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# On the Convergence of an Approximation Scheme of Fractional-Step Type, Associated to a Nonlinear Second-Order System with Coupled In-Homogeneous Dynamic Boundary Conditions

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**Abstract:** The paper concerns a nonlinear second-order system of coupled PDEs, having the principal part in *divergence* form and subject to in-homogeneous dynamic boundary conditions, for both $\theta(t,x)$ and $\varphi(t,x)$. Two main topics are addressed here, as follows. First, under a certain hypothesis on the input data, $f_1$, $f_2$, $w_1$, $w_2$, $\alpha$, $\xi$, $\theta_0$, $\alpha_0$, $\varphi_0$, and $\xi_0$, we prove the well-posedness of a solution $\theta, \alpha, \varphi, \xi$, which is $(\theta(t,x), \alpha(t,x)) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $(\varphi(t,x), \xi(t,x)) \in W_\nu^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $\nu = \min\{q, \mu\}$. According to the new formulation of the problem, we extend the previous results, allowing the new mathematical model to be even more complete to describe the diversity of physical phenomena to which it can be applied: interface problems, image analysis, epidemics, etc. The main goal of the present paper is to develop an iterative scheme of fractional-step type in order to approximate the unique solution to the nonlinear second-order system. The convergence result is established for the new numerical method, and on the basis of this approach, a conceptual algorithm, **alg-frac_sec-ord_u+varphi_dbc,** is elaborated. The benefit brought by such a method consists of simplifying the computations so that the time required to approximate the solutions decreases significantly. Some conclusions are given as well as new research topics for the future.

**Keywords:** boundary value problems for nonlinear parabolic PDE; dynamic boundary conditions; fractional step method; convergence of numerical scheme; numerical algorithm; phase changes

**MSC:** 35K55; 35K60; 65N06; 65N12; 80A99



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## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$, be a bounded domain with a $C^2$ boundary $\partial\Omega$ and $[0,T]$ as a generic time interval. We consider the nonlinear second-order system of coupled PDEs

$$\begin{cases} p_1 \dfrac{\partial}{\partial t}\theta(t,x) + q_1 \dfrac{\partial}{\partial t}\varphi(t,x) - p_2 \mathrm{div}\Big(K_1\left(t,x,\theta(t,x)\right) \nabla\theta(t,x)\Big) \\ \qquad = p_3 f_1(t,x) \\ q_2 \dfrac{\partial}{\partial t}\varphi(t,x) - q_3 \mathrm{div}\Big(K_2\left(t,x,\varphi(t,x)\right) \nabla\varphi(t,x)\Big) \\ \qquad = q_4\left[\varphi(t,x) - \varphi^3(t,x)\right] + p_4\theta(t,x) + q_5 f_2(t,x) \end{cases} \quad \text{in } Q, \qquad (1)$$

subject to in-homogeneous dynamic boundary conditions in both unknown functions $\theta$ and $\varphi$, i.e.,

$$\begin{cases} p_2 \dfrac{\partial}{\partial \mathbf{n}}\theta + p_1 \dfrac{\partial}{\partial t}\theta - \Delta_\Gamma \theta + p_5 \theta = w_1(t,x) \\ q_3 \dfrac{\partial}{\partial \mathbf{n}}\varphi + q_2 \dfrac{\partial}{\partial t}\varphi - \Delta_\Gamma \varphi + q_6 \varphi = w_2(t,x) \end{cases} \quad \text{on } \Sigma, \tag{2}$$

and with the initial conditions

$$\theta(0,x) = \theta_0(x), \qquad \varphi(0,x) = \varphi_0(x) \qquad \text{in } \Omega, \tag{3}$$

where $Q = (0,T] \times \Omega$, $\Sigma = (0,T] \times \partial\Omega$, $\theta(t,x)$, $\varphi(t,x)$, $\dfrac{\partial}{\partial s}\theta(s,\cdot)$ ($\theta_s$, in short), $\nabla\theta = \theta_x$, $\nabla\varphi(t,x) = \varphi_x(t,x)$ ($\nabla\varphi = \varphi_x$, in short) $p, q$, $\mathbf{n} = \mathbf{n}(x)$, which are the same as in [1], while

- $p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4, q_5$, and $q_6$ are positive values;
- $K_1(s,y,\theta(s,y))$ and $K_2(s,y,\varphi(s,y))$ are the mobility functions (attached to the solution $\theta(s,y), \varphi(s,y)$, $(s,y) \in Q$, of $(1)_1$ and $(1)_2$, respectively; see [2] for more details);
- $f_1(t,x) \in L^p(Q)$ and $f_2(t,x) \in L^q(Q)$ are given functions (see [1,3–16] for more details).
- $w_1(t,x), w_2(t,x) \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, $p \geq 2$ are given functions;
- $\theta_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$, with $p_2 \dfrac{\partial}{\partial \mathbf{n}}\theta_0 - \Delta_\Gamma \theta_0 + p_5\theta_0 = w_1(0,x)$,

  and $\varphi_0 \in W_\infty^{2-\frac{2}{q}}(\Omega)$, with $q_3 \dfrac{\partial}{\partial \mathbf{n}}\varphi_0 - \Delta_\Gamma \varphi_0 + q_6\varphi_0 = w_2(0,x)$.

**Remark 1.** *Besides classical meanings, like the density of heat sources or sinks of heat, the pairs of given functions $\{f_1, f_2\}$ and $\{w_1, w_2\}$ in (1) and (2), respectively, can be also interpreted as distributed and boundary control, respectively, which opens a wide field of applicability for the nonlinear parabolic systems (1) and (3), such as optimal control problems.*

The basic tools in our approach are as follows:

- The Leray–Schauder degree theory (see [17] and references therein);
- The $L^p$-theory of linear and quasi-linear parabolic equations [18];
- Green's first identity

$$-\int_\Omega y \operatorname{div} z \, dx = \int_\Omega \nabla y \cdot z \, dx - \int_{\partial\Omega} y \frac{\partial}{\partial \mathbf{n}} z \, d\gamma$$

  for any scalar-valued function $y$ and $z$, a continuously differentiable vector field in $n$ dimensional space;
- The Lions and Peetre embedding Theorem (see [17], p. 18) to ensure the existence of a continuous embedding $W_p^{1,2}(Q) \subset L^{\mu_1}(Q)$, $p \geq 2$, where the real number $\mu_1$ is defined as follows:

$$\mu_1 = \begin{cases} \text{any positive number } \geq 3p & \text{if } \dfrac{1}{p} - \dfrac{2}{n+2} \leq 0, \\ \dfrac{p\,(n+2)}{n+2-2p} & \text{if } \dfrac{1}{p} - \dfrac{2}{n+2} > 0, \end{cases}$$

and, for $k \in \{1, 2, \cdots\}$ and $1 \le p \le \infty$, $W_p^{k,2k}(Q)$ denotes the Sobolev space on $Q$:

$$W_p^{k,2k}(Q) = \left\{ y \in L^p(Q) : \frac{\partial^r}{\partial t^r} \frac{\partial^q}{\partial x^q} \, y \in L^p(Q), \text{ for } 2r + q \le 2k \right\},$$

i.e., the spaces of functions whose $t$-derivatives and $x$-derivatives up to the order $k$ and $2k$, respectively, belong to $L^p(Q)$;

- Also, we shall use the set $C^{1,2}(\bar{Q})$ $(C^{1,2}(Q))$ of all continuous functions in $\bar{Q}$ (in $Q$) having continuous derivatives $u_t$, $u_x$, $u_{xx}$ in $\bar{Q}$ (in $Q$), as well as the Sobolev spaces $W_p^{\ell}(\Omega)$, $W_p^{\ell,\ell/2}(\Sigma)$ (see [17,19] and reference therein);
- As far as the techniques used in the paper are concerned, it should be noted that we derive the a priori estimates in $L^p(Q)$ and $L^p(\Sigma)$.

In the following, we denote by $C$ several positive constants, being understood that the extra dependencies are set out on occurrence.

## 2. Well-Posedness of Solutions to the Nonlinear Second-Order System (1)–(3)

In order to approach the nonlinear second-order systems (1)–(3), we use the same idea as in V. Berinde, A. Miranville, and C. Moroşanu [1]. In this regard, let $\zeta = \theta$ and $\xi = \varphi$ be further variables such that $\zeta(0, x) = \theta_0$, $\xi(0, x) = \varphi_0$ on $\partial\Omega$, while for the remaining data in (1)–(3), we keep the same meanings formulated at the beginning. Correspondingly, the boundary conditions in (2) are approached in the sequel by

$$\begin{cases} \theta = \alpha \\ p_2 \dfrac{\partial}{\partial \mathbf{n}} \theta + p_1 \dfrac{\partial}{\partial t} \alpha - \Delta_\Gamma \alpha + p_5 \alpha = w_1(t, x) \end{cases} \quad \text{on } \Sigma, \tag{4}$$

$$\begin{cases} \varphi = \xi \\ q_3 \dfrac{\partial}{\partial \mathbf{n}} \varphi + q_2 \dfrac{\partial}{\partial t} \xi - \Delta_\Gamma \xi + q_6 \xi = w_2(t, x) \end{cases} \quad \text{on } \Sigma, \tag{5}$$

where $\zeta(0, x) = \zeta_0(x)$, $\xi(0, x) = \xi_0(x), x \in \partial\Omega$, and $\zeta_0, \xi_0 \in W_\infty^{2-\frac{2}{p}}(\partial\Omega)$, $p \ge 2$.

