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Iteration of Operators with Contractive Mutual Relations of Kannan Type

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Abstract: Inspired by the ideas of R. Kannan, we define the new concepts of mutual Kannan contractivity and mutual contractivity between two self-maps on a metric space that generalize the concepts of the Kannan map and contraction. We give some examples and deduce the properties of the operators satisfying this type of condition; in particular, we study the case where the space is normed, and the maps are linear. Then we generalize some theorems proposed by this author on the existence of a fixed point of one operator or a common fixed point for two operators. Our results first prove the existence of a common fixed point of a set of self-maps of any cardinal number (countable or uncountable) satisfying the conditions of Kannan type in metric spaces. The same is proved for a set of maps satisfying the mutual relations of classical contractivity. We prove in both cases the convergence of iterative schemes involving operators with mutual relations of contractivity, proposing sufficient conditions for the iteration of the operators on any element of the space to converge to the common fixed point when a different operator is taken in each step. The results obtained are applied to operators acting on real functions, coming from the fractal convolution with the null function.

Keywords: iteration; fixed point; discrete dynamical systems; attractors; Kannan's mappings

MSC: 26A18; 47H10; 47J26; 54H25; 37C25

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1. Introduction

Iterative schemes are ubiquitous in applied mathematics and in numerical methods and algorithmics in general. Usually they are described by a recurrence formula such as $x_{k+1} = T(x_k)$, where x_k belongs to some space E, and T is a self-map on E.

These types of iterations are found in classical numerical procedures, such as Newton–Raphson and fixed point methods to solve nonlinear equations, as well as in fractal theory to define fractal sets and attractors. In the framework of the fractal interpolation, there is an operator T defined in a space of functions E (usually E is a subspace of continuous or integrable real functions), giving rise to a fractal interpolant f^* of a set of data (see for instance [1–4]). The fractal function is obtained as attracting fixed point of an iterative process of the aforementioned type:

$$f_{k+1} = T f_k.$$

The operator T is a Banach contraction, and it depends on a vectorial parameter, called a scale vector, that measures the contractivity ratio of the self-map T. If the scale vector is constant (it does not depend on the step k), the iterations of any map $f_0 \in E$ converge to the fractal function f^*. This scheme may be generalized in different ways. For instance, if the scale vector (that is to say, the operator T) is different at each step, do we obtain also an attracting fixed map? What happens if T (or T_k) is a more general contraction (not necessarily Banach)?

We substitute here the mapping T with a set of operators on a metric space of any cardinal number (countable or uncountable) and establish Kannan and contractivity conditions for the existence of a common fixed point and the convergence of the scheme. For similar results on classical Banach contractions, see the reference [5].

In the 1960s, R. Kannan ([6,7]) introduced a new type of mappings, which are a form of contraction, now known as Kannan's mappings. They are defined as follows:

Definition 1. *Let E be a metric space and $T : E \to E$. If there exists a number β, $0 < \beta < \frac{1}{2}$, such that, for all $x, y \in E$,*

$$d(T(x), T(y)) \leq \beta[d(x, T(x)) + d(y, T(y))]$$

then T is called a Kannan mapping.

Remark 1. *The concepts of Kannan map and contractivity are independent (see [6,7]). Let $f : [0,1] \to [0,1]$ be defined by $f(x) = \frac{x}{3}$. This function f is a contractive mapping but it is not a Kannan mapping. On the other hand, the function $g : [0,1] \to [0,1]$ defined by*

$$g(x) = \begin{cases} \frac{x}{4}, & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{x}{5}, & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

is a Kannan mapping with $\beta = \frac{4}{9}$ but it is not a contractive mapping due to its discontinuity. Both examples are taken from the reference [7].

In general, the maps of Kannan type need not be continuous and, in this sense, are more general than contractions. For applications of these kinds of mappings, the reader may consult the references [8,9].

In Section 2, we define the concept of mutual Kannan contractivity for any collection of mappings and show the existence of a common fixed point for the given collection under some specific conditions. We also show that the common fixed point of this collection is asymptotically stable. In Section 3, we obtain some results on a common fixed point for a set of mutually contractive mappings. Therein, we also prove some results regarding a common fixed point for a set of mappings satisfying weaker assumptions than the classical contractivity. In both cases, we study the convergence of discrete iterative processes defined by the operators. In Section 4, we illustrate some applications of our results on fractal convolution.

2. Sets of Mutually Kannan Operators

We consider in this section the existence of common fixed points for a set of operators with mutual relations of Kannan type. We will prove the existence of a common fixed point of a set of self-maps on a metric space E. The results generalize the next theorem of R. Kannan ([6], Theorem 1):

Theorem 1. *If T_1 and T_2 are two operators, each mapping a complete metric space (E, d) into itself, and if*

$$d(T_1(x), T_2(y)) \leq \beta[d(x, T_1(x)) + d(y, T_2(y))] \;\; \forall x, y \in E, \tag{1}$$

where $0 < \beta < \frac{1}{2}$, then T_1 and T_2 have a unique fixed point.

Based on this result we propose the following definition describing a condition involving two self-maps on a metric space.

