Article

# On Unique Solvability of a Multipoint Boundary Value Problem for Systems of Integro-Differential Equations with Involution 

Kairat I. Usmanov *, ${ }^{\text {© (D) }}$, Batirkhan Kh. Turmetov ${ }^{\dagger(1)}$ and Kulzina Zh. Nazarova ${ }^{\dagger(\mathbb{D})}$<br>Department of Mathematics, Faculty of Natural Science, Khoja Akhmet Yassawi International Kazakh-Turkish University, Turkistan 161200, Kazakhstan; batyrkhan.turmetov@ayu.edu.kz (B.K.T.); kulzina.nazarova@ayu.edu.kz (K.Z.N.)<br>* Correspondence: kairat.usmanov@ayu.edu.kz<br>$\dagger$ These authors contributed equally to this work.

Citation: Usmanov, K.I; Turmetov, B.K.; Nazarova, K.Z. On Unique Solvability of a Multipoint Boundary Value Problem for Systems of Integro-Differential Equations with Involution. Symmetry 2022, 14, 1626. https:/ /doi.org/10.3390/ sym14081626

Academic Editor: Calogero Vetro

Received: 13 July 2022
Accepted: 4 August 2022
Published: 7 August 2022
Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/).


#### Abstract

In this paper, a multipoint boundary value problem for systems of integro-differential equations with involution has been studied. To solve the studied problem, the parameterization method is used. Based on the parametrization method, the studied problem is decomposed into two parts, i.e., into the Cauchy problem and a system of linear equations. Necessary and sufficient conditions for the unique solvability of the studied problem are determined.


Keywords: multipoint boundary value problem; integro-differential equations; involution; parametrization method; loaded equations

## 1. Introduction

Many applied problems describing processes with aftereffects are known to be determined by integro-differential equations. For example, Volterra's torsional oscillation problem [1]

$$
\omega^{\prime}(t)=k[f(t)-m \omega(t)]+\int_{0}^{t} K(t, s)[f(s)-m \omega(s)] d s,
$$

proctor's problem of elastic beam equilibrium [2]

$$
a_{1} y^{I V}(x)+y(x)=a_{2} \int_{0}^{1} K(t, s) y^{I V}(s) d s
$$

The issues of solvability of initial and boundary value problems for integro-differential equations are discussed in the works of many authors [3-10].

The main methods for studying the unique solvability of a boundary value problem for integro-differential equations are the Green's function method, the Nekrasov method and its analogues. Green's method assumes the unique solvability of a boundary value problem for a differential equation without an integral term. This condition is very stringent, so this method is rarely used. One of the frequently used methods is the Nekrasov method and its analogues. The essence of the Nekrasov method is the reduction of the original equation to an integral equation of the Fredholm type, and its unique solvability is required. In [11], an example was given that shows that the condition of the Nekrasov method is not always satisfied, although the problem under study has a unique solution and this solution is easily determined by the parameterization method.

Recently, to study the problem of unique solvability, the parameterization method proposed by Professor D. Dzhumabaev [12] has been used. In [13-18], this method was
applied in the study of the unique solvability of boundary value problems for various integro-differential equations.

As is known, differential and integro-differential equations with deflecting arguments play an important role in the study of problems in medicine, biology, economics, etc. For example, in [19], an economic model is considered that describes the relationship between population growth and agricultural production. It is shown that if we consider the delay model with positive dispersion, then the dynamics of the economy are determined by a system of integro-differential equations with delay.

Some of these deviations have properties $\alpha:[0, T] \rightarrow[0, T]$ and $\alpha^{2}(t)=\alpha(\alpha(t))=t$. Differential and integro-differential equations, which together with the desired function $x(t)$ include the values $x(\alpha(t))$ and $\dot{x}(\alpha(t))$, are called equations with Carleman shifts [20] or equations with involutive transformations. On the segment $[0, T]$, as such a transformation, we can consider a transformation of the form $\alpha(t)=T-t$.

Solvability of various differential equations with involution was considered in the monographs of D. Przeworska-Rolewicz [21] and J. Wiener [22]. J. Wiener investigated the existence of a solution to a partial differential equation with involution by the method of separation of variables. The properties of such transformations were also considered in the works of N.Karapetiants and S.Samko [23]. The work of Alberto Cabada and F. Tojo is devoted to the construction of the Green's function for one-dimensional differential equations with involution [24] .

The correctness of boundary and initial-boundary value problems for differential equations with various types of involution, qualitative properties of their solution, as well as their spectral issues were quite well studied in [25-28]. Spectral problems for the second-order differential operator were studied in [26,27]. In [28], the eigenfunctions and eigenvalues of the boundary value problem for the nonlocal Laplace equation with multiple involutions were studied.

Multipoint boundary value problems for various differential and integro-differential equations and their applications are considered in [29-31]. It is known that multipoint boundary conditions are important in terms of applications, as they are directly related to the theory of splines and interpolation, and are also used in the study of problems with multi-support beams. For example, in [32], multipoint boundary conditions are applied in the design of bridges.

Therefore, in this paper we decided to investigate a multipoint boundary value problem for integro-differential equations with involutive properties. To determine the unique solvability of the studied problem, the parameterization method was applied.

Consider a multipoint boundary value problem with an involutive transformation

$$
\begin{gather*}
\frac{d x(t)}{d t}+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \cdot \frac{d x(\alpha(t))}{d t}=\int_{0}^{T} K_{1}(t, s) x(s) d s \\
+\int_{0}^{T} K_{2}(t, s) \dot{x}(s) d s+f(t), \quad t \in[0, T], \quad x \in R^{n}  \tag{1}\\
\sum_{i=0}^{m} B_{i} x\left(\theta_{i}\right)=d, d \in R^{n}  \tag{2}\\
0=\theta_{0}<\theta_{1}<\ldots<\theta_{m-1}<\theta_{m}=T
\end{gather*}
$$

where matrices $K_{1}(t, s)$ and $K_{2}(t, s)$ are continuous on $[0, T] \times[0, T]$, and $n$-dimensional vector function $f(t)$ is continuous, respectively, on $[0, T] . a_{j} \in R, j=\overline{1, n} . B_{i}, j=\overline{0, m}$ are constant matrices.

Remark 1. In Equation (1) and further, the expression $\frac{d x(\alpha(t))}{d t}$ will mean

$$
\frac{d x(\alpha(t))}{d t}=\left.\frac{d x(\xi)}{d t}\right|_{\xi=\alpha(t)}
$$

Note that problem (1) and (2) in the case when $a_{j}=0, j=\overline{1, n}, K_{2}(t, s) \equiv 0$ and $B_{i}=0$, $j=\overline{1, m-1}$ was studied in [13,14].