Accordingly, problems (1)–(3) can be rewritten suitably as follows:

$$\begin{cases} p_1 \dfrac{\partial}{\partial t} \theta(t, x) - p_2 \dfrac{\partial}{\partial u_{x_j}} \left[ K_1(t, x, \theta) \theta_{x_i} \right] \theta_{x_j x_i} & \\ \quad = A_1(t, x, \theta, \theta_{x_i}) - q_1 \dfrac{\partial}{\partial t} \varphi + p_3 f_1(t, x) & \text{in } Q \\ \theta(t, x) = \alpha(t, x) & \text{on } \Sigma \\ p_2 \dfrac{\partial}{\partial \mathbf{n}} \theta + p_1 \dfrac{\partial}{\partial t} \alpha - \Delta_\Gamma \alpha + p_5 \alpha = w_1(t, x) & \text{on } \Sigma \\ \theta(0, x) = \theta_0(x) & \text{on } \Omega \\ \alpha(0, x) = \alpha_0(x) & x \in \partial\Omega, \end{cases} \tag{6}$$

$$
\begin{cases}
q_2 \dfrac{\partial}{\partial t}\varphi(t,x) - q_3 \dfrac{\partial}{\partial \varphi_{x_j}}\Big[(K_2(t,x,\varphi)\varphi_{x_i}\Big]\varphi_{x_j x_i} \\
\qquad = A_2(t,x,\varphi,\varphi_{x_i}) + q_4\big[\varphi - \varphi^3\big] + p_4\theta(t,x) + q_5 f_2(t,x) & \text{in } Q \\
\varphi(t,x) = \xi(t,x) & \text{on } \Sigma \\
q_3 \dfrac{\partial}{\partial \mathbf{n}}\varphi + q_2 \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + q_6 \xi = w_2(t,x) & \text{on } \Sigma \\
\varphi(0,x) = \varphi_0(x) & \text{on } \Omega \\
\xi(0,x) = \xi_0(x) & x \in \partial\Omega,
\end{cases}
\tag{7}
$$

where (see [18])

$$
\theta_{x_j x_i} = \frac{\partial^2}{\partial x_j \partial x_i}\theta(t,x),\ i,j = 1,\dots,n,
$$

$$
A_1(t,x,\theta(t,x),\theta_{x_i}(t,x)) = \frac{\partial}{\partial \theta}\Big[K_1(t,x,\theta)\theta_{x_i}\Big]\theta_{x_i} + \frac{\partial}{\partial x_i}\Big[K_1(t,x,\theta)\theta_{x_i}\Big],\ i = 1,\dots,n,
$$

and

$$
\varphi_{x_j x_i} = \frac{\partial^2}{\partial x_j \partial x_i}\varphi(t,x),\ i,j = 1,\dots,n,
$$

$$
A_2(t,x,\varphi(t,x),\varphi_{x_i}(t,x)) = \frac{\partial}{\partial \varphi}\Big[K_2(t,x,\varphi)\varphi_{x_i}\Big]\varphi_{x_i} + \frac{\partial}{\partial x_i}\Big[K_2(t,x,\varphi)\varphi_{x_i}\Big],\ i = 1,\dots,n.
$$

*The Validity of an Auxiliary Nonlinear Second-Order Boundary Value Problem*

We consider the following auxiliary nonlinear parabolic problem derived from (7):

$$
\begin{cases}
q_2 \dfrac{\partial}{\partial t}\Phi(t,x) - q_3 \operatorname{div}\Big(K_2\big(t,x,\Phi(t,x)\big)\ \nabla\Phi(t,x)\Big) \\
\qquad = q_4\big[\Phi(t,x) - \Phi^3(t,x)\big] + h(t,x) & \text{in } \ Q \\
\Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\
q_3 \dfrac{\partial}{\partial \mathbf{n}}\Phi + q_2 \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + q_6 \xi = w_2(t,x) & \text{on } \Sigma \\
\Phi(0,x) = \Phi_0(x) & \text{on } \ \Omega \\
\xi(0,x) = \xi_0(x) & x \in \partial\Omega.
\end{cases}
\tag{8}
$$

**Definition 1.** *Any solution* $\big(\Phi(t,x), \xi(t,x)\big)$ *of problem* (8) *is called the* `classical solution` *if it is continuous in* $\bar{Q}$*, has continuous derivatives* $\Phi_t$*,* $\Phi_x$*,* $\Phi_{xx}$ *in* $Q$ *and* $\zeta_t$*,* $\zeta_x$*,* $\zeta_{xx}$ *on* $\Sigma$*, satisfies the equation* $(8)_1$ *at all points* $(t,x) \in Q$*, and satisfies conditions* $(8)_{2,3}$ *and* $(8)_{4,5}$ *on the lateral surface* $\Sigma$ *of the cylinder* $Q$ *and for* $t = 0$*, respectively.*

Our main results regarding the existence, uniqueness, and regularity of solutions to problem (8) (practically, well-posedness of the solutions to the nonlinear second-order boundary value problem (1) or (7)) are as follows.

**Theorem 1.** *Suppose* $\big(\Phi(t,x), \xi(t,x)\big) \in C^{1,2}(Q) \times C^{1,2}(\Sigma)$ *is a classical solution of problem* (8)*, and for positive numbers* $M$*,* $M_0$*,* $m_1$*,* $M_1$*,* $M_2$*,* $M_3$*,* $M_4$*, and* $M_5$*, one has the following:*

$\mathbf{I}_1$. $|\Phi(t,x)| < M$ *for any* $(t,x) \in Q$, *and for any* $z(t,x)$, *the map* $K_2(t,x,z)$ *is continuous and differentiable in* $x$; *its* $x$-*derivatives are measurable bounded, and it satisfies the uniformly parabolic conditions* (see [18]), *and*

$$0 < K2_m \le K_2\big(t,x,\Phi(t,x)\big) < K2_M, \quad for \ \ (t,x) \in Q, \tag{9}$$

$$\begin{aligned}
&\sum_{i=1}^{n}\left[|a_i(t,x,\Phi(t,x),z(t,x))| + \left|\frac{\partial}{\partial\Phi}a_i(t,x,\Phi(t,x),z(t,x))\right|\right](1+|z|)\\
&+\sum_{i,j=1}^{n}\left|\frac{\partial}{\partial x_j}a_i(t,x,\Phi(t,x),z(t,x))\right|+|\Phi(t,x)|\le M_0(1+|z|)^2.
\end{aligned} \tag{10}$$

$\mathbf{I}_2$. *For any sufficiently small* $\varepsilon > 0$, *functions* $\Phi(t,x)$ *and* $K_2(t,x,\Phi(t,x))$ *satisfy the relations*

$$\|\Phi\|_{L^s(Q)} \le M_2, \qquad \|K_2(t,x,\Phi(t,x))\Phi_{x_i}\|_{L^r(Q)} < M_3, \ \ i = 1,\ldots,n,$$

*where*

$$r = \begin{cases} \max\{p,4\} & p \ne 4 \\ 4+\varepsilon & p = 4, \end{cases} \qquad s = \begin{cases} \max\{p,2\} & p \ne 2 \\ 2+\varepsilon & p = 2. \end{cases}$$

*Then,* $\forall h(t,x) \in L^p(Q)$, $\Phi_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$, $\xi_0(x) \in W_\infty^{2-\frac{2}{p}}(\Gamma)$, $w_2 \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, *with* $p \ne \frac{3}{2}$, *and there exists a unique solution* $(\Phi,\xi) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$ *to* (8) *that satisfies*

$$\begin{aligned}
&\|\Phi\|_{W_p^{1,2}(Q)} + \|\xi\|_{W_p^{1,2}(\Sigma)}\\
&\quad\le C\Big\{1 + \|\Phi_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\xi_0\|_{W_\infty^{2-\frac{2}{p}}(\partial\Omega)} + \|\Phi_0\|_{L^{3p-2}(\Omega)}^{\frac{3p-2}{p}} + \|\xi_0\|_{L^{3p-2}(\partial\Omega)}^{\frac{3p-2}{p}}\\
&\qquad+\|h\|_{L^{3p-2}(Q)}^{\frac{3p-2}{p}} + \|w_2\|_{L^{3p-2}(\Sigma)}^{\frac{3p-2}{p}} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Big\},
\end{aligned} \tag{11}$$

*where* $C > 0$ *is independent on* $\Phi$, $\zeta$, $h$, *and* $w_2$.

*If* $(\Phi^1,\xi^1)$, $(\Phi^2,\xi^2)$ *are solutions to* (8), *corresponding to* $(\Phi_0^1,\xi_0^1)$, $(\Phi_0^2,\xi_0^2) \in W_\infty^{2-\frac{2}{p}}(\Omega) \times W_\infty^{2-\frac{2}{p}}(\partial\Omega)$, $h^1$, $h^2$, $w_2^1$, *and* $w_2^2$, *respectively, such that*

$$\|\Phi^1\|_{W_p^{1,2}(Q)}, \quad \|\Phi^2\|_{W_p^{1,2}(Q)} \le M_4, \tag{12}$$

$$\|\xi^1\|_{W_p^{1,2}(\Sigma)}, \quad \|\xi^2\|_{W_p^{1,2}(\Sigma)} \le M_5, \tag{13}$$

*then the following holds*

$$\begin{aligned}
\max_{(t,x)\in Q}|\Phi^1-\Phi^2| \quad &+ \max_{(t,x)\in\Sigma}|\xi^1-\xi^2|\\
&\le C_1e^{CT}\max\Big\{\max_{(t,x)\in\Omega}|\Phi_0^1-\Phi_0^2|, \max_{(t,x)\in\partial\Omega}|\xi_0^1-\xi_0^2|,\\
&\qquad\max_{(t,x)\in Q}|h^1-h^2|, \max_{(t,x)\in\Sigma}|w_2^1-w_2^2|\Big\},
\end{aligned} \tag{14}$$

*where* $C_1 > 0$, $C > 0$ *are independent on* $\left\{\Phi^1,\xi^1,h^1,w_2^1,\Phi_0^1,\xi_0^1\right\}$ *and* $\left\{\Phi^2,\xi^2,h^2,w_2^2,\Phi_0^2,\xi_0^2\right\}$. *In particular, the uniqueness of the solution to* (8) *holds.*

Corresponding to a different formulation than the one presented in (2), results similar to those in Theorem 1 were established in [1,2,13,17–19]. Here, we omit details of the proof.

