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Coercive and Noncoercive Mixed Generalized Complementarity Problems

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Abstract: Impressed with the very recent developments of noncoercive complementarity problems and the use of recession sets in complementarity problems, here, we discuss mixed generalized complementarity problems in Hausdorff topological vector spaces. We used the Tikhonov regularization procedure, as well as arguments from the recession analysis, to establish the existence of solutions for mixed generalized complementarity problems without coercivity assumptions in Banach spaces.

Keywords: complementarity problem; Tikhonov regularization procedure; copositive mapping; recession function; recession cone; variational inequality

MSC: 47H04; 47H10; 46N10

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1. Introduction

In the year 1965, Lemke [1] initiated the complementarity concept, followed by Cottle and Dantzig [2], in order to study linear and quadratic programming problems, as well as the bimatrix game problems. Nowadays, it provides a reliable platform to analyze a wide variety of unrelated problems in physics, optimization, transportation, etc; see, for instance, [2–4]. Initially, complementarity problem formulation was studied in the framework of finite dimensional spaces (see [3,5,6]); out of which, the work of Habetler and Price [3] is significant. Habetler and Price [3] have replaced the usual non-negative partial ordering generated by $\mathbb{R}^n$ for finite dimensional form of complementarity problems by partial orderings induced by some given cone along with its polar. Karamardian [7] have extended the work of Habetler and Price [3] into the framework of locally convex Hausdorff topological vector spaces. Saigal [6] has extended the problem of Habetler and Price [3] for multivalued mappings. Schaible and Yao [8] considered the problem of Habetler and Price [3] in the setting of Banach lattices and studied the equivalence of complementarity problems, least-element problems, and variational inequalities. The results in [8] were then extended by Ansari, Lai, and Yao [9] and by Zeng, Ansari, and Yao [10] for multivalued mappings by using pseudomonotonicity of operators in the sense of Karamardian.

Most of the techniques used to obtain the existence of solutions for the complementarity problems are usually based on the Karamardian [4] type of monotonicity. On the other hand, there is an another kind of monotonicity property of operators that is used to evaluate equilibrium problems, as well as variational inequalities, known as pseudomonotonicity in the topological sense. This concept of monotonicity was initiated by Brézis [11] in the year 1968, and then later it became known as pseudomonotonicity in the sense of Brézis. It is a hybridization of both the monotonicity as well as continuity property of operators, and thus it distinguishes itself from the others. Aubin [12] has considered this concept of pseudomonotonicity with a relaxing of the continuity properties of operators while formulating minimax problems in game theory, as well as fixed point problems. The unique feature of Brézis pseudomonotonicity is that it provides a unified approach to both monotonicity and compactness. The class of pseudomonotone operators in the sense of Brézis is quite large, for example, Kien et al. [13] proved by means of examples that there exists an operator which is pseudomonotone in the sense of Brézis but not pseudomonotone in the sense of Karamardian. Further, Browder [14] had proved that every maximal monotone operator is Brézis pseudomonotone [14]. The pseudomonotonicity in the sense of Brézis is rich in applications, and it is very useful for the study of coercive and noncoercive hemivariational inequalities see [15,16].

The concept of recession function was introduces by Brézis and Nirenberg [17] in 1978 for a nonlinear operator, in the context of Hilbert spaces, in order to find an analytic description of the range of sum of linear and nonlinear operators while solving nonlinear partial differential equations. On the other hand, the concept of recession cone was initiated by Baocchi, Gastaldi, and Tomarelli [18] in the year 1986 while establishing the existence of solutions for the results on noncoercive variational inequalities. A bridge between the recession function and recession cone was established by Baiocchi et al. [19] in 1988. Later on, the concept of recession functions, as well as recession cones, was enormously used by various authors while evaluating noncoercive equilibrium problems, as well as variational inequalities [20–22].

Very recently, Sahu et al. [23] used both the concepts of Brézis pseudomonotonicity as well as recession sets and studied the noncoercive complementarity problems with copositivity assumptions.

Motivated by the above works, in this paper, we defined a very general kind of complementarity problem, known as the mixed generalized complementarity problem in the framework of Hausdorff topological vector spaces. We first establish an equivalence between our mixed generalized complementarity problem with a class of variational inequality problems for multivalued mappings and then find the existence of solutions for the complementarity problem. We then used the Tikhonov regularization procedure as well as arguments from the recession analysis and found some existing results on mixed generalized complementarity problems without coercivity assumptions in Banach spaces. Our results improve many existing results in the literature, including the results of Sahu et al. [23], Chadli et al. [24], Karamardian [7], and Park [25].

The rest of this paper is organized as follows. We introduce the mixed generalized complementarity problem and give some definitions and preliminaries in Section 2. Section 3 is devoted to studying the solvability for the mixed generalized complementarity problem in the context of Hausdorff topological vector spaces. In Section 4, our focus is on noncoercive complementarity problems in the case of reflexive Banach spaces.

2. Preliminaries

Let $K \subset X$ be a closed convex cone and $\langle \cdot, \cdot \rangle$ be a bilinear form on $X \times Y$ to $\mathbb{R}$. Here, X is a real topological vector space, and Y is a real vector space. The bilinear form is defined in a way such that the points of Y are separated by the family of linear functionals $\{\langle x, \cdot \rangle\}_{x \in X}$. Let us assume that the family of linear functionals $\{\langle x, \cdot \rangle\}_{x \in X}$ generates weak topology denoted by $\sigma(Y, X)$ on Y. For any subset $A \subset X$, let us denote the convex hull of A by $co(A)$, the closure of A by $cl(A)$, and the collection of all finite subsets of A by $\mathcal{F}(A)$. Let I be an index set, then for any net $\{x_\alpha\}_{\alpha \in I}$ in X, the set of all cluster points is denoted by $\mathcal{C}_{x_\alpha}$. For the multivalued mapping $F : K \to 2^Y \setminus \{\emptyset\}$ and for any $x \in K$, let the upper support function of $F(x)$ at $z \in K$ be $\sigma^\sharp_{F(x)}(z)$, where $\sigma^\sharp_{F(x)}(z) = \sup\limits_{y \in F(x)} \langle z, y \rangle$ (see [23]).

The mixed generalized complementarity problem that we considered in this paper is as follows:

Definition 1. *Let $F : K \to 2^Y$ be a multivalued mapping and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction, then the mixed generalized complementarity problem (MGCP) is to find an $x \in X$ and a $\varpi \in F(x)$ such that*

$$x \in K,\ \langle y, \varpi \rangle + \Phi(x, y) \geq 0, \text{ and } \langle x, \varpi \rangle = 0, \text{ for all } y \in K. \tag{1}$$

Let $MGCP(K, F)$ denote the solution set of the problem (1).

Remark 1. *If $\Phi \equiv 0$, then the problem (1) reduces to the extended generalized complementarity problem (EGCP):*
Find an $x \in X$ and a $\varpi \in F(x)$ such that

$$x \in K,\ \varpi \in K^*, \text{ and } \langle x, \varpi \rangle = 0, \tag{2}$$

where K^* is the polar of K. The problem (2) was considered by Sahu et al. [23] in 2021. If we consider both X and Y as $\mathbb{R}^n$, $\langle \cdot, \cdot \rangle$ as the usual inner product in $\mathbb{R}^n$ and $\Phi \equiv 0$, then (1) boils down to the complementarity problem studied by Saigal [6]. Let F be a single-valued mapping from K into Y, then the problem (1) becomes to the mixed complementarity problem (MCP):
Find an $x \in X$ such that

$$x \in K,\ \langle y, F(x) \rangle + \Phi(x, y) \geq 0,\ \langle x, F(x) \rangle = 0, \text{ for all } y \in K. \tag{3}$$

It may be observed that the complementarity problem considered by Karamardian [7] can be obtained from (3), if $\Phi \equiv 0$.

In order to find the existence of solutions for the problem (1), we need the following class of variational inequalities called mixed generalized variational inequalities.

Definition 2. *Let $F : K \to 2^Y$ be a multivalued mapping and $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction, then the mixed generalized variational inequality problem (MGVI) is to find an $x \in K$ and a $\varpi \in F(x)$ such that*

$$\langle y - x, \varpi \rangle + \Phi(x, y) \geq 0, \quad \forall y \in K. \tag{4}$$

Let $MGVI(K, F)$ denote the solution set of (4).

The notion of pseudomonotonicity in the sense of Brézis as defined in [11] is defined below.

Definition 3 ([11])**.** *Let K be a nonempty closed and convex subset of X. A single-valued mapping $T : K \to Y$ is said to be pseudomonotone in the sense of Brézis (in short B-pseudomonotone) if, for any net $\{x_\alpha\}_{\alpha \in I}$ satisfying $\{x_\alpha\}_{\alpha \in I}$ that stays in a compact set and converges to $\bar{x}$ and $\limsup_{\alpha \in I} \langle x_\alpha - \bar{x}, T x_\alpha \rangle \leq 0$, its limit $\bar{x}$ satisfies*

$$\langle \bar{x} - z, T\bar{x} \rangle \leq \liminf_{\alpha \in I} \langle x_\alpha - z, T x_\alpha \rangle, \text{ for all } z \in K.$$

The pseudomonotonicity in the sense of Brézis was then extended to bifunctions by J. Gwinner [26] in the year 1978 and further by himself in a couple of papers [27,28].