## 2. Using Involution Properties

Let us determine the value of Equation (1) at the point $t^{*}=\alpha(t)$

$$
\begin{aligned}
& \frac{d x(\alpha(t))}{d t}+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \cdot \frac{d x(t)}{d t}=\int_{0}^{T} K_{1}(\alpha(t), s) x(s) d s \\
& \quad+\int_{0}^{T} K_{2}(\alpha(t), s) \dot{x}(s) d s+f(\alpha(t)) .
\end{aligned}
$$

Then, we obtain the following system of equations

$$
\left\{\begin{aligned}
\frac{d x(t)}{d t}+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \cdot \frac{d x(\alpha(t))}{d t}= & \int_{0}^{T} K_{1}(t, s) x(s) d s+\int_{0}^{T} K_{2}(t, s) \dot{x}(s) d s+f(t), \\
\frac{d x(\alpha(t))}{d t}+\operatorname{diag}\left(a_{1},\right. & \left.a_{2}, \ldots, a_{n}\right) \cdot \frac{d x(t)}{d t}=\int_{0}^{T} K_{1}(\alpha(t), s) x(s) d s \\
& +\int_{0}^{T} K_{2}(\alpha(t), s) \dot{x}(s) d s+f(\alpha(t)) .
\end{aligned}\right.
$$

Multiplying the second equation by the matrix $-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ on the left side, and adding both equations we obtain

$$
\begin{gathered}
\operatorname{diag}\left(1-a_{1}^{2}, 1-a_{2}^{2}, \ldots, 1-a_{n}^{2}\right) \frac{d x(t)}{d t} \\
=\int_{0}^{T}\left[K_{1}(t, s)-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{1}(\alpha(t), s)\right] x(s) d s \\
+\int_{0}^{T}\left[K_{2}(t, s)-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{2}(\alpha(t), s)\right] \dot{x}(s) d s \\
\quad+\left[f(t)-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) f(\alpha(t))\right] .
\end{gathered}
$$

Let $a_{i} \neq \pm 1, i=\overline{1, n}$, then the original boundary value problem can be written as

$$
\begin{gather*}
\frac{d x}{d t}=\int_{0}^{T} \tilde{K}_{1}(t, s) x(s) d s+\int_{0}^{T} \tilde{K}_{2}(t, s) \dot{x}(s) d s+\tilde{f}(t), \quad t \in[0, T]  \tag{3}\\
\sum_{i=0}^{m} B_{i} x\left(\theta_{i}\right)=d, d \in R^{n}  \tag{4}\\
0=\theta_{0}<\theta_{1}<\ldots<\theta_{m-1}<\theta_{m}=T
\end{gather*}
$$

where

$$
\begin{gathered}
\tilde{K}_{1}(t, s)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)\left[K_{1}(t, s)\right. \\
\left.\quad-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{1}(\alpha(t), s)\right] \\
\tilde{K}_{2}(t, s)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)\left[K_{2}(t, s)\right.
\end{gathered}
$$

$$
\begin{gathered}
\left.-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{2}(\alpha(t), s)\right] \\
\tilde{f}(t)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)[f(t) \\
\left.-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) f(\alpha(t))\right]
\end{gathered}
$$

It is important to note that the condition $a_{i} \neq \pm 1, i=\overline{1, n}$ is significant.
Indeed, let us consider the following homogeneous boundary value problem with involution $a=1$

$$
\begin{gathered}
\frac{d x(t)}{d t}+\frac{d x(-t)}{d t}=\int_{-\pi}^{\pi} x(s) d s+\int_{-\pi}^{\pi} \dot{x}(s) d s, \quad t \in[-\pi, \pi] \\
x(-\pi)-x(\pi)=0
\end{gathered}
$$

This problem has a solution $x(t)=\cos (k t)$. It turns out that the homogeneous boundary value problem has a set of nonzero solutions. In the case $a=-1$, as a nonzero solution, we can take the function $x(t)=\sin (k t)$. The boundary value problems (1)-(4) are equivalent in the sense that if $x(t)$ is a solution to the multipoint boundary value problem (3) and (4), then it also satisfies the multipoint boundary value problem (1), (2) and vice versa.

Suppose that $a_{i} \neq \pm 1, i=\overline{1, n}$ and let $x^{*}(t)$ be a solution to problem (3) and (4), then $x^{*}(t)$ also satisfies (2). As $x^{*}(t)$ is the solution to (3), then substituting $x^{*}(t)$ into the right-hand side of Equation (3), we obtain:

$$
\begin{equation*}
\frac{d x^{*}(t)}{d t}=\int_{0}^{T} \tilde{K}_{1}(t, s) x^{*}(s) d s+\int_{0}^{T} \tilde{K}_{2}(t, s) \dot{x}^{*}(s) d s+\tilde{f}(t), \quad t \in[0, T] . \tag{5}
\end{equation*}
$$

Consider the value of Equation (5) in the point $t=\alpha(t) \in[0, T]$

$$
\begin{equation*}
\frac{d x^{*}(\alpha(t))}{d t}=\int_{0}^{T} \tilde{K}_{1}(\alpha(t), s) x^{*}(s) d s+\int_{0}^{T} \tilde{K}_{2}(\alpha(t), s) \dot{x}^{*}(s) d s+\tilde{f}(\alpha(t)), \quad \alpha(t) \in[0, T] . \tag{6}
\end{equation*}
$$

Multiplying Equation (6) by $\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and adding it to Equation (5) we obtain:

$$
\begin{align*}
& \frac{d x^{*}(t)}{d t}+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \frac{d x^{*}(\alpha(t))}{d t}  \tag{7}\\
& =\int_{0}^{T}\left[\tilde{K}_{1}(t, s)+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \tilde{K}_{1}(\alpha(t), s)\right] x^{*}(s) d s \\
& +\int_{0}^{T}\left[\tilde{K}_{2}(t, s)+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \tilde{K}_{2}(\alpha(t), s)\right] \dot{x}^{*}(s) d s \\
& +\left[\tilde{f}(t)+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \tilde{f}(\alpha(t))\right], \quad t \in[0, T] .
\end{align*}
$$

Substituting

$$
\begin{gathered}
\tilde{K}_{1}(t, s)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)\left[K_{1}(t, s)\right. \\
\left.\quad-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{1}(\alpha(t), s)\right] \\
\tilde{K}_{2}(t, s)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)\left[K_{2}(t, s)\right. \\
\left.\quad-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) K_{2}(\alpha(t), s)\right] \\
\tilde{f}(t)=\operatorname{diag}\left(1 /\left(1-a_{1}^{2}\right), 1 /\left(1-a_{2}^{2}\right), \ldots, 1 /\left(1-a_{n}^{2}\right)\right)[f(t) \\
\left.-\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) f(\alpha(t))\right]
\end{gathered}
$$

in (7), we get

$$
\begin{align*}
& \frac{d x^{*}(t)}{d t}+\operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right) \frac{d x^{*}(\alpha(t))}{d t}=\int_{0}^{T} K_{1}(t, s) x^{*}(s) d s  \tag{8}\\
& \\
& \quad+\int_{0}^{T} K_{2}(t, s) \dot{x}^{*}(s) d s+f(t), \quad t \in[0, T] .
\end{align*}
$$

Hence, it follows that $x^{*}(t)$ satisfies (1).
Vice versa, let $\tilde{x}(t)$ be a solution to Equations (1) and (2), then it is easy to show that $\tilde{x}(t)$ satisfies (3) and (4).

## 3. Parameterization Method

In [33], it was assumed that the Fredholm integral equation of the second kind

$$
z(t)=\int_{0}^{T} \tilde{K}_{2}(t, s) z(s) d s+\Phi(t)
$$

has a unique solution for any function $\Phi(t) \in C\left([0, T], R^{n}\right)$.
However, it is known that the resolvent of an integral equation cannot always be determined unambiguously.