## 3. The Validity of the Problem (6) and (7) in the Class $W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $W_v^{1,2}(Q) \times W_p^{1,2}(\Sigma)$

**Definition 2.** *Any solution* $(\theta, \alpha, \varphi, \xi)$ *of the nonlinear second-order boundary value problem* (6) and (7) *is called the* `classical solution` *if it is continuous in* $Q$*, has continuous derivatives* $\theta_t$, $\theta_x$, $\theta_{xx}$, $\varphi_t$, $\varphi_x$, $\varphi_{xx}$ *in* $Q$ *and* $\alpha_t$, $\alpha_x$, $\alpha_{xx}$, $\xi_t$, $\xi_x$, $\xi_{xx}$ *on* $\Sigma$*, satisfies the equation* $(6)_1$ *and* $(7)_1$ *at all points* $(t,x) \in Q$ *as well as the conditions* $(6)_{2,3}$-$(7)_{2,3}$ *and* $(6)_{4,5}$-$(7)_{4,5}$ *for* $(t,x) \in \Sigma$ *and for* $t = 0$*, respectively.*

Here, we approach the systems (6) and (7) in the spirit given by Hadamard's well-posedness conditions (see [17], p. 46). Therefore, the main results regarding the existence, uniqueness, and regularity of solutions to (6) and (7) (practically, the well-posedness of the solutions to the problem (1)–(3)) are as follows:

**Theorem 2.** *Suppose* $\{(\theta, \alpha), (\varphi, \xi)\} \in \left[C^{1,2}(Q) \times C^{1,2}(\Sigma)\right]^2$ *is a classical solution of problems* (6) and (7)*, and for positive numbers*

$$M, M_0, M_1, M_2, M_3, M_4, \quad and \quad N, N_0, N_1, N_2, N_3, N_4,$$

*one has the following:*

$\mathbf{I}_1$. $|\theta(t,x)| < M$*, and for any* $t, x, z$*, the function* $K_1(t,x,\theta)$ *is continuous and differentiable with respect to* $x$*,* $\theta$*; its* $x$*-derivatives and* $\theta$*-derivatives are bounded-measurable, it satisfies the uniformly parabolic conditions* (see [19])*, and*

$$\begin{aligned} &0 < K1_m \le K_1(t,x,\theta) < K1_M, \quad for \ \ (t,x) \in Q, \\ &\sum_{i=1}^{n}\left[|K_1(t,x,\theta)\theta_{x_i}| + \left|\frac{\partial}{\partial\theta}(K_1(t,x,\theta)\theta_{x_i})\right|\right](1+|z|) \\ &\qquad + \sum_{i,j=1}^{n}\left|\frac{\partial}{\partial x_j}(K_1(t,x,\theta)\theta_{x_i})\right| \le M_0(1+|z|)^2. \end{aligned} \tag{15}$$

$\mathbf{I}_2$. *For every* $\varepsilon > 0$*, functions* $\theta(t,x)$ *and* $K_1(t,x,\theta)$ *satisfy*

$$\|\theta\|_{L^s(Q)} \le M_1, \qquad \|K_1(t,x,\theta)\theta_{x_i}\|_{L^r(Q)} < M_2, \ \ i = 1, \dots, n,$$

*where*

$$r = \begin{cases} \max\{p,4\} & p \ne 4 \\ 4+\varepsilon & p = 4, \end{cases} \qquad s = \begin{cases} \max\{p,2\} & p \ne 2 \\ 2+\varepsilon & p = 2. \end{cases}$$

$\mathbf{J}_1$ . $|\varphi(t,x)| < N$*, and for any* $t, x, z$*, function* $K_2(t,x,\varphi)$ *is continuous and differentiable with respect to* $x$*,* $\varphi$*; its* $x$*-derivatives and* $\varphi$*-derivatives are bounded-measurable, it satisfies the uniformly parabolic conditions* (see [19])*, and*

$$\begin{aligned} &0 < K2_m \le K_2(t,x,\varphi) < K2_M, \quad for \ \ (t,x) \in Q, \\ &\sum_{i=1}^{n}\left[|K_2(t,x,\varphi)\varphi_{x_i}| + \left|\frac{\partial}{\partial\varphi}(K_2(t,x,\varphi)\varphi_{x_i})\right|\right](1+|z|) \\ &\qquad + \sum_{i,j=1}^{n}\left|\frac{\partial}{\partial x_j}(K_2(t,x,\varphi)\varphi_{x_i})\right| \le N_0(1+|z|)^2. \end{aligned} \tag{16}$$

$\mathbf{J}_2$. *For every* $\varepsilon > 0$*, the functions* $\varphi(t,x)$ *and* $K_2(t,x,\varphi)$ *satisfy*

$$\|\varphi\|_{L^s(Q)} \le N_1, \qquad \|K_2(t,x,\varphi)\varphi_{x_i}\|_{L^r(Q)} < N_2, \ \ i = 1, \dots, n,$$

*where the quantities r and s are defined in* $\mathbf{I}_2$ .

*Then,* $\forall f_1 \in L^p(Q)$, $\theta_0 \in W_\infty^{2-\frac{2}{p}}(\Omega)$, $\alpha_0(x) \in W_\infty^{2-\frac{2}{p}}(\partial\Omega)$, $w_1 \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, *and* $\forall f_2 \in L^q(Q)$, $\varphi_0 \in W_\infty^{2-\frac{2}{q}}(\Omega)$, $\xi_0(x) \in W_\infty^{2-\frac{2}{p}}(\partial\Omega)$, $w_2 \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, *and there exists a unique solution* $\theta \in W_p^{1,2}(Q)$, $\varphi \in W_\nu^{1,2}(Q)$ $(\nu = min\{q,\mu\})$, $\alpha, \xi \in W_p^{1,2}(\Sigma)$ *to* (6) and (7), $p, q \neq \frac{3}{2}$ *that satisfies*

$$
\begin{aligned}
&\|\theta\|_{W_p^{1,2}(Q)} + \|\varphi\|_{W_\nu^{1,2}(Q)} + \|\alpha\|_{W_p^{1,2}(\Sigma)} + \|\xi\|_{W_p^{1,2}(\Sigma)} \\
&\leq C\Bigg[1 + \|\theta_0\|_{W_\infty^{2-\frac{2}{p}}(\Omega)} + \|\varphi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{q}}(\Omega)} + \|\alpha_0\|_{W_\infty^{2-\frac{2}{p}}(\partial\Omega)} + \|\xi_0\|^{\frac{3p-2}{p}}_{W_\infty^{2-\frac{2}{p}}(\partial\Omega)} \\
&+ \|f_1\|_{L^{p'}(Q)} + \|f_2\|_{L^q(Q)} + \|w_1\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)} + \|w_2\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)}\Bigg],
\end{aligned} \tag{17}
$$

*where the constant* $C > 0$ *is independent of* $\theta$, $\varphi$, $\zeta$, $\xi$, $f_1$, $f_2$, $w_1$, *and* $w_2$.

*If* $(\theta^1, \alpha^1, \varphi^1, \xi^1)$, $(\theta^2, \alpha^2, \varphi^2, \xi^2)$ *are two solutions to* (6) and (7) *corresponding to* $(\theta_0^1, \alpha_0^1, \varphi_0^1, \xi_0^1)$, $(\theta_0^2, \alpha_0^2, \varphi_0^2, \xi_0^2) \in W_\infty^{2-\frac{2}{p}}(\Omega) \times W_\infty^{2-\frac{2}{p}}(\partial\Omega) \times W_\infty^{2-\frac{2}{q}}(\Omega) \times W_\infty^{2-\frac{2}{p}}(\partial\Omega)$, $(f_1^a, f_2^a)$, $(f_1^b, f_2^b) \in L^p(Q) \times L^q(Q)$, $w_1^a, w_2^a, w_1^b, w_2^b \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$, *respectively, such that*

$$
\begin{cases}
\|\theta^1\|_{W_p^{1,2}(Q)}, \ \|\theta^2\|_{W_p^{1,2}(Q)} \leq M_3, \quad \|\alpha^1\|_{W_p^{1,2}(\Sigma)}, \ \|\alpha^2\|_{W_p^{1,2}(\Sigma)} \leq M_4, \\
\|\varphi^1\|_{W_p^{1,2}(Q)}, \ \|\varphi^2\|_{W_\nu^{1,2}(Q)} \leq N_3, \quad \|\xi^1\|_{W_p^{1,2}(\Sigma)}, \ \|\xi^2\|_{W_p^{1,2}(\Sigma)} \leq N_4,
\end{cases} \tag{18}
$$

