mathematics

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Inverse Problem to Determine Two Time-Dependent Source Factors of Fractional Diffusion-Wave Equations from Final Data and Simultaneous Reconstruction of Location and Time History of a Point Source

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Abstract: In this paper, two inverse problems for the fractional diffusion-wave equation that use final data are considered. The first problem consists in the determination of two time-dependent source terms. Uniqueness for this inverse problem is established under an assumption that given space-dependent factors of these terms are "sufficiently different". The proof uses asymptotical properties of Mittag–Leffler functions. In the second problem, the aim is to reconstruct a location and time history of a point source. The uniqueness for this problem is deduced from the uniqueness theorem for the previous problem in the one-dimensional case.

Keywords: inverse problem; fractional diffusion-wave equation; final overdetermination; source reconstruction

MSC: 35R30; 35R11

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Citation: Janno, J. Inverse Problem to Determine Two Time-Dependent Source Factors of Fractional Diffusion-Wave Equations from Final Data and Simultaneous Reconstruction of Location and Time History of a Point Source. *Mathematics* **2023**, *11*, 456. https://doi.org/10.3390/math11020456

Academic Editor: Dongfang Li

Received: 10 December 2022
Revised: 8 January 2023
Accepted: 11 January 2023
Published: 14 January 2023

1. Introduction

Fractional diffusion-wave equations (FDWE) are widely used to model anomalous diffusion, and wave processes in physics, chemistry, and biology [1–4]. Applications of FDWE also occur in engineering sciences [5].

Often parameters of models are unknown. To determine the parameters, inverse problems that use measurements of states of the processes are solved [6–11]. Depending on possibilities, the state can be measured at a boundary of a domain where the process is going on, in a subdomain or at fixed time values, e.g., at a final time moment.

An important practical inverse problem for a diffusion equation is the determination of the location or time history of groundwater or atmospheric pollution sources [12,13]. In case the medium is accessible, then it is possible to measure the concentration of the pollution at a final time moment over the domain and use this information in the inverse problem.

The problems to determine space-dependent components of source factors of FDWE from final measurements of states are well-studied [14–20]. On the other hand, problems to determine time-dependent components of sources of FDWE from final data have received very little attention. In [9], these problems were discussed from a general viewpoint and in [21] the uniqueness of a solution to a problem to determine a single time-dependent source factor of FDWE was proved. The method of the paper [21] is based on power-type asymptotic expansions of Mittag–Leffler functions occurring in the formula of solution of a corresponding direct problem. However, these expansions can be used in the inverse problem only if the time-dependent source factor is a priori known in an arbitrarily small left neighborhood $(T-\varepsilon, T)$ of the final time value T. We also mention that a similar uniqueness result was obtained in [22] for a problem that involves unknown time-dependent boundary conditions instead of the time-dependent source factor.

In this paper, we will consider two inverse problems. In the first problem, the source function of FDWE consists of two addends of the form of products of known space-dependent and unknown time-dependent functions. We will prove that two time-dependent factors are uniquely recovered by the final values of the state. The basic idea of the proof is that if the given two space-dependent components are "sufficiently different" in a certain sense, then from the series of Fourier coefficients of the final state, we can extract 2 subseries that can be asymptotically used to construct two separate families of integral equations for both unknown time-dependent terms. These equations lead to the uniqueness result. This technique is a further development of the method presented in [21], where a problem with a single unknown was considered.

In the second inverse problem, the aim is to construct a location and time history of a point source in FDWE. We deduce the uniqueness of this problem from the uniqueness theorem for the previous problem in the one-dimensional case. We emphasize that this is the first time when a problem of simultaneous reconstruction of unknown time- and space-dependent source terms from final data is studied.

As in [21], the results of the present paper are obtained under the additional assumption that the unknown time-dependent terms are a priori known in an arbitrarily small left neighborhood $(T-\varepsilon, T)$ of the final time value T.

The plan for the paper is as follows. In Section 2, we will formulate the inverse problems. Section 3 has mainly a referative character. There we introduce some concepts and refer to mathematical sentences we need in our further analysis. Sections 4 and 5 contain the main results of the paper. There we prove the uniqueness of solutions to the posed inverse problems. In Section 6, we will give conclusions.

2. Formulation of Problems

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be an open bounded domain. We assume that in case $d \geq 2$ the boundary $\partial\Omega$ of Ω is of the class C^2. Moreover, let $T > 0$ and consider the following equation (cf. [21,23]):

$$ {}^{C}D_t^{\alpha} u(t,x) - \varkappa \Delta u(t,x) = F(t,x), \quad x \in \Omega, \quad t \in (0,T), \tag{1} $$

where ${}^{C}D_t^{\alpha}$ is the Caputo–Djrbashian fractional derivative of the order $\alpha \in (0,1) \cup (1,2)$ defined by the formula [24]

$$ {}^{C}D_t^{\alpha} w(t) = \int_0^t \frac{(t-\tau)^{\lfloor \alpha \rfloor - \alpha}}{\Gamma(1 + \lfloor \alpha \rfloor - \alpha)} w^{(\lfloor \alpha \rfloor + 1)}(\tau) d\tau, \tag{2} $$

$\lfloor \cdot \rfloor$ denotes the floor function, Δ is the Laplacian and $\varkappa$ is a positive number.

Equation (1) is called the fractional diffusion-wave equation. It governs subdiffusion (the case $\alpha \in (0,1)$) and superdiffusion or fractional wave (the case $\alpha \in (1,2)$) processes. The function u is a state of the process, and F is a source function. There are two methods to derive Equation (1). One approach is based on modeling continuous time random walk processes of particles at the micro-level. The obtained master equation leads to (1) after homogenization [25,26]. Another approach is partly classical and consists in inserting a constitutive relation with memory to a conservation equation [27].

Let us bring in (2) one derivative from the function w to the front of the integral. Then we obtain the following expression:

$$ {}^{C}D_t^{\alpha} w(t) = {}^{R}D_t^{\alpha - \lfloor \alpha \rfloor} \left[w^{(\lfloor \alpha \rfloor)}(t) - w^{(\lfloor \alpha \rfloor)}(0) \right], \tag{3} $$

where ${}^{R}D_t^{\beta} w(t)$ is the Rieman–Liouville fractional derivative of the order $\beta \in (0,1)$ [24] given by the formula

$$ {}^{R}D_t^{\beta} w(t) = \frac{d}{dt} \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} w(\tau) d\tau. \tag{4} $$

The representation (3) requires less regularity of w than (2).

Further, let us introduce initial and boundary conditions and modify the Equation (1) according to the formula (3). Then we obtain the following direct problem for the state function u:

$$ {}^{R}D_t^{\alpha-\lfloor\alpha\rfloor}\left[\frac{\partial^{\lfloor\alpha\rfloor}}{\partial t^{\lfloor\alpha\rfloor}}u-\varphi_{\lfloor\alpha\rfloor}\right](t,x)-\varkappa\Delta u(t,x)=F(t,x),\quad x\in\Omega,\ \ t\in(0,T), \tag{5} $$

$$ \frac{\partial^j}{\partial t^j}u(t,x)|_{t=0}=\varphi_j(x),\ x\in\Omega,\ j\in\{0;\lfloor\alpha\rfloor\},\quad u(t,x)=0,\ t\in(0,T),\ x\in\partial\Omega. \tag{6} $$

In this paper, we will consider two inverse problems for Equation (5). In the first problem, we assume that the source function F has the form

$$ F(t,x)=g(t)f(x)+\widetilde{g}(t)\widetilde{f}(x), \tag{7} $$

where $f,\widetilde{f}$ and given but $g,\widetilde{g}$ are unknown. To recover the unknowns, the final condition

$$ u(T,x)=\psi(x) \tag{8} $$

is prescribed with a given function ψ. Thus, the inverse problem consists in finding the triplet $(g,\widetilde{g},u)$ satisfying (5)–(8).