Definition 2. *The self-maps $T_1, T_2 : E \to E$, where E is a metric space with respect to d are mutually Kannan with constant β if there exists $\beta \in \mathbb{R}$ such that $0 < \beta < \frac{1}{2}$ satisfying the condition (1).*

Remark 2. *T is Kannan (Definition 1) if and only if is mutually Kannan with respect to itself. In this sense a mutual Kannan contraction generalizes the Kannan contractivity.*

Example 1. *Let $f, g : [0,1] \to [0,1]$ be defined by*

$$f(x) = \frac{x}{4} \text{ and } g(x) = \frac{x}{6}.$$

It is simple to check that both functions f and g are mutually Kannan mappings since

$$d(f(x), g(y)) = \left|\frac{x}{4} - \frac{y}{6}\right| \leq \left|\frac{x}{4}\right| + \left|\frac{y}{6}\right| \leq \frac{1}{3}\left\{\left|x - \frac{x}{4}\right| + \left|y - \frac{y}{6}\right|\right\} =$$
$$\beta\{d(x, f(x)) + d(y, g(y))\},$$

where $\beta = \frac{1}{3}$.

Two maps with mutual relation of Kannan type need not be contractive, as explained in Remark 1. However, if the maps are linear on a normed space, we can establish some relations between contractivity and mutual Kannan contractivity. The next result is classical in functional analysis.

Lemma 1. *If L is a linear operator from a Banach space into itself such that $\|L\| < 1$, then $(I - L)^{-1}$ exists, is bounded and*

$$(I - L)^{-1} = I + L + L^2 + \ldots,$$

where I is the identity.

Proposition 1. *If E is a normed space, $L, S : E \to E$ are linear and mutually Kannan with constant β, then L and S are bounded, contractive and such that*

$$\max\{\|L\|, \|S\|\} \leq \frac{\beta}{1 - \beta}.$$

If E is Banach, $I - L$ and $I - S$ are invertible.

Proof. Applying the property of Kannan (1), for $x \in E$ and $y = 0$,

$$\|L(x)\| \leq \beta \|L(x) - x\| \leq \beta \|x\| + \beta \|L(x)\|.$$

Thus

$$\|L\| \leq \frac{\beta}{1 - \beta} < 1$$

consequently L is contractive. The invertibility of $I - L$ comes from the previous lemma. The same is true for S. □

In this case, the mutual Kannan property implies contractivity.

The following result can be found in the reference [10].

Lemma 2. *If L is a linear operator from a Banach space into itself, and there exist $c_1, c_2 \in R$, such that $0 \leq c_1, c_2 < 1$ and*

$$\|L(x) - x\| \leq c_1 \|x\| + c_2 \|L(x)\|. \tag{2}$$

Then L is a topological isomorphism and

$$\frac{1-c_1}{1+c_2}\|x\| \leq \|L(x)\| \leq \frac{1+c_1}{1-c_2}\|x\|, \tag{3}$$

$$\frac{1-c_2}{1+c_1}\|x\| \leq \|L^{-1}(x)\| \leq \frac{1+c_2}{1-c_1}\|x\|. \tag{4}$$

Proposition 2. *If E is a Banach space, $L : E \to E$ is linear and mutually Kannan with $S : E \to E$ linear and invertible, such that $\|I - S^{-1}\| < 1$, then, L is a topological isomorphism, and*

$$\frac{1-2\beta}{1+\beta}\|x\| \leq \|L(x)\| \leq \frac{1+2\beta}{1-\beta}\|x\|,$$

$$\frac{1-\beta}{1+2\beta}\|x\| \leq \|L^{-1}(x)\| \leq \frac{1+\beta}{1-2\beta}\|x\|.$$

where β is the constant of the mutual contractivity.

Proof. It is a consequence of the previous lemma. Let us take $y = S^{-1}x$ in the inequality (1); then, for any $x \in E$,

$$\|L(x) - x\| \leq \beta(\|L(x) - x\| + \|x - S^{-1}(x)\|) \leq \beta\|x\| + \beta\|L(x)\| + \beta\|x\|.$$

We obtain the inequality (2) for $c_1 = 2\beta < 1$ and $c_2 = \beta < 1$. Consequently, L is invertible, and

$$\frac{1-2\beta}{1+\beta}\|x\| \leq \|L(x)\| \leq \frac{1+2\beta}{1-\beta}\|x\|,$$
$$\frac{1-\beta}{1+2\beta}\|x\| \leq \|L^{-1}(x)\| \leq \frac{1+\beta}{1-2\beta}\|x\|.$$

Thus, the proof of the proposition is completed. □

The following simple note can be seen in [11]. However, we include its details for the reader's convenience.

Note 1. *Let $T : E \to E$ be a contraction of a metric space (X, d) with constant $c < \frac{1}{3}$. Then T is Kannan contractive relative to d.*

Since T is a contraction, we have

$$d(T(x), T(y)) \leq cd(x, y) \leq cd(x, T(x)) + cd(T(x), T(y)) + cd(T(y), y),\ \forall\, x, y \in E.$$

This yields

$$d(T(x), T(y)) \leq \frac{c}{1-c}[d(x, T(x)) + d(y, T(y))],\ \forall\, x, y \in E.$$

Since $0 < \beta := \frac{c}{1-c} < \frac{1}{2}$, T is a Kannan mapping.

Proposition 3. *Let d_1 and d_2 be equivalent metrics on a complete metric space E, i.e., there exist positive constants c_1, c_2 such that*

$$c_1 d_1(x, y) \leq d_2(x, y) \leq c_2 d_1(x, y),\quad x, y \in E.$$

If T is a contraction on E with respect to the metric d_1 then there exists an $m \in \mathbb{N}$, such that T^m is a contraction with respect to the metric d_2. Moreover, for some $n_0 \in \mathbb{N}$, T^n is a Kannan map with respect to the metric d_2 for all $n \geq n_0$.