Definition 4 ([26])**.** *Let K be a nonempty closed and convex subset of X. A bifunction $\Psi : K \times K \to \mathbb{R}$ is pseudomonotone in the sense of Brézis if, for any generalized sequence $\{x_\alpha\}_{\alpha \in I}$ satisfying $\{x_\alpha\}_{\alpha \in I}$ that stays in a compact set and converges to $\bar{x}$ and $\liminf_{\alpha \in I} \Psi(x_\alpha, \bar{x}) \geq 0$, its limit $\bar{x}$ satisfies*

$$\Psi(\bar{x}, z) \geq \limsup_{\alpha \in I} \Psi(x_\alpha, z), \text{ for all } z \in K.$$

Remark 2. *The operator $T : K \to Y$ is B-pseudomonotone if and only if the bifunction $\Psi : K \times K \to \mathbb{R}$ defined by $\Psi(x, y) = \langle y - x, Tx \rangle$ is B-pseudomonotone. Further, if Ψ is upper semicontinuous with respect to the first argument, then it is B-pseudomonotone. The converse of which is also true was asserted by Sadeqi and Paydar [29] in 2015. But later, in 2019, Steck [30] proved that the assertion of Sadeqi and Paydar was wrong by providing a counter example.*

In 2021, the pseudomonotonicity in the sense of Brézis for the single-valued mappings was then extended to the case of multivalued mappings by Sahu, Chadli, Mohapatra, and Pani in [23].

Definition 5 ([23])**.** *Consider $K \subset X$ to be a nonempty convex and closed subset of X and $F : K \to 2^Y \setminus \{\emptyset\}$ be a multivalued mapping. F is defined to be pseudomonotone in the sense of Brézis if $\Phi(x, z) = \sigma^{\sharp}_{F(x)}(z - x)$ is pseudomonotone in the sense of Brézis. Here, Φ is a real-valued bifunction on $K \times K$.*

Definition 6 ([24])**.** *Suppose $K \subset X$ is nonempty and Φ is a bifunction from $K \times K$ to $\mathbb{R}$. Φ is monotone, if for each $x, z \in K$, we have $\Phi(x, z) + \Phi(z, x) \leq 0$.*

Definition 7 ([31])**.** *A single-valued function f defined from a topological space X to $[-\infty, \infty]$ is lower semicontinuous at a point $\bar{x}$ in X, if for any net $\{x_\alpha\}_{\alpha \in I} \subset X$ converging to $\bar{x}$ we have*

$$f(\bar{x}) \leq \liminf_{\alpha \in I} f(x_\alpha).$$

The single-valued function f is lower semicontinuous on X if f is lower semicontinuous for each x in X.

Definition 8 ([32])**.** *Suppose X, Y are topological spaces. The mapping $F : X \to 2^Y$ is upper semicontinuous at x in X if there is a neighborhood V of x for which $F(V)$ is in G for every open set $G \supset F(x)$. If F is upper semicontinuous for each x in X, then it is upper semicontinuous on X .*

We need the following results to prove the solvability for the complementarity problems.

Proposition 1 ([32] (Proposition 2, Page 41))**.** *Consider a multivalued map $F : X \to 2^Y$, where X and Y are Hausdorff topological spaces. Let $F(x) \subset Y$ be closed for every x in X and F be upper semicontinuous on X, then the graph of F*

$$\mathcal{G}(\mathcal{F}) = \{(x, y) \in X \times Y : y \in T(x)\}$$

is closed.

Proposition 2 ([32] (Proposition 3, Page 42))**.** *Let X and Y be two Hausdorff topological spaces. Let K be a compact subset of X and $F : K \to 2^Y$ be a multivalued mapping. Let $F(x) \subset Y$ be compact for each x in K and F be upper semicontinuous on K, then $F(K)$ is compact in Y.*

Proposition 3 ([33] (Proposition 15, Page II.14))**.** *Suppose X is a Hausdorff topological vector space. For a finite number of compact convex sets A_i, in X, where i ranges from 1 to n, co$\left(\bigcup_{i=1}^{n} A_i\right)$ is compact.*

Consider a Hausdorff topological vector space X. For a nonempty set $K \subset X$, the multivalued mapping $F : K \to 2^X$ is said to be a Knaster–Kuratowski–Mazurkiewicz

(KKM) mapping if for any finite set $\{x_1, x_2, \cdots, x_n\}$ in K, $co(\{x_1, x_2, \cdots, x_n\})$ is a subset of $\bigcup_{i=1}^{n} F(x_i)$.

Lemma 1 ([34])**.** *Suppose X is a Hausdorff topological vector space. Consider $K \subset X$ to be nonempty and the multivalued mapping $F : K \to 2^X$ to be KKM. Let $F(x)$ be closed in X for all x in K and compact in X for some x in K, then*

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Lemma 2 ([35])**.** *Let D be a convex and compact set and K be a convex set. Let $\Phi : D \times K \to \mathbb{R}$ be a real-valued bifunction such that Φ is convex and lower semicontinuous with respect to first argument and concave with respect to second argument. If $\min_{\xi \in D} \Phi(\xi, y) \leq 0$ for all $y \in K$, then there exists $\bar{\xi} \in D$ such that $\Phi(\bar{\xi}, y) \leq 0$ for all $y \in K$.*

3. Mixed Generalized Complementarity Problems in Topological Spaces

Throughout this section, unless otherwise stated, we assume that X is a Hausdorff topological vector space, Y is a real vector space, $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ is a bilinear form such that the points of Y are separated by $\{\langle x, \cdot \rangle\}_{x \in X}$, and the family of linear functionals and Y is equipped with the weak topology $\sigma(Y, X)$ generated by the family of linear functionals $\{\langle x, \cdot \rangle\}_{x \in X}$.

Theorem 1. *Suppose $K \subset X$ is a convex closed cone and $F : K \to 2^Y$. Let Φ be a real-valued bifunction on $K \times K$ for which $\Phi(x, x) = 0$ for all x in K. In addition, let Φ be monotone, and positively homogeneous in the first argument. Then, $x \in MGCP(K, F)$ if and only if $x \in MGVI(K, F)$.*

Proof. Let $x \in MGVI(K, F)$, then there is $\varpi \in F(x)$ such that

$$\langle y - x, \varpi \rangle + \Phi(x, y) \geq 0, \text{ for each } y \text{ in } K. \tag{5}$$

Using monotonicity of Φ, we obtain

$$\langle y - x, \varpi \rangle \geq \Phi(y, x), \text{ for every } y \in K. \tag{6}$$

Since Φ is positively homogeneous in the first argument, taking $y = \mathbf{0}$ in (6), we obtain $\langle x, \varpi \rangle \leq 0$. Again, considering $y = \lambda x$, $\lambda > 1$ in (6) and using the assumption that $\Phi(x, x) = 0$, we obtain $\langle x, \varpi \rangle \geq 0$. Thus, we obtain a vector $\varpi \in F(x)$ for which $\langle x, \varpi \rangle = 0$. Further, (5), implies

$$\langle y, \varpi \rangle + \Phi(x, y) \geq \langle x, \varpi \rangle = 0, \text{ for all } y \in K.$$

Thus, $x \in MGCP(K, F)$.

Conversely, let $x \in MGCP(K, F)$, then we have a vector $\varpi \in F(x)$, such that $\langle y, \varpi \rangle + \Phi(x, y) \geq 0$, $\forall\, y \in K$ and $\langle x, \varpi \rangle = 0$. Thus, we have

$$\langle y - x, \varpi \rangle + \Phi(x, y) = \langle y, \varpi \rangle - \langle x, \varpi \rangle + \Phi(x, y) = \langle y, \varpi \rangle + \Phi(x, y) \geq 0, \ \forall\, y \in K.$$

Thus, $x \in MGVI(K, F)$, and the proof is complete. □

Remark 3. *Theorem 1 generalizes Theorem 1 of Sahu et al. [23], Lemma 3.1 of Karamardian [7], and Theorem 2.3.1 in Chang's book [36].*

Lemma 3. *Suppose K is a closed convex cone in X and $x \mapsto \langle x, y\rangle$ is continuous for every y in Y. Let Φ be a real-valued bifunction on $K \times K$ for which $\Phi(x, x) = 0$, $\forall\, x \in K$, and $F : K \to 2^Y \setminus \{\emptyset\}$ be a multivalued map such that $F(x)$ is convex, for each $x \in K$. Assume the following:*

(i) *Φ to be monotone;*
(ii) *Φ to be convex in the second argument and lower semicontinuous;*
(iii) *For every $A \in \mathcal{F}(K)$, F is upper semicontinuous on $co(A)$ and $F(x) \subset Y$ is compact, $\forall\, x \in K$.*

Then, for every A in $\mathcal{F}(K)$, we have x in $co(A)$ and ϖ in $F(x)$ such that

$$\langle y - x, \varpi\rangle \geq \Phi(y, x), \quad \forall\, y \in co(A). \tag{7}$$

Proof. Proceeding step by step, we prove the following:

(a) For every A in $\mathcal{F}(K)$, there is $x \in co(A)$ such that for all $y \in co(A)$, $\exists \varpi \in F(x)$ satisfying

$$\langle y - x, \varpi\rangle \geq \Phi(y, x). \tag{8}$$

(b) For each $A \in \mathcal{F}(K)$, there exist $x \in co(A)$ and ϖ in $F(x)$ such that

$$\langle y - x, \varpi\rangle \geq \Phi(y, x), \quad \forall\, y \in co(A). \tag{9}$$

Proof of (a). For each $A \in \mathcal{F}(K)$, define $G : co(A) \to 2^{co(A)}$ by

$$G(y) = \{x \in co(A) : \exists \varpi \in F(x) \text{ such that } \langle y - x, \varpi\rangle \geq \Phi(y, x)\}.$$

It is clear that for each $y \in co(A)$, $G(y) \neq \emptyset$. Let $\{x_\alpha\}_{\alpha \in I}$ be a net in $G(y)$ converging to $\bar{x}$ in $co(A)$. For every α in I, there exists $\varpi_\alpha \in F(x_\alpha)$ such that

$$\langle y - x_\alpha, \varpi_\alpha\rangle \geq \Phi(y, x_\alpha). \tag{10}$$

By Proposition 3, $co(A)$ is compact. Using assumption (iii), we conclude from Proposition 2 that $F(co(A))$ is compact in Y. Therefore, the net $\{\varpi_\alpha\}_{\alpha \in I}$ has a convergent subnet $\{\varpi_\beta\}_{\beta \in J}$ in $F(co(A))$. Suppose the net $\{\varpi_\beta\}_{\beta \in J}$ converges to ϖ. Since the graph of F on $co(A)$ is closed by Proposition 1, $\varpi \in F(\bar{x})$.

Now, by the triangle inequality, for each $y \in co(A)$, we have

$$|\langle y - x_\beta, \varpi_\beta\rangle - \langle y - \bar{x}, \varpi\rangle| \leq |\langle y - \bar{x}, \varpi_\beta - \varpi\rangle| + |\langle \bar{x} - x_\beta, \varpi\rangle| + |\langle \bar{x} - x_\beta, \varpi_\beta - \varpi\rangle|. \tag{11}$$

The first two expressions of the right hand side of the relation (11) are easily seen to be zero. The last one is zero due the equicontinuity of the family $\{\langle ., \varpi_\beta - \varpi\rangle\}_{\beta \in J}$ on $co(A)$ by topological form of Banach–Steinhaus theorem. Therefore, from (11), we have

$$\langle y - x_\beta, \varpi_\beta\rangle \to \langle y - \bar{x}, \varpi\rangle. \tag{12}$$

Since $\Phi(x, \cdot)$ is lower semicontinuous, taking the lower limit in (10) and using relation (12), we obtain

$$\langle y - \bar{x}, \varpi\rangle \geq \Phi(y, \bar{x}).$$

This proves $\bar{x} \in G(y)$, and consequently, $G(y)$ is closed in $co(A)$, $\forall\, y \in co(A)$.

Now, we claim that the multivalued mapping G is KKM. Assuming the contradiction, let us suppose G is not KKM, then we have a finite set $\{y_1, \cdots, y_p\}$ in $co(A)$ and an $y \in co(\{y_1, \cdots, y_p\})$, $y = \sum_{i=1}^{p} \lambda_i y_i$, where $\lambda_i \geq 0$, $1 \leq i \leq p$ and $\sum_{i=1}^{p} \lambda_i = 1$, but $y \notin \bigcup_{i=1}^{p} G(y_i)$. By definition of G, for each $\varpi \in F(y)$, we have

$$\langle y_i - y, \varpi\rangle < \Phi(y_i, y), \text{ for each } i = 1, \cdots, p.$$

Using the monotonicity of Φ , we obtain

$$\langle y_i - y, \varpi \rangle + \Phi(y, y_i) < 0, \text{ for each } i = 1, \cdots, p. \tag{13}$$

Since $\Phi(x, \cdot)$ is convex, from (13), we have

$$0 = \langle z - z, y \rangle + \Phi(y, y) \leq \sum_{i=1}^{p} \lambda_i \langle z_i - z, y \rangle + \sum_{i=1}^{p} \lambda_i \Phi(y, y_i) < 0,$$

which is clearly a contradiction. Thus, G is a KKM mapping. Finally, since $co(A)$ is compact and $G(y)$ in $co(A)$ is closed for every $y \in co(A)$, $G(y)$ is compact. Hence, using Lemma 1, we conclude

$$\bigcap_{y \in co(A)} \Gamma(y) \neq \emptyset.$$

Therefore, for every $A \in \mathcal{F}(K)$, there is $x \in co(A)$, for which $\exists \varpi \in F(x)$, satisfying

$$\langle y - x, \varpi \rangle \geq \Phi(y, x), \ \forall y \in co(A).$$

Proof of (b). For every A in $\mathcal{F}(K)$, define a mapping $\Psi : F(x) \times co(Z) \to \mathbb{R}$ by

$$\Psi(\varpi, y) = \langle x - y, \varpi \rangle - \Phi(x, y).$$

By part (a),

$$\min_{\varpi \in F(x)} \Psi(\varpi, y) \leq 0, \text{ for all } y \in co(Z).$$

By the assumptions, $F(x)$ is compact in Y and convex. Since, $\langle \cdot, \cdot \rangle$ is a real-valued bilinear form on $X \times Y$ and $\Phi(x, \cdot)$ is convex, Ψ is convex in the first argument and concave in the second argument. Furthermore, since Y is equipped with $\sigma(Y, X)$ topology generated by the family of linear functionals $\{\langle x, \cdot \rangle\}_{x \in X}$, Ψ is lower semicontinuous in the first argument. Thus, all the conditions of Lemma 2 hold well, and hence, by Lemma 2, there exists $\varpi \in F(x)$ such that $\Psi(\varpi, y) \leq 0$ for all $y \in co(A)$. Therefore, we have, for every $A \in \mathcal{F}(K)$, that there is $x \in co(A)$ and ϖ in $F(x)$, which satisfies

$$\langle y - x, \varpi \rangle \geq \Phi(y, x), \quad \text{for all } y \in co(A).$$

This completes the proof. □

Lemma 4. *Let $K \subset X$ be a convex closed cone . Let us assume that $x \mapsto \langle x, y \rangle$ is continuous for each y in Y, and $F : K \to 2^Y \setminus \{\emptyset\}$ is a multivalued mapping. Consider a real-valued bifunction Φ on $K \times K$ for which $\Phi(x, x) = 0$ for every x in K. Let $\Phi(x, \cdot)$ be convex and, for every A in $\mathcal{F}(K)$, Φ is continuous on $co(A)$ with respect to the first argument. If there exists $x_0 \in K$ and $\varpi \in F(x_0)$ satisfying*

$$\langle x - x_0, \varpi \rangle \geq \Phi(x, x_0), \ \forall x \in K,$$

then

$$\langle x - x_0, \varpi \rangle + \Phi(x_0, x) \geq 0, \ \forall x \in K.$$

Proof. Consider a sequence $\{t_i\}_{i \in I}$ in $(0, 1)$ converging to zero, and for each $x \in K$, let $x_i = t_i x + (1 - t_i) x_0$. Then, for each $i \in I$, x_i is in K as well as in $co(\{x, x_0\})$ and further $x_i \to x_0$. Thus, by assumption

$$\langle x_i - x_0, \varpi \rangle \geq \Phi(x_i, x_0). \tag{14}$$

Since $\Phi(x, \cdot)$ is convex, we have

$$0 = \Phi(x_i, x_i) \leq t_i \Phi(x_i, x) + (1 - t_i) \Phi(x_i, x_0).$$

Thus, from (14), we have

$$t_i[\Phi(x_i, x_0) - \Phi(x_i, x)] \le \Phi(x_i, x_0) \le \langle x_i - x_0, \varpi \rangle = t_i \langle x - x_0, \varpi \rangle.$$

Since $t_i > 0$, for each $i \in I$, we have

$$\Phi(x_i, x_0) \le \langle x - x_0, \varpi \rangle + \Phi(x_i, x).$$

Further, since $\{x, x_0\} \in \mathcal{F}(K)$, by taking the limit and then using assumption (iii), we obtain

$$\langle x - x_0, \varpi \rangle + \Phi(x_0, x) \ge 0.$$

Thus, there is $x_0 \in K$ and $\varpi \in F(x_0)$ such that

$$\langle x - x_0, \varpi \rangle + \Phi(x_0, x) \ge 0, \text{ for all } x \in K.$$

□

Theorem 2. *Consider a closed convex cone $K \subset X$. Let $x \mapsto \langle x, y \rangle$ be continuous for all y in Y, and the multivalued mapping $F : K \to 2^Y \setminus \{\emptyset\}$ be defined in a way such that for every $x \in K$, $F(x)$ is convex. Assume Φ to be a real-valued bifunction on $K \times K$, where $\Phi(x, x) = 0$ for every x in K and $\Phi(\cdot, y)$ is positively homogeneous, for each $y \in K$, such that the following hold:*

(i) *Φ is monotone;*
(ii) *$\Phi(x, \cdot)$ is lower semicontinuous;*
(iii) *For each $A \in \mathcal{F}(K)$, Φ is continuous with respect to the first argument on $co(A)$;*
(iv) *Φ is convex with respect to the second argument;*
(v) *F is upper semicontinuous, for all A in $\mathcal{F}(K)$, on $co(A)$ and for every $x \in K$, $F(x)$ is compact in Y;*
(vi) *F is B-pseudomonotone;*
(vii) *Coercivity: For a nonempty compact set D in K and a nonempty convex and compact set C in K, there exists y in C such that*

$$\langle y - x, \varpi \rangle + \Phi(x, y) < 0,$$

for every $x \in K \setminus D$ and for every $\varpi \in F(x)$.