In the second inverse problem we assume that $d=1$, $\Omega=(0,1)$ and

$$ F(t,x)=g(t)\delta(x-x_0), \tag{9} $$

where δ is the Dirac delta distribution and $x_0\in(0,1)$. The aim is to find the triplet (g,x_0,u) that satisfies (5), (6), (8), and (9).

Since the second inverse problem involves $\delta(x-x_0)$, we need to treat the direct problem in the distributional sense with respect to the space variable. We will introduce the concept of a generalized solution to the direct problem (Section 3.3) and prove the uniqueness results for the inverse problems in the generalized setting.

3. Preliminaries

3.1. Spaces Related to the Operator $L=-\varkappa\Delta$

Let us introduce the operator

$$ L=-\varkappa\Delta\quad\text{with the domain } \mathcal{D}(L)=\{z\in W_2^2(\Omega)\,:\,z|_{\partial\Omega}=0\}\text{ in } L_2(\Omega). $$

Let (λ_k,v_k), $k\in\mathbb{N}$ denote the pairs of eigenvalues and eigenfunctions of L, i.e., $Lv_k=\lambda_k v_k$, $k\in\mathbb{N}$. We assume that the eigenvalues are ordered nondecreasingly, i.e., $0<\lambda_1\le\lambda_2\le\ldots$ and the eigenfunctions are orthonormed in $L_2(\Omega)$. Then $(v_k)_{k\in\mathbb{N}}$ forms a basis in $L_2(\Omega)$.

For any $z\in L_2(\Omega)$, we will denote the k-s Fourier coefficient of z by means of the subscript k, i.e.,

$$ z_k=\langle z,v_k\rangle_{L_2(\Omega)},\ \ k\in\mathbb{N}. \tag{10} $$

It is well-known that the following quantity:

$$ \|z\|_{\mathcal{D}(L)}=\left[\sum_{k=1}^{\infty}\lambda_k^2|z_k|^2\right]^{\frac{1}{2}} $$

is a norm in the space $\mathcal{D}(L)$, the operator L is a bijection from $\mathcal{D}(L)$ to $L_2(\Omega)$ and the ormulas

$$Lz = \sum_{k=1}^{\infty} \lambda_k z_k v_k,\ z \in \mathcal{D}(L), \quad L^{-1}z = \sum_{k=1}^{\infty} \lambda_k^{-1} z_k v_k,\ z \in L_2(\Omega)$$

are valid.

Let us define nonnegative powers of L and their domains as follows:

$$L^{\gamma} z(x) = \sum_{k=1}^{\infty} \lambda_k^{\gamma} z_k v_k(x),$$
$$\mathcal{D}_{\gamma} = \left\{ z \in L_2(\Omega) \,:\, \|z\|_{\mathcal{D}_{\gamma}} := \left[\sum_{k=1}^{\infty} \lambda_k^{2\gamma} |z_k|^2 \right]^{\frac{1}{2}} < \infty \right\}, \quad \gamma \geq 0.$$

Evidently, $\mathcal{D}_0 = L_2(\Omega)$ and $\mathcal{D}_1 = \mathcal{D}(L)$. The set $\mathcal{D}_{\gamma}$ is a Hilbert space endowed by the inner product

$$\langle y, z \rangle_{\mathcal{D}_{\gamma}} = \sum_{k=1}^{\infty} \lambda_k^{2\gamma} y_k \overline{z_k}.$$

Moreover,

$$\mathcal{D}_{\gamma_2} \hookrightarrow \mathcal{D}_{\gamma_1}, \quad 0 \leq \gamma_1 < \gamma_2.$$

Let us introduce the family of Gelfand triples $\mathcal{D}_{\gamma} \hookrightarrow L_2(\Omega) \hookrightarrow \mathcal{D}'_{\gamma}$, $\gamma > 0$, define

$$\mathcal{D}_{-\gamma} := \mathcal{D}'_{\gamma}, \ \ \gamma > 0,$$

and denote the value of the functional $z \in \mathcal{D}_{-\gamma}$, $\gamma > 0$, applied to $\zeta \in \mathcal{D}_{\gamma}$ by $\langle z, \zeta \rangle_{-\gamma}$.

Then it holds that

$$\langle z, \zeta \rangle_{-\gamma} = \langle z, \zeta \rangle_{L_2(\Omega)} \quad \text{for } z \in L_2(\Omega),\ \zeta \in \mathcal{D}_{\gamma}, \gamma > 0. \tag{11}$$

Moreover,

$$\mathcal{D}_{-\gamma_1} \hookrightarrow \mathcal{D}_{-\gamma_2}, \ \ 0 < \gamma_1 < \gamma_2, \tag{12}$$
$$\langle z, \zeta \rangle_{-\gamma_1} = \langle z, \zeta \rangle_{-\gamma_2}, \ \ z \in \mathcal{D}_{-\gamma_1},\ \zeta \in \mathcal{D}_{\gamma_2},\ 0 < \gamma_1 < \gamma_2. \tag{13}$$

Since v_k, $k \in \mathbb{N}$, belongs to $\cap_{s>0} \mathcal{D}_s$, it is an argument of any distribution $z \in \mathcal{D}_{-\gamma}$, $\gamma > 0$. Let $\gamma > 0$ and $z \in \mathcal{D}_{-\gamma}$. We define the generalized Fourier coefficients of the distribution z by the formula

$$z_{-\gamma,k} = \langle z, v_k \rangle_{-\gamma}, \ \ k \in \mathbb{N}. \tag{14}$$

Due to (11) and (13), $z_{-\gamma_1,k} = z_{-\gamma_2,k}$ for $z \in \mathcal{D}_{-\gamma_1}$, $0 < \gamma_1 < \gamma_2$ and $z_{-\gamma,k} = z_k$ for $z \in L_2(\Omega)$. Therefore, $z_{-\gamma,k}$ is an extension of z_k, and we will omit the first subscript $-\gamma$ in its notation.

The quantity $\langle z, \zeta \rangle_{-\gamma}$ can expressed as

$$\langle z, \zeta \rangle_{-\gamma} = \sum_{k=1}^{\infty} z_k \overline{\zeta_k}, \quad \zeta \in \mathcal{D}_{\gamma}, \ \ z \in \mathcal{D}_{-\gamma},$$

and the norm in the space $\mathcal{D}_{-\gamma}$ has the formula

$$\|z\|_{\mathcal{D}_{-\gamma}} = \left[\sum_{k=1}^{\infty} \lambda_k^{-2\gamma} |z_k|^2 \right]^{\frac{1}{2}}.$$

Finally, let us extend the operator L outside $\mathcal{D}(L)$. For any $\gamma < 1$ we define L_γ as an operator that maps $\mathcal{D}_\gamma$ to $\mathcal{D}_{\gamma-1}$ by the formula

$$\langle L_\gamma z, \zeta \rangle_{\gamma-1} = \sum_{k=1}^{\infty} \lambda_k z_k \overline{\zeta_k}, \ \ \zeta \in \mathcal{D}_{1-\gamma}, \ \ z \in \mathcal{D}_\gamma.$$

We note that $L_{\gamma_1} z = L_{\gamma_2} z$ for $z \in \mathcal{D}_{\gamma_2}$, $\gamma_1 < \gamma_2 < 1$ and $L_\gamma z = Lz$ for $z \in \mathcal{D}(L)$, $\gamma < 1$. This means that L_γ is an extension of the operator L and we will omit the subscript γ in further formulas.