Proof. Since T is contraction with respect to metric d_1, there exists $0 \leq c < 1$ such that

$$d_1(T(x), T(y)) \leq c d_1(x, y), \quad x, y \in E.$$

Then $d_2(T(x), T(y)) \leq c_2 d_1(T(x), T(y)) \leq c_2 c d_1(x, y) \leq \left(\frac{c_2}{c_1} c\right) d_2(x, y)$. Choose $m \in \mathbb{N}$ such that $\frac{c_2}{c_1} c^m < 1$. Then

$$d_2(T^m(x), T^m(y)) \leq c_2 d_1(T^m(x), T^m(y)) \leq c_2 c^m d_1(x, y) \leq \left(\frac{c_2}{c_1} c^m\right) d_2(x, y).$$

This completes the proof of the first part. According to the previous note, taking n large enough to have $c_2 c^n / c_1 < 1/3$, T^n is Kannan as well. □

Definition 3. *Given a set of self-maps $\mathcal{F} = \{T_i : E \to E; i \in \mathcal{I}\}$, where E is a metric space, $\bar{x} \in E$ is a fixed point of $\mathcal{F}$, if $T_i(\bar{x}) = \bar{x}$, $\forall i \in \mathcal{I}$.*

We will consider now the iterative scheme

$$x_k = T_{i_k}(x_{k-1}), \tag{5}$$

$\forall k \geq 1$, $x_0 \in E$, $T_{i_k} \in \mathcal{F}$.

Definition 4. *$x^* \in E$ is a global attractor for the scheme (5) if $\lim_{n \to \infty} \tau_n(x) = x^*$, $\forall x \in E$, where $\tau_n := T_{i_n} \circ T_{i_{n-1}} \circ \ldots T_{i_2} \circ T_{i_1}$.*

Theorem 2. *Let E be a complete metric space, and $\mathcal{F} = \{T_i : E \to E, i \in \mathcal{I}\}$, such that $\forall i, j \in \mathcal{I}$, T_i, T_j are mutually Kannan with constant β_{ij}; $0 < \beta_{ij} < \frac{1}{2}$ and $\beta = \sup_{i,j} \beta_{ij} < \frac{1}{2}$. Then:*

1. *$\mathcal{F}$ has a unique fixed point $\bar{x} \in E$.*
2. *$\bar{x}$ is the only fixed point of each T_i $\forall i \in \mathcal{I}$.*
3. *The point $\bar{x} \in E$ is a global attractor for any scheme of type (5).*

Proof. According to Kannan's theorem $\forall i, j \in \mathcal{I}$, T_i, T_j have a unique fixed point $\bar{x}_{ij} \in E$. Let us see now that $\bar{x}_{ij} = \bar{x}$ $\forall i, j$. Let us consider the maps T_i, T_k $(k \neq j)$ applied to $\bar{x}_{ij}$:

$$d(T_i(\bar{x}_{ij}), T_k(\bar{x}_{ij})) \leq \beta_{i_k} [d(\bar{x}_{ij}, T_i(\bar{x}_{ij})) + d(\bar{x}_{ij}, T_k(\bar{x}_{ij}))].$$

Since $\bar{x}_{ij}$ is a fixed point of T_i and $\beta_{ik} < \beta < 1/2$:

$$d(\bar{x}_{ij}, T_k(\bar{x}_{ij})) \leq d(\bar{x}_{ij}, T_k(\bar{x}_{ij}))/2.$$

Consequently $\bar{x}_{ij}$ is a fixed point of T_k and $\bar{x}_{ij} = \bar{x}_{ik} = \bar{x}$. According to Kannan's Theorem 1 $\bar{x}$ is unique.

For the second item, let us assume that T_i has another fixed point $\bar{y} \in E$, then for $k \in \mathcal{I}$,

$$d(\bar{x}, \bar{y}) = d(T_k(\bar{x}), T_i(\bar{y})) \leq \beta [d(\bar{x}, T_k(\bar{x})) + d(\bar{y}, T_i(\bar{y}))] = 0$$

and $\bar{x} = \bar{y}$. Accordingly, the fixed point of T_i is unique.

For the third item, let us define, for any $x \in E$, $x_0 = x$, $x_1 = T_{i_1}(x_0)$, $x_2 = T_{i_2}(x_1), \ldots x_n = T_{i_n}(x_{n-1})$, then

$$d(x_1, x_2) = d(T_{i_1}(x_0), T_{i_2}(x_1)) \leq \beta [(d(x_0, T_{i_1}(x_0)) + d(x_1, T_{i_2}(x_1))]$$

thus

$$d(x_1, x_2) \leq \beta [d(x_0, x_1) + d(x_1, x_2)]$$

hence,

$$d(x_1, x_2) \leq \frac{\beta}{1-\beta} d(x_0, x_1).$$

In the same way,

$$d(x_2, x_3) \leq \frac{\beta}{1-\beta} d(x_1, x_2) \leq \left(\frac{\beta}{1-\beta}\right)^2 d(x_0, x_1).$$

In general,

$$d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1), \tag{6}$$

where $\gamma = \beta/(1-\beta) < 1$. Let us check that (x_n) is a Cauchy sequence. For $q \geq 1$:

$$d(x_n, x_{n+q}) \leq \sum_{j=0}^{q-1} d(x_{n+j}, x_{n+j+1}) \leq \left(\sum_{k=n}^{n+q-1} \gamma^k\right) d(x_0, x_1).$$

Since $\sum_{k=0}^{\infty} \gamma^k$ is convergent, (x_n) is a Cauchy sequence, and there exists $x^* \in E$, such that $x^* = \lim_{n\to\infty} x_n$.