*then the following estimate holds:*

$$
\begin{aligned}
&\max_{(t,x)\in Q}|\theta^1 - \theta^2| + \max_{(t,x)\in\Sigma}|\alpha^1 - \alpha^2| + \max_{(t,x)\in Q}|\varphi^1 - \varphi^2| + \max_{(t,x)\in\Sigma}|\xi^1 - \xi^2| \\
&\leq C_1 e^{CT} max\Big\{ \max_{(t,x)\in\Omega}|\theta_0^1 - \theta_0^2|, \ \max_{(t,x)\in\partial\Omega}|\alpha_0^1 - \alpha_0^2|, \\
&\qquad \max_{(t,x)\in\Omega}|\varphi_0^1 - \varphi_0^2|, \ \max_{(t,x)\in\partial\Omega}|\xi_0^1 - \xi_0^2|, \\
&\qquad \max_{(t,x)\in Q}|f_1^a - f_1^b|, \ \max_{(t,x)\in Q}|f_2^a - f_2^b|, \\
&\qquad \max_{(t,x)\in\Sigma}|w_1^a - w_1^b|, \ \max_{(t,x)\in\Sigma}|w_2^a - w_2^b|\Big\},
\end{aligned} \tag{19}
$$

*where the positive constants* $C_1 > 0$, $C > 0$ *are independent of* $\left\{\theta^1, \alpha^1, \varphi^1, \xi^1, f_1^a, w_1^a, \theta_0^1, \alpha_0^1, \varphi_0^1, \xi_0^1\right\}$ *and* $\left\{\theta^2, \alpha^2, \varphi^2, \xi^2, f_2^a, w_2^a, \theta_0^2, \alpha_0^2, \varphi_0^2, \xi_0^2\right\}$. *In particular, the uniqueness of the solution to problems* (6) and (7) *holds.*

**Proof of the Theorem 2.** Here, we apply the Leray–Schauder principle in order to prove the first part of the result established by Theorem 2. On this line, we consider suitable the Banach space

$$B^S = W_p^{0,1}(Q) \times L^p(\Sigma),$$

endowed with the norm $\|\cdot\|_{B^S}$, given by

$$\|(v, \bar{v})\|_{B^S} = \|v\|_{L^p(Q)} + \|v_x\|_{L^p(Q)} + \|\bar{v}\|_{L^p(\Sigma)},$$

and a nonlinear operator $S : B^S \times [0,1] \to B^S$, defined by

$$(\theta, \alpha) = S(v, \bar{v}, \lambda) = \Big(\theta(v, \bar{v}, \lambda), \alpha(v, \bar{v}, \lambda)\Big), \quad \forall (v, \bar{v}) \in B^S,\ \forall \lambda \in [0,1], \tag{20}$$

where $(\theta, \alpha)$ is the unique solution to the following linear boundary value problem (see (6)):

$$\begin{cases} p_1 \dfrac{\partial}{\partial t}\theta(t,x) - \left[\lambda p_2 \dfrac{\partial}{\partial v_{x_j}}(K_1(t,x,v)v_{x_i}) - (1-\lambda)\delta_i^j\right]\theta_{x_i x_j} & \\ \qquad = \lambda\left[A_1(t,x,v,v_{x_i}) - q_1 \dfrac{\partial}{\partial t}\Phi(t,x) + p_3 f_1(t,x)\right] & \text{in } Q \\ \theta(t,x) = \alpha(t,x) & \text{on } \Sigma \\ p_2 \dfrac{\partial}{\partial \mathbf{n}}\theta + p_1 \dfrac{\partial}{\partial t}\alpha - \Delta_\Gamma \alpha + p_5 \alpha = \lambda w_1(t,x) & \text{on } \Sigma \\ \theta(0,x) = \lambda \theta_0(x) & \text{on } \Omega \\ \alpha(0,x) = \lambda \alpha_0(x) & x \in \Gamma, \end{cases} \tag{21}$$

where $\Phi$ represents the unique solution to the nonlinear parabolic boundary value problem (8) corresponding to $h(t,x) = p_4 v(t,x) + q_5 f_2(t,x)$, i.e.,

$$\begin{cases} q_2 \dfrac{\partial}{\partial t}\Phi(t,x) - q_3 \dfrac{\partial}{\partial \Phi_{x_j}}\Big(K_2(t,x,\Phi)\Phi_{x_i}\Big)\Phi_{x_j x_i} & \\ \qquad = A_2(t,x,\Phi,\Phi_{x_i}) + q_4\left[\Phi - \Phi^3\right] + p_4 v(t,x) + q_5 f_2(t,x) & \text{in } Q, \\ \Phi(t,x) = \xi(t,x) & \text{on } \Sigma \\ q_3 \dfrac{\partial}{\partial \mathbf{n}}\Phi + q_2 \dfrac{\partial}{\partial t}\xi - \Delta_\Gamma \xi + q_6 \xi = w_2(t,x) & \text{on } \Sigma \\ \Phi(0,x) = \varphi_0(x) & \text{on } \Omega \\ \xi(0,x) = \xi_0(x) & x \in \Gamma. \end{cases} \tag{22}$$

Let us recall that

$$f_1(t,x) \in L^p(Q),\ f_2(t,x) \in L^q(Q) \ \text{ and } \ w_1(t,x),\ w_2(t,x) \in W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)$$

are given functions, while $p$ and $q$ satisfy the relation (4) in [1].

Since $p \le q$ (see [1]), then $h(t,x) = p_4 v(t,x) + q_5 f_2(t,x) \in L^p(Q)$. Using Theorem 1 (see (22)), we obtain that $\Phi \in W_p^{2,1}(Q)$ and, thus, $-q_1 \dfrac{\partial}{\partial t}\Phi(t,x) + p_3 f_1(t,x) \in L^p(Q)$. The $L_p$-theory guarantees that the linear parabolic equation (21) has a unique solution $\theta \in W_p^{2,1}(Q)$. Accordingly, the operator $S$ introduced in (20) is well defined.

Subsequently, following the same steps as in [1,2,17,18], we obtain (17) and (19) in Theorem 2.

The uniqueness of solution $\{\theta, \varphi\}$ follows from (19) by taking $f_1^a = f_1^b$, $f_2^a = f_2^b$, $w_1^a = w_1^b$, and $w_2^a = w_2^b$, and thus, the proof of Theorem 2 is complete. $\square$

## 4. Approximating Scheme—Convergence

Following the same steps as in [17,18], we associate to the nonlinear system (6) and (7) the following numerical scheme:

$$\begin{cases} p_1 \dfrac{\partial}{\partial t}\theta^{\varepsilon}(t,x) + q_1 \dfrac{\partial}{\partial t}\varphi^{\varepsilon}(t,x) - p_2 \mathrm{div}\Big(K_1\left(t,x,\theta^{\varepsilon}(t,x)\right) \nabla\theta^{\varepsilon}(t,x)\Big) & \\ \qquad\qquad = p_3 f_1(t,x) & \text{in } \ Q_i^{\varepsilon} \\ p_2 \dfrac{\partial}{\partial \mathbf{n}}\theta^{\varepsilon} + p_1 \dfrac{\partial}{\partial t}\alpha^{\varepsilon} - \Delta_{\Gamma}\alpha^{\varepsilon} + p_5 \alpha^{\varepsilon} = w_1(t,x) & \text{on } \ \Sigma_i^{\varepsilon} \\ \theta_+^{\varepsilon}(i\varepsilon,x) = \theta_-^{\varepsilon}(i\varepsilon,x), \quad \theta^{\varepsilon}(0,x) = \theta_0(x) & \text{on } \ \Omega, \\ \alpha^{\varepsilon}(i\varepsilon,x) = \theta^{\varepsilon}(i\varepsilon,x) & \text{on } \ \partial\Omega, \end{cases} \tag{23}$$

$$\begin{cases} q_2 \dfrac{\partial}{\partial t}\varphi^{\varepsilon}(t,x) - q_3 \mathrm{div}\Big(K_2\left(t,x,\varphi^{\varepsilon}(t,x)\right) \nabla\varphi^{\varepsilon}(t,x)\Big) & \\ \qquad\qquad = q_4 \varphi^{\varepsilon}(t,x) + p_4 \theta^{\varepsilon}(t,x) + q_5 f_2(t,x) & \text{in } \ Q_i^{\varepsilon} \\ q_3 \dfrac{\partial}{\partial \mathbf{n}}\varphi^{\varepsilon} + q_2 \dfrac{\partial}{\partial t}\xi^{\varepsilon} - \Delta_{\Gamma}\xi^{\varepsilon} + q_6 \xi^{\varepsilon} = w_2(t,x) & \text{in } \ \Sigma_i^{\varepsilon} \\ \varphi^{\varepsilon}(i\varepsilon,x) = z(\varepsilon, \varphi_-^{\varepsilon}(i\varepsilon,x)) & \text{on } \ \Omega, \\ \xi^{\varepsilon}(i\varepsilon,x) = \varphi^{\varepsilon}(i\varepsilon,x) & \text{on } \ \partial\Omega, \end{cases} \tag{24}$$

with $z(\varepsilon, \varphi_-^{\varepsilon}(i\varepsilon,x))$ being the solution of Cauchy problem:

$$\begin{cases} z'(s) + q_4 z^3(s) = 0 & s \in [0,\varepsilon] \\ z(0) = \varphi_-^{\varepsilon}(i\varepsilon,x) & \text{on } \ \Omega \\ \quad \varphi_-^{\varepsilon}(0,x) = \varphi_0(x) & \text{on } \ \Omega \\ \quad \varphi_-^{\varepsilon}(0,x) = \xi_0(x) & \text{on } \ \partial\Omega, \end{cases} \tag{25}$$

for $i = 0, 1, \cdots, M_{\varepsilon} - 1$, where $\varphi_-^{\varepsilon}$ stands for the left-hand limit of $\varphi^{\varepsilon}$.

Detailed discussions with respect to the advantage of (23)–(25) can be found in the works [3,4,15,17,18].