3.2. Abstract Functional Spaces: Riemann–Liouville Fractional Integral and Derivative

Let X be a complex Banach space and $G \subseteq \mathbb{R}$. We introduce the following spaces of Bochner's measurable functions in G with values in X:

$$L_p(G;X) = \left\{ w \,:\, \int_G \|w(t)\|_X^p dt < \infty \right\}, \ \ p \in [1,\infty),$$

$$L_\infty(G;X) = \left\{ w \,:\, \operatorname*{ess\,sup}_{t \in G} \|w(t)\|_X < \infty \right\},$$

$$W_p^l(G;X) = \left\{ w \,:\, w^{(j)} \in L_p(G;X),\, j = 0,\ldots,l \right\}, \ \ p \in [1,\infty], \ l \in \mathbb{N},$$

$$C(G;X) = \{ w \,:\, w \text{ is continuous in } G \},$$

$$C^l(G;X) = \left\{ w \,:\, w^{(j)} \in C(G;X),\, j = 0,\ldots,l \right\}, \ \ l \in \mathbb{N},$$

$$H_p^s(\mathbb{R};X) = \left\{ w \in L_p(\mathbb{R};X) \,:\, \mathcal{F}^{-1}|\xi|^s \mathcal{F} w \in L_p(\mathbb{R};X) \right\}, \ \ p \in (1,\infty),\, s > 0,$$

where $\mathcal{F}$ denotes the Fourier transform with the argument ξ,

$$H_p^s((0,T);X) = \left\{ w|_{(0,T)} \,:\, w \in H_p^s(\mathbb{R};X) \right\}, \ \ p \in (1,\infty),\, s > 0,$$

$${}_0H_p^s((0,T);X) = \left\{ w|_{(0,T)} \,:\, w \in H_p^s(\mathbb{R};X),\, \operatorname{supp} w \subseteq [0,\infty) \right\}, \ \ p \in (1,\infty),\, s > 0.$$

In case $X = \mathbb{C}$, we drop the value space $\mathbb{C}$ in these notations.

The Riemann–Liouville fractional integral I_t^β and Riemann–Lioville fractional derivative ${}^R D_t^\beta$ of the order $\beta \in (0,1)$ of a Bochner measurable function $w \,:\, (0,T) \mapsto X$ are formally given by the relations

$$I_t^\beta w(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} w(\tau) d\tau,$$

$${}^R D_t^\beta w(t) = \frac{d}{dt} \int_0^t \frac{(t-\tau)^{-\beta}}{\Gamma(1-\beta)} w(\tau) d\tau,$$

respectively.

Let us formulate a lemma that gives a relationship between I_t^β and ${}^R D_t^\beta$.

Lemma 1 ([28]). *Let X be a complex Hilbert space, $\beta \in (0,1)$ and $p \in (1,\infty)$. The operator I_t^β is a bijection from $L_p((0,T);X)$ onto ${}_0H_p^\beta((0,T);X)$, the operator ${}^R D_t^\beta$ is the left inverse of I_t^β and*

$$\|w\|_{{}_0H_p^\beta((0,T);X)} = \|{}^R D_t^\beta w\|_{L_p((0,T);X)}$$

is a norm in the space ${}_0H_p^\beta((0,T);X)$. Moreover, in the case $p \in (\frac{1}{\beta},\infty)$ it hold $H_p^\beta((0,T);X) \hookrightarrow C([0,T];X)$ and $w(0) = 0$ for $w \in {}_0H_p^\beta((0,T);X)$.

3.3. Generalized Solution of Direct Problem

To generalize the direct problem, we define the following spaces:

$$u \in \mathcal{U}_{\alpha,s,\gamma} = \Big\{ u \in C^{\lfloor \alpha \rfloor}([0,T]; \mathcal{D}_\gamma) \cap L_s((0,T); \mathcal{D}_{\gamma+1}) \ : \\ u^{(\lfloor \alpha \rfloor)} - u^{(\lfloor \alpha \rfloor)}(0) \in {}_0H_s^{\alpha - \lfloor \alpha \rfloor}((0,T); \mathcal{D}_\gamma) \Big\}, \\ \alpha \in (0,1) \cup (1,2),\ s > 1,\ \gamma \in \mathbb{R}.$$

Let $\alpha \in (0,1) \cup (1,2)$, $F \in L_s((0,T); \mathcal{D}_\gamma)$ and $\varphi_j \in \mathcal{D}_\gamma$, $j \in \{0; \lfloor \alpha \rfloor\}$, for some $s > 1$, $\gamma \in \mathbb{R}$. We call a function $u \in \mathcal{U}_{\alpha,s,\gamma}$ a *generalized solution of the direct problem* if

$${}^R D_t^{\alpha - \lfloor \alpha \rfloor}(u^{(\lfloor \alpha \rfloor)} - \varphi_{\lfloor \alpha \rfloor})(t) + Lu(t) = F(t) \quad \text{a.e. } t \in (0,T), \tag{15}$$

and

$$u^{(j)}(0) = \varphi_j, \quad j \in \{0; \lfloor \alpha \rfloor\}. \tag{16}$$

Note that if u is a generalized solution of the direct problem, then due to the definition of the extension of L in Section 3.1 and Lemma 1, all addends included in the Equation (15) belong to $L_s((0,T); \mathcal{D}_\gamma)$.

We also mention that if $\gamma \geq 0$ then all terms in (15) become regular distributions in Ω and the function $u(t,x) = (u(t))(x)$ is a strong solution of the direct problem (5) and (6).

Theorem 1. *Let $\alpha \in (0,1) \cup (1,2)$.*

(i) Let the direct problem have a generalized solution $u \in \mathcal{U}_{\alpha,s,\gamma}$ for some $s > 1$, $\gamma \in \mathbb{R}$. If $\varphi_j = 0$, $j \in \{0; \lfloor \alpha \rfloor\}$, and $F = 0$ then $u = 0$.

(ii) Let $F(t) = g(t)f + \widetilde{g}(t)\widetilde{f}$, where $f, \widetilde{f} \in \mathcal{D}_\gamma$ and $g, \widetilde{g} \in L_p(0,T)$ for some $\gamma \in \mathbb{R}$ and $p > \frac{1}{\alpha - \lfloor \alpha \rfloor}$. Moreover, assume that $\varphi_j \in \mathcal{D}_{\gamma + r_j}$, $j \in \{0; \lfloor \alpha \rfloor\}$, for some $r_j > 1 - \frac{j}{\alpha} - \frac{1}{p\alpha}$. Then the direct problem has a generalized solution u that belongs to any space $\mathcal{U}_{\alpha,p,\gamma'}$, $\gamma' < \gamma$, and its Fourier coefficients have the formulas

$$\begin{aligned} u_k(t) &= \sum_{j=1}^{\lfloor \alpha \rfloor} \varphi_{j,k} t^j E_{\alpha,j+1}(-\lambda_k t^\alpha) \\ &+ f_k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) g(\tau) d\tau \\ &+ \widetilde{f}_k \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-\tau)^\alpha) \widetilde{g}(\tau) d\tau, \quad t \in [0,T], \quad k \in \mathbb{N}. \end{aligned} \tag{17}$$

Here $E_{\alpha,\beta}$ is the two-parametric Mittag–Leffler function.

Proof. The assertion (i) and the assertion (ii) in case $\widetilde{g}\widetilde{f} = 0$ (i.e., $F(t) = g(t)f$) were proved in [21]. The generalization of (ii) to the case $\widetilde{g}\widetilde{f} \neq 0$ is immediate. □

Let us denote

$$\mathcal{D} = \bigcup_{\gamma \in \mathbb{R}} \mathcal{D}_\gamma.$$

By means of this notation, we can reformulate the statement (ii) of Theorem 1 in the following simpler form:

Corollary 1. *Let $\alpha \in (0,1) \cup (1,2)$, $\varphi_j \in \mathcal{D}$, $j \in \{0; \lfloor \alpha \rfloor\}$, and $F(t) = g(t)f + \widetilde{g}(t)\widetilde{f}$, where $f, \widetilde{f} \in \mathcal{D}$ and $g, \widetilde{g} \in L_p(0,T)$ for some $p > \frac{1}{\alpha - \lfloor \alpha \rfloor}$. Then the direct problem has a generalized* *solution in the space $\mathcal{U}_{\alpha,p,\gamma}$ for some $\gamma \in \mathbb{R}$ and its Fourier coefficients are expressed by the formulas* (17).