Then, there exists at least one solution to the MGCP (1), and the solution set for $MGCP(K, F)$ is compact in X.

Proof. We prove the following:

(a) The solvability of the problem (1) for a compact set K.
(b) The solvability of the problem (1) for an arbitrary K.
(c) The solution set $MGCP(K, F)$ in X is compact.

Proof of (a). Define a multivalued set function $H : \mathcal{F}(K) \to 2^K$ by

$$H(A) = \{x \in K : \text{ there exists } \varpi \in F(x) \text{ satisfying } \langle y - x, \varpi \rangle \ge \Phi(y, x), \ \forall y \in co(A)\}.$$

By Lemma 3, $H(A) \neq \emptyset$, for each $A \in \mathcal{F}(K)$. Next, we claim that $\bigcap_{A \in \mathcal{F}(K)} cl(H(A)) \neq \emptyset$. Suppose $\{A_1, A_2, \cdots, A_n\}$ is a finite subcollection of $\mathcal{F}(K)$ and let $A = \bigcup_{i=1}^n A_i$. Then, $A \in \mathcal{F}(K)$, and hence, $cl(H(A)) \neq \emptyset$. But by definition of H, $H(A) \subset H(A_i)$, for i ranging from 1 to n. Thus, $H(A) \subset \bigcap_{i=1}^n H(A_i)$ and thus

$$\emptyset \neq cl(H(A)) \subset cl\left(\bigcap_{i=1}^n H(A_i)\right) \subset \bigcap_{i=1}^n cl(H(A_i)).$$

Thus, the collection $\{cl(H(A))\}_{A\in\mathcal{F}(K)}$ satisfies the finite intersection property. Since K is compact,

$$\bigcap_{A\in\mathcal{F}(K)} cl(H(A)) \neq \emptyset.$$

Consider $x_0 \in \bigcap_{A\in\mathcal{F}(K)} cl(H(A))$. Further suppose $x \in K$ is any element. If $B = \{x_0, x\} \in \mathcal{F}(K)$, then $B \in \mathcal{F}(K)$, and hence, $x_0 \in cl(H(B))$. Thus, we have a sequence $\{x_\alpha\}$, where $\alpha \in I$ in $H(B)$ converging to x_0. Hence, for every α in I, we have $\varpi_\alpha \in F(x_\alpha)$ satisfying

$$\langle y - x_\alpha, \varpi_\alpha\rangle \geq \Phi(y, x_\alpha),\ \forall y \in co(B).$$

Thus,

$$\sigma^\sharp_{F(x_\alpha)}(y - x_\alpha) \geq \Phi(y, x_\alpha),\ \forall y \in co(B). \tag{15}$$

By considering $y = x_0$ in relation (15), we obtain

$$\sigma^\sharp_{F(x_\alpha)}(x_0 - x_\alpha) \geq \Phi(x_0, x_\alpha),\ \forall \alpha \in I. \tag{16}$$

Since $\Phi(x, \cdot)$ is lower semicontinuous, by taking lower limit in (16), we obtain

$$\liminf_{\alpha\in I} \sigma^\sharp_{F(x_\alpha)}(x_0 - x_\alpha) \geq \liminf_{\alpha\in I} \Phi(x_0, x_\alpha) \geq \Phi(x_0, x_0) = 0.$$

Using B-pseudomonotonicity of F, we have

$$\limsup_{\alpha\in I} \sigma^\sharp_{F(x_\alpha)}(z - x_\alpha) \leq \sigma^\sharp_{F(x_0)}(z - x_0),\ \text{for all } z \in K.$$

In particular, for $z = x$,

$$\limsup_{\alpha\in I} \sigma^\sharp_{F(x_\alpha)}(x - x_\alpha) \leq \sigma^\sharp_{F(x_0)}(x - x_0). \tag{17}$$

Now, putting $y = x$ and taking upper limit in relation (15), we obtain

$$\limsup_{\alpha\in I} \sigma^\sharp_{F(x_\alpha)}(x - x_\alpha) \geq \limsup_{\alpha\in I} \Phi(x, x_\alpha).$$

Since $\Phi(x, \cdot)$ is lower semicontinuous, by using relation (17), we obtain

$$\sigma^\sharp_{F(x_0)}(x - x_0) \geq \Phi(x, x_0).$$

Thus, there are $\varpi \in F(x_0)$ and $x_0 \in K$ satisfying

$$\langle x - x_0, \varpi\rangle \geq \Phi(x, x_0),\ \forall x \in K. \tag{18}$$

Invoking Lemma 4, we conclude that there exists $x_0 \in K$ and $\varpi \in F(x_0)$ such that

$$\langle x - x_0, \varpi\rangle + \Phi(x_0, x) \geq 0,\ \forall x \in K. \tag{19}$$

Therefore, by Theorem 1, it may be observed that, a solution to MGCP (1) is x_0, when K is compact.

Proof of (b). Consider a finite subset $\{x_1, x_2, \cdots, x_n\}$ of K and let $L = co(C \cup \{x_1, x_2, \cdots, x_n\})$. Then, by Proposition 3, L is a compact in X. Thus, by relation (19), there exists $x_0 \in L$ and $\varpi \in F(x_0)$ such that

$$\langle x - x_0, \varpi\rangle + \Phi(x_0, x) \geq 0 \text{ for all } x \in L.$$

By coercivity condition, $x_0 \in D$. Now, we define the multivalued set function $\mathcal{H} : \mathcal{F}(K) \to 2^D$ by

$$\mathcal{H}(A) = \{x \in D : \exists \varpi \in F(x) \text{ satisfying } \langle y - x, \varpi \rangle + \Phi(x, y) \geq 0, \forall y \in co(C \cup A)\}.$$

Since $co(C \cup A)$ is compact, $\forall A \in \mathcal{F}(K)$, $\mathcal{H}(A) \neq \emptyset$. Repeating the same argument as in (a), and using the compactness of D, we see that

$$\bigcap_{A \in \mathcal{F}(K)} cl(\mathcal{H}(A)) \neq \emptyset.$$

Suppose $x \in K$ is any element and $x_0 \in \bigcap_{A \in \mathcal{F}(K)} cl(\mathcal{H}(A))$. If $B = \{x_0, x\}$, then $B \in \mathcal{F}(K)$, and hence, $x_0 \in cl(\mathcal{H}(B))$. Thus, we have a sequence $\{x_\alpha\}_{\alpha \in I} \subset H(B)$ converging to x_0. Hence, for every α in I, there is $\varpi_\alpha \in F(x_\alpha)$ satisfying

$$\langle y - x_\alpha, \varpi_\alpha \rangle + \Phi(x_\alpha, y) \geq 0, \forall y \in co(C \cup B).$$

Since Φ is monotone, we can write it as

$$\sigma^\sharp_{F(x_\alpha)}(y - x_\alpha) \geq \Phi(y, x_\alpha), \forall y \in co(C \cup B). \tag{20}$$

In particular,

$$\sigma^\sharp_{F(x_\alpha)}(x_0 - x_\alpha) \geq \Phi(x_0, x_\alpha), \forall \alpha \in I.$$

Since $\Phi(x, \cdot)$ is lower semicontinuous, we have

$$\liminf_{\alpha \in I} \sigma^\sharp_{F(x_\alpha)}(x_0 - x_\alpha) \geq 0.$$

Using B-pseudomonotonicity of F, we have

$$\limsup_{\alpha \in I} \sigma^\sharp_{F(x_\alpha)}(z - x_\alpha) \leq \sigma^\sharp_{F(x_0)}(z - x_0), \text{ for all } z \in K.$$

In particular, for $z = x$,

$$\limsup_{\alpha \in I} \sigma^\sharp_{F(x_\alpha)}(x - x_\alpha) \leq \sigma^\sharp_{F(x_0)}(x - x_0). \tag{21}$$

Now, putting $y = x$ and taking upper limit in relation (20), we obtain

$$\limsup_{\alpha \in I} \sigma^\sharp_{F(x_\alpha)}(x - x_\alpha) \geq \limsup_{\alpha \in I} \Phi(x, x_\alpha). \tag{22}$$

Again using the fact that $\Phi(x, \cdot)$ lower semicontinuous, from (21) and (22), we obtain

$$\sigma^\sharp_{F(x_0)}(x - x_0) \geq \limsup_{\alpha \in I} \Phi(x, x_\alpha) \geq \liminf_{\alpha \in I} \Phi(x, x_\alpha) \geq \Phi(x, x_0).$$

Once again invoking Lemma 4, we obtain that, there exists $x_0 \in K$ and $\varpi \in F(x_0)$ satisfying

$$\langle x - x_0, \varpi \rangle + \Phi(x_0, x) \geq 0, \text{ for all } x \in K. \tag{23}$$

Therefore, by Theorem 1, it is clear that MGCP (1) is solvable, and x_0 is the solution.