Let us prove that x^* is the fixed point of $\mathcal{F}$. For $i \in \mathcal{I}$ and $n \in \mathbb{N}$:

$$d(x^*, T_i(x^*)) \leq d(x^*, x_n) + d(x_n, T_i(x^*)).$$

Applying the condition (1) for T_{i_n} and T_i,

$$d(x^*, T_i(x^*)) \leq d(x^*, x_n) + \beta\,[d(x_{n-1}, T_{i_n}(x_{n-1})) + d(x^*, T_i(x^*))]$$

and

$$(1-\beta)\, d(x^*, T_i(x^*)) \leq d(x^*, x_n) + \beta\, d(x_{n-1}, x_n).$$

The terms of the right hand tend to zero; consequently $x^* = T_i(x^*)$, and $x^* = \bar{x}$. Then the limit does not depend on x. □

Let us remind two definitions on dynamical systems.

Definition 5. *$B \subseteq E$ is a forward invariant set of $\mathcal{F}$ if $T_i(B) \subseteq B$ $\forall i \in \mathcal{I}$.*

Definition 6. *$\hat{x} \in E$ is Lyapunov stable for the system (5), if $\forall\,\varepsilon > 0$ $\exists\,\delta > 0$, such that if $d(x, \hat{x}) < \delta$, then $d(\tau_n(x), \tau_n(\hat{x})) < \varepsilon$, where $\tau_n := T_{i_n} \circ T_{i_{n-1}} \circ \cdots \circ T_{i_1}$ for all n. An element $\hat{x} \in E$ is asymptotically stable, if it is stable and attractor.*

Proposition 4. *In the conditions of Theorem 2, any ball $B_r = B(\bar{x}, r)$, where $\bar{x}$ is the fixed point of $\mathcal{F}$ and $r > 0$, is a forward invariant set of $\mathcal{F}$.*

Proof. Let $y \in B_r$, and $i, k \in \mathcal{I}$,

$$d(T_i(y), \bar{x}) \leq \beta\,[d(y, T_i(y)) + d(\bar{x}, T_k(\bar{x}))]$$

and

$$d(T_i(y), \bar{x}) \leq \beta\,[d(y, \bar{x}) + d(\bar{x}, T_i(y))].$$

Consequently,

$$d(T_i(y), \bar{x}) \leq \frac{\beta}{1-\beta} d(y, \bar{x}) < d(y, \bar{x}) < r$$

and $T_i(y) \in B_r$. □

Proposition 5. *In the conditions of Theorem 2, $\bar{x}$ is asymptotically stable.*

Proof. $\bar{x}$ is a fixed point of τ_n for all n. For any $\varepsilon > 0$, let us take $\delta = \varepsilon$. If $d(x, \bar{x}) < \delta$, then, according to the previous proposition $T_{i_1}(x) \in B(\bar{x}, \delta)$, $\tau_2(x) = T_{i_2} \circ T_{i_1}(x) \in B(\bar{x}, \delta)$, etc. In general $d(\tau_n(x), \bar{x}) < \delta = \varepsilon$.

Since $\bar{x}$ is a global attractor and stable, then, $\bar{x}$ is asymptotically stable. □

Let us now deduce the rate of convergence of the orbits $\tau_n(x)$ to $\bar{x}$:

$$d(\tau_n(x), \bar{x}) = d(T_{i_n}(x_{n-1}), T_{i_n}(\bar{x})) \leq \beta\,[d(x_{n-1}, T_{i_n}(x_{n-1})) + d(\bar{x}, T_{i_n}(\bar{x}))],$$

then, according to (6),

$$d(\tau_n(x), \bar{x}) \leq \ \beta d(x_{n-1}, x_n) \leq \beta\, \gamma^{n-1} d(x_0, x_1)$$

and the convergence is of exponential type. The last inequality illustrates also the convergence of different orbits:

$$\forall x, y \in E;\ d(\tau_n(x), \tau_n(y)) \leq \beta\, \gamma^{n-1} \left[d(x, T_{i_1}(x)) + d(y, T_{i_1}(y))\right].$$

3. Mutually Contractive Operators

We consider in this section the existence of common fixed points for systems of operators with mutual relations of "classical" contractivity. Let $\mathcal{F}$ be a set of self-maps on a metric space:

$$\mathcal{F} = \{T_i : E \to E; i \in \mathcal{I}\},$$

where $\mathcal{I}$ may be finite or infinite.

Our aim is the generalization of the following theorem by R. Kannan ([6], Theorem 3) for a type of operators different from the previous sections.

Theorem 3. *If T_1 and T_2 are two operators mapping a complete metric space (E, d) into itself, and if*

1. $d(T_1(x), T_2(y)) \leq \alpha\, d(x, y), 0 < \alpha < 1,\ x, y \in E, x \neq y.$
2. *T_2 is a contraction mapping, i.e., there exists $\beta, 0 < \beta < 1$, such that*

$$d(T_2(x), T_2(y)) \leq \beta\, d(x, y),\ \forall\, x, y \in E.$$

3. *There exists $x \in E$, such that the sequence $x_1 = T_1(x)$, $x_2 = T_2(x_1)$, $x_3 = T_1(x_2)$, $x_4 = T_2(x_3) \ldots$, is such that $x_r \neq x_s$ if $r \neq s$.*

   Then, T_1, T_2 have a unique common fixed point.

On the basis of the first condition, we introduce the concept of mutual (classical) contractivity.