Next, we are interested in the convergence of the sequence $\Big\{(\theta^{\varepsilon},\alpha^{\varepsilon}),(\varphi^{\varepsilon},\xi^{\varepsilon})\Big\}$ of solutions to (23) and (24) to $\Big\{(\theta,\alpha),(\varphi,\xi)\Big\}$—the solution of problems (6) and (7) (see [3,17,18,20] for more details).

For later use, we set

$$\begin{aligned} W_Q &= L^2([0,T];H^1(\Omega)) \cap W^{1,2}([0,T];(H^1(\Omega))') \quad \text{and} \\ W_{\Sigma} &= L^2([0,T];H^1(\partial\Omega)) \cap W^{1,2}([0,T];(H^1(\partial\Omega))'). \end{aligned}$$

**Definition 3.** *By a weak solution to the nonlinear system* (6) and (7)*, we mean the pair of functions* $\left\{(\theta,\alpha),(\varphi,\xi)\right\} \in W_Q \times W_\Sigma$*,* $\theta=\alpha$ *and* $\varphi=\xi$ *on* $\Sigma$*, which satisfy* (6) and (7) *in the following sense:*

$$\begin{aligned}
&p_1\int_Q \left(\frac{\partial}{\partial t}\theta,\phi_1\right)dt\,dx + q_1\int_Q \left(\frac{\partial}{\partial t}\varphi,\phi_1\right)dt\,dx + p_2\int_Q K_1(t,x,\theta)\nabla\theta\cdot\nabla\phi_1\,dt\,dx\\
&\quad + p_1\int_\Sigma \left(\frac{\partial}{\partial t}\alpha,\phi_2\right)dt\,d\gamma + \int_\Sigma \nabla\alpha\cdot\nabla\phi_2\,dt\,d\gamma + q_6\int_\Sigma \alpha\phi_2\,dtd\gamma\\
&\quad = p_3\int_Q f_1\phi_1\,dtdx + \int_\Sigma w_1\phi_2\,dt\,d\gamma,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&q_1\int_Q \left(\frac{\partial}{\partial t}\varphi,\phi_1\right)dt\,dx + q_3\int_Q K_2(t,x,\varphi)\nabla\varphi\cdot\nabla\phi_1\,dt\,dx\\
&\quad + q_1\int_\Sigma \left(\frac{\partial}{\partial t}\xi,\phi_2\right)dt\,d\gamma + \int_\Sigma \nabla\xi\cdot\nabla\phi_2\,dt\,d\gamma + q_6\int_\Sigma \xi\phi_2\,dtd\gamma\\
&\quad = q_4\int_Q (\varphi-\varphi^3)\phi_1\,dt\,dx + p_4\int_Q \theta\phi_1\,dt\,dx + q_5\int_Q f_2\phi_1\,dtdx + \int_\Sigma w_2\phi_2\,dtd\gamma,
\end{aligned} \tag{27}$$

$$\forall(\phi_1,\phi_2)\in L^2([0,T];H^1(\Omega))\times L^2([0,T];H^1(\partial\Omega)),$$

*with* $\phi_1=\phi_2$ *on* $\Sigma$ *and* $\theta(0,x)=\theta_0(x),\ \varphi(0,x)=\varphi_0(x)$ *on* $\Omega$*.*

**Definition 4.** *By a weak solution to the nonlinear system* (23) and (24)*, we mean the pair of functions* $\left\{(\theta^\varepsilon,\alpha^\varepsilon),(\varphi^\varepsilon,\xi^\varepsilon)\right\} \in W_{Q_i^\varepsilon}\times W_{\Sigma_i^\varepsilon}$*,* $\theta_i^\varepsilon=\alpha_i^\varepsilon$ *and* $\varphi_i^\varepsilon=\xi_i^\varepsilon$ *on* $\Sigma_i^\varepsilon$*,* $i\in\{0,1,\dots,M_\varepsilon-1\}$*, which satisfy* (23) and (24) *in the following sense:*

$$\begin{aligned}
&p_1\int_Q \left(\frac{\partial}{\partial t}\theta^\varepsilon,\phi_1\right)dt\,dx + q_1\int_Q \left(\frac{\partial}{\partial t}\varphi^\varepsilon,\phi_1\right)dt\,dx\\
&\quad + p_2\int_Q K_1(t,x,\theta^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\phi_1\,dt\,dx\\
&\quad + p_1\int_\Sigma \left(\frac{\partial}{\partial t}\alpha^\varepsilon,\phi_2\right)dt\,d\gamma + \int_\Sigma \nabla\alpha^\varepsilon\cdot\nabla\phi_2\,dt\,d\gamma + q_6\int_\Sigma \alpha^\varepsilon\phi_2\,dtd\gamma\\
&\quad = p_3\int_Q f_1\phi_1\,dtdx + \int_\Sigma w_1\phi_2\,dt\,d\gamma,
\end{aligned} \tag{28}$$

$$
\begin{aligned}
& q_2 \int\limits_Q \left( \frac{\partial}{\partial t} \varphi^\varepsilon, \phi_1 \right) dt\, dx + q_3 \int\limits_Q K_2(t, x, \varphi^\varepsilon) \nabla \varphi^\varepsilon \cdot \nabla \phi_1 \, dt\, dx \\
& \quad + q_2 \int\limits_\Sigma \left( \frac{\partial}{\partial t} \xi^\varepsilon, \phi_2 \right) dt\, d\gamma + \int\limits_\Sigma \nabla \xi^\varepsilon \cdot \nabla \phi_2 \, dt\, d\gamma + q_6 \int\limits_\Sigma \xi^\varepsilon \phi_2 \, dt d\gamma \\
& \quad = q_4 \int\limits_Q \varphi^\varepsilon \phi_1 \, dt\, dx + p_4 \int\limits_Q \theta^\varepsilon \phi_1 \, dt\, dx + q_5 \int\limits_Q f_2 \phi_1 \, dt dx + \int\limits_\Sigma w_2 \phi_2 \, dt d\gamma, \\
& \qquad \forall (\phi_1, \phi_2) \in L^2([0,T]; H^1(\Omega)) \times L^2([0,T]; H^1(\partial\Omega)),
\end{aligned} \tag{29}
$$

*and* $\theta^\varepsilon_-(0,x) = \theta_0(x), \ \varphi^\varepsilon_-(0,x) = \varphi_0(x)$ *on* $\Omega$.

In (26)–(29), we denote by the same symbol $\int\limits_Q$ the duality between

$$L^2([0,T]; H^1(\Omega)) \text{ and } L^2([0,T]; (H^1(\Omega))').$$

*Convergence of the Numerical Scheme (23) and (24)*

Here, we prove the convergence of the solution to the numerical scheme (23) and (24), associated with the nonlinear systems (6) and (7). Therefore, the following holds:

**Theorem 3.** *Assume that* $\theta_0, \varphi_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $p \geq 2$, *with* $p_2 \frac{\partial}{\partial \mathbf{n}} \theta_0 + \Delta_\Gamma \theta_0 + p_5 \theta_0 = w_1(0,x)$, $q_3 \frac{\partial}{\partial \mathbf{n}} \varphi_0 + \Delta_\Gamma \varphi_0 + q_6 \varphi_0 = w_2(0,x)$ *on* $\partial\Omega$ *and* $w_1, w_2 \in W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\Sigma)$. *Let* $\left\{ (\theta^\varepsilon, \alpha^\varepsilon), (\varphi^\varepsilon, \xi^\varepsilon) \right\}$ *be the solution of the approximating scheme* (23) and (24). *As* $\varepsilon \to 0$, *one has*

$$
\begin{aligned}
& \left\{ (\theta^\varepsilon(s), \alpha^\varepsilon(s)), (\varphi^\varepsilon(s), \xi^\varepsilon(s)) \right\} \to \left\{ (\theta^*(s), \alpha^*(s)), (\varphi^*(s), \xi^*(s)) \right\} \\
& \qquad \qquad \text{strongly in } L^2(\Omega) \times L^2(\partial\Omega) \ \text{ for any } s \in (0,T],
\end{aligned} \tag{30}
$$

*where* $\left\{ (\theta^*(s), \alpha^*(s)), (\varphi^*(s), \xi^*(s)) \right\} \in W_Q \times W_\Sigma$ *is the weak solution of the nonlinear systems* (6) *and* (7).

The inequalities (31)–(34) (listed below) are essential in proving the main result of the present work—Theorem 3.

$$\|\varphi^\varepsilon(i\varepsilon, x)\|^2_{L^2(\Omega)} \leq \|\varphi^\varepsilon_-(i\varepsilon, x)\|^2_{L^2(\Omega)}, \tag{31}$$

$$z^2(\varepsilon, \varphi^\varepsilon_-(i\varepsilon, x)) \leq \varphi^\varepsilon_-(i\varepsilon, x)^2, \quad a.e \ \ x \in \Omega, \tag{32}$$

$$\|\nabla \varphi^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \leq \|\nabla \varphi^\varepsilon_-(i\varepsilon, x)\|_{L^2(\Omega)}, \tag{33}$$

$$\|z(\varepsilon, x) - \varphi^\varepsilon_-(i\varepsilon, x)\|_{L^2(\Omega)} \leq \varepsilon L, \tag{34}$$

$i = 0, 1, ..., M_\varepsilon - 1$.

**Proof of Theorem 3.** Following the same steps as in [17], we obtain the solution to problem (24) as $\varphi^\varepsilon \in W_p^{1,2}(Q_i^\varepsilon) \cap L^\infty(Q_i^\varepsilon)$, $\forall i \in \{0, 1, ..., M_\varepsilon - 1\}$.