Proof of (c). Let $x \in cl(MGCP(K, F))$. Then, there is a sequence $\{x_\alpha\}_{\alpha \in I}$ in $MGCP(K, F)$ which converges to x. Thus, by Theorem 1 for every α in I, we have $\varpi_\alpha \in F(x_\alpha)$ satisfying

$$\langle y - x_\alpha, \varpi_\alpha \rangle + \Phi(x_\alpha, y) \geq 0, \forall y \in K. \tag{24}$$

Now for arbitrary z in K, the segment $[v, x] \subset K$ and hence the relation (24) becomes

$$\langle y - x_\alpha, \varpi_\alpha \rangle + \Phi(x_\alpha, y) \geq 0, \forall y \in [z, x]. \tag{25}$$

As F is B-pseudomonotone, by using condition (iii) and repeating the same argument as in (b), we conclude the following,

$$\exists \varpi \in F(x) : \langle z - x, \varpi \rangle + \Phi(x, z) \geq 0.$$

Since $z \in K$ is arbitrary, deploying Theorem 1 once again, we conclude that $x \in MGCP(K, F)$. Hence, $MGCP(K, F) \subset D$ is closed, and since D is compact, $MGCP(K, F)$ is compact in X. $\square$

Remark 4. *If $\Phi \equiv 0$, then Theorem 2 boils down to Theorem 2 of Sahu et al. [23]. Therefore, Theorem 2 is a proper generalization of Theorem 2 in Sahu et al. [23]. Theorem 2 also generalizes and improves the Theorem 3.1 of Chadli et al. [24] and Theorem 3.1 of Karamardian [7].*

When the multivalued mapping F is a single-valued mapping, we obtain the following consequences of Theorem 2.

Corollary 1. *Suppose $K \subset X$ is a closed convex cone and for every y in Y, and the mapping $x \mapsto \langle x, y \rangle$ is continuous. Let us consider F to be a single-valued map from K to Y and Φ to be a real-valued bifunction on $K \times K$, where $\Phi(x, x) = 0$ for all x in K and is positively homogeneous with respect to first argument. Furthermore, we assume the conditions mentioned below:*

(i) Φ *is monotone;*
(ii) $\Phi(x, \cdot)$ *is lower semicontinuous;*
(iii) *For every $A \in \mathcal{F}(K)$, $\Phi(\cdot, y)$ is continuous on $co(A)$;*
(iv) $\Phi(x, \cdot)$ *is convex;*
(v) *For every A in $\mathcal{F}(K)$, F is continuous on $co(A)$;*
(vi) *F is B-pseudomonotone;*
(vii) *Coercivity: For a nonempty compact set $D \subset K$ and a nonempty convex and compact set C in K, there exists $y \in C$ satisfying*

$$\langle y - x, F(x) \rangle + \Phi(x, y) < 0, \ \forall x \in K \setminus D.$$

Then, the mixed complementarity problem (3) has a solution.

4. Mixed Generalized Complementarity Problems in Reflexive Banach Spaces

In this section, we use the Tikhonov regularization procedure as well as arguments from recession analysis and prove the solvability of the mixed generalized complementarity problem in the framework of reflexive Banach spaces.

Otherwise stated, in this section, we consider $\mathbb{B}^*$ as the dual space of a reflexive Banach space $\mathbb{B}$. We assume that x^* in $\mathbb{B}^*$ takes the value $\langle x, x^* \rangle$ at $x \in \mathbb{B}$. The family of linear functionals $\{\langle x, \cdot \rangle\}_{x \in \mathbb{B}}$ generates weak topology, denoted as $\sigma(\mathbb{B}^*, \mathbb{B})$ on the space $\mathbb{B}^*$. The strong, weak, and weak* convergence are denoted by the symbols $\rightarrow$, $\rightharpoonup$, and $\overset{*}{\rightharpoonup}$, respectively. Let $K^* \subset B^*$ be the polar of K, where $K \subset \mathbb{B}$ is a nonempty closed convex cone having a vertex at the origin $\mathbf{0}$.

In this section, our principal aim is to establish the solvability of the following mixed generalized complementarity problem in Banach spaces.

Definition 9. *Consider a multivalued mapping $F : K \to 2^{\mathbb{B}^*}$ and a real-valued bifunction Φ on $K \times K$, then the mixed generalized complementarity problem in Banach spaces is to obtain an x in $\mathbb{B}$ and ϖ in $F(x)$ satisfying*

$$x \in K, \ \langle y, \varpi \rangle + \Phi(x, y) \geq 0, \ \ \forall y \in K, \textit{ and } \langle x, \varpi \rangle = 0. \tag{26}$$

Let $\mathcal{S}$ be the solution set of (26).

In the case of Banach spaces, the pseudomonotonicity in the sense of Brézis given in Definition 3 becomes the following (see Steck [30]).

Definition 10 ([30])**.** *Let T from $\mathbb{B}$ to $\mathbb{B}^*$ be a single-valued mapping. The pseudomonotone in the sense of Brézis, in case for any net $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}$ if $x_n \rightharpoonup x$ along with $\limsup_{n\to\infty}\langle x_n - x, T(x_n)\rangle \leq 0$, then*

$$\langle x - z, T(x)\rangle \leq \liminf_{n\to\infty}\langle x_n - z, T(x_n)\rangle, \text{ for all } z \in K.$$

Definition 11. *A multivalued mapping $F : \mathbb{B} \to 2^{\mathbb{B}^*}$ is said to be*

(i) copositive on K, if there is a $z^* \in F(\mathbf{0})$ such that for all $x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^*\rangle \geq 0$;

(ii) strictly copositive on K, if there is a $z^* \in F(\mathbf{0})$ such that for all $0 \neq x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^*\rangle > 0$;

(iii) strongly copositive on K, if there is a scalar $\alpha > 0$ and a vector $z^* \in F(\mathbf{0})$ such that for all $x \in K$ and $y^* \in F(x)$, $\langle x, y^* - z^*\rangle \geq \alpha\|x\|^2$.

Given a single-valued function $f : \mathbb{B} \to \mathbb{R}\cup\{+\infty\}$, the recession function of f defined by Baocchi et al. [19] in 1988 is as follows:

$$f_\infty(x) := \liminf_{\substack{t\to+\infty\\ v\to x}}[f(tv)/t] = \inf\left\{\liminf_{n\to+\infty}[f(t_n v_n)/t_n] : t_n \to +\infty,\ v_n \to x\right\}.$$

The recession function of Baocchi et al. [19] was then extended by Goeleven [21] in 1996 to a general single-valued operator F from $\mathbb{B}$ to $\mathbb{B}^*$ with respect to $z_0 \in \mathbb{B}$ as

$$\underline{r}_{z_0,F}(x) := \liminf_{\substack{t\to+\infty\\ z\to x}}\left[\frac{1}{t}\langle F(tz), tz - z_0\rangle\right].$$

The recession function of Goeleven [21] was then further extended by Sahu, Chadli, Mohapatra, and Pani [23] in 2021 to the case of multivalued mappings as given below.

Definition 12 ([23])**.** *The recession function for a multivalued mapping $F : \mathbb{B} \to 2^{\mathbb{B}^*}$ with respect to $z_0 \in \mathbb{B}$ is*

$$\begin{aligned} f^F_{\infty,z_0}(x) &:= \limsup_{\substack{t\to+\infty\\ z\to x}}\left[\frac{1}{t}\sigma^\sharp_{F(tz)}(z_0 - tz)\right] \\ &= \sup\left\{\limsup_{n\to+\infty}\left[\frac{1}{t_n}\sigma^\sharp_{F(t_n z_n)}(z_0 - t_n z_n)\right] : t_n \to +\infty,\ z_n \to x\right\}. \end{aligned}$$

The *recession cone* defined by Adly et al. [20] in 1996 is given by

$$\begin{aligned} K_\infty =&\{x \in \mathbb{B} : \exists \text{ nets } \{t_n\}_{n\in\mathbb{N}} \text{ in } \mathbb{R}^+ \text{ and } \{x_n\}_{n\in\mathbb{N}} \text{ in } K \\ &\text{satisfying } \lim_{n\to\infty} t_n = +\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{t_n}x_n = w\}. \end{aligned}$$

For a given family of sets $\{S_n\}_{n\in\mathbb{N}}$ of K, the recession set or set of asymptotic directions of $\{S_n\}_{n\in\mathbb{N}}$, defined by Adly et al. [20] is as follows:

$$\begin{aligned} \mathcal{R}(\{S_n\}) =&\{w \in K_\infty : \text{there exists } x_n \text{ in } S_n \text{ for which } \|x_n\| \to +\infty, \\ &w_n = \frac{1}{\|x_n\|}x_n \rightharpoonup w\}. \end{aligned}$$