Definition 7. *The operators $T, S : E \to E$, where E is a metric space, are mutually contractive, if there exists $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and for $x \neq y$, $x, y \in E$,*

$$d(T(x), S(y)) \leq \alpha\, d(x, y). \tag{7}$$

Remark 3. *It is clear that T is contractive if and only if T is mutually contractive with itself. As a consequence, the concept of mutual contractivity generalizes that of the classical contraction.*

Remark 4. *Kannan mutual contractivity and mutual contractivity are idependent concepts. For instance the map g of Remark 1 is mutually Kannan with itself, but it is not contractive.*

Remark 5. *Two contractions need not be mutually contractive. For instance, let us consider $f(x) = (1/2)x$ and $g(x) = (1/6)x$ in $\mathbb{R}$ with the usual metric, taking $x = 1, y = 1.01$ in the inequality (7)*

$$|f(1) - g(1.01)| = 1.99/6 \leq \alpha\, 0.01,$$

and this contradicts the condition $\alpha < 1$.

The following results provide a way of constructing mutually contractive linear operators.

Proposition 6. *Let E be a normed space with norm $\|\cdot\|$, and let us define the distance*

$$d(x,y) := \|x-y\| + \|x\| + \|y\|. \tag{8}$$

Let $L, L' : E \to E$ be two linear and bounded operators such that $k := \max\{\|L - L'\|, \|L\|, \|L'\|\} < 1/2$. Then L, L' are mutually contractive with constant $\alpha = 2k < 1$ with respect to the distance d.

Proof. Let us prove the triangular inequality for the map d.

$$d(x,z) = \|x-z\| + \|x\| + \|z\| \leq \|x-y\| + \|y-z\| + \|x\| + \|z\|.$$

This is less than or equal to $d(x,y) + d(y,z)$.

Now, let us prove the property of being mutually contractive:

$$d(L(x), L'(y)) = \|L(x) - L'(y)\| + \|L(x)\| + \|L'(y)\| \leq$$

$$\|L(x) - L'(x)\| + \|L'(x) - L'(y)\| + \|L(x)\| + \|L'(y)\|$$

and

$$d(L(x), L'(y)) \leq \|L - L'\|\|x\| + \|L'\|\|x-y\| + \|L\|\|x\| + \|L'\|\|y\|.$$

This quantity is less than or equal to

$$k\|x-y\| + 2k\|x\| + k\|y\|,$$

and thus

$$d(L(x), L'(y)) \leq \alpha d(x,y),$$

taking $\alpha = 2k$. □

Example 2. *The maps $f(x) = (1/4)x$ and $g(x) = (1/6)x$ are mutually contractive with respect to the metric d in $\mathbb{R}$, according to the previous proposition.*

We now give the inverses of the previous result.

Proposition 7. *Let E be a normed space with norm $\|\cdot\|$ and $L, L' : E \to E$ be two linear and bounded operators. If L, L' are mutually contractive with constant α, and $k := \max\{\|L - L'\|, \|L\|\} < \alpha/2$, Then, L is contractive with respect to the distance d defined in (8).*

Proof. Applying the contractivity condition of L, L' for $x \neq y$:

$$\|L(x) - L(y)\| \leq \|L'(x) - L(y)\| + \|L(x) - L'(x)\| \leq \alpha\|x-y\| + \|L - L'\|\|x\|,$$

then,

$$d(L(x), L(y)) = \|L(x) - L(y)\| + \|L(x)\| + \|L(y)\| \leq \alpha\|x-y\| + 2k\|x\| + k\|y\| \leq$$

$$\alpha(\|x-y\| + \|x\| + \|y\|).$$

□

Proposition 8. *Let E be a normed space and $L : E \to E$ be a linear operator. If L is mutually contractive with the identity with constant α, then L is bounded and contractive, and $\|L\| \leq \alpha$.*

Proof. Take $y = 0$ in the inequality (7). $\square$

In the following results, we provide conditions for the existence of a common fixed point of a family of mutually contracting operators (not necessarily linear).

Theorem 4. *Let E be a complete metric space and $\mathcal{F}$ be a set of self-maps $\mathcal{F} = \{T_i : E \to E, i \in \mathcal{I}\}$. If there exists $T_{i_0} \in \mathcal{F}$, such that, $\forall i \in \mathcal{I}$, T_{i_0} and T_i are mutually contractive with factor $\alpha_i \in \mathbb{R}$, such that $\alpha := \sup \alpha_i < 1/3$, then,*

1. *T_i is contractive $\forall i \in \mathcal{I}$.*
2. *$\mathcal{F}$ has a unique fixed point.*

Proof. T_{i_0} is contractive then it owns a unique fixed point $\bar{x}$. For $i \neq i_0$, if $x \neq y$,

$$d(T_i(x), T_i(y)) \leq d(T_i(x), T_{i_0}(y)) + d(T_{i_0}(y), T_{i_0}(x)) + d(T_{i_0}(x), T_i(y)) \leq 3\alpha d(x, y),$$

consequently, T_i is also contractive. Let $\bar{x}_i$ be its fixed point. If $\bar{x} \neq \bar{x}_i$ then,

$$d(\bar{x}, \bar{x}_i) = d(T_{i_0}(\bar{x}), T_i(\bar{x}_i)) \leq \alpha d(\bar{x}, \bar{x}_i).$$

Since $\alpha < 1$ both agree. $\square$

According to the last result, if the constants α and β of Kannan's Theorem 3 are lower than $1/3$ then T_1 is a contraction too, and the condition 3 is no longer needed.

