*, 113. [CrossRef]
77. Obukhov, A. Structure of the temperature field in turbulent flows. *Isv. Geogr. Geophys. Ser.* **1949**, *13*, 58.
78. Corrsin, S. On the spectrum of isotropic temperature fluctuations in an isotropic turbulence. *J. Appl. Phys.* **1951**, *22*, 469. [CrossRef]
79. Roberts, P. Analytical theory of turbulent diffusion. *J. Fluid Mech.* **1961**, *11*, 257. [CrossRef]
80. Newman, G.R.; Herring, J. A test field model of a passive scalar in isotropic turbulence. *J. Fluid Mech.* **1979**, *94*, 163. [CrossRef]
81. Vignon, J.M.; Cambon, C. Thermal spectral calculation using eddy-damped quasi-normal Markovian theory. *Phys. Fluids* **1980**, *23*, 1935–1937. [CrossRef]
82. Herring, J.; Schertzer, D.; Lesieur, M.; Newman, G.; Chollet, J.P.; Larcheveque, M. A comparative assessment of spectral closures as applied to passive scalar diffusion. *J. Fluid Mech.* **1982**, *124*, 411. [CrossRef]
83. Rubinstein, R.; Clark, T.T. Reassessment of the classical closures for scalar turbulence. *J. Turbul.* **2013**, *14*, 71–98. [CrossRef]
84. Bos, W.J.T.; Rubinstein, R. Dissipation in unsteady turbulence. *Phys. Rev. Fluids* **2017**, *2*, 022601. [CrossRef]
85. Vassilicos, J.C. Dissipation in turbulent flows. *Ann. Rev. Fluid Mech.* **2015**, *47*, 95–114. [CrossRef]
86. Goto, S.; Vassilicos, J.C. Energy dissipation and flux laws for unsteady turbulence. *Phys. Lett. A* **2015**, *379*, 1144–1148. [CrossRef]
87. Hearst, R.J.; Lavoie, P. Decay of turbulence generated by a square-fractal-element grid. *J. Fluid Mech.* **2014**, *741*, 567–584. [CrossRef]
88. Zheng, Y.; Nakamura, K.; Nagata, K.; Watanabe, T. Unsteady dissipation scaling in static-and active-grid turbulence. *J. Fluid Mech.* **2023**, *956*, A20. [CrossRef]
89. Zheng, Y.; Koto, N.; Nagata, K.; Watanabe, T. Unsteady dissipation scaling of grid turbulence in the near-field region. *Phys. Fluids* **2023**, *35*, 095131. [CrossRef]
90. Wacławczyk, M.; Nowak, J.L.; Siebert, H.; Malinowski, S.P. Detecting nonequilibrium states in atmospheric turbulence. *J. Atmos. Sci.* **2022**, *79*, 2757–2772. [CrossRef]
91. Bos, W.J.T.; Rubinstein, R. Mixing in modulated turbulence. analytical results. *Comput. Fluids* **2017**, *151*, 102–107. [CrossRef]
92. Yang, Y.; Chahine, R.; Rubinstein, R.; Bos, W.J.T. Passive scalar mixing in modulated turbulence. *Fluid Dyn. Res.* **2019**, *51*, 045501. [CrossRef]
93. Liu, F.; Lu, L.P.; Bos, W.J.T.; Fang, L. Assessing the nonequilibrium of decaying turbulence with reversed initial fields. *Phys. Rev. Fluids* **2019**, *4*, 084603. [CrossRef]
94. Shao, X.; Fang, J.; Fang, L. Non-equilibrium dissipation laws in a minimal two-scale wake model. *Phys. Fluids* **2023**, *35*, 085105. [CrossRef]
95. Sagaut, P.; Cambon, C. *Homogeneous Turbulence Dynamics*; Cambridge University Press: Cambridge, UK, 2008.
96. Davidson, P.A. *Turbulence in Rotating, Stratified and Electrically Conducting Fluids*; Cambridge University Press: Cambridge, UK, 2013.
97. Lumley, J. Similarity and the turbulent energy spectrum. *Phys. Fluids* **1967**, *10*, 855. [CrossRef]
98. Yoshida, K.; Ishihara, T.; Kaneda, Y. Anisotropic spectrum of homogeneous turbulent shear flow in a Lagrangian renormalized approximation. *Phys. Fluids* **2003**, *15*, 2385–2397. [CrossRef]
99. Ishihara, T.; Yoshida, K.; Kaneda, Y. Anisotropic Velocity Correlation Spectrum at Small Scales in a Homogeneous Turbulent Shear Flow. *Phys. Rev. Lett.* **2002**, *88*, 154501. [CrossRef] [PubMed]
100. Kaneda, Y.; Yoshida, K. Small-scale anisotropy in stably stratified turbulence. *New J. Phys.* **2004**, *6*, 34. [CrossRef]
101. Ishida, T.; Kaneda, Y. Small-scale anisotropy in magnetohydrodynamic turbulence under a strong uniform magnetic field. *Phys. Fluids* **2007**, *19*, 075104. [CrossRef]
102. Kaneda, Y.; Yamamoto, Y.; Tsuji, Y. Linear response theory for one-point statistics in the inertial sublayer of wall-bounded turbulence. *Phys. Rev. Lett.* **2019**, *122*, 194502. [CrossRef] [PubMed]
103. Kaneda, Y. Linear response theory of turbulence. *J. Stat. Mech. Theory Exp.* **2020**, *2020*, 034006. [CrossRef]
104. Alexakis, A.; Biferale, L. Cascades and transitions in turbulent flows. *Phys. Rep.* **2018**, *767–769*, 1–101. [CrossRef]
105. Marino, R.; Sorriso-Valvo, L. Scaling laws for the energy transfer in space plasma turbulence. *Phys. Rep.* **2023**, *1006*, 1–144. [CrossRef]
106. Zilitinkevich, S.S.; Elperin, T.; Kleeorin, N.; Rogachevskii, I. Energy-and flux-budget (EFB) turbulence closure model for stably stratified flows. Part I: Steady-state, homogeneous regimes. *Bound.-Layer Meteorol.* **2007**, *125*, 167–191. [CrossRef]
107. Rogachevskii, I.; Kleeorin, N.; Zilitinkevich, S. Energy-and flux-budget theory for surface layers in atmospheric convective turbulence. *Phys. Fluids* **2022**, *34*, 116602. [CrossRef]