Suppose that $K_{2}(t, s)$ has continuous partial derivatives with respect to $s$, then

$$
\begin{aligned}
& \int_{0}^{T} \tilde{K}_{2}(t, s) \dot{x}(s) d s=\left.\tilde{K}_{2}(t, s) x(s)\right|_{0} ^{T}-\int_{0}^{T} \frac{\partial \tilde{K}_{2}(t, s)}{\partial s} x(s) d s \\
& \quad=\tilde{K}_{2}(t, T) x(T)-\tilde{K}_{2}(t, 0) x(0)-\int_{0}^{T} \frac{\partial \tilde{K}_{2}(t, s)}{\partial s} x(s) d s
\end{aligned}
$$

Hence, Equations (3) and (4) can be written as:

$$
\begin{gather*}
\frac{d x}{d t}=\int_{0}^{T} K(t, s) x(s) d s+K_{20}(t) x\left(\theta_{0}\right)+K_{21}(t) x\left(\theta_{m}\right)+\tilde{f}(t)  \tag{9}\\
\sum_{i=0}^{m} B_{i} x\left(\theta_{i}\right)=d, d \in R^{n}  \tag{10}\\
0=\theta_{0}<\theta_{1}<\ldots<\theta_{m-1}<\theta_{m}=T
\end{gather*}
$$

where

$$
\begin{gathered}
K_{20}(t)=-\tilde{K}_{2}(t, 0) \\
K_{21}(t)=\tilde{K}_{2}(t, T) \\
\int_{0}^{T} K(t, s) x(s) d s=\int_{0}^{T} \tilde{K}_{1}(t, s) x(s) d s-\int_{0}^{T} \frac{\partial \tilde{K}_{2}(t, s)}{\partial s} x(s) d s
\end{gathered}
$$

Let us apply the parametrization method to the boundary value problem (9) and (10), for this we take a natural number $l \in \mathrm{~N}$ and make a partition with respect to this number: $[0, T)=\bigcup_{r=1}^{m(l+1)}\left[t_{r-1}, t_{r}\right)$, where $t_{i(l+1)+j}=t_{i(l+1)}+\frac{h_{i+1}}{l}, \quad h_{i}=\theta_{i}-\theta_{i-1}$, $i=\overline{0, m-1}, \quad j=\frac{r=1}{1, l+1}$. Denote $h=\max \left\{h_{1}, h_{1}, \ldots, h_{m}\right\}, \beta=\max _{t, s \in[0, T]}\|K(t, s)\|$. Let
us use $x_{r}(t), \quad r=\overline{1, m(l+1)}$ to denote the narrowing of the function $x(t)$ on the intervals $\left[t_{r-1}, t_{r}\right), \quad r=\overline{1, m(l+1)}$. Then, the multipoint boundary value problem for systems of loaded integro-differential Equations (9) on (10) can be written as:

$$
\begin{gather*}
\frac{d x_{r}}{d t}=\sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(t, s) x_{i}(s) d s+K_{20}(t) x_{1}\left(t_{0}\right)  \tag{11}\\
+K_{21}(t) \lim _{t \rightarrow T-0} x_{m(l+1)}(t)+\tilde{f}(t), t \in\left[t_{r-1}, t_{r}\right), r=\overline{1, m(l+1)}, \\
\sum_{i=0}^{m-1} B_{i} x_{j(l+1)+1}\left(t_{j(l+1)}\right)+B_{m} \lim _{t \rightarrow T-0} x_{m(l+1)}(t)=d, d \in R^{n},  \tag{12}\\
\lim _{t \rightarrow t_{s}-0} x_{s}(t)=x_{s+1}\left(t_{s}\right), s=\overline{1, m(l+1)-1}, \tag{13}
\end{gather*}
$$

here (13) provides conditions for continuity of the solution in the points of partition. Let us introduce the notation $\lambda_{r}=x_{r}\left(t_{r-1}\right), r=\overline{1, m(l+1)}, \lambda_{m(l+1)+1}=\lim _{t \rightarrow T-0} x_{m(l+1)}(t)$ make a substitution $x_{r}(t)=u_{r}(t)+\lambda_{r}, r=\overline{1, m(l+1)}$ in each of the intervals $t \in\left[t_{r-1}, t_{r}\right)$. Then, problem (11)-(13) is reduced to the equivalent multipoint boundary value problem with the parameter

$$
\begin{gather*}
\frac{d u_{r}}{d t}=\sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(t, s)\left[u_{i}(s)+\lambda_{i}\right] d s+K_{20}(t) \lambda_{1}  \tag{14}\\
+K_{21}(t) \lambda_{m(l+1)+1}+\tilde{f}(t), \quad t \in\left[t_{r-1}, t_{r}\right), r=\overline{1, m(l+1)}, \\
u_{r}\left(t_{r-1}\right)=0, \quad r=\overline{1, m(l+1)},  \tag{15}\\
\sum_{i=0}^{m} B_{i} \lambda_{j(l+1)+1}=d, d \in R^{n},  \tag{16}\\
\lambda_{s}+\lim _{t \rightarrow t_{s}-0} u_{s}(t)=\lambda_{s+1}, \quad s=\overline{1, m(l+1)} . \tag{17}
\end{gather*}
$$

The initial conditions $u_{r}\left(t_{r-1}\right)=0, r=\overline{1, m(l+1)}$ make it possible to determine functions $u_{r}(t)$ for fixed values $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m(l+1)+1}\right)$ from the systems of integral equations

$$
\begin{align*}
& u_{r}(t)=\int_{t_{r-1}}^{t} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) u_{i}(s) d s d \tau+\int_{t_{r-1}}^{t} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) d s d \tau \lambda_{i}  \tag{18}\\
& +\int_{t_{r-1}}^{t} K_{20}(\tau) d \tau \lambda_{1}+\int_{t_{r-1}}^{t} K_{21}(\tau) d \tau \lambda_{m(l+1)+1}+\int_{t_{r-1}}^{t} \tilde{f}(\tau) d \tau, \quad t \in\left[t_{r-1}, t_{r}\right) .
\end{align*}
$$

In (18), assuming that $t=\tau$, we multiply both sides of the equation by $K(t, \tau)$ and integrate with respect to $\tau$ in the interval $\left[t_{r-1}, t_{r}\right)$. Then,

$$
\begin{align*}
\int_{t_{r-1}}^{t_{r}} K(t, \tau) & u_{r}(\tau) d \tau=\int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K\left(\tau_{1}, s\right) u_{i}(s) d s d \tau_{1} d \tau  \tag{19}\\
& +\int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K\left(\tau_{1}, s\right) d s d \tau_{1} d \tau \lambda_{i}
\end{align*}
$$

$$
+\int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau}\left[K_{20}\left(\tau_{1}\right) \lambda_{1}+K_{21}\left(\tau_{1}\right) \lambda_{m(l+1)+1}+\tilde{f}\left(\tau_{1}\right)\right] d \tau_{1} d \tau, \quad r=\overline{1, m(l+1)}
$$

Summing up the left and right sides of Equation (19), we obtain:

$$
\begin{align*}
& \sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) u_{r}(\tau) d \tau=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K\left(\tau_{1}, s\right) u_{i}(s) d s d \tau_{1} d \tau  \tag{20}\\
& +\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau}\left[\sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K\left(\tau_{1}, s\right) d s d \tau_{1} d \tau \lambda_{i}\right. \\
& \left.\left.\quad+K_{20}\left(\tau_{1}\right) \lambda_{1}+K_{21}\left(\tau_{1}\right) \lambda_{m(l+1)+1}+\tilde{f}\left(\tau_{1}\right)\right] d \tau_{1} d \tau,\right] \quad t \in[0, T]
\end{align*}
$$