3.4. Inverse Problem to Determine a Single Time-Dependent Source Factor

In this subsection, we present a previously published uniqueness result for an inverse problem to determine a time-dependent component g of the source function of the form

$$F(t) = g(t)f, \tag{18}$$

where the element f is given.

Let us define the following class of degenerate elements of $L_2(\Omega)$:

$$\mathcal{F}_0 = \{f \in L_2(\Omega) \,:\, \exists k_0 \in \mathbb{N} \,:\, f_k = 0,\, k \geq k_0\}.$$

Evidently, $\mathcal{F}_0 \subset \bigcap\limits_{\gamma\in\mathbb{R}} \mathcal{D}_\gamma$.

Theorem 2 ([21])**.** *Let $\alpha \in (0,1) \cup (1,2)$, $f \in \mathcal{D} \setminus \mathcal{F}_0$, and $g \in L_p(0,T)$ for some $p > \frac{1}{\alpha-\lfloor\alpha\rfloor}$. If $\varphi_j = 0$, $j \in \{0; \lfloor\alpha\rfloor\}$, $g(t) = 0$ a.e. $t \in (T-\varepsilon, T)$ for some $\varepsilon \in (0,T)$ and the generalized solution of the direct problem with F of the form* (18) *satisfies $u(T) = 0$ then $g = 0$ and $u = 0$.*

4. Inverse Problem to Determine Two Time-Dependent Source Factors

In this section, we will treat the problem to determine two time-dependent source factors in a generalized setting.

Let $\alpha \in (0,1) \cup (1,2)$ and suppose that F has the form

$$F(t) = g(t)f + \widetilde{g}(t)\widetilde{f}, \tag{19}$$

where $g, \widetilde{g} \in L_s(0,T)$ for some $s > 1$ and $f, \widetilde{f} \in \mathcal{D}$. Assume that the elements $f, \widetilde{f}$ and initial data $\varphi_j \in \mathcal{D}$, $j \in \{0; \lfloor\alpha\rfloor\}$, are given but $g, \widetilde{g}$ are unknown. In addition, we assume that the generalized solution of the direct problem satisfies the final condition

$$u(T) = \psi, \tag{20}$$

where ψ is a prescribed element of $\mathcal{D}$. The aim is to determine the triplet $(g, \widetilde{g}, u)$.

To handle this problem, we need to introduce some additional sets. With any pair of non-degenerate elements $f, \widetilde{f} \in \mathcal{D} \setminus \mathcal{F}_0$ we associate the set

$$K_{f,\widetilde{f}} = \{k \in \mathbb{N} \,:\, |f_k| + |\widetilde{f}_k| \neq 0\}.$$

We also define the following set of pairs of non-degenerate elements $f, \widetilde{f}$ that have "similar" asymptotics of Fourier coefficients:

$$\mathcal{P}_{\mathrm{sim}} = \Big\{(f,\widetilde{f}) \in (\mathcal{D}\setminus\mathcal{F}_0)^2 \,:\, \exists k_1 \in \mathbb{N},\, m \in \mathbb{Z},\, c \in \mathbb{C}\setminus\{0\} \,: \\ f_k \neq 0,\, \widetilde{f}_k \neq 0 \text{ for } k \geq k_1,\, k \in K_{f,\widetilde{f}} \text{ and } \lim_{\substack{k\to\infty \\ k\in K_{f,\widetilde{f}}}} \lambda_k^m \frac{\widetilde{f}_k}{f_k} = c\Big\}.$$

In the following theorem, we will prove the uniqueness of the posed inverse problem in the case f and $\widetilde{f}$ are non-degenerate and have not a "similar" asymptotics in the sense of the definition of $\mathcal{P}_{\mathrm{sim}}$.

Theorem 3. *Let $\alpha \in (0,1) \cup (1,2)$, $(f,\widetilde{f}) \in (\mathcal{D}\setminus\mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$ and $g, \widetilde{g} \in L_p(0,T)$ for some $p > \frac{1}{\alpha-\lfloor\alpha\rfloor}$. If $\varphi_j = 0$, $j \in \{0; \lfloor\alpha\rfloor\}$, $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (T-\varepsilon, T)$ for some $\varepsilon \in (0,T)$ and* *the generalized solution of the direct problem with F of the form (19) satisfies $u(T) = 0$ then $g = 0$, $\widetilde{g} = 0$ and $u = 0$.*

Proof of this theorem uses the following two lemmas:

Lemma 2 ([22]). *Let $\alpha \in (0,1) \cup (1,2)$ and $g \in L_1(0, T-\varepsilon)$ for some $\varepsilon \in (0,T)$. Then, for any $N \in \mathbb{N}$,*

$$\begin{aligned} &\int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) g(\tau) d\tau \\ &= \sum_{n=1}^{N} \frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} \frac{1}{\lambda_k^{n+1}} + O\left(\frac{1}{\lambda_k^{N+2}}\right) \quad as \quad k \to \infty. \end{aligned} \tag{21}$$

Lemma 3 ([22]). *Let $\alpha \in (0,1) \cup (1,2)$, $g \in L_1(0, T-\varepsilon)$ for some $\varepsilon \in (0,T)$ and*

$$\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = 0, \; n \in \mathbb{N}.$$

Then, $g(t) = 0$ a.e. $t \in (0, T-\varepsilon)$.

Proof of Theorem 3. Let $\varphi_j = 0$, $j \in \{0; \lfloor \alpha \rfloor\}$, $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (T-\varepsilon, T)$ for some $\varepsilon \in (0,T)$ and the generalized solution of the direct problem with F of the form (19) satisfy $u(T) = 0$. We aim to show that $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (0, T-\varepsilon)$. Then due to the assumption $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (T-\varepsilon, T)$ we have $F = 0$ and Theorem 1 (i) implies $u = 0$; hence the assertion of the theorem follows.

Due to (17) and the imposed assumptions we have

$$\begin{aligned} &f_k \int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) g(\tau) d\tau \\ &+ \widetilde{f}_k \int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \widetilde{g}(\tau) d\tau = 0, \;\; k \in \mathbb{N}. \end{aligned} \tag{22}$$

The assumption $(f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\text{sim}}$ implies that for any $m \in \mathbb{Z}$ at least one of the following statements (23)–(25) is valid:

$$\left.\begin{aligned} &\text{there exists a subsequence } (k[m]_i)_{i\in\mathbb{N}} \subseteq K_{f,\widetilde{f}} \text{ such that} \\ &f_{k[m]_i} \neq 0,\, i \in \mathbb{N}, \text{ and } \lim_{i\to\infty} \lambda^m_{k[m]_i} \frac{\widetilde{f}_{k[m]_i}}{f_{k[m]_i}} = 0; \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned} &\text{there exist subsequences } (k[m]_{1,i})_{i\in\mathbb{N}}, (k[m]_{2,i})_{i\in\mathbb{N}} \subset K_{f,\widetilde{f}}, \\ &\text{and } c_1, c_2 \in \mathbb{C} \setminus \{0\},\, c_1 \neq c_2, \text{ such that} \\ &f_{k[m]_{j,i}} \neq 0,\, \widetilde{f}_{k[m]_{j,i}} \neq 0,\, i \in \mathbb{N},\, j = 1,2, \\ &\text{and } \lim_{i\to\infty} \lambda^m_{k[m]_{j,i}} \frac{\widetilde{f}_{k[m]_{j,i}}}{f_{k[m]_{j,i}}} = c_j, \quad j = 1,2; \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} &\text{there exists a subsequence } (\hat{k}[m]_i)_{i\in\mathbb{N}} \subseteq K_{f,\widetilde{f}} \text{ such that} \\ &\widetilde{f}_{\hat{k}[m]_i} \neq 0,\, i \in \mathbb{N}, \text{ and } \lim_{i\to\infty} \lambda^{-m}_{\hat{k}[m]_i} \frac{f_{\hat{k}[m]_i}}{\widetilde{f}_{\hat{k}[m]_i}} = 0. \end{aligned}\right\} \tag{25}$$

Since $\lim_{k\to\infty} \lambda_k = \infty$, we can draw the following conclusions. If (23) is valid for some $m = m_0$ then (23) holds for all $m \leq m_0 - 1$. If (25) is valid for some $m = m_0$ then (25) holds for all $m \geq m_0 + 1$. Consequently, at least one of the following four cases occurs:

1° (23) is valid for all $m \in \mathbb{Z}$;

2° there exists $m_0 \in \mathbb{Z}$ such that (24) is valid for $m = m_0$;

3° there exists $m_0 \in \mathbb{Z}$ such that (23) is valid for $m = m_0$ and (25) is valid for $m = m_0 + 1$;

4° (25) is valid for all $m \in \mathbb{Z}$.