Theorem 5. *Let E be a complete metric space and $\mathcal{F}$ a set of self-maps $\mathcal{F} = \{T_i : E \to E, i \in \mathcal{I}\}$. Let $T_{i_0} \in \mathcal{F}$ be such that $\forall i \in \mathcal{I}$, T_{i_0} and T_i are mutually contractive with factor $\alpha_i \in \mathbb{R}$ and $\alpha := \sup \alpha_i$ be such that $1/3 \leq \alpha < 1$. Let us assume that there exists $y \in E$, such that the sequence $y_n = (T_{i_0})^n(y), y_0 = y$, tends to $\bar{x}$ and satisfies the inequalities $y_n \neq \bar{x}$, for all $n > n_0$, where n_0 is a natural number, such that $n_0 \geq 1$, and $\bar{x}$ is the fixed point of T_{i_0}, then:*

1. *$\mathcal{F}$ has a unique fixed point $\bar{x} \in E$.*
2. *$\bar{x}$ is the only fixed point of every $T_i \in \mathcal{F}$.*

Proof. Since T_{i_0} is a contraction and E is complete, there exists $\bar{x} \in E$, such that $\bar{x}$ is the fixed point of T_{i_0}. Given $y \in E$ such that the sequence $y_n = (T_{i_0})^n(y)$, $y_0 = y$ satisfies the conditions described, let us consider for $i \neq i_0$ and $n > n_0$:

$$d(\bar{x}, T_i(\bar{x})) \leq d(\bar{x}, y_n) + d(y_n, T_i(\bar{x})) \leq d(\bar{x}, y_n) + d(T_{i_0}(y_{n-1}), T_i(\bar{x})) \leq d(\bar{x}, y_n) + \alpha\, d(y_{n-1}, \bar{x}).$$

Both right summands tend to zero; then $\bar{x} = T_i(\bar{x})$ and $\bar{x}$ is a fixed point of T_i, $\forall i \in \mathcal{I}$. T_{i_0} has a unique fixed point; consequently, $\mathcal{F}$ has only the fixed point $\bar{x}$.

For $i \neq i_0$, if $\bar{x}_i$ is another fixed point of T_i and $\bar{x}_i \neq \bar{x}$, then

$$d(\bar{x}_i, \bar{x}) = d(T_i(\bar{x}_i), T_{i_0}(\bar{x})) \leq \alpha\, d(\bar{x}_i, \bar{x}),$$

where $\alpha < 1$. Hence, $\bar{x}_i = \bar{x}$, and T_i has only a fixed point (equal to $\bar{x}$). $\square$

Theorem 6. *Let E be a complete metric space and $\mathcal{F} = \{T_i : E \to E; i \in \mathcal{I}\}$ satisfying the conditions of Theorem 4 or Theorem 5. Let us define $\forall\, x \in E$ the sequence $x_0 = x$, and for $k \geq 1$,*

$$x_k = T_{i_k}(x_{k-1})$$

where $T_{i_k} \in \mathcal{F}$. Let $\bar{x}$ be the fixed point of $\mathcal{F}$. Then:

1. *For any $x \in E$*
$$\lim_{n\to\infty} \tau_n(x) = \lim_{n\to\infty} T_{i_n} \circ T_{i_{n-1}} \circ \cdots \circ T_{i_1}(x) = \bar{x}.$$
2. *$\bar{x}$ is globally asymptotically stable.*

Proof. Let us consider any $x \in E$ and define the sequence $x_n = \tau_n(x) = T_{i_n} \circ T_{i_{n-1}} \circ \cdots \circ T_{i_1}(x)$, $x_0 = x$. If $x_n \neq \bar{x}\ \forall n \geq 0$, then

$$d(x_n, \bar{x}) = d(T_{i_n}(x_{n-1}), T_{i_0}(\bar{x})) \leq \alpha d(x_{n-1}, \bar{x}) \leq \ldots \alpha^n d(x_0, \bar{x}). \tag{9}$$

If there exists $m > 0$ such that $x_m = \bar{x}$ then $x_{m+1} = T_{i_{m+1}}(x_m) = x_m = \bar{x}$ and so on. In any case, $d(x_n, \bar{x}) \leq \alpha^n d(x_0, \bar{x})$. Consequently, $\lim\limits_{n \to \infty} \tau_n(x) = \bar{x}$ and therefrom the attraction.

For any $\varepsilon > 0$ if $d(x, \bar{x}) < \delta$ then by (9), $d(\tau_n(x), \bar{x}) \leq \alpha^n\, d(x, \bar{x}) < \alpha^n\, \delta < \delta$. The election $\delta = \varepsilon$ satisfies the definition of stability. Hence, $\bar{x}$ is asymptotically stable. □

Theorem 7. *If E is a Banach space, $\mathcal{F} = \{L_i : E \to E, i \in \mathcal{I}\}$ a family of linear and bounded operators, and there exists $j \in \mathcal{I}$ such that the constants $k_i := \max\{\|L_i - L_j\|, \|L_i\|, \|L_j\|\} < 1/2$ satisfy the condition $k := \sup k_i < 1/2$, then L_i, L_j are mutually contractive for any $i \in \mathcal{I}$ with respect to the distance d defined in (8), and*

1. *0 is an equilibrium asymptotically stable for the system $x_k = L_{i_k}(x_{k-1})$ for $k \geq 1$.*
2. *1 does not belong to the point spectrum of L_i for any i.*

Proof. According to Proposition 6, L_i, L_j are mutually contractive with respect to the distance d with constant $\alpha_i = 2k_i$. The set $\mathcal{F}$ owns the fixed point zero. If any L_i has another fixed point $\bar{x}_i \neq 0$, then

$$d(\bar{x}_i, 0) = d(L_i(\bar{x}_i), L_j(0)) \leq 2k_i d(\bar{x}_i, 0).$$

Since $2k_i < 1$, then $d(\bar{x}_i, 0) = 0$, and L_i owns a single fixed point. The proof for stability is similar to Theorem 6. □

Remark 6. *The convergence of the system holds in norm also due to the definition of d (8).*

Proposition 9. *In the hypothesis of Theorems 4 or 5, any ball $B(\bar{x}, r)$, where $\bar{x}$ is the fixed point of $\mathcal{F}$ and $r > 0$, is an invariant set.*

Proof. If $x \in B(\bar{x}, r)$ then if $x \neq \bar{x}$, $\forall i \in \mathcal{I}$,

$$d(T_i(x), \bar{x}) = d(T_i(x), T_{i_0}(\bar{x})) \leq \alpha\, d(x, \bar{x}) < \alpha\, r < r$$

then, $T_i(x) \in B(\bar{x}, r)$.