Let us introduce the notations:

$$
\begin{gathered}
\Phi_{l}(t)=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) u_{r}(\tau) d \tau \\
H_{r}(l, t)=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) X(\tau) \int_{t_{r-1}}^{\tau} X^{-1}\left(\tau_{1}\right) \int_{t_{j-1}}^{t_{j}} K\left(\tau_{1}, s\right) d s d \tau_{1} d \tau \\
P_{1}(l, t)=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} K_{20}\left(\tau_{1}\right) d \tau_{1} d \tau \\
P_{2}(l, t)=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} K_{21}\left(\tau_{1}\right) d \tau_{1} d \tau \\
F(l, t)=\sum_{r=1}^{m(l+1)} \int_{t_{r-1}}^{t_{r}} K(t, \tau) \int_{t_{r-1}}^{\tau} \tilde{f}\left(\tau_{1}\right) d \tau_{1} d \tau
\end{gathered}
$$

Then, Equation (17) can be written as:

$$
\begin{align*}
\Phi_{l}(t)= & \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(t, \tau) \int_{t_{i-1}}^{\tau} \Phi_{l}\left(\tau_{1}\right) d \tau_{1} d \tau+\sum_{r=1}^{m(l+1)} H_{r}(l, t) \lambda_{r}  \tag{21}\\
& +P_{1}(l, t) \lambda_{1}+P_{2}(l, t) \lambda_{m(l+1)+1}+F(l, t) .
\end{align*}
$$

Let us take such $l_{0}$ that $q\left(l_{0}\right)=\beta T \frac{h}{l_{0}}<1$. Then, from the estimation

$$
\begin{equation*}
\left\|\sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(t, \tau) \int_{t_{i-1}}^{\tau} \Phi_{l}\left(\tau_{1}\right) d \tau_{1} d \tau\right\| \leq \beta T \frac{h}{l_{0}} \max _{t \in[0, T]}\left\|\Phi_{l}(t)\right\|, \quad t \in[0, T] \tag{22}
\end{equation*}
$$

it follows that for any $l \geq l_{0}$ Equation (21) has a unique solution.
The set of all $l$ for which the Cauchy problem (14), (15) has a unique solution is called a regular partition and is denoted by $\Delta_{l}$. As can be seen from (22), this set is nonempty.

Let $l \in \Delta_{l}$. Using the successive approximation method, we determine

$$
\Phi_{l}(t)=\sum_{i=1}^{m(l+1)} H_{i}^{*}(l, t) \lambda_{i}+P_{1}^{*}(l, t) \lambda_{1}+P_{2}^{*}(l, t) \lambda_{m(l+1)+1}+F^{*}(l, t)
$$

the unique solution to Equation (21). Substituting the obtained expression for $\Phi_{l}(t)$ into the right-hand side of (18), we get:

$$
\begin{align*}
& u_{r}(t)=\int_{t_{r-1}}^{t}\left[\sum_{i=1}^{m(l+1)} H_{i}^{*}(l, \tau) \lambda_{i}+P_{1}^{*}(l, \tau) \lambda_{1}+P_{2}^{*}(l, \tau) \lambda_{m(l+1)+1}+F^{*}(l, \tau)\right] d \tau  \tag{23}\\
& +\int_{t_{r-1}}^{t} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) d s d \tau \lambda_{i}+\int_{t_{r-1}}^{t} K_{20}(\tau) d \tau \lambda_{1} \\
& \quad+\int_{t_{r-1}}^{t} K_{21}(\tau) d \tau \lambda_{m(l+1)+1}+\int_{t_{r-1}}^{t} \tilde{f}(\tau) d \tau, \quad t \in\left[t_{r-1}, t_{r}\right), r=\overline{1, m(l+1)} .
\end{align*}
$$

Determining the limits $\lim _{t \rightarrow t_{s}-0} u_{s}(t), s=\overline{1, m(l+1)}$ from (23) and substituting them into the boundary conditions (17), we obtain a system of linear equations with respect to the introduced parameters $\lambda_{r}, r=1, m(l+1)+1$

$$
\begin{gather*}
\sum_{i=0}^{m} B_{i} \lambda_{j(l+1)+1}=d  \tag{24}\\
\lambda_{s}+\int_{t_{s-1}}^{t_{s}} \sum_{i=1}^{m(l+1)}\left[H_{i}^{*}(l, \tau)+\int_{t_{i-1}}^{t_{i}} K(\tau, s) d s\right] d \tau \lambda_{i}+\int_{t_{s-1}}^{t_{s}}\left[P_{1}^{*}(l, \tau)+K_{20}(\tau)\right] d \tau \lambda_{1}  \tag{25}\\
+\int_{t_{s-1}}^{t_{s}}\left[P_{2}^{*}(l, \tau)+K_{21}(\tau)\right] d \tau \lambda_{m(l+1)+1}-\lambda_{s+1} \\
=-\int_{t_{s-1}}^{t_{s}}\left[F^{*}(l, \tau)+\tilde{f}(\tau)\right] d \tau, s=\overline{1, m(l+1)}
\end{gather*}
$$

The matrix corresponding to the right-hand side of the system of algebraic equations is denoted by $Q_{*}(l)$. Then, the system of Equations (24) and (25) can be written in the following matrix form:

$$
\begin{equation*}
Q_{*}(l) \lambda=F_{*}(l), \quad \lambda \in R^{n[m(l+1)+1]}, \tag{26}
\end{equation*}
$$

where

$$
F_{*}(l)=\left(d, \quad-\int_{0}^{t_{1}}\left(F^{*}(l, \tau)+f(\tau)\right) d \tau, \ldots,-\int_{T-h_{m}}^{T}\left(F^{*}(l, \tau)+f(\tau)\right) d \tau\right)
$$

Lemma 1. For $l \in \Delta_{l}$ the following statements are valid:

1. Vector $\lambda^{*}=\left(\lambda_{1}^{*}, \lambda_{2}^{*}, \ldots, \lambda_{m(l+1)+1}^{*}\right) \in R^{n[m(l+1)+1]}$ composed of the values of the solution to problem (9), (10)-functions $x^{*}(t)$ in the points $\lambda_{r}^{*}=x^{*}\left(t_{r-1}\right), \quad r=\overline{1, m(l+1)+1}$ satisfies the system of Equation (26);
2. If $\tilde{\lambda}=\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)+1}\right) \in R^{n[m(l+1)+1]}$ is a solution to system (26), and the system of functions $\tilde{u}[t]=\left(\tilde{u}_{1}(t), \tilde{u}_{2}(t), \ldots, \tilde{u}_{m(l+1)}(t)\right)$ is a solution to the special Cauchy problem (14), (15) for $\lambda_{r}=\tilde{\lambda}_{r}, \quad r=\overline{1, m(l+1)+1}$, then the function $\tilde{x}(t), \tilde{x}(t)=\tilde{\lambda}_{r}+\tilde{u}_{r}(t)$, $t \in\left[t_{r-1}, t_{r}\right), r=\overline{1, m(l+1)}, \tilde{x}(T)=\tilde{\lambda}_{m(l+1)+1}$ is a solution to problem (9), (10).