Firstly, let us consider the case 2° and suppose that $m_0 \geq 0$. From (22), we have

$$\int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{k_{j,i}}(T-\tau)^\alpha) g(\tau) d\tau$$
$$+\lambda_{k_{j,i}}^{m_0} \frac{\widetilde{f}_{k_{j,i}}}{f_{k_{j,i}}} \frac{1}{\lambda_{k_{j,i}}^{m_0}} \int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{k_{j,i}}(T-\tau)^\alpha) \widetilde{g}(\tau) d\tau = 0, \ \ i \in \mathbb{N}, \ j = 1, 2,$$

where $k_{j,i} = k[m_0]_{j,i}$. Let us choose some $N \in \mathbb{N}$, $N \geq m_0 + 1$. Using Lemma 2 for the term $\int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{k_{j,i}}(T-\tau)^\alpha) g(\tau) d\tau$ and Lemma 2 with N replaced by $N - m_0$ for the term $\int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{k_{j,i}}(T-\tau)^\alpha) \widetilde{g}(\tau) d\tau$ and observing the relation $\lim_{i\to\infty} \lambda_{k_{j,i}} = \infty$ as well as the boundedness of the sequence $\left(\lambda_{k_{j,i}}^{m_0} \frac{\widetilde{f}_{k_{j,i}}}{f_{k_{j,i}}}\right)_{i\in\mathbb{N}}$, we deduce

$$\Theta_{m_0} \sum_{n=1}^{m_0} \underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau) d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n} \frac{1}{\lambda_{k_{j,i}}^{n+1}}$$
$$+ \sum_{n=m_0+1}^{N} \Big[\underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau) d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n}$$
$$+ \underbrace{\frac{(-1)^{n-m_0}}{\Gamma(-(n-m_0)\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau) d\tau}{(T-\tau)^{(n-m_0)\alpha+1}}}_{\widetilde{I}_{n-m_0}} \lambda_{k_{j,i}}^{m_0} \frac{\widetilde{f}_{k_{j,i}}}{f_{k_{j,i}}} \Big]$$
$$\times \frac{1}{\lambda_{k_{j,i}}^{n+1}} = O\left(\frac{1}{\lambda_{k_{j,i}}^{N+2}}\right) \quad \text{as} \quad i \to \infty, \ \ j = 1, 2, \tag{26}$$

where

$$\Theta_{m_0} = 1 \text{ if } m_0 \geq 1 \text{ and } \Theta_{m_0} = 0 \text{ if } m_0 = 0. \tag{27}$$

If any of the quantities I_n, $n = 1, \ldots, m_0$, differs from zero, then the left-hand side of (26) has lower order asymptotics than the right-hand side of (26) in the process $i \to \infty$. Consequently, $I_n = 0$, $n = 1, \ldots, m_0$, i.e.

$$\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau) d\tau}{(T-\tau)^{n\alpha+1}} = 0, \ \ n = 1, \ldots, m_0. \tag{28}$$

This means that the first sum at the left-hand side of (26) vanishes. Now we multiply (26) by $\lambda_{k_{j,i}}^{m_0+2}$ and send i to ∞. Taking the relations $\lim_{i\to\infty} \lambda_{k_{j,i}}^{m_0} \frac{\widetilde{f}_{k_{j,i}}}{f_{k_{j,i}}} = c_j$, $j = 1, 2$, into account, we reach the following system of equations:

$$I_{m_0+1} + c_j \widetilde{I}_1 = 0 \ \ j = 1, 2.$$

Since $c_1 \neq c_2$, we obtain $I_{m_0+1} = \widetilde{I}_1 = 0$. Thus,

$$\frac{1}{\Gamma(-(m_0+1)\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{(m_0+1)\alpha+1}} = 0, \quad \frac{1}{\Gamma(-\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{\alpha+1}} = 0.$$

Multiplying (26) next by $\lambda_{k_{j,i}}^{m_0+3}$ and sending i to ∞ we reach a system of equations that yields $I_{m_0+2} = \widetilde{I}_2 = 0$. Hence

$$\frac{1}{\Gamma(-(m_0+2)\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{(m_0+2)\alpha+1}} = 0, \quad \frac{1}{\Gamma(-2\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{2\alpha+1}} = 0.$$

Continuing this process, we deduce the relations

$$\begin{aligned} &\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = 0, \quad n = m_0+1, \ldots, N, \\ &\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = 0, \quad n = 1, \ldots, N-m_0. \end{aligned} \tag{29}$$

Since $N \geq m_0+1$ was chosen arbitrarily, (28) and (29) imply

$$\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = \frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = 0, \quad n \in \mathbb{N}. \tag{30}$$

Lemma 3 implies $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (0, T-\varepsilon)$.

The case 2°, $m_0 < 0$, reduces to the case 2° with m_0 replaced by $-m_0$ if we exchange the roles of f_k and $\widetilde{f}_k$ and rewrite the limit relations in (24) in the form $\lim\limits_{i\to\infty} \lambda_{k[m]_{j,i}}^{-m} \frac{f_{k[m]_{j,i}}}{\widetilde{f}_{k[m]_{j,i}}} = \frac{1}{c_j}$, $j = 1, 2$, with $m = m_0$.

Secondly, we consider the case 3°, $m_0 \geq 0$. Let us choose $N \in \mathbb{N}$, $N \geq m_0+2$ and denote $k_i = k[m_0]_i$, $\hat{k}_i = \hat{k}[m_0+1]_i$. Using the boundedness of the sequences $\left(\lambda_{k_i}^{m_0} \frac{\widetilde{f}_{k_i}}{f_{k_i}}\right)_{i\in\mathbb{N}}$ and $\left(\lambda_{\hat{k}_i}^{-m_0-1} \frac{f_{\hat{k}_i}}{\widetilde{f}_{\hat{k}_i}}\right)_{i\in\mathbb{N}}$ and applying arguments similar to the ones used in the previous case, we deduce the relations

$$\begin{aligned} &\Theta_{m_0} \sum_{n=1}^{m_0} \underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n} \frac{1}{\lambda_{k_i}^{n+1}} \\ &+ \sum_{n=m_0+1}^{N} \Bigg[\underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n} \\ &+ \underbrace{\frac{(-1)^{n-m_0}}{\Gamma(-(n-m_0)\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{(n-m_0)\alpha+1}}}_{\widetilde{I}_{n-m_0}} \lambda_{k_i}^{m_0} \frac{\widetilde{f}_{k_i}}{f_{k_i}} \Bigg] \\ &\times \frac{1}{\lambda_{k_i}^{n+1}} = O\left(\frac{1}{\lambda_{k_i}^{N+2}}\right) \quad \text{as} \quad i \to \infty \end{aligned} \tag{31}$$

and

$$\Theta_{m_0+1}\sum_{n=1}^{m_0+1}\underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)}\int_0^{T-\varepsilon}\frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n}\lambda_{\hat{k}_i}^{-m_0-1}\frac{f_{\hat{k}_i}}{\widetilde{f}_{\hat{k}_i}}\frac{1}{\lambda_{\hat{k}_i}^{n+1}}$$

$$+\sum_{n=m_0+2}^{N}\Big[\underbrace{\frac{(-1)^n}{\Gamma(-n\alpha)}\int_0^{T-\varepsilon}\frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}}}_{I_n}\lambda_{\hat{k}_i}^{-m_0-1}\frac{f_{\hat{k}_i}}{\widetilde{f}_{\hat{k}_i}}$$

$$+\underbrace{\frac{(-1)^{n-m_0-1}}{\Gamma(-(n-m_0-1)\alpha)}\int_0^{T-\varepsilon}\frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{(n-m_0-1)\alpha+1}}\Big]}_{\widetilde{I}_{n-m_0-1}}$$

$$\times\frac{1}{\lambda_{\hat{k}_i}^{n+1}}=O\left(\frac{1}{\lambda_{\hat{k}_i}^{N+2}}\right)\quad\text{as}\quad i\to\infty. \tag{32}$$