Next, we give a priori estimates in $Q_i^\varepsilon$, $\forall i \in \{0, 1, ..., M_\varepsilon - 1\}$. Multiplying $(23)_1$ by $\frac{p_4}{q_1} \theta^\varepsilon$ and $(24)_1$ by $\varphi^\varepsilon_t$ and using integration by parts, Green's formula, and the relations (28) and (29), we obtain

$$\frac{p_4}{q_1}\frac{p_1}{2}\frac{d}{dt}\int_{\Omega}|\theta^{\varepsilon}|^2dx+\frac{p_4}{q_1}\frac{p_1}{2}\frac{d}{dt}\int_{\partial\Omega}|\alpha^{\varepsilon}|^2d\gamma+p_4\int_{\Omega}\theta^{\varepsilon}\varphi_t^{\varepsilon}dx$$
$$+\frac{p_4}{q_1}p_2\int_{\Omega}K_1(t,x,\theta^{\varepsilon})|\nabla\theta^{\varepsilon}|^2dx+\frac{p_4}{q_1}\int_{\partial\Omega}|\nabla_{\Gamma}\alpha^{\varepsilon}|^2d\gamma+\frac{p_4}{q_1}p_5\int_{\partial\Omega}|\alpha^{\varepsilon}|^2d\gamma \tag{35}$$
$$=\frac{p_4}{q_1}p_3\int_{\Omega}f_1\theta^{\varepsilon}\,dx+\frac{p_4}{q_1}\int_{\partial\Omega}w_1\theta^{\varepsilon}\,d\gamma,$$

$$q_2\int_{\Omega}|\varphi_t^{\varepsilon}|^2dx+q_2\int_{\partial\Omega}|\xi_t^{\varepsilon}|^2d\gamma$$
$$+\frac{q_3}{2}\int_{\Omega}K_2(t,x,\varphi^{\varepsilon})\frac{d}{dt}|\nabla\varphi^{\varepsilon}|^2dx+\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}|\nabla_{\Gamma}\xi^{\varepsilon}|^2d\gamma+\frac{q_6}{2}\frac{d}{dt}\int_{\partial\Omega}|\xi^{\varepsilon}|^2d\gamma \tag{36}$$
$$=\frac{q_4}{2}\frac{d}{dt}\int_{\Omega}|\varphi^{\varepsilon}|^2dx+p_4\int_{\Omega}\theta^{\varepsilon}\varphi_t^{\varepsilon}dx+q_5\int_{\Omega}f_2\,\varphi_t^{\varepsilon}\,dx+\int_{\partial\Omega}w_2\xi_t^{\varepsilon}d\gamma.$$

Using Hölder's inequality for the right-side terms $\frac{p_4}{q_1}p_3\int_{\Omega}f_1\theta^{\varepsilon}\,dx$, $\frac{p_4}{q_1}\int_{\partial\Omega}w_1\theta^{\varepsilon}\,d\gamma$, $q_5\int_{\Omega}f_2\,\varphi_t^{\varepsilon}\,dx$, and $\int_{\partial\Omega}w_2\xi_t^{\varepsilon}d\gamma$, we obtain

$$\frac{p_4}{q_1}p_3\int_{\Omega}f_1\theta^{\varepsilon}\,dx\leq\frac{1}{2}\int_{\Omega}|\theta^{\varepsilon}|^2\,dx+\frac{p_4}{q_1}\frac{p_3}{2}\int_{\Omega}|f_1|^2\,dx,$$
$$\frac{p_4}{q_1}\int_{\partial\Omega}w_1\theta^{\varepsilon}\,d\gamma\leq\frac{p_4}{q_1}p_5\int_{\partial\Omega}|\theta^{\varepsilon}|^2d\gamma+\frac{p_4}{q_1}\frac{1}{p_5}\int_{\partial\Omega}|w_1|^2d\gamma,$$
$$q_5\int_{\Omega}f_2\,\varphi_t^{\varepsilon}\,dx\leq\frac{q_2}{2}\int_{\Omega}|\varphi_t^{\varepsilon}|^2\,dx+\frac{q_5}{2q_2}\int_{\Omega}|f_2|^2\,dx,$$
$$\int_{\partial\Omega}w_2\xi_t^{\varepsilon}\,d\gamma\leq\frac{q_2}{2}\int_{\partial\Omega}|\xi_t^{\varepsilon}|^2d\gamma+\frac{1}{2q_2}\int_{\partial\Omega}|w_2|^2d\gamma.$$

Adding (35) and (36) and making use of the above, we obtain

$$\begin{aligned}
&\frac{p_4}{q_1}\frac{p_1}{2}\frac{d}{dt}\int_{\Omega}|\theta^{\varepsilon}|^2dx+\frac{p_4}{q_1}\frac{p_1}{2}\frac{d}{dt}\int_{\partial\Omega}|\alpha^{\varepsilon}|^2d\gamma+\frac{p_4}{q_1}p_2K1_m\int_{\Omega}|\nabla\theta^{\varepsilon}|^2dx\\
&+\frac{q_2}{2}\int_{\Omega}|\varphi_t^{\varepsilon}|^2\,dx+\frac{q_2}{2}\int_{\partial\Omega}|\xi_t^{\varepsilon}|^2\,d\gamma+\frac{q_3}{2}K2_m\frac{d}{dt}\int_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx\\
&+\frac{p_4}{q_1}\int_{\partial\Omega}|\nabla_{\Gamma}\alpha^{\varepsilon}|^2d\gamma+\frac{1}{2}\frac{d}{dt}\int_{\partial\Omega}|\nabla_{\Gamma}\xi^{\varepsilon}|^2d\gamma+\frac{q_6}{2}\frac{d}{dt}\int_{\partial\Omega}|\xi^{\varepsilon}|^2d\gamma\\
&\leq\frac{q_4}{2}\frac{d}{dt}\int_{\Omega}|\varphi^{\varepsilon}|^2dx+\frac{1}{2}\int_{\Omega}|u^{\varepsilon}|^2\,dx\\
&\quad+\frac{p_4}{q_1}\frac{p_3}{2}\int_{\Omega}|f_1|^2\,dx+\frac{q_5}{2q_2}\int_{\Omega}|f_2|^2\,dx\\
&\quad+\frac{p_4}{q_1}\frac{1}{p_5}\int_{\partial\Omega}|w_1|^2(t,x)d\gamma+\frac{1}{2q_2}\int_{\partial\Omega}|w_2|^2d\gamma,
\end{aligned} \tag{37}$$

where the inequalities $(15)_1$ and $(16)_1$ are used, too.

Multiplying now $(24)_1$ by $\dfrac{2q_4}{q_2}\varphi^\varepsilon$ as shown above, we obtain

$$\begin{aligned} & q_4 \frac{d}{dt}\int_\Omega |\varphi^\varepsilon|^2 dx + q_4 \frac{d}{dt}\int_{\partial\Omega} |\xi^\varepsilon|^2 d\gamma + \frac{2q_4}{q_2} q_3 \int_\Omega K_2(t,x,\varphi^\varepsilon)|\nabla\varphi^\varepsilon|^2 dx \\ & \quad + \frac{2q_4}{q_2}\int_{\partial\Omega} |\nabla_\Gamma \xi^\varepsilon|^2 d\gamma + \frac{2q_4}{q_2} q_6 \int_{\partial\Omega} |\xi^\varepsilon|^2 d\gamma \\ & \quad = \frac{2q_4}{q_2} q_4 \int_\Omega |\varphi^\varepsilon|^2 dx + \frac{2q_4}{q_2} p_4 \int_\Omega \theta^\varepsilon \varphi^\varepsilon dx + \frac{2q_4}{q_2} q_5 \int_\Omega f_2\, \varphi^\varepsilon\, dx + \frac{2q_4}{q_2}\int_{\partial\Omega} w_2\, \varphi^\varepsilon\, d\gamma. \end{aligned} \tag{38}$$

Again, using Hölder's inequality for the right-side terms $\int_\Omega \theta^\varepsilon \varphi^\varepsilon dx$, $\int_\Omega f_2\, \varphi^\varepsilon\, dx$, and $\int_{\partial\Omega} w_2\, \varphi^\varepsilon\, d\gamma$, we have

$$\begin{aligned} \frac{2q_4}{q_2} p_4 \int_\Omega \theta^\varepsilon \varphi^\varepsilon dx &\le \frac{2q_4}{2q_2} p_4 \int_\Omega |\theta^\varepsilon|^2\, dx + \frac{2q_4}{2q_2} p_4 \int_\Omega |\varphi^\varepsilon|^2\, dx, \\ \frac{2q_4}{q_2} q_5 \int_\Omega f_2\, \varphi^\varepsilon\, dx &\le \frac{2q_4}{2q_2} q_5 \int_\Omega |\varphi^\varepsilon|^2\, dx + \frac{2q_4}{2q_2} q_5 \int_\Omega |f_2|^2\, dx, \\ \frac{2q_4}{q_2}\int_{\partial\Omega} w_2\, \varphi^\varepsilon\, d\gamma &\le \frac{2q_4}{2q_2}\int_{\partial\Omega} |\varphi^\varepsilon|^2\, d\gamma + \frac{2q_4}{2q_2}\int_{\partial\Omega} |w_2|^2\, d\gamma, \end{aligned}$$

and then, from (38), we obtain

$$\begin{aligned} & q_4 \frac{d}{dt}\int_\Omega |\varphi^\varepsilon|^2 dx + q_4 \frac{d}{dt}\int_{\partial\Omega} |\xi^\varepsilon|^2 d\gamma + \frac{2q_4}{q_2} q_3 K2_m \int_\Omega |\nabla\varphi^\varepsilon|^2 dx \\ & \quad + \frac{2q_4}{q_2}\int_{\partial\Omega} |\nabla_\Gamma \xi^\varepsilon|^2 d\gamma + \frac{2q_4}{q_2} q_6 \int_{\partial\Omega} |\xi^\varepsilon|^2 d\gamma \\ & \quad \le C(q_2,q_3,q_4,p_4,q_5)\left[\int_\Omega |\theta^\varepsilon|^2 dx + \int_\Omega |\varphi^\varepsilon|^2 dx + \int_\Omega |f_2|^2 dx + \int_{\partial\Omega} |w_2|^2 d\gamma\right], \end{aligned} \tag{39}$$

where the inequality $(16)_1$ is used, too.