Proof. 1. Let $x^{*}(t)$ be a solution to problem (9), (10). Then, a pair $\left(\lambda^{*}, u^{*}[t]\right)$ with elements $\lambda_{j}^{*}=x^{*}\left(t_{j-1}\right), t \in\left[t_{j-1}, t_{j}\right), j=\overline{1, m(l+1)+1}$ è $u_{r}^{*}(t)=x^{*}(t)-x^{*}\left(t_{r-1}\right)$, $t \in\left[t_{r-1}, t_{r}\right), \quad r=\overline{1, m(l+1)}$ will be a solution to a problem equivalent to the boundary value problem with parameter (14)-(17). Taking into account the assumption $l \in \Delta_{l}$ and repeating the above reasoning, we find that $\lambda^{*}=\left(\lambda_{1}^{*}, \lambda_{2}^{*}, \ldots, \lambda_{m(l+1)+1}^{*}\right) \in R^{n[m(l+1)+1]}$ satisfies the system of Equations (26).
2. Let $\tilde{\lambda}=\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)+1}\right) \in R^{n[m(l+1)+1]}$ be a solution to systems of Equation (26). As $l \in \Delta_{l}$, the special Cauchy problem (14), (15) has a unique solution for any $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m(l+1)+1}\right) \in R^{n[m(l+1)+1]}$. Denoteits solution for any $\tilde{\lambda}=\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)+1}\right)$ $\in R^{n[m(l+1)+1]}$ as $\tilde{u}[t]=\left(\tilde{u}_{1}(t), \tilde{u}_{2}(t), \ldots, \tilde{u}_{m(l+1)}(t)\right)$. Let us show that the pair $(\tilde{\lambda}, \tilde{u}[t])$ is a solution to problem (14)-(17). If $\tilde{\lambda}=\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)+1}\right) \in R^{n[m(l+1)+1]}$ satisfies (26), then (24) and (25) are valid for it, i.e.,

$$
\begin{gather*}
\sum_{i=0}^{m} B_{i} \tilde{\lambda}_{j(l+1)+1}=d  \tag{27}\\
\tilde{\lambda}_{s}+\int_{t_{s-1}}^{t_{s}} \sum_{i=1}^{m(l+1)}\left[H_{i}^{*}(l, \tau)+\int_{t_{i-1}}^{t_{i}} K(\tau, s) d s\right] d \tau \tilde{\lambda}_{i}+\int_{t_{s-1}}^{t_{s}}\left[P_{1}^{*}(l, \tau)+K_{20}(\tau)\right] d \tau \tilde{\lambda}_{1}  \tag{28}\\
+\int_{t_{s-1}}^{t_{s}}\left[P_{2}^{*}(l, \tau)+K_{21}(\tau)\right] d \tau \tilde{\lambda}_{m(l+1)+1}-\tilde{\lambda}_{s+1} \\
=-\int_{t_{s-1}}^{t_{s}}\left[F^{*}(l, \tau)+\tilde{f}(\tau)\right] d \tau, s=\overline{1, m(l+1)}
\end{gather*}
$$

Condition (16) follows from (27). We rewrite (28) as

$$
\begin{align*}
& \tilde{\lambda}_{s}+\left\{\int_{t_{s-1}}^{t_{s}}\left[\sum_{i=1}^{m(l+1)} H_{i}^{*}(l, \tau) \tilde{\lambda}_{i}+P_{1}^{*}(l, \tau) \tilde{\lambda}_{1}+P_{2}^{*}(l, \tau) \tilde{\lambda}_{m(l+1)+1}+F^{*}(l, \tau)\right] d \tau\right. \\
&+\int_{t_{s-1}}^{t_{s}} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) d s d \tau \tilde{\lambda}_{i}+\int_{t_{s-1}}^{t_{s}} K_{20}(\tau) d \tau \lambda_{1} \\
&\left.+\int_{t_{s-1}}^{t_{s}} K_{21}(\tau) d \tau \lambda_{m(l+1)+1}+\int_{t_{s-1}}^{t_{s}} \tilde{f}(\tau) d \tau\right\}=\lambda_{s+1}, \quad s=\overline{1, m(l+1)} . \tag{29}
\end{align*}
$$

From (23) we get that

$$
u_{s}(t)=\int_{t_{s-1}}^{t}\left[\sum_{i=1}^{m(l+1)} H_{i}^{*}(l, \tau) \tilde{\lambda}_{i}+P_{1}^{*}(l, \tau) \tilde{\lambda}_{1}+P_{2}^{*}(l, \tau) \tilde{\lambda}_{m(l+1)+1}+F^{*}(l, \tau)\right] d \tau
$$

$$
\begin{aligned}
& +\int_{t_{s-1}}^{t} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) d s d \tau \tilde{\lambda}_{i}+\int_{t_{s-1}}^{t} K_{20}(\tau) d \tau \tilde{\lambda}_{1} \\
+ & \int_{t_{s-1}}^{t} K_{21}(\tau) d \tau \tilde{\lambda}_{m(l+1)+1}+\int_{t_{s-1}}^{t} \tilde{f}(\tau) d \tau, \quad t \in\left[t_{s-1}, t_{s}\right) .
\end{aligned}
$$

Because

$$
\begin{aligned}
\lim _{t \rightarrow t_{s}-0} u_{s}(t)= & \int_{t_{s-1}}^{t_{s}}\left[\sum_{i=1}^{m(l+1)} H_{i}^{*}(l, \tau) \tilde{\lambda}_{i}+P_{1}^{*}(l, \tau) \tilde{\lambda}_{1}+P_{2}^{*}(l, \tau) \tilde{\lambda}_{m(l+1)+1}+F^{*}(l, \tau)\right] d \tau \\
& +\int_{t_{s-1}}^{t_{s}} \sum_{i=1}^{m(l+1)} \int_{t_{i-1}}^{t_{i}} K(\tau, s) d s d \tau \tilde{\lambda}_{i}+\int_{t_{s-1}}^{t_{s}} K_{20}(\tau) d \tau \tilde{\lambda}_{1} \\
& +\int_{t_{s-1}}^{t_{s}} K_{21}(\tau) d \tau \tilde{\lambda}_{m(l+1)+1}+\int_{t_{s-1}}^{t_{s}} \tilde{f}(\tau) d \tau .
\end{aligned}
$$

Since the expression in curly bracket (29) determines $\lim _{t \rightarrow t_{s}-0} u_{s}(t)$, then from (29) the validity of relation (17) follows. Then, the function $\tilde{x}(t)$ constructed using the pair $\left[\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)}\right),\left(\tilde{u}_{1}(t), \tilde{u}_{2}(t), \ldots, \tilde{u}_{m(l+1)}(t)\right)\right]$, i.e., $\tilde{x}(t)=\tilde{u}_{r}(t)+\tilde{\lambda}_{r}, t \in\left[t_{r-1}, t_{r}\right)$, $r=\overline{1, m(l+1)}, \tilde{x}(T)=\tilde{\lambda}_{m(l+1)+1}$ will be a solution to problem (9) and (10). The lemma is proved.

Definition 1. Problem (1) and (2) is called uniquely solvable if for any pair $(f(t), d)$ it has a unique solution $x(t)$.

Theorem 1. Let the conditions $a_{i} \neq \pm 1, i=\overline{1, n}$ be satisfied. Then, the boundary value problem (1), (2) is uniquely solvable if and only if for any $l \in \Delta_{l}$ matrix $Q_{*}(l)$ is reversible.