Like in the previous case, from (31) we deduce $I_n=0, n=1,\dots,m_0$, that implies (28). Thus, the first sum at the left-hand side of (31) vanishes. Now we multiply (31) by $\lambda_{k_i}^{m_0+2}$ and send i to ∞. Since $\lim\limits_{i\to\infty}\lambda_{k_i}^{m_0}\frac{\widetilde{f}_{k_i}}{f_{k_i}}=0$, we obtain $I_{m_0+1}=0$. Therefore,

$$\frac{1}{\Gamma(-(m_0+1)\alpha)}\int_0^{T-\varepsilon}\frac{g(\tau)d\tau}{(T-\tau)^{(m_0+1)\alpha+1}}=0.$$

Next, we turn to (32). Since $I_n=0, n=1,\dots,m_0+1$, the first sum at the left-hand side of (32) vanishes. We multiply (32) by $\lambda_{\hat{k}_i}^{m_0+3}$ and send i to ∞. Since $\lim\limits_{i\to\infty}\lambda_{\hat{k}_i}^{-m_0-1}\frac{f_{\hat{k}_i}}{\widetilde{f}_{\hat{k}_i}}=0$, we obtain $\widetilde{I}_1=0$ that implies

$$\frac{1}{\Gamma(-\alpha)}\int_0^{T-\varepsilon}\frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{\alpha+1}}=0.$$

Having performed these operations, we return to (31). Since $I_{m_0+1}=\widetilde{I}_1=0$, the first addend of the second sum at the left-hand side of (31) vanishes. Multiplying (31) by $\lambda_{k_i}^{m_0+3}$ and sending i to ∞ we obtain $I_{m_0+2}=0$ that implies

$$\frac{1}{\Gamma(-(m_0+2)\alpha)}\int_0^{T-\varepsilon}\frac{g(\tau)d\tau}{(T-\tau)^{(m_0+2)\alpha+1}}=0.$$

Then we turn again to (32). In view of $I_{m_0+2}=\widetilde{I}_1=0$, the first addend of the second sum at the left-hand side of (32) vanishes. Multiplying (32) by $\lambda_{\hat{k}_i}^{m_0+4}$ and sending i to ∞ we deduce $\widetilde{I}_2=0$. Therefore,

$$\frac{1}{\Gamma(-2\alpha)}\int_0^{T-\varepsilon}\frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{2\alpha+1}}=0.$$

Continuing this process, we obtain the family of relations

$$\begin{aligned}&\frac{1}{\Gamma(-n\alpha)}\int_0^{T-\varepsilon}\frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}}=0,\quad n=m_0+1,\dots,N,\\&\frac{1}{\Gamma(-n\alpha)}\int_0^{T-\varepsilon}\frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{n\alpha+1}}=0,\quad n=1,\dots,N-m_0-1.\end{aligned} \tag{33}$$

Since $N \geq m_0 + 2$ was chosen arbitrarily, (28) and (33) imply (30). Lemma 3 yields $g(t) = \widetilde{g}(t) = 0$ a.e. $t \in (0, T-\varepsilon)$.

The case 3°, $m_0 < 0$, reduces to the case 3° with m_0 replaced by $-m_0$ if we exchange the roles of f_k and $\widetilde{f}_k$. Then (25) with $m = m_0 + 1$ becomes (23) with $m = -m_0 - 1$ and (23) with $m = m_0$ becomes (25) with $m = -m_0$.

Thirdly, we consider the case 1°. Let $N \in \mathbb{N}$ and denote $k_i = k[N]_i$. From (22) due Lemma 2 we have

$$\begin{aligned}&\sum_{n=1}^{N} \frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} \frac{1}{\lambda_{k_i}^{n+1}} \\ &+ \sum_{n=1}^{N} \frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{n\alpha+1}} \lambda_{k_i}^N \frac{\widetilde{f}_{k_i}}{f_{k_i}} \frac{1}{\lambda_{k_i}^{n+N+1}} \\ &= O\left(\frac{1}{\lambda_{k_i}^{N+2}}\right) \quad \text{as} \quad i \to \infty. \end{aligned} \tag{34}$$

Since $\lim_{i\to\infty} \lambda_{k_i}^N \frac{\widetilde{f}_{k_i}}{f_{k_i}} = 0$, we deduce from (34)

$$\frac{1}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{g(\tau)d\tau}{(T-\tau)^{n\alpha+1}} = 0 \tag{35}$$

for $n = 1, \ldots, N$. Since $N \in \mathbb{N}$ was chosen arbitrarily, (35) is valid for all $n \in \mathbb{N}$. Lemma 3 yields $g(t) = 0$ a.e. $t \in (0, T-\varepsilon)$. Now (22) takes the form

$$\widetilde{f}_k \int_0^{T-\varepsilon} (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-\tau)^\alpha) \widetilde{g}(\tau) d\tau = 0, \quad k \in \mathbb{N}.$$

Since $\widetilde{f} \notin \mathcal{F}_0$, there exists a sequence $(\widetilde{k}_i)_{i\in\mathbb{N}}$ such that $\widetilde{f}_{\widetilde{k}_i} \neq 0$, $i \in \mathbb{N}$. By means of Lemma 2, we deduce

$$\sum_{n=1}^{N} \frac{(-1)^n}{\Gamma(-n\alpha)} \int_0^{T-\varepsilon} \frac{\widetilde{g}(\tau)d\tau}{(T-\tau)^{n\alpha+1}} \frac{1}{\lambda_{\widetilde{k}_i}^{n+1}} = O\left(\frac{1}{\lambda_{\widetilde{k}_i}^{N+2}}\right) \quad \text{as} \quad i \to \infty$$

for any $N \in \mathbb{N}$. Handling this relation similarly to (34) we reach $\widetilde{g}(t) = 0$ a.e. $t \in (0, T-\varepsilon)$.

Finally, case 4° reduces to case 1° if we exchange the roles of $\widetilde{f}_k$ and f_k there. □

Example 1. *Let $d = 1$, $\Omega = (0,1)$, $\varkappa = 1$ and*

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x < 1, \end{cases} \qquad \widetilde{f}(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x < 1. \end{cases}$$

Then $\lambda_k = (\pi k)^2$, $v_k(x) = \sqrt{2} \sin k\pi x$, $k \in \mathbb{N}$, and

$$f_k = \frac{\sqrt{2}}{k\pi}\left(1 - \cos\frac{k\pi}{2}\right), \quad \widetilde{f}_k = \frac{\sqrt{2}}{k\pi}\left(\cos\frac{k\pi}{2} - \cos k\pi\right), \quad k \in \mathbb{N}.$$

Note that

$$\left(1 - \cos\frac{k\pi}{2}\right)_{k\in\mathbb{N}} = (1, 2, 1, 0, \ldots), \quad \left(\cos\frac{k\pi}{2} - \cos k\pi\right)_{k\in\mathbb{N}} = (1, -2, 1, 0, \ldots).$$

Therefore, $f, \widetilde{f} \in \mathcal{D} \setminus \mathcal{F}_0$*,* $K_{f,\widetilde{f}} = \mathbb{N} \setminus \{4j : j \in \mathbb{N}\}$ *and* $f_k \neq 0$*,* $\widetilde{f}_k \neq 0$ *for any* $k \in K_{f,\widetilde{f}}$*. Clearly, the sequence* $\left(\lambda_k^m \frac{\widetilde{f}_k}{f_k}\right)_{k \in K_{f,\widetilde{f}}}$ *has not a limit in* $\mathbb{C} \setminus \{0\}$ *for any* $m \in \mathbb{Z}$*. Consequently,* $(f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{sim}$ *and assumptions of Theorem 3 are satisfied for the pair* $f, \widetilde{f}$*.*

5. Inverse Problem to Determine Location and Time History of a Point Source

In this section, we will study the problem of simultaneous reconstruction of the location and time history of a point source in the one-dimensional case.