If $x = \bar{x}$, $T_i(x) = T_i(\bar{x}) = \bar{x} \in B(\bar{x}, r)$. □

In fractal theory, the backward orbits of type

$$\tilde{\tau}_n(y) = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}(y)$$

are as important as the forward orbits, that acquire full sense in algorithmics.

Let us study now the stability of the point $\bar{x}$ with respect to the backward orbits $\tilde{\tau}_n$.

Proposition 10. *If $E, \mathcal{F}$ satisfy the conditions of Theorem 4, then the fixed point $\bar{x}$ is asymptotically stable with respect to backward orbits $\tilde{\tau}_n$.*

Proof. For $y \in E$, let us define $y_n = \tilde{\tau}_n(y) = T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_n}(y)$. According to Theorem 4, the operators T_i are contractive with factor 3α, then,

$$d(\tilde{\tau}_n(y), \bar{x}) = d(\tilde{\tau}_n(y), \tilde{\tau}_n(\bar{x})) \leq (3\alpha)^n d(y, \bar{x}).$$

From this inequality the conclusion is easily verified. □

Variant of Banach's theorem

The following theorem proves the existence of fixed point of a single operator on a metric space not necessarily complete (see the reference [7]).

Theorem 8. *Let E be a metric space (not necessarily complete), with $T : E \to E$ continuous at $x_0 \in E$. If there exists $x \in E$, such that the sequence of iterations $(T^n(x))$ converges to x_0, then x_0 is a fixed point of T. If, in addition, $\exists\, \alpha \in \mathbb{R}$, such that $0 < \alpha < 1$ and $\forall\, \xi \in E$*

$$d(T(x_0), T(\xi)) \leq \alpha\, d(x_0, \xi),$$

then, x_0 is the unique fixed point of T.

We generalize the former result to a set of operators acting on a metric space not necessarily complete, proving the existence of a common fixed point, which is the unique fixed point of every element of the family.

Theorem 9. *Let E be a metric space (not necessarily complete) and $\mathcal{F} = \{T_i : E \to E; i \in \mathcal{I}\}$ a collection of self-maps. For $i_0 \in \mathcal{I}$, let us assume that T_{i_0} is continuous at $\bar{x} \in E$. If there exists $x \in E$, such that $T_{i_0}^n(x) \to \bar{x}$ (as $n \to \infty$), and $\forall \xi \in E$, $\forall i \in \mathcal{I}$*

$$d(T_i(\bar{x}), T_{i_0}(\xi)) \leq \alpha_i\, d(\bar{x}, \xi),$$

where $\alpha = \sup \alpha_i < 1$, then $\bar{x}$ is the unique fixed point of each T_i.

Proof. Due to continuity, $\bar{x}$ is a fixed point of T_{i_0}. Let us take $i \neq i_0$ and, for the element x of the statement, define $x_n = T_{i_0}^n(x)$, then

$$d(\bar{x}, T_i(\bar{x})) \leq d(\bar{x}, x_n) + d(x_n, T_i(\bar{x})) \leq d(\bar{x}, x_n) + \alpha\, d(x_{n-1}, \bar{x}) \to 0.$$

Consequently, $\bar{x} = T_i\bar{x}$, and $\bar{x}$ is a fixed point of $\mathcal{F}$.

If $\bar{x}_i$ is another fixed point of T_i

$$d(\bar{x}, \bar{x}_i) = d(T_{i_0}(\bar{x}), T_i(\bar{x}_i)) \leq \alpha\, d(\bar{x}, \bar{x}_i).$$

Since $\alpha < 1$ then $\bar{x} = \bar{x}_i$. □

4. An Application to Fractal Convolution

In this section, we apply the previous results to iterations of a side operator related to fractal convolution [12]. We introduce first the formalism of a type of Iterated Function System (IFS).

Let us define in $E = [a, b] \times \mathbb{R}$ an IFS $\{w_n : n = 1, \ldots, N\}$ associated with a partition of the interval, $a = t_0 < t_1 < t_2 < \ldots t_N = b$, where $N > 1$, and a set of scale functions (or constants) $\{\alpha_n(t) : n = 1, \ldots, N\}$, such that the sup norm is $\|\alpha_n\|_\infty < 1$ for all n .

The IFS is composed of the mappings $w_n(t, x) = (H_n(t), F_n(t, x))$, where H_n are affine, and $H_n(t_0) = t_{n-1}, H_n(t_N) = t_n$, and $F_n(t, x) = \alpha_n(x - b(t)) + f \circ H_n(t)$, for $n = 1, \ldots N$, where $f, b \in \mathcal{L}^p([a, b])$. This system induces an operator W on the set $K(E)$ of compact sets of E, defined for $A \in K(E)$ as:

$$W(A) = \cup_{n=1}^N w_n(A).$$

The self-map W owns an attractor G. The set G is the graph of a function on the interval called in previous papers α-fractal function and denoted as $f^\alpha \in \mathcal{L}^p([a, b])$. The notation describes the dependence of the map on the so-called scale vector $(\alpha_n)_{n=1}^N$.