Proof. Let the matrix $Q_{*}(l)$ be reversible for $l \in \Delta_{l}$ and $f(t) \in C\left([0, T], \Delta_{m(l+1)}, R^{n}\right)$, $d \in R^{n}$. Using the reversibility of the matrix $Q_{*}(l)$, we find a unique solution to the system of linear algebraic equations:

$$
\lambda^{*}=-\left[Q_{*}(l)\right]^{-1} \cdot F_{*}(l), \quad \lambda^{*} \in R^{n[m(l+1)+1]}
$$

The solution of the special Cauchy problem (14), (15) for $\lambda=\lambda^{*}$ defines the system of functions $u^{*}[t]=\left(u_{1}^{*}(t), u_{2}^{*}(t), \ldots, u_{m(l+1)}^{*}(t)\right)$. Due to the regularity of the partition $l \in \Delta_{l}$, there must exist systems of functions $u^{*}[t]$ with elements $u_{r}^{*}(t), r=\overline{1, m(l+1)}$ determined by the right-hand part of (23) for $\lambda=\lambda^{*}=\left(\lambda_{1}^{*}, \lambda_{2}^{*}, \ldots, \lambda_{m(l+1)+1}^{*}\right) \in R^{n[m(l+1)+1]}$. Then, according to lemma the function $x^{*}(t)$ defined by the equalities $x^{*}(t)=\lambda_{r}^{*}+$ $u_{r}^{*}(t), \quad t \in\left[t_{r-1}, t_{r}\right), \quad r=\overline{1, m(l+1)}, x^{*}(T)=\lambda_{m(l+1)+1}^{*}$ is a solution to problem (9), (10). Therefore, $x^{*}(t)$ satisfies (10). Equation (9) can be written as:

$$
\begin{align*}
& \frac{d x^{*}}{d t}=\int_{0}^{T} \tilde{K}_{1}(t, s) x^{*}(s) d s-\int_{0}^{T} \frac{\partial \tilde{K}_{2}(t, s)}{\partial s} x^{*}(s) d s \\
& \quad-\tilde{K}_{2}(t, 0) x^{*}\left(\theta_{0}\right)+\tilde{K}_{2}(t, T) x^{*}\left(\theta_{m}\right)+\tilde{f}(t) \tag{30}
\end{align*}
$$

As

$$
\int_{0}^{T} \tilde{K}_{2}(t, s) \dot{x}^{*}(s) d s=-\int_{0}^{T} \frac{\partial \tilde{K}_{2}(t, s)}{\partial s} x^{*}(s) d s-\tilde{K}_{2}(t, 0) x^{*}(0)+\tilde{K}_{2}(t, T) x^{*}(T)
$$

$x^{*}(t)$ satisfies

$$
\frac{d x^{*}}{d t}=\int_{0}^{T} \tilde{K}_{1}(t, s) x^{*}(s) d s+\int_{0}^{T} \tilde{K}_{2}(t, s) \dot{x}^{*}(s) d s+\tilde{f}(t)
$$

i.e., $x^{*}(t)$ satisfies (9).

From the equivalence of boundary value problems (9), (10) and (1), (2) under the condition $a_{i} \neq \pm 1, i=\overline{1, n}$ it follows that $x^{*}(t)$ is a solution to problem (1) and (2).

Let us show the uniqueness of the solution.
Suppose that problem (1), (2) besides solution $x^{*}(t)$ has the other solution $\widetilde{x}(t)$. Then, the boundary value problem (14)-(17) has, respectively, solutions $\left(\lambda^{*}, u^{*}[t]\right)$ and $(\tilde{\lambda}, \tilde{u}[t])$.

According to Lemma 1, the system of Equation (26) is satisfied for $\lambda^{*}$, as well as for $\tilde{\lambda}$, i.e.,

$$
Q_{*}(l) \cdot \lambda^{*}=-F_{*}(l), \quad Q_{*}(l) \cdot \tilde{\lambda}=-F_{*}(l) .
$$

However, the matrix $Q_{*}(l)$ is reversible, hence it follows that $\lambda^{*}=\tilde{\lambda}$. For our regular partitioning, the special Cauchy problem has a unique solution, therefore $u^{*}{ }_{r}[t]=\tilde{u}_{r}[t]$, $t \in\left[t_{r-1}, t_{r}\right), r=1, m(l+1)$. Hence, $x^{*}(t)=\widetilde{x}(t)$.

Let us prove the necessity. Fulfillment of the condition $a_{i} \neq \pm 1, i=\overline{1, n}$ makes it possible to pass from the boundary value problem (1), (2) to the boundary value problem (9), (10). Let problem (9), (10) be uniquely solvable and $l \in \Delta_{l}$. Suppose that, if the problem (9), (10) is uniquely solvable, the matrix $Q_{*}(l)$ is not reversible. This is possible only in case of a nonzero solution of the homogeneous system $Q_{*}(l) \cdot \lambda=0, \quad \lambda \in R^{n[m(l+1)+1]}$.

Suppose that $\tilde{\lambda}=\left(\tilde{\lambda}_{1}, \tilde{\lambda}_{2}, \ldots, \tilde{\lambda}_{m(l+1)+1}\right) \in R^{n[m(l+1)+1]}$ is a nonzero solution of (26). This means that for $f(t)=0, d=0$ the homogeneous boundary value problem (1), (2) has a nonzero solution. However, this contradicts the unique solvability of the boundary value problem (1) and (2).

## 4. An Illustrative Example

Example 1. Consider the following three-point boundary value problem in the segment $[0,1]$ :

$$
\begin{align*}
\dot{x}(t)-2 \dot{x}(1-t)= & \int_{0}^{1}(3 t+1) x(s) d s+3 \int_{0}^{1}(t+s) \dot{x}(s) d s-3 t-1  \tag{31}\\
& x(0)-2 x\left(\frac{1}{2}\right)+x(1)=0 . \tag{32}
\end{align*}
$$

Consider the values of Equation (31) in the point $t^{*}=1-t$

$$
\begin{equation*}
\dot{x}(1-t)-2 \dot{x}(t)=\int_{0}^{1}(4-3 t) x(s) d s+3 \int_{0}^{1}(1-t+s) \dot{x}(s) d s+3 t-4 . \tag{33}
\end{equation*}
$$

Then, from the system of Equations (31) and (33) we get

$$
\dot{x}(t)=\int_{0}^{1}(t-3) x(s) d s+3 \int_{0}^{1}(t-3 s-2) \dot{x}(s) d s-t+3 .
$$

Integrating the second integral by parts and grouping the corresponding terms, we obtain the following equivalent boundary value problem:

$$
\begin{gather*}
\dot{x}(t)=\int_{0}^{1} t x(s) d s+(t-5) x(1)-(t-2) x(0)-t+3  \tag{34}\\
x(0)-2 x\left(\frac{1}{2}\right)+x(1)=0 . \tag{35}
\end{gather*}
$$

Let us divide the segment $[0,1)$ into two parts $[0,1)=\left[0, \frac{1}{2}\right) \cup\left[\frac{1}{2}, 1\right)$. Introduce parameters $\lambda_{1}=x_{1}(0), \quad \lambda_{2}=x_{1}(1 / 2), \quad \lambda_{3}=\lim _{t \rightarrow 1} x_{2}(t)$ and make the substitution $x_{r}(t)=u_{r}(t)+\lambda_{r}, \quad r=\overline{1,2}$.