Let us formulate this problem in a generalized setting. Let $\alpha \in (0,1) \cup (1,2)$, $d = 1$, $\Omega = (0,1)$ and

$$F(t) = g(t)\delta(\cdot - x_0),$$

where $g \in L_s(0,T)$ for some $s > 1$ and $x_0 \in (0,1)$. Note that $\delta(\cdot - x_0) \in \mathcal{D}^{-1}$. Suppose that g and x_0 are unknown. Further, assume that $\varphi_j \in \mathcal{D}$, $j \in \{0; \lfloor \alpha \rfloor\}$, are given, and the generalized solution of the direct problem satisfies the final condition (20), where ψ is a prescribed element of $\mathcal{D}$. The inverse problem consists of the determination of the triplet (g, x_0, u).

The next theorem provides uniqueness for this inverse problem.

Theorem 4. *Let* $\alpha \in (0,1) \cup (1,2)$*,* $d = 1$*,* $\Omega = (0,1)$*,* $x_0, \widetilde{x}_0 \in (0,1)$ *and* $g, \widetilde{g} \in L_p(0,T)$ *for some* $p > \frac{1}{\alpha - \lfloor \alpha \rfloor}$*. Let* $\varphi_j \in \mathcal{D}$*,* $j \in \{0; \lfloor \alpha \rfloor\}$*,* $g(t) = \widetilde{g}(t) = 0$ *a.e.* $t \in (T - \varepsilon, T)$ *for some* $\varepsilon \in (0,T)$*. Moreover, define* $F(t) = g(t)\delta(\cdot - x_0)$*,* $\widetilde{F}(t) = \widetilde{g}(t)\delta(\cdot - \widetilde{x}_0)$ *and denote the generalized solutions of the direct problems corresponding to the data vectors* $(F, \varphi_j|_{j \in \{0; \lfloor \alpha \rfloor\}})$ *and* $(\widetilde{F}, \varphi_j|_{j \in \{0; \lfloor \alpha \rfloor\}})$ *by* u *and* $\widetilde{u}$*, respectively. If* $u(T) = \widetilde{u}(T)$ *and* $g|_{(0,T-\varepsilon)} \neq 0$ *then* $g = \widetilde{g}$*,* $x_0 = \widetilde{x}_0$ *and* $u = \widetilde{u}$*.*

Proof. The spectral data of $L = -\varkappa\Delta$ with the domain $\mathcal{D}(L) = \{z \in W_2^2(0,1), z(0) = z(1) = 0\}$ are $\lambda_k = \varkappa(k\pi)^2$, $v_k(x) = \sqrt{2}\sin k\pi x$, $k \in \mathbb{N}$.

The function $U = u - \widetilde{u}$ is a generalized solution of the direct problem with zero initial conditions and the source term of the form

$$F(t) = g(t)f + (-\widetilde{g}(t))\widetilde{f},$$

where

$$f = \delta(\cdot - x_0), \quad \widetilde{f} = \delta(\cdot - \widetilde{x}_0). \tag{36}$$

Moreover, if $u(T) = \widetilde{u}(T)$ then $U(T) = 0$.

Evidently, $f, \widetilde{f} \in \mathcal{D} \setminus \mathcal{F}_0$. The basic idea of the proof consists in showing that the implication

$$x_0 \neq \widetilde{x}_0 \implies (f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}} \tag{37}$$

is true. Suppose that (37) is valid. Then if $x_0 \neq \widetilde{x}_0$, the assumptions of Theorem 3 are satisfied for $f, \widetilde{f}$. Theorem 3 yields $(g, -\widetilde{g}, U) = 0$. This contradicts to the assumption $g|_{(0,T-\varepsilon)} \neq 0$. Thus, $x_0 = \widetilde{x}_0$, and F reduces to the form $F(t) = (g(t) - \widetilde{g}(t))\delta(\cdot - x_0)$. Now Theorem 2 implies $g - \widetilde{g} = 0$. Finally, Theorem 1 (i) yields $U = u - \widetilde{u} = 0$.

So, it remains to prove (37). Firstly, we consider the case $x_0 = \frac{1}{2}$ and $\widetilde{x}_0 \in [(0,1) \cap \mathbb{Q}] \setminus \{\frac{1}{2}\}$. We express $\widetilde{x}_0$ as $\widetilde{x}_0 = \frac{r}{s}$ where $r, s \in \mathbb{N}$ and consider the subsequences f_k, $\widetilde{f}_k$, $k = 2(\bar{k}s + 1)$, where $\bar{k} \in \mathbb{N}$. Noting that

$$f_k = \sqrt{2}\sin k\pi x_0, \quad \widetilde{f}_k = \sqrt{2}\sin k\pi\widetilde{x}_0,$$

we have $f_k = \sqrt{2}\sin(\bar{k}s+1)\pi = 0$ but $\widetilde{f}_k = \sqrt{2}\sin(2\bar{k}r\pi + \frac{2r}{s}\pi) = \sqrt{2}\sin\frac{2r}{s}\pi \neq 0$ because $\frac{r}{s} \in (0,1) \setminus \{\frac{1}{2}\}$. Therefore, there exist arbitrarily large values of k such that $f_k = 0$ but $\widetilde{f}_k \neq 0$. This is in contradiction with the requirement $\exists k_1 \in \mathbb{N},:\ f_k \neq 0,\ \widetilde{f}_k \neq 0$ for $k \geq k_1$, $k_1 \in K_{f,\widetilde{f}}$, in the definition of $\mathcal{P}_{\rm sim}$. Thus, $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\rm sim}$.

Similarly, we reach the relation $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\rm sim}$ in case $\widetilde{x}_0 = \frac{1}{2}$ and $x_0 \in [(0,1) \cap \mathbb{Q}] \setminus \{\frac{1}{2}\}$.

Next, let us consider the case $x_0 \in (0,1) \cap \mathbb{Q}$ and $\widetilde{x}_0 \in (0,1) \cap \mathbb{I}$. Then $x_0 = \frac{p}{q}$ where $p, q \in \mathbb{N}$. For the subsequences f_k, $\widetilde{f}_k$, $k = \bar{k}q$, where $\bar{k} \in \mathbb{N}$, it holds $f_k = 0$ but $\widetilde{f}_k \neq 0$. As before, this leads to the conclusion that $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\rm sim}$.

Similarly, we obtain $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\rm sim}$ in case $\widetilde{x}_0 \in (0,1) \cap \mathbb{Q}$ and $x_0 \in (0,1) \cap \mathbb{I}$.