Adding (37) and (39), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Bigg[\frac{p_4}{q_1}\frac{p_1}{2}\int\limits_{\Omega}|\theta^{\varepsilon}|^2dx+\frac{p_4}{q_1}\frac{p_1}{2}\int\limits_{\partial\Omega}|\alpha^{\varepsilon}|^2d\gamma+\frac{q_4}{2}\int\limits_{\Omega}|\varphi^{\varepsilon}|^2dx+\left(q_4+\frac{q_6}{2}\right)\int\limits_{\partial\Omega}|\xi^{\varepsilon}|^2d\gamma \\
&\quad+\frac{q_3}{2}K2_m\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx+\frac{1}{2}\int\limits_{\partial\Omega}|\nabla_{\Gamma}\xi^{\varepsilon}|^2d\gamma\Bigg] \\
&\quad+\frac{q_2}{2}\int\limits_{\Omega}|\varphi_t^{\varepsilon}|^2dx+\frac{q_2}{2}\int\limits_{\partial\Omega}|\xi_t^{\varepsilon}|^2d\gamma \\
&\quad+\frac{p_4}{q_1}\int\limits_{\partial\Omega}|\nabla_{\Gamma}\alpha^{\varepsilon}|^2d\gamma+\frac{2q_4}{q_2}\int\limits_{\partial\Omega}|\nabla_{\Gamma}\xi^{\varepsilon}|^2d\gamma+\frac{2q_4}{q_2}q_6\int\limits_{\partial\Omega}|\xi^{\varepsilon}|^2d\gamma \\
&\quad+\frac{p_4}{q_1}p_2K1_m\int\limits_{\Omega}|\nabla\theta^{\varepsilon}|^2dx+\frac{2q_4}{q_2}q_3K2_m\int\limits_{\Omega}|\nabla\varphi^{\varepsilon}|^2dx \\
&\le C(p_1,p_2,p_3,p_4,p_5,q_1,q_2,q_3,q_4,q_5,q_6)\Bigg[\int\limits_{\Omega}|\theta^{\varepsilon}|^2dx+\int\limits_{\Omega}|\varphi^{\varepsilon}|^2dx \\
&\quad+\int\limits_{\Omega}|f_1|^2\,dx+\int\limits_{\Omega}|f_2|^2dx+\int\limits_{\partial\Omega}|w_1|^2d\gamma+\int\limits_{\partial\Omega}|w_2|^2d\gamma\Bigg].
\end{aligned}
$$

Integrating the preceding on $Q_i^{\varepsilon}$, $i=0,1,2,...,M_{\varepsilon}-1$ (i.e., on $[i\varepsilon,(i+1)\varepsilon]$, $i=0,1,2,...,M_{\varepsilon}-1$) and summing the inequalities obtained, we derive (see [18])

$$
\begin{aligned}
&\frac{p_4}{q_1}\frac{p_1}{2}\|\theta_{-}^{\varepsilon}(T,x)\|^2_{L^2(\Omega)}+\frac{p_4}{q_1}\frac{p_1}{2}\|\alpha_{-}^{\varepsilon}(T,x)\|^2_{L^2(\partial\Omega)} \\
&+\frac{q_4}{2}\|\varphi_{-}^{\varepsilon}(T,x)\|^2_{L^2(\Omega)}+\left(q_4+\frac{q_6}{2}\right)\|\xi_{-}^{\varepsilon}(T,x)\|^2_{L^2(\partial\Omega)} \\
&+\frac{q_3}{2}K2_m\|\nabla\varphi_{-}^{\varepsilon}(T,x)\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla_{\Gamma}\xi_{-}^{\varepsilon}(T,x)\|^2_{L^2(\partial\Omega)} \\
&+\int_0^T\Bigg[\frac{q_2}{2}\|\varphi_t^{\varepsilon}\|^2_{L^2(\Omega)}+\frac{q_2}{2}\|\xi_t^{\varepsilon}\|^2_{L^2(\partial\Omega)} \\
&\qquad+\frac{p_4}{q_1}\|\nabla_{\Gamma}\alpha^{\varepsilon}\|^2_{L^2(\partial\Omega)}+\frac{2q_4}{q_2}\|\nabla_{\Gamma}\xi^{\varepsilon}\|^2_{L^2(\partial\Omega)}+\frac{2q_4}{q_2}q_6\|\xi^{\varepsilon}\|^2_{L^2(\partial\Omega)} \\
&\qquad+\frac{p_4}{q_1}p_2K1_m\|\nabla\theta^{\varepsilon}\|^2_{L^2(\Omega)}+\frac{2q_4}{q_2}q_3K2_m\|\nabla\varphi^{\varepsilon}\|^2_{L^2(\Omega)}\Bigg]dt \\
&\le\frac{p_4}{q_1}\frac{p_1}{2}\|\theta_0\|^2_{L^2(\Omega)}+\frac{p_4}{q_1}\frac{p_1}{2}\|\alpha_0\|^2_{L^2(\partial\Omega)}+\frac{q_4}{2}\|\varphi_0\|^2_{L^2(\Omega)}+\frac{q_4}{2}\|\xi_0\|^2_{L^2(\partial\Omega)} \\
&+\frac{p_2}{2}\|\nabla\theta_0\|^2_{L^2(\Omega)}+\frac{q_3}{2}K_{min}\|\nabla\varphi_0\|^2_{L^2(\Omega)} \\
&+C(p_1,p_2,p_3,p_4,p_5,q_1,q_2,q_3,q_4,q_5,q_6)\Bigg\{\int_0^T\left[\|\theta^{\varepsilon}\|^2_{L^2(\Omega)}+\|\varphi^{\varepsilon}\|^2_{L^2(\Omega)}\right]dt \\
&+\|f_1\|^2_{L^2(Q)}+\|f_2\|^2_{L^2(Q)}+\|w_1\|^2_{L^2(\Sigma)}+\|w_2\|^2_{L^2(\Sigma)}\Bigg\},
\end{aligned}
$$

where the inequalities (31) and (33) are used.

Applying the Gronwall inequality to the above inequality, we finally deduce

$$\int_0^T \Big\{ \|\varphi_t^\varepsilon\|^2_{L^2(\Omega)} + \|\xi_t^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\nabla\theta^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\alpha^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\nabla_\Gamma\xi^\varepsilon\|^2_{L^2(\partial\Omega)} \Big\} dt \le C, \tag{40}$$

where $C > 0$ is independent of $\varepsilon$ and $M_\varepsilon$.

Owing to (23)$_3$, (24)$_3$, and (34), we obtain

$$\sum_{i=0}^{M_\varepsilon - 1} \|\theta^\varepsilon(i\varepsilon, x) - \theta_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Omega)} \le TL = C_1, \tag{41}$$

$$\sum_{i=0}^{M_\varepsilon - 1} \|\varphi^\varepsilon(i\varepsilon, x) - \varphi_-^\varepsilon(i\varepsilon, x)\|_{L^2(\Gamma)} \le C_2, \tag{42}$$

where $C_1 > 0$, $C_2 > 0$ are independent of $M_\varepsilon$ and $\varepsilon$. Adding (40)–(42), we derive

$$\overset{T}{\underset{0}{V1}}\,\theta^\varepsilon + \overset{T}{\underset{0}{V2}}\,\varphi^\varepsilon + \int_0^T \Big\{ \|\varphi_t^\varepsilon\|^2_{L^2(\Omega)} + \|\xi_t^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\nabla\theta^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla_\Gamma\alpha^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\nabla_\Gamma\xi^\varepsilon\|^2_{L^2(\partial\Omega)} \Big\} dt \le C, \tag{43}$$

where $C > 0$ is independent on $M_\varepsilon$ and $\varepsilon$, while $\overset{T}{\underset{0}{V1}}\,\theta^\varepsilon$ and $\overset{T}{\underset{0}{V2}}\,\varphi^\varepsilon$ stand for the variation of $\theta^\varepsilon : [0, T] \to L^2(\Omega)$ and $\varphi^\varepsilon : [0, T] \to L^2(\Omega)$, respectively.

Now, multiplying (23)$_1$ by $\theta_t^\varepsilon$, integrating over $[i\varepsilon, (i+1)\varepsilon]$, $i = 0, 1, \cdots, M_\varepsilon - 1$, and involving Cauchy–Schwartz's inequalities, Hölder's inequality, Cauchy's inequality, Gronwall–Bellman's inequality, Green's formula, as well as the relations (15)$_1$ and (40), we finally obtain the estimate

$$\int_0^T \Bigg[ \frac{p_1}{2} \int_\Omega (\theta_t^\varepsilon)^2 dx + \frac{p_1}{2} \int_{\partial\Omega} (\alpha_t^\varepsilon)^2 d\gamma + \frac{p_2}{2} K1_m \frac{d}{dt} \int_\Omega |\nabla\theta^\varepsilon|^2 dx + \frac{1}{2}\frac{d}{dt} \int_{\partial\Omega} |\nabla_\Gamma\alpha^\varepsilon|^2 d\gamma + \frac{p_5}{2}\frac{d}{dt} \int_{\partial\Omega} |\alpha^\varepsilon|^2 d\gamma \Bigg] ds \le C, \tag{44}$$

for all $\varepsilon > 0$, where the constant $C > 0$ does not depend on $M_\varepsilon$ and $\varepsilon$.