Definition 13 ([20]). *Given a family of sets $\{S_n\}_{n\in\mathbb{N}}$ of K, the recession set $\mathcal{R}(\{S_n\})$ of $\{S_n\}_{n\in\mathbb{N}}$ is defined to be asymptotically compact if for every $w \in \mathcal{R}(\{S_n\})$, the net $\{w_n\}_{n\in\mathbb{N}}$ that is given in $\mathcal{R}(\{S_n\})$ converges strongly to w.*

Let us define a multivalued map $F_\varepsilon : K \to 2^{\mathbb{B}^*}$ as $F_\varepsilon(x) = F(x) + \varepsilon J(x)$ for $x \in K$, and $\varepsilon > 0$, where $J : \mathbb{B} \to 2^{\mathbb{B}^*}$ is the duality map given by

$$J(x) = \left\{x^* \in \mathbb{B}^* : \langle x, x^* \rangle = \|x^*\|^2 \text{ and } \|x^*\| = \|x\|\right\}.$$

Lemma 5. *Suppose $\mathbb{B}^*$ is the topological dual of a reflexive Banach space $\mathbb{B}$. Consider $K \subset \mathbb{B}$ to be a nonempty convex and closed cone and a multivalued map $F : K \to 2^{\mathbb{B}^*} \setminus \{\emptyset\}$. If F is B-pseudomonotone, then the multivalued map F_ε is B-pseudomonotone.*

Proof. Consider a net $\{x_n\}_{n\in\mathbb{N}} \subset K$ for which $x_n \rightharpoonup x$ in K. Suppose that

$$\begin{aligned} &\liminf_{n\in\mathbb{N}} \sigma^\sharp_{F_\varepsilon(x_n)}(x - x_n) \geq 0 \\ \implies &\liminf_{n\in\mathbb{N}} \left[\sigma^\sharp_{F(x_n)}(x - x_n) + \varepsilon\langle x - x_n, J(x_n)\rangle\right] \geq 0. \end{aligned} \tag{27}$$

Then, we must have

$$\liminf_{n\in\mathbb{N}} \sigma^\sharp_{F(x_n)}(x - x_n) \geq 0 \quad \text{and} \quad \liminf_{n\in\mathbb{N}} \langle x - x_n, J(x_n)\rangle \geq 0. \tag{28}$$

Indeed, suppose there exists subsequences $\{x_k\}_{k\in\mathbb{N}}$ and $\{x_l\}_{l\in\mathbb{N}}$ of the sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\left.\begin{aligned} \text{either} \quad \text{(a)} \quad &\lim_{k\to+\infty} \sigma^\sharp_{F(x_k)}(x - x_k) \leq r < 0 \\ \text{or} \quad \text{(b)} \quad &\lim_{l\to+\infty} \langle x - x_l, J(x_l)\rangle \leq s < 0 \end{aligned}\right\}, \tag{29}$$

for some $r, s \in \mathbb{R}$ hold.

If (a) of (29) holds, then from (27), we must have

$$\liminf_{k\in\mathbb{N}} \langle x - x_k, J(x_k)\rangle \geq -\frac{r}{\epsilon} > 0. \tag{30}$$

Since J is monotone and continuous, by Steck [30], J is B-pseudomonotone, and therefore, from (30), we have

$$0 < -\frac{r}{\epsilon} \leq \liminf_{k\in\mathbb{N}} \langle x - x_k, J(x_k)\rangle \leq \limsup_{k\in\mathbb{N}} \langle x - x_k, J(x_k)\rangle \leq \langle x - x, J(x)\rangle = 0.$$

Thus, we reach a contradiction.

Again, if (b) of (29) holds, then from (27), we must have

$$\liminf_{l\in\mathbb{N}} \sigma^\sharp_{F(x_l)}(x - x_l) \geq -\epsilon s > 0.$$

Since F is B-pseudomonotone, we have

$$0 < -\epsilon s \leq \liminf_{l\in\mathbb{N}} \sigma^\sharp_{F(x_l)}(x - x_l) \leq \limsup_{l\in\mathbb{N}} \sigma^\sharp_{F(x_l)}(x - x_l) \leq \sigma^\sharp_{F(x)}(x - x) = 0,$$

again a contradiction and thus we obtain the relation (28). Therefore, by B-pseudomonotonicity of F and J, we have

$$\limsup_{n\in\mathbb{N}}\Big[\sigma^{\sharp}_{F(x_n)}(z-x_n)+\varepsilon\langle z-x_n,J(x_n)\rangle\Big]$$
$$\leq \sigma^{\sharp}_{F(x)}(z-x)+\varepsilon\langle z-x,J(x)\rangle,\ \forall\, z\in K.$$

Therefore, F_ε is B-pseudomonotone. □

With the aim of solving (26), we need the following regularized mixed generalized complementarity problem in Banach spaces, which is also of general interest.

Definition 14. *Suppose $F : K \to 2^{\mathbb{B}^*}$ is a multivalued map, Φ is a real-valued bifunction on $K\times K$, and $\varepsilon > 0$ is a given number. The regularized mixed generalized complementarity problem in Banach spaces is to obtain x_ε in $\mathbb{B}$ and $\varpi_\varepsilon \in F_\varepsilon(x_\varepsilon)$ satisfying*

$$x_\varepsilon \in K,\ \langle y,\varpi_\varepsilon\rangle+\Phi(x_\varepsilon,y)\geq 0,\ \ \forall y\in K,\ \text{and}\ \langle x_\varepsilon,\varpi_\varepsilon\rangle=0. \tag{31}$$

Let $\mathcal{S}_\varepsilon$ be the set of solutions of (31).

Theorem 3. *Suppose $\mathbb{B}^*$ is the dual of a reflexive Banach space $\mathbb{B}$ and $\emptyset \neq K \subset \mathbb{B}$ is a convex and closed cone. Assume that $F : K \to 2^{\mathbb{B}^*}\setminus\{\emptyset\}$ is a multivalued map. Let $F(x)$ in $\mathbb{B}^*$ be bounded, closed, and convex for every x in K. Let Φ be a real-valued bifunction on $K\times K$ for which $\Phi(x,x) = 0$ for each x in K and Φ be positively homogeneous with respect to the first argument. In addition, we assume the following:*

(i) *Φ is monotone;*
(ii) *Φ is lower semicontinuous with respect to the second argument;*
(iii) *For each $A\in\mathcal{F}(K)$, $\Phi(\cdot,y)$ is continuous on $co(A)$;*
(iv) *$\Phi(x,\cdot)$ is convex;*
(v) *F is upper semicontinuous for every A in $\mathcal{F}(K)$, on $co(A)$;*
(vi) *F is B-pseudomonotone;*
(vii) *Coercivity: For a nonempty compact subset D in K, we have a weakly compact and convex $C_\varepsilon \subset K$ for an arbitrarily small $\varepsilon>0$, for which there exists $y \in C_\varepsilon$ such that*

$$\langle y-x,\varpi\rangle+\varepsilon\langle y-x,J(x)\rangle+\Phi(x,y)<0,$$

for every x in $K\setminus D$ and $\varpi\in F(x)$.

Then, the following hold:

(a) *The regularized mixed generalized complementarity problem* (31) *has a solution, for every $\varepsilon>0$.*
(b) *$\bigcup_{\varepsilon>0}\mathcal{S}_\varepsilon$ is bounded.*
(c) *If $x\in\mathcal{C}_{x_{\varepsilon_n}}$ in the weak topology $\sigma(\mathbb{B},\mathbb{B}^*)$, where $x_{\varepsilon_n}\in\mathcal{S}_{\varepsilon_n}$ such that the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ converges to zero, then $x\in\mathcal{S}$.*

Proof. (a) Since J is continuous by assumption (v), F_ε is upper semicontinuous on $co(A)$ for each $A\in\mathcal{F}(A)$. By Lemma 5 and assumption (vi), F_ε is B-pseudomonotone. Thus, the assumptions in Theorem 2 hold for $\mathbb{B}$ and $\mathbb{B}^*$ equipped with the weak topologies $\sigma(\mathbb{B},\mathbb{B}^*)$ and $\sigma(\mathbb{B}^*,\mathbb{B})$, respectively. Therefore, by Theorem 2, the regularized mixed generalized complementarity problem (31) has a solution for every $\varepsilon>0$.

(b) For any $\varepsilon>0$, if $x_\varepsilon\in\mathcal{S}_\varepsilon$, then by Theorem 2, $x_\varepsilon\in D$. Since D is compact in $\mathbb{B}$, $\bigcup_{\varepsilon>0}\mathcal{S}_\varepsilon$ is bounded.