Then, from the boundary value problem (34), (35) we transfer to the following equivalent problem

$$
\begin{gather*}
\dot{u}_{1}(t)=t \int_{0}^{1 / 2} u_{1}(s) d s+t \int_{1 / 2}^{1} u_{2}(s) d s \\
-\left(\frac{t}{2}-2\right) \lambda_{1}+\frac{t}{2} \lambda_{2}+(t-5) \lambda_{3}-t+3, \quad t \in\left[0, \frac{1}{2}\right)  \tag{36}\\
u_{1}(0)=0  \tag{37}\\
-\left(\frac{t}{2}-2\right) \lambda_{1}+\frac{t}{2} \lambda_{2}+(t-5) \lambda_{3}-t+3, \quad t \in\left[\frac{1}{2}, 1\right), \\
u_{0}^{1 / 2} u_{1}(s) d s+t \int_{1 / 2}^{1} u_{2}(s) d s  \tag{38}\\
u_{1}-2\left(\frac{1}{2}\right)=0  \tag{39}\\
\lambda_{1}+\lim _{t \rightarrow 1}+\lambda_{1}(t)=\lambda_{2}  \tag{40}\\
\lambda_{2}+\lim _{t \rightarrow 1}  \tag{41}\\
\lambda_{2}(t)=\lambda_{3} \tag{42}
\end{gather*}
$$

For fixed values of parameters $\lambda_{1}, \lambda_{2}, \lambda_{3}$ determine solution of the Cauchy problem (36)-(39)

$$
\begin{gather*}
u_{1}(t)=2 t \lambda_{1}+\frac{12}{43} t^{2} \lambda_{2}-\left(\frac{6}{43} t^{2}+5 t\right) \lambda_{3}-\frac{6}{43} t^{2}+3 t, \quad t \in\left[0, \frac{1}{2}\right)  \tag{43}\\
u_{2}(t)=(2 t-1) \lambda_{1}+\left(\frac{12}{43} t^{2}-\frac{3}{43}\right) \lambda_{2} \\
-\left(\frac{6}{43} t^{2}+5 t-\frac{109}{43}\right) \lambda_{3}-\frac{6}{43} t^{2}+3 t-\frac{63}{43}, \quad t \in\left[\frac{1}{2}, 1\right) \tag{44}
\end{gather*}
$$

Substituting the obtained solution into the boundary conditions (40)-(42), we obtain a system of linear algebraic equations with respect to the introduced parameters

$$
\left\{\begin{array}{c}
\lambda_{1}-2 \lambda_{2}+\lambda_{3}=0  \tag{45}\\
2 \lambda_{1}-\frac{40}{43} \lambda_{2}-\frac{109}{43} \lambda_{3}=-\frac{63}{43} \\
\lambda_{1}+\frac{52}{43} \lambda_{2}-\frac{155}{43} \lambda_{3}=-\frac{60}{43}
\end{array}\right.
$$

From (45), we obtain that the matrix $Q_{*}$ is revertible and $\lambda_{1}=\lambda_{2}=\lambda_{3}=1$. Substituting the obtained values in (43), (44) we determine that $u_{1}(t)=0, u_{2}(t)=0$. As the matrix $Q_{*}$ is revertible, the unique solution to the boundary value problem (31), (32) is found in the form of the sum $x_{1}(t)=u_{1}(t)+\lambda_{1}=1, x_{2}(t)=u_{2}(t)+\lambda_{2}=1$ or $x(t)=1$.

Example 2. On the segment $[0,1]$, consider the following three-point boundary value problem:

$$
\begin{gather*}
\dot{x}(t)-3 \dot{x}(1-t)=2 \int_{0}^{1} x(s) d s+\int_{0}^{1} \dot{x}(s) d s-2  \tag{46}\\
x(0)-3 x\left(\frac{1}{3}\right)+x(1)=0 \tag{47}
\end{gather*}
$$

Applying the property of the involutive transformation, we obtain the following boundary value problem:

$$
\begin{gather*}
\dot{x}(t)=-\int_{0}^{1} x(s) d s+\frac{1}{2} \int_{0}^{1} \dot{x}(s) d s+1,  \tag{48}\\
x(0)-3 x\left(\frac{1}{3}\right)+x(1)=0 . \tag{49}
\end{gather*}
$$

Let us divide the segment $[0,1)$ into two parts $[0,1)=\left[0, \frac{1}{3}\right) \cup\left[\frac{1}{3}, 1\right)$. Introduce parameters $\lambda_{1}=x_{1}(0), \quad \lambda_{2}=x_{2}(1 / 3), \quad \lambda_{3}=\lim _{t \rightarrow 1} x_{2}(t)$ and make the substitution $x_{r}(t)=u_{r}(t)+\lambda_{r}, r=\overline{1,2}$.

Then, from the boundary value problem (48), (49), we obtain the following equivalent problem:

$$
\begin{gather*}
\dot{u}_{1}(t)=-\int_{0}^{1 / 3} u_{1}(s) d s-\int_{1 / 3}^{1} u_{2}(s) d s-\frac{5}{6} \lambda_{1}-\frac{2}{3} \lambda_{2}+\frac{1}{2} \lambda_{3}+1, \quad t \in\left[0, \frac{1}{3}\right)  \tag{50}\\
u_{1}(0)=0  \tag{51}\\
\dot{u}_{2}(t)=-\int_{0}^{1 / 3} u_{1}(s) d s-\int_{1 / 3}^{1} u_{2}(s) d s-\frac{5}{6} \lambda_{1}-\frac{2}{3} \lambda_{2}+\frac{1}{2} \lambda_{3}+1, \quad t \in\left[\frac{1}{3}, 1\right)  \tag{52}\\
u_{2}\left(\frac{1}{3}\right)=0  \tag{53}\\
\lambda_{1}-\frac{1}{3} \lambda_{2}+\lambda_{3}=0  \tag{54}\\
\lambda_{1}+\lim _{t \rightarrow 1} u_{1}(t)=\lambda_{2}  \tag{55}\\
\lambda_{2}+\lim _{t \rightarrow 1} u_{2}(t)=\lambda_{3} \tag{56}
\end{gather*}
$$

For fixed values of the parameter $\lambda_{1}, \quad \lambda_{2}, \lambda_{3}$ we find the solution of the Cauchy problem (50)-(53)

$$
\begin{gather*}
u_{1}(t)=\left[-15 \lambda_{1}-\frac{36}{69} \lambda_{2}+\frac{54}{138} \lambda_{3}+\frac{18}{23}\right] t, \quad t \in\left[0, \frac{1}{3}\right),  \tag{57}\\
u_{2}(t)=\left[-15 \lambda_{1}-\frac{36}{69} \lambda_{2}+\frac{54}{138} \lambda_{3}+\frac{18}{23}\right]\left(t-\frac{1}{3}\right), \quad t \in\left[\frac{1}{3}, 1\right) . \tag{58}
\end{gather*}
$$

Substituting the obtained solution of the Cauchy problem (50)-(53) into the boundary conditions (54)-(56), we obtain a system of linear algebraic equations with respect to the introduced parameters:

$$
\left\{\begin{array}{c}
\lambda_{1}-3 \lambda_{2}+\lambda_{3}=0  \tag{59}\\
-12 \lambda_{1}-\frac{243}{69} \lambda_{2}+\frac{54}{138} \lambda_{3}=-\frac{18}{23} \\
-30 \lambda_{1}+\frac{135}{69} \lambda_{2}-\frac{306}{138} \lambda_{3}=-\frac{36}{23}
\end{array}\right.
$$

The matrix $Q_{*}$ is invertible and $\lambda_{1}=0, \quad \lambda_{2}=\frac{1}{3}, \quad \lambda_{3}=1$. Substituting the obtained values into (57) and (58) we obtain $u_{1}(t)=t, u_{2}(t)=t-\frac{1}{3}$. As the matrix $Q_{*}$ is invertible, the boundary value problem (46), (47) has a unique solution $x_{1}(t)=u_{1}(t)+\lambda_{1}=t, \quad t \in\left[0, \frac{1}{3}\right)$, $x_{2}(t)=u_{2}(t)+\lambda_{2}=t-\frac{1}{3}+\frac{1}{3}=t, \quad t \in\left[\frac{1}{3}, 1\right), \quad \lim _{t \rightarrow 1} x_{2}(t)=1$ or $x(t)=t, \quad t \in[0,1]$.