Moreover, f^α can be seen as the result of an operation between f and b. This operation has been called fractal convolution of f and b (for instance, see the reference [12]), though it has nothing to do with other types of convolution between maps. Thus,

$$f^\alpha = f * b.$$

Beginning from this association, we defined the side or partial convolutions with the null function:

$$L_0(b) = f_0 * b,$$

$$R_0(f) = f * f_0,$$

where f_0 is the null function.

Figure 1 represents the outcome of the action of L_0 on the map $b(x) = \sin(\pi x)$ in the interval $[0, 2\pi]$, an evenly sampled partition with $N = 10$ subintervals and $\alpha_n(t) = t/8$, for any $n = 1, \ldots, 10$.

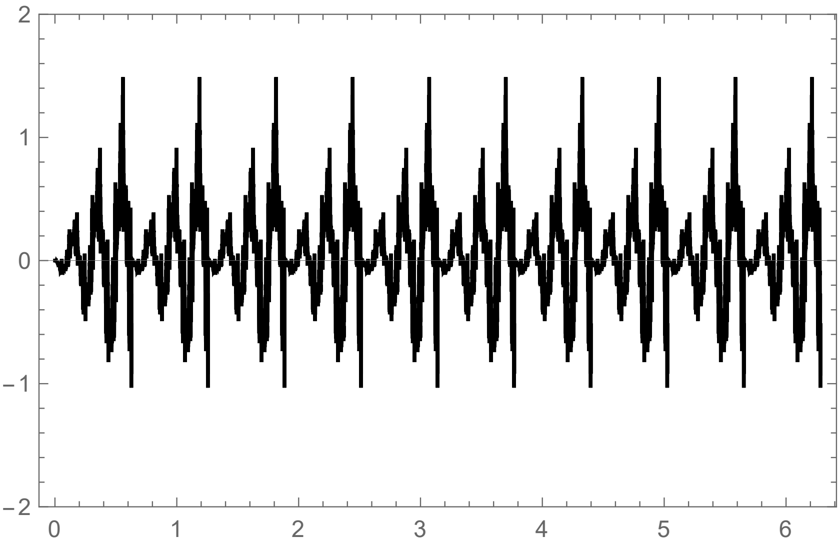

Figure 1. Graph of the image of the function $b(x) = \sin(\pi x)$ by the operator L_0 in the interval $I = [0, 2\pi]$ (fractal convolution of the null function with a sinus).

We studied the properties of these operators. For instance, they are linear and bounded, and

$$\|L_0\| \leq \frac{\Lambda}{1-\Lambda}, \tag{10}$$

$$\|R_0\| \leq \frac{1}{1-\Lambda}, \tag{11}$$

where $\Lambda = \max\{\|\alpha_n\|_\infty : n = 1, \ldots, N\} < 1$. Consequently, if $\Lambda < 1/2$ then L_0 is a contraction, and if $\Lambda = 0$ then $L_0 = 0$, where 0 represents the null operator.

We consider now the iteration of the left convolution with zero but associated with different scale vectors, defining $\mathcal{F} = \{L_0^{\alpha^i} : i \in \mathcal{I}\}$ and

$$g_k = L_0^{\alpha^k}(g_{k-1}),$$

for $k \geq 1$, $L_0^{\alpha^k} \in \mathcal{F}$ and $g_0 \in \mathcal{L}^p(I)$. That is to say, in each step, the convolution with zero is perfprmed with respect to the scale vector α^k in the IFS described.

According to Theorem 7, if $\Lambda^i = \max\{\|\alpha_n^i\|_\infty : n = 1, \ldots, N\}$, and there exists $j \in \mathcal{I}$ such that

$$\sup_{i \in \mathcal{I}} \left\{ \frac{\Lambda^i}{1-\Lambda^i} + \frac{\Lambda^j}{1-\Lambda^j} \right\} < 1/2$$

then the sequence g_k converges to the null function for any $g_0 \in \mathcal{L}^p(I)$ as k tends to infinity.

5. Conclusions

For two self-maps T_1, T_2 defined on a metric space (E, d), we have defined two new concepts (Definitions 2 and 7): Mutual Kannan contractivity and mutual contractivity. We proved some relations between mutual Kannan contractivity and classical Banach contraction and also with isomorphism (in the case where E is a normed space, and the self-maps are linear).

In Theorem 2, we provided sufficient conditions for the existence of a common fixed point of a set $\mathcal{F}$ of operators satisfying mutual relations of Kannan type. Under the hypotheses given, we proved that this equilibrium is a global attractor for any iterative scheme of type

$$x_k = T_{i_k}(x_{k-1}),$$

where $x_k \in E$ and $T_{i_k} \in \mathcal{F}$.We observed some other stability and invariance aspects of the stationary point.

We also described some properties related to mutual contractivity, in particular, if E is a normed space, and the operators are linear. Theorems 4 and 5 establish sufficient conditions for the existence of a common fixed point of a set of operators (linear or nonlinear) showing mutual contractivity. With the conditions given, the equilibrium is also asymptotically stable for the same system.

In Theorem 9, we studied the existence of a common fixed point of a set of operators acting on a metric space E, non necessarily complete.

In the last section, we considered the fractal convolution operation in $\mathcal{L}^p(I)$ defined in previous papers (see for instance [12]). We found that under some conditions on the scale vectors, the iteration of the left fractal convolution with the null function converges to zero for any initial map in $\mathcal{L}^p(I)$ even if the scale vector is changed at each step.

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