(c) Let the subsequence $\{x_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of the sequence $\{x_{\varepsilon_n}\}_{n\in\mathbb{N}}$ converges weakly to x. Since $x_{\varepsilon_k} \in \mathbb{S}_{\varepsilon_k}$, by Theorem 1, we deduce from relation (23) that

$$\exists \xi_k \in F_{\varepsilon_k}(x_{\varepsilon_k}) \text{ such that } \langle z - x_{\varepsilon_k}, \xi_k\rangle + \Phi(x_{\varepsilon_k}, z) \geq 0,\ \forall z \in K. \tag{32}$$

By monotonicity of Φ, we have

$$\sigma^{\sharp}_{F(x_{\varepsilon_k})}(z - x_{\varepsilon_k}) + \varepsilon_k\langle z - x_{\varepsilon_k}, J(x_{\varepsilon_k})\rangle \geq \Phi(z, x_{\varepsilon_k}),\ \forall z \in K.$$

Since J is monotone, we have

$$\sigma^{\sharp}_{F(x_{\varepsilon_k})}(x - x_{\varepsilon_k}) + \varepsilon_k\langle x - x_{\varepsilon_k}, J(x)\rangle \geq \Phi(x, x_{\varepsilon_k}). \tag{33}$$

Since $\Phi(x,\cdot)$ is lower semicontinuous, we have

$$\liminf_{k\in\mathbb{N}} \sigma^{\sharp}_{F(x_{\varepsilon_k})}(x - x_{\varepsilon_k}) \geq 0.$$

By B-pseudomonotonicity of F and lower semicontinuity of $\Phi(x,\cdot)$, we have from (33) that

$$\sigma^{\sharp}_{F(x)}(z - x) \geq \limsup_{k\in\mathbb{N}} \Phi(x, x_{\varepsilon_k}) \geq \liminf_{k\in\mathbb{N}} \Phi(x, x_{\varepsilon_k}) \geq \Phi(z, x),\ \text{for each } z \text{ in } K.$$

Hence, there exists $\varpi \in F(\bar{x})$ such that

$$\langle z - x, \varpi\rangle \geq \Phi(z, x),\ \text{for every } z \text{ in } K.$$

Using Lemma 4, it follows that

$$\langle z - x, \varpi\rangle + \Phi(x, z) \geq 0,\ \text{for all } z \in K.$$

Therefore, by Theorem 1, $x \in \mathcal{S}$. □

Remark 5. *Theorem 3 generalizes Theorem 6 of Sahu et al. [23].*

So far, we have obtained the solutions of the mixed generalized complementarity problems under the coercivity assumptions. On the other hand, the variational formulations of most of the engineering problems do not have coercivity due to boundary conditions. In order to tackle such problems, various authors have studied noncoercive problems with different approaches; see, for instance, [16,20,21]. Now, we are interested in solving the mixed generalized complementarity problems without the coercivity assumption.

Theorem 4. *Suppose $\mathbb{B}^*$ is the dual of a reflexive Banach space $\mathbb{B}$ and $\emptyset \neq K \subset \mathbb{B}$ is a convex and closed cone. Consider $F : K \to 2^{\mathbb{B}^*} \setminus \{\emptyset\}$ as a strongly copositive multivalued map for which $F(x)$ is closed, bounded, and convex for every x in K. Let Φ be a real-valued bifunction on $K \times K$ such that $\Phi(x, x) = 0$ for all x in K and positively homogeneous with respect to the first argument. Furthermore, assume the following:*

(i) Φ is monotone;
(ii) Φ is lower semicontinuous in the second argument;
(iii) For every $A \in \mathcal{F}(K)$, $\Phi(\cdot, y)$ is continuous on $co(A)$;
(iv) $\Phi(x,\cdot)$ is convex;
(v) For every A in $\mathcal{F}(K)$, F is upper semicontinuous on $co(A)$;
(vi) F is B-pseudomonotone.

Then, the mixed generalized complementarity problem (26) has a solution.

Proof. Using strong copositivity condition on F, we can find $\alpha > 0$ and z^* in $F(\mathbf{0})$ such that for every x in K and y^* in $F(x)$, we have

$$\langle x, y^* - z^* \rangle \geq \alpha \|x\|^2. \tag{34}$$

If $z^* = \mathbf{0}$, then $x = \mathbf{0}$ solves (26). Indeed, by using the assumption that Φ is positively homogeneous with respect to the first argument, we have

$$\langle y, z^* \rangle + \Phi(\mathbf{0}, y) \geq 0, \text{ for each } y \text{ in } K.$$

Since $\langle \mathbf{0}, z^* \rangle = 0$, we see that $x = \mathbf{0}$ is the solution of (26). Now, suppose that $z^* \neq \mathbf{0}$, and consider a subset D of K defined by $D = \left\{ x \in K : \|x\| \leq \frac{\|z^*\|}{\alpha} \right\}$. Then, D is nonempty and weakly compact in $\mathbb{B}$. For every $x \in K \setminus D$, we have from relation (34) that

$$\begin{aligned} \langle x, y^* \rangle \geq & \ \alpha \|x\|^2 + \langle x, z^* \rangle, \forall y^* \in F(x) \\ \geq & \ \alpha \|x\|^2 - \|x\| \|z^*\|, \forall y^* \in F(x) \\ > & \ 0. \end{aligned}$$

Taking $\{\mathbf{0}\} = C \subset K$, which is nonempty, weakly compact, and convex, we see that for each x in $K \setminus D$ and $y^* \in F(x)$, there exists $\mathbf{0}$ in C satisfying

$$\langle \mathbf{0} - x, y^* \rangle + \Phi(x, \mathbf{0}) < 0 - \Phi(\mathbf{0}, x) = 0.$$

Thus, all the conditions of Theorem 2 are satisfied for $\mathbb{B}$ equipped with the weak topology $\sigma(\mathbb{B}, \mathbb{B}^*)$ and $\mathbb{B}^*$ equipped with the weak topology $\sigma(\mathbb{B}^*, \mathbb{B})$, and hence, by Theorem 2, x is a solution of the mixed generalized complementarity problem (26). □

Theorem 5. *Suppose $\mathbb{B}^*$ is the dual of a reflexive Banach space $\mathbb{B}$ and $\emptyset \neq K \subset \mathbb{B}$ is a convex and closed cone. Consider $F : K \to 2^{\mathbb{B}^*} \setminus \{\emptyset\}$ as a copositive multivalued map for which $F(x)$ is a closed, bounded, and convex subset of $\mathbb{B}^*$ for every x in K. Let Φ be a real-valued bifunction on $K \times K$ such that $\Phi(x, x) = 0$ for all x in K and is positively homogeneous with respect to the first argument. Furthermore, suppose that the following hold:*

(i) *Φ is monotone;*
(ii) *$\Phi(x, \cdot)$ is lower semicontinuous;*
(iii) *For every $A \in \mathcal{F}(K)$, $\Phi(\cdot, y)$ is continuous on $co(A)$;*
(iv) *$\Phi(x, \cdot)$ is convex;*
(v) *For every A in $\mathcal{F}(K)$, F is upper semicontinuous on $co(A)$;*
(vi) *F is B-pseudomonotone.*

Then, the regularized mixed generalized complementarity problem (31) *has at least one solution.*

Proof. In order to apply Theorem 4, we need only to show that F_ε is strongly copositive and B-pseudomonotone, and that for each $A \in \mathcal{F}(K)$, F_ε is upper semicontinuous on $co(A)$. Since, for every A in $\mathcal{F}(K)$, F is upper semicontinuous on $co(A)$ and J is continuous on $\mathbb{B}$, F_ε is upper semicontinuous on $co(A)$ for every $A \in \mathcal{F}(K)$. By the B-pseudomonotonicity of F and Lemma 5, we see that F_ε is B-pseudomonotone. Finally, using copositivity of F on K, we have z^* in $F(\mathbf{0})$ satisfying

$$\langle x, y^* - z^* \rangle \geq 0, \ \forall \ x \in K, y^* \in F(x). \tag{35}$$

Since $\langle x, J(x) \rangle = \|x\|^2$ and J is positively homogeneous, we have

$$\begin{aligned} \langle x, y^* - z^* \rangle + \epsilon \langle x, J(x) - J(\mathbf{0}) \rangle & \geq \epsilon \langle x, J(x) - J(\mathbf{0}) \rangle \\ & = \epsilon \|x\|^2. \end{aligned}$$

Thus, F_ε is strongly copositive on K. Therefore, by Theorem 4, the regularized mixed generalized complementarity problem (31) has a solution. □

Now, we are in a position to find the existence of solutions for the problem (26) using the solutions of the problem (31).

Theorem 6. *Suppose $\mathbb{B}^*$ is the dual of a reflexive Banach space $\mathbb{B}$ and $\emptyset \neq K \subset \mathbb{B}$ is a convex and closed cone. Consider $F : K \to 2^{\mathbb{B}^*} \setminus \{\emptyset\}$ as a copositive multivalued map for which $F(x)$ is a closed, bounded, and convex subset of $\mathbb{B}^*$ for every x in K. Let Φ be a real-valued bifunction on $K \times K$ such that $\Phi(x,x) = 0$ for all x in K and is positively homogeneous with respect to the first argument. Furthermore, assume the following:*

(i) *Φ is monotone;*
(ii) *Φ is lower semicontinuous in the second argument;*
(iii) *For every A in $\mathcal{F}(K)$, $\Phi(\cdot, y)$ is continuous on $co(A)$;*
(iv) *$\Phi(x, \cdot)$ is convex;*
(v) *For every A in $\mathcal{F}(K)$,, F is upper semicontinuous on $co(A)$;*
(vi) *F is B-pseudomonotone;*
(vii) *$\mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\}) = \emptyset$, where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ converging to zero.*

Then, the mixed generalized complementarity problem (26) is solvable.