## 5. Conclusions

In this paper, the parametrization method was used to solve a multipoint boundary value problem for systems of integro-differential equations with involution transformations. Introduction of new parameters and a successful change of variables enables us to split the problem into two parts: the Cauchy problem for systems of integro-differential equations and a system of linear equations with respect to the introduced parameters. Applying the theory of integral equations, the solution to the problem is reduced to the reversibility of the matrix, depending on the initial data. Thus, necessary and sufficient conditions for the unique solvability of the studied problem have been established. The effectiveness and accuracy of the method is demonstrated by an illustrative example. In the future, it is planned to apply the parametrization method to multipoint boundary value problems for integro-differential equations with fractional derivatives.

Author Contributions: Conceptualization, K. I. U. and B.K.T.; methodology, K.I.U.; validation, K.I.U., B.K.T. and K.Z.N.; formal analysis, B.K.T.; investigation, K.I.U.; resources, B.K.T.; writing-original draft preparation, K.I.U.; writing-review and editing, K.Z.N.; supervision, B.K.T.; project administration, K.I.U. All authors have read and agreed to the published version of the manuscript.
Funding: This research has been partially supported by the Ministry of Educationand Science of the Republic of Kazakhstan, Grants No. AP09259137.

Institutional Review Board Statement: Not applicable.
Informed Consent Statement: Not applicable.
Data Availability Statement: All the data is present within the manuscript.
Conflicts of Interest: The authors declare no conflict of interest.

## References

1. Volterra, V. Theory of Functionals and of Integrals and Integro-Differential Equations; Dover Publications: New York, NY, USA, 1959.
2. Shishkin, G. Linear integro-differential Fredholm equations. Ulan-Ude, 2007. Availiable online: https:/ /www.bsu.ru/content/ page/1416/06.pdf (accessed on: 12 July 2022)
3. Brunner, H. Collocation Methods for Volterra Integral and Related Functional Equations; Cambridge University Press: Cambridge, UK, 2004.
4. Du, H.; Zhao, G.; Zhao, C. Reproducing kernel method for solving Fredholm integro-differential equations withweakly singularity. J. Comput. Appl. Math. 2014, 255, 122-132. [CrossRef]
5. Maleknejad, K.; Attary, M. An efficient numerical approximation for the linear Fredholm integro-differential equations based on Cattani's method. Commun. Nonlinear Sci. Numer. Simulat. 2011, 16, 2672-2679. [CrossRef]
6. Yuzbasi, S. Numerical solutions of system of linear Fredholm-Volterra integro-differential equations by the Bessel collocation method and error estimation. Appl Math Comput. 2015, 250, 320-338.
7. Yuldashev, T.K. On the solvability of a boundary value problem for the ordinary Fredholm integro-differential equation with a degenerate kernel. Comput. Math. Math. Phys. 2019, 59, 241-252. [CrossRef]
8. Yuldashev, T.K. Nonlocal inverse problem for a pseudohyperbolic-pseudoelliptic type integro-differential equations. Axioms 2020, 9, 45. [CrossRef]
9. Abdella, K. ; Ross, G. Solving integro-differential boundary value problems using sinc-derivative collocation. Mathematics 2020, 8, 1637. [CrossRef]
10. Turmetov, B.; Usmanov, K.; Nazarova, K. On the operator method for solving linear integro-differential equations with fractional conformable derivatives. Fractal Fract. 2021, 5, 109. [CrossRef]
11. Dzhumabaev, D.S. Necessary and sufficient conditions for the solvability of linear boundary-value problems for the Fredholm integrodifferential equations. Ukr. Math. J. 2015, 66, 1200-1219. [CrossRef]
12. Dzhumabayev, D.S. Criteria for the unique solvability of a linear boundary-value problem for an ordinary differential equation. U.S.S.R. Comput. Maths. Math. Phys. 1989, 29, 34-46. [CrossRef]
13. Dzhumabayev, D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations. J. Comput. Appl. Math. 2016, 294, 342-357. [CrossRef]
14. Dzhumabayev, D.S. Computational methods of solving the boundary value problems for the loaded differential and Fredholm integro-differential equations. Math. Methods Appl. Sci. 2018, 41, 1439-1462. [CrossRef]
15. Assanova, A.T.; Bakirova, E.A.; Kadirbayeva, Z.M. Numerical solution to a control problem for integro-differential equations. Comput. Math. Math. Phys. 2020, 60, 203-221. [CrossRef]
16. Nazarova, K.; Usmanov, K. On a boundary value problem for systems of integro-differential equations with involution. Int. J. Appl. Math. 2021, 34, 225-235. [CrossRef]
17. Nazarova, K.; Usmanov, K. Unique solvability of the boundary value problem for integro-differential equations with involution. AIP Conf. Proc. 2021, 2365, 070012.
18. Usmanov, K.I.; Nazarova, K.Z.; Yerkisheva, Z.S. On the unique solvability of a boundary value problem for systems of loaded integro-differential equations with involution. Lobachevskii J. Math. 2021, 12, 3022-3034. [CrossRef]
19. Ciano, T.; Ferrara, M.; Guerrini, L. Qualitative analysis of a model of renewable resources and population with distributed delays. Mathematics 2022, 10, 1247. [CrossRef]
20. Carleman, T. La théorie des équations intégrales singuliéres et ses applications. Ann. De L'institut Henri Poincaré 1932, 1, 401-430.
21. Przeworska-Rolewicz, D. Equations with Transformed Argument, An Algebraic Approach, 1st ed.; Elsevier Scientific: Amsterdam, The Netherlands, 1973; ISBN 0-444-41078-3.
22. Wiener, J. Generalized Solutions of Functional Differential Equations, 1st ed.; World Scientific: Singapore; River Edge, NJ, USA; London, UK; Hong Kong, China, 1993; ISBN 981-02-1207-0.
23. Karapetiants N.; Samko, S. Equations with Involutive Operators, 1st ed.; World Boston: Boston, MA, USA, 2001; ISBN 978-1-4612-0183-0.
24. Cabada, A.; Tojo, F.A.F. Differential Equations with Involutions, 1st ed.; Atlantis Press: Paris, France, 2015; ISBN 978-94-6239-120-8
25. Al-Salti, N.; Kerbal, S.; Kirane, M. Initial-boundary value problems for a time-fractional differential equation with involution perturbation. Math. Model. Nat. Phenom. 2019, 14, 1-15. [CrossRef]
26. Kritskov, L.V.; Sadybekov, M.A.; Sarsenbi, A.M. Properties in Lp of root functions for a nonlocal problem with involution. Turk. J. Math. 2019, 43, 393-401. [CrossRef]
27. Sarsenbi, A.A.; Sarsenbi, A.A. On eigenfunctions of the boundary value problems for second order differential equations with involution. Symmetry 2021, 13, 1972. [CrossRef]
28. Turmetov, B.K.; Karachik, V.V. On eigenfunctions and eigenvalues of a nonlocal Laplace operator with multiple involution. Symmetry 2021, 13, 1981. [CrossRef]
29. Nahushev, A.M. Loaded Equations and Applications; Nauka: Moscow, Russia, 2012.
30. Il'in, V.A.; Moiseev, E.I. Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects. Differ. Equ. 1987, 23, 803-810.
31. Il'in, V.A.; Moiseev, E.I. Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator. Differ. Equ. 1987, 23, 979-987.
32. Geng, F.; Cui, M. Multi-point boundary value problem for optimal bridge design. Int. J. Comput. Math. 2010, 87, 1051-1056. [CrossRef]
33. Nazarova, K.Z.; Usmanov, K.I. Unique solvability of b oundary value problem for functional-differential equations with involution. Bull. Karaganda Univ. Math. 2020, 103, 68-75. [CrossRef]