Combining (43) with (44), we obtain

$$\overset{T}{\underset{0}{V1}}\,\theta^\varepsilon + \overset{T}{\underset{0}{V2}}\,\varphi^\varepsilon + \int_0^T \Big[ \|\theta_t^\varepsilon\|^2_{L^2(\Omega)} + \|\alpha_t^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\varphi_t^\varepsilon\|^2_{L^2(\Omega)} + \|\xi_t^\varepsilon\|^2_{L^2(\partial\Omega)} + \|\nabla\theta^\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\varphi^\varepsilon\|^2_{L^2(\Omega)} \Big] dt \le C. \tag{45}$$

Since the injection of $L^2(\Omega)$ into $H^{-1}(\Omega)$ is compact and $\{\theta_s^\varepsilon(s)\}$, $\{\varphi_s^\varepsilon(s)\}$ are bounded in $L^2(\Omega)$ $\forall s \in [0, T]$, we conclude that there exists a bounded variation function: $\theta^*(s) \in$

$BV([0,T];H^{-1}(\Omega))$, $\varphi^*(s) \in BV([0,T];H^{-1}(\Omega))$, respectively, and the subsequences $\theta^\varepsilon(s)$, $\varphi^\varepsilon(s)$ (see [17]) such that

$$\begin{cases} \theta^\varepsilon(s) \to \theta^*(s) \\ \varphi^\varepsilon(s) \to \varphi^*(s) \end{cases} \quad \text{strongly in} \quad H^{-1}(\Omega) \quad \forall s \in [0,T]. \tag{46}$$

A similar reasoning carried out for $\{\alpha_s^\varepsilon(s)\}$ and $\{\xi_s^\varepsilon(s)\}$ allows us to conclude the convergence

$$\begin{cases} \alpha^\varepsilon(s) \to \alpha^*(s) \\ \xi^\varepsilon(s) \to \xi^*(s) \end{cases} \quad \text{strongly in} \quad H^{-1}(\partial\Omega) \quad \forall s \in [0,T]. \tag{47}$$

Furthermore, from (45) we deduce that

$$\begin{aligned} &\begin{cases} \theta^\varepsilon \to \theta^* \\ \varphi^\varepsilon \to \varphi^* \end{cases} \quad \text{weakly in} \quad H^{-1}(\Omega) \quad \forall s \in [0,T], \\ &\begin{cases} \alpha^\varepsilon \to \alpha^* \\ \xi^\varepsilon \to \xi^* \end{cases} \quad \text{weakly in} \quad H^{-1}(\partial\Omega) \quad \forall s \in [0,T]. \end{aligned} \tag{48}$$

By the well-known embeddings,

$$H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega) \text{ and } H^1(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-1}(\partial\Omega),$$

standard interpolation inequalities (see [17], p. 17) yield that $\forall \ell > 0$, $\exists C(\ell) > 0$ such that

$$\begin{aligned} &\begin{cases} \|\theta^\varepsilon(s) - \theta^*(s)\|_{L^2(\Omega)} \le \ell \|\theta^\varepsilon(s) - \theta^*(s)\|_{H^1(\Omega)} + C(\ell)\|\theta^\varepsilon(s) - \theta^*(s)\|_{H^{-1}(\Omega)} \\ \|\varphi^\varepsilon(s) - \varphi^*(s)\|_{L^2(\Omega)} \le \ell \|\varphi^\varepsilon(s) - \varphi^*(s)\|_{H^1(\Omega)} + C(\ell)\|\varphi^\varepsilon(s) - \varphi^*(s)\|_{H^{-1}(\Omega)}, \end{cases} \\ &\begin{cases} \|\alpha^\varepsilon(s) - \alpha^*(s)\|_{L^2(\partial\Omega)} \le \ell \|\alpha^\varepsilon(s) - \alpha^*(s)\|_{H^1(\partial\Omega)} + C(\ell)\|\alpha^\varepsilon(s) - \alpha^*(s)\|_{H^{-1}(\partial\Omega)} \\ \|\xi^\varepsilon(s) - \xi^*(s)\|_{L^2(\partial\Omega)} \le \ell \|\xi^\varepsilon(s) - \xi^*(s)\|_{H^1(\partial\Omega)} + C(\ell)\|\xi^\varepsilon(s) - \xi^*(s)\|_{H^{-1}(\partial\Omega)}, \end{cases} \end{aligned} \tag{49}$$

$\forall \varepsilon > 0$ and $\forall s \in [0,T]$, where $C(\ell) \to 0$ as $\ell \to 0$.

Finally, relations (46)–(49) permit us to conclude that the assertion conducted in (30) holds true, ending the proof of Theorem 3. □

**Corollary 1.** *Assume $\theta_0, \varphi_0 \in W_p^{2-\frac{2}{p}}(\Omega)$, $p \ge 2$, with $p_2 \frac{\partial}{\partial \mathbf{n}}\theta_0(x) - \Delta_\Gamma \theta_0 + p_5 \theta_0(x) = w_1(0,x)$, $q_3 \frac{\partial}{\partial \mathbf{n}}\varphi_0(x) - \Delta_\Gamma \varphi_0 + q_6 \varphi_0(x) = w_2(0,x)$ on $\partial\Omega$ and $w_1, w_2 \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\Sigma)$. Then, $\left\{(\theta^*, \alpha^*), (\varphi^*, \xi^*)\right\} \in W_Q \times W_\Sigma$, $\theta^* = \alpha^*$ and $\varphi^* = \xi^*$ on $\Sigma$, is a weak solution of the nonlinear second-order parabolic systems* (6) and (7).

The general framework of the numerical algorithm to compute the approximate solution of problems (6) and (7) (practically, the approximate solution to the nonlinear second-order boundary value problem (1)–(3)) via the *fractional-step scheme* may be demonstrated as follows:

Begin **alg-frac _sec-ord** _`u+varphi`_**dbc**
$i := 0 \ \to \theta_0$ from $(23)_3$ and $\varphi_0$ from $(25)_3$;
For $i := 0$ to $M_\varepsilon - 1$ do
Compute $z(\varepsilon, \cdot)$ from (25);
$\varphi^\varepsilon(i\varepsilon, \cdot) := z(\varepsilon, \cdot)$;

$\alpha^{\varepsilon}(i\varepsilon, \cdot) := \theta^{\varepsilon}(i\varepsilon, \cdot)$;
$\xi^{\varepsilon}(i\varepsilon, \cdot) := \varphi^{\varepsilon}(i\varepsilon, \cdot)$;
Compute $(\theta^{\varepsilon}((i+1)\varepsilon, \cdot), \varphi^{\varepsilon}((i+1)\varepsilon, \cdot))$ solving the linear system $(23)_{1-2}$ + $(24)_{1-2}$;
End-for;
End.

An example of numerical implementation to **alg-frac _sec-ord** _u+varphi_**dbc** , considering a particular case of parameters $p_1, p_2, p_3, p_4, p_5, q_1, q_2, q_3, q_4, q_5, q_6, K_1 = K_2 = 1$, can be found in [18].

## 5. Conclusions

The main problem studied in this paper is a nonlinear second-order parabolic system of coupled PDEs (1), with the principal part in *divergence* form for both unknown functions $u$, $\varphi$ and subject to in-homogeneous dynamic boundary conditions (2). Provided that the initial and boundary data meet appropriate regularity as well as compatibility conditions, it is proven the well-posedness of a classical solution to the nonlinear problem in this new formulation (Theorem 2). Precisely, the Leray–Schauder principle, as well as the $L^p$ theory of linear and quasi-linear parabolic equations, via Lemma 7.4 (see [18] and reference therein), is applied to prove the qualitative properties of solutions $\theta(t,x), \alpha(t,x)$, $\varphi(t,x), \xi(t,x)$. Moreover, the a priori estimates are made in $L^p(Q)$ and $L^p(\Sigma)$, which permit us to derive regularity properties of higher order for $\theta, \alpha, \varphi, \xi$, that is, $\big(\theta(t,x), \alpha(t,x)\big) \in W_p^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $\big(\varphi(t,x), \xi(t,x)\big) \in W_\nu^{1,2}(Q) \times W_p^{1,2}(\Sigma)$, $\nu = \min\{q, \mu\}$ (see [17]).

Let us remark that, because of the presence of the terms $K_1(t,x,\theta(t,x))$ and $K_2(t,x,\varphi(t,x))$, the nonlinear operator $S$ in (20) does not represent the gradient of the energy functional. Therefore, the new proposed second-order nonlinear systems (6) and (7) cannot be obtained from the minimization of any energy cost functional, i.e., (1) is not a variational PDE model.

Next, an iterative scheme of fractional-step type is introduced to approximate the problems (6) and (7). The convergence result is established for the proposed numerical scheme, and a conceptual numerical algorithm, **alg-frac _sec-ord** _u+varphi_**dbc** , is formulated in the end. See [17] and references therein for an example of numerical implementation to the conceptual algorithm **alg-frac_sec-ord**_u+varphi_**dbc** .

The qualitative results obtained here can be used later in the quantitative approaches of the mathematical model (1)–(3) as well as in the study of distributed and/or boundary nonlinear optimal control problems governed by such a nonlinear problem. Numerical implementation of the conceptual algorithm, **alg-frac _sec-ord** _u+varphi_**dbc** , as well as various simulations regarding the physical phenomena described by nonlinear second-order parabolic system (1), correspondingly, especially, to the different choice of mobility functions $K_1(t,x,\theta(t,x))$ and $K_2(t,x,\varphi(t,x))$, (see [2]), represent a matter for further investigation.

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