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ in K such that $x_n \in \mathcal{S}_{\varepsilon_n}$ for each $n \in \mathbb{N}$. Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. More precisely, suppose there is a subsequence $\{x_k\}_{k\in\mathbb{N}}$ of the sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\|x_k\| \to +\infty$. Consider the sequence $\{w_k\}_{k\in\mathbb{N}}$ such that $w_k = \frac{1}{\|x_k\|} x_k$. Then, $\{w_k\}_{k\in\mathbb{N}}$ is bounded and therefore, there is a subsequence $\{w_l\}_{l\in\mathbb{N}}$ of $\{w_k\}_{k\in\mathbb{N}}$ such that $w_l \rightharpoonup w$. As a result, we have $w \in \mathcal{R}(\{\mathbb{S}_{\varepsilon_n}\})$, which contradicts vii. Let $x \in \mathcal{C}_{x_{\varepsilon_n}}$ in the weak topology $\sigma(\mathbb{B}, \mathbb{B}^*)$. Then, there exists a subsequence $\{x_k\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $x_k \rightharpoonup x$. Since $x_n \in \mathcal{S}_{\varepsilon_n}$, from Theorem 1 and relation (23), we have

$$\exists y_k \in F_{\varepsilon_k}(x_k) \text{ such that } \langle z - x_k, y_k \rangle + \Phi(x_k, z) \geq 0, \ \forall z \in K.$$

By monotonicity of Φ, we deduce that

$$\sigma^{\sharp}_{F(x_k)}(z - x_k) + \varepsilon_k \langle z - x_k, J(x_k) \rangle \geq \Phi(z, x_k), \ \forall z \in K.$$

Sine J is monotone, we have

$$\sigma^{\sharp}_{F(x_k)}(z - x_k) + \varepsilon_k \langle z - x_k, J(z) \rangle \geq \Phi(z, x_k). \tag{36}$$

Since Φ is lower semicontinuous in second argument, by replacing z with x and then taking the lower limit, we obtain

$$\liminf_{k\in\mathbb{N}} \sigma^{\sharp}_{F(x_k)}(x - x_k) \geq 0.$$

Using B-pseudomonotonicity of F, from (36), we have

$$\sigma^{\sharp}_{F(\bar{x})}(z - x) \geq \Phi(z, x), \ \text{for each } z \in K.$$

Thus, there is ϖ in $F(x)$ such that

$$\langle z - x, \varpi \rangle \geq \Phi(z, x), \ \text{for each } z \in K.$$

By Lemma 4, we obtain

$$\langle z - x, \varpi \rangle + \Phi(x, z) \geq 0, \ \text{for each } z \in K.$$

Invoking Theorem 1 once again, it is found that $x \in \mathcal{S}$. □

The following theorem provides a very good method for establishing the solvability of (26) when it is not possible to show that recession set of the family $\{\mathcal{S}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ of solution sets for the problem (31) is empty.

Theorem 7. *Suppose $\mathbb{B}^*$ is the dual of a reflexive Banach space $\mathbb{B}$ and $\emptyset \neq K \subset \mathbb{B}$ is a convex and closed cone. Consider $F : K \to 2^{\mathbb{B}^*} \setminus \{\emptyset\}$ as a copositive multivalued map for which $F(x)$ is a closed, bounded, and convex subset of $\mathbb{B}^*$ for every x in K. Let Φ be a real-valued bifunction on $K \times K$ such that $\Phi(x, x) = 0$ for all x in K and is positively homogeneous with respect to the first argument. Furthermore, suppose the following:*

(i) Φ is monotone;
(ii) Φ is lower semicontinuous in second argument;
(iii) For every A in $\mathcal{F}(K)$, $\Phi(\cdot, y)$ is continuous on $co(A)$;
(iv) $\Phi(x, \cdot)$ is convex;
(v) For every A in $\mathcal{F}(K)$, F is upper semicontinuous on $co(A)$;
(vi) F is B-pseudomonotone;
(vii) $\mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\})$ is asymptotically compact, where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ converging to zero;
(viii) There is $\emptyset \neq D \subset \mathbb{B} \setminus \{\mathbf{0}\}$, for which $R(\{\mathcal{S}_{\varepsilon_n}\}) \subset D$, and we have a $z_0 \in K$ satisfying

$$f^F_{\infty,z_0}(\varpi) < 0, \quad \forall \varpi \in D.$$

Then, the mixed generalized complementarity problem (26) is solvable.

Proof. Clearly, all but assumption vii of Theorem 6 hold. In order to apply Theorem 6, we have only to establish that $\mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\}) = \emptyset$. Suppose, on the contrary, $\mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\}) \neq \emptyset$, and let $w \in \mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\})$ be any element. Then, we have $\{x_n\}_{n\in\mathbb{N}}$ in K, $x_n \in \mathcal{S}_{\varepsilon_n}$, for which the sequence $\{t_n\}_{n\in\mathbb{N}}$, $t_n = \|x_n\|$ converges to $+\infty$, and the sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathbb{B}$, $w_n = \frac{x_n}{t_n}$ converges weakly to w. Since $x_n \in \mathcal{S}_{\varepsilon_n}$, from Theorem 1 and relation (23), we have

$$\exists y_n \text{ in } F_{\varepsilon_n}(x_n) \text{ satisfying } \langle z - x_n, y_n\rangle + \Phi(x_n, z) \geq 0, \text{ for all } z \in K.$$

Thus,

$$\sigma^\sharp_{F(x_n)}(z - x_n) + \varepsilon_n\langle z - x_n, J(x_n)\rangle + \Phi(x_n, z) \geq 0, \text{ for all } z \in K.$$

Since J is monotone, we have

$$\sigma^\sharp_{F(x_n)}(z - x_n) + \varepsilon_n\langle z - x_n, J(z)\rangle + \Phi(x_n, z) \geq 0, \ \forall z \in K. \tag{37}$$

In particular, for $z = w$, we can write relation (37) as

$$\sigma^\sharp_{F(t_n w_n)}(w - t_n w_n) + \varepsilon_n\langle w - t_n w_n, J(w)\rangle + \Phi(t_n w_n, w) \geq 0.$$

Since, Φ is positively homogeneous with respect to the first argument, we have

$$\frac{1}{t_n}\sigma^\sharp_{F(t_n w_n)}(w - t_n w_n) + \varepsilon_n\left\langle \frac{1}{t_n}w - w_n, J(w)\right\rangle + \Phi(w_n, w) \geq 0.$$

Since Φ is monotone, we have

$$\frac{1}{t_n}\sigma^\sharp_{F(t_n w_n)}(w - t_n w_n) + \varepsilon_n\left\langle \frac{1}{t_n}w - w_n, J(w)\right\rangle \geq \Phi(w, w_n). \tag{38}$$

By taking account of the fact that $\varepsilon_n \to 0$, $t_n \to +\infty$ and the weak topology $\sigma(\mathbb{B}, \mathbb{B}^*)$ of $\mathbb{B}$, we have

$$\lim_{n\to\infty} \varepsilon_n\left\langle \frac{1}{t_n}w - w_n, J(w)\right\rangle = 0. \tag{39}$$

Since $\mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\})$ is asymptotically compact, we obtain $w_n \to w$. As $\Phi(x, \cdot)$ is lower semicontinuous, taking limit supremum in (38) and then using relation (39), we would have

$$f^F_{\infty,w}(w) \geq 0.$$

Since $w \in \mathcal{R}(\{\mathcal{S}_{\varepsilon_n}\})$ is arbitrary, there is no subset D of $\mathbb{B} \setminus \{\mathbf{0}\}$ satisfying the conditions of (viii). Thus, we reach a contradiction, and hence, condition vii of Theorem 6 holds well. The result now follows from Theorem 6. $\square$

Remark 6. *Theorems 4–7 generalize, respectively, Theorem 4, Theorem 5, Theorem 7, and Theorem 8 of Sahu et al. [23].*

5. Discussion

In this paper, we introduced the mixed generalized complementarity problem (MGCP) in the Hausdorff topological vector space X. Under the coercivity assumption, we proved that the problem MGCP has solution on an arbitrary topological vector space using the notion of pseudomonotonicity in the sense of Brézis, which is obviously a weaker assumption than that of monotonicity in the Karamardian sense, as was proved by Kien et al. [13] in 2009. We then proceed to use the Tikhonov regularization procedure in order to establish the solution of the problem MGCP in reflexive Banach spaces. Finally, we proved the existence of the solution to the MGCP in reflexive Banach spaces using the notion recession sets, as well as asymptotical compactness without coercivity assumptions. In the future, we plan to investigate some iterative schemes in order to find the numerical solutions for the above model of complementarity problems. Moreover, we plan to carry out some more existence results on mixed generalized complementarity problems related to our results.

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Abbreviations

The following abbreviations are used in this manuscript:

MGCP	mixed generalized complementarity problem
EGCP	extended generalized complementarity problem
MGVI	mixed generalized variational inequality problem
KKM	Knaster–Kuratowski–Mazurkiewicz

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