Now let us deal with the case $x_0, \widetilde{x}_0 \in [(0,1) \cap \mathbb{Q}] \setminus \{\frac{1}{2}\}$, $x_0 \neq \widetilde{x}_0$. We express $x_0 = \frac{p}{q}$ and $\widetilde{x}_0 = \frac{r}{s}$ where $p, q, r, s \in \mathbb{N}$. For the subsequences f_k, $\widetilde{f}_k$, $k = 2\bar{k}qs + 1$, where $\bar{k} \in \mathbb{N}$, we have $f_k = \sqrt{2}\sin(2\bar{k}ps\pi + \frac{p}{q}\pi) = \sqrt{2}\sin x_0\pi \neq 0$ and $\widetilde{f}_k = \sqrt{2}\sin(2\bar{k}rq\pi + \frac{r}{s}\pi) = \sqrt{2}\sin\widetilde{x}_0\pi \neq 0$. Therefore, $\frac{\widetilde{f}_k}{f_k} = c_1$ for $k = 2\bar{k}qs + 1$, $\bar{k} \in \mathbb{N}$, where $c_1 = \frac{\sin\widetilde{x}_0\pi}{\sin x_0\pi}$. Moreover, for the subsequences f_k, $\widetilde{f}_k$, $k = 2\bar{k}qs + 2$, where $\bar{k} \in \mathbb{N}$, we have $f_k = \sqrt{2}\sin(2\bar{k}ps\pi + 2\frac{p}{q}\pi) = \sqrt{2}\sin 2x_0\pi \neq 0$ and $\widetilde{f}_k = \sqrt{2}\sin(2\bar{k}rq\pi + 2\frac{r}{s}\pi) = \sqrt{2}\sin 2\widetilde{x}_0\pi \neq 0$. Therefore, $\frac{\widetilde{f}_k}{f_k} = c_2$ for $k = 2\bar{k}qs + 2$, $\bar{k} \in \mathbb{N}$, where $c_2 = \frac{\sin 2\widetilde{x}_0\pi}{\sin 2x_0\pi} = c_1\frac{\cos\widetilde{x}_0\pi}{\cos x_0\pi} \neq c_1$. This shows that the requirement $\exists m \in \mathbb{Z},\ c \in \mathbb{C} \setminus \{0\} :\ \lim_{\substack{k\to\infty \\ k\in K_{f,\widetilde{f}}}} \lambda_k^m \frac{\widetilde{f}_k}{f_k} = c$ in $\mathcal{P}_{\rm sim}$ is not satisfied. Consequently, $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\rm sim}$.

Finally, we consider the case $x_0, \widetilde{x}_0 \in (0,1) \cap \mathbb{I}$, $x_0 \neq \widetilde{x}_0$. Then $f_k \neq 0$, $\widetilde{f}_k \neq 0$ for all $k \in \mathbb{N}$, hence $K_{f,\widetilde{f}} = \mathbb{N}$. Suppose that $(f,\widetilde{f}) \in \mathcal{P}_{\rm sim}$. Then $\exists m \in \mathbb{Z},\ c \in \mathbb{C} \setminus \{0\} :\ \lim_{k\to\infty} \lambda_k^m \frac{\widetilde{f}_k}{f_k} = c$. Firstly, let $m > 0$. Then in view of $|f_k| = |\sqrt{2}\sin kx_0\pi| \leq \sqrt{2}$ and $\lim_{k\to\infty} \lambda_k = \infty$ it holds $\lim_{k\to\infty} \widetilde{f}_k = 0$. Let us choose $\epsilon > 0$ such that $\arcsin\epsilon < \frac{\pi}{2}\min\{\widetilde{x}_0; 1 - \widetilde{x}_0\}$. There exists k_ϵ such that for $k > k_\epsilon$ it holds $|\widetilde{f}_k| = |\sqrt{2}\sin k\widetilde{x}_0\pi| < \sqrt{2}\epsilon$. From the latter inequality, we deduce that for any $k > k_\epsilon$ there exists $n_k \in \mathbb{Z}$ such that

$$-\arcsin\epsilon < k\widetilde{x}_0\pi + n_k\pi < \arcsin\epsilon. \tag{38}$$

This yields also

$$-\arcsin\epsilon < -k\widetilde{x}_0\pi - n_k\pi < \arcsin\epsilon. \tag{39}$$

Applying (38) with $k+1$ instead of k we obtain

$$-\arcsin\epsilon < (k+1)\widetilde{x}_0\pi + n_{k+1}\pi < \arcsin\epsilon. \tag{40}$$

Adding (39) and (40), we have

$$-2\arcsin\epsilon < \widetilde{x}_0\pi + n\pi < 2\arcsin\epsilon \tag{41}$$

with some $n \in \mathbb{Z}$. On the other hand, the inequality $\arcsin\epsilon < \frac{\pi}{2}\min\{\widetilde{x}_0; 1 - \widetilde{x}_0\}$ implies $(\widetilde{x}_0 - 1)\pi < -2\arcsin\epsilon$ and $2\arcsin\epsilon < \widetilde{x}_0\pi$. Thus, from (41) we deduce $(\widetilde{x}_0 - 1)\pi < \widetilde{x}_0\pi + n\pi < \widetilde{x}_0\pi$. This implies the relation $-1 < n < 0$ that contradicts to $n \in \mathbb{Z}$.

Consequently, the supposition $(f,\widetilde{f}) \in \mathcal{P}_{\mathrm{sim}}$ was wrong and we have $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$. The case $m < 0$ reduces to the case $m > 0$ if we interchange the roles of f_k and $\widetilde{f}_k$. It remains to consider the case $m = 0$. Then $\lim\limits_{k\to\infty} \frac{\widetilde{f}_k}{f_k} = \lim\limits_{k\to\infty} \frac{\sin k\widetilde{x}_0\pi}{\sin kx_0\pi} = c \neq 0$ yields $c = \lim\limits_{k\to\infty} \frac{\sin 2k\widetilde{x}_0\pi}{\sin 2kx_0\pi} = c \lim\limits_{k\to\infty} \frac{\cos k\widetilde{x}_0\pi}{\cos kx_0\pi}$. Thus, $\lim\limits_{k\to\infty} \frac{\cos k\widetilde{x}_0\pi}{\cos kx_0\pi} = 1$. Note that we can exclude the case $x_0 + \widetilde{x}_0 = 1$. Indeed, if $x_0 + \widetilde{x}_0 = 1$ then the quotient $\frac{\cos k\widetilde{x}_0\pi}{\cos kx_0\pi}$ equals $\cos k\pi$ and its limit is not 1. Thus, in the sequel, we assume $x_0 + \widetilde{x}_0 \neq 1$. We have

$$\frac{\cos k\widetilde{x}_0\pi}{\cos kx_0\pi} = 1 + \eta_k, \tag{42}$$

where $\lim\limits_{k\to\infty} \eta_k = 0$. The relation (42) implies

$$k\widetilde{x}_0\pi \in \{2n\pi \pm \arccos((1+\eta_k)\cos kx_0\pi) \,:\, n \in \mathbb{Z}\}. \tag{43}$$

Due to the mean value theorem,

$$\arccos((1+\eta_k)\cos kx_0\pi) = \arccos(\cos kx_0\pi) + \epsilon_k, \tag{44}$$

where $\epsilon_k = -\frac{1}{\sqrt{1-\xi_k^2}}\eta_k \cos kx_0\pi$ and ξ_k is between $\cos kx_0\pi$ and $(1+\eta_k)\cos kx_0\pi$. The relation $\lim\limits_{k\to\infty} \eta_k = 0$ implies $\lim\limits_{k\to\infty} \epsilon_k = 0$. Since $\arccos(\cos kx_0\pi) \in \{2n\pi \pm kx_0\pi \,:\, n \in \mathbb{Z}\}$, from (43) and (44) we obtain

$$k\widetilde{x}_0\pi \in \{2n\pi \pm kx_0\pi + \varepsilon_k \,:\, n \in \mathbb{Z}\},$$

where $\varepsilon_k \in \{\pm\epsilon_k\}$. Therefore, for any $k \in \mathbb{N}$, there exist $n_k \in \mathbb{Z}$ and $\theta_k \in \{-1;1\}$ such that

$$k\frac{\widetilde{x}_0 - \theta_k x_0}{2} = n_k + \frac{\varepsilon_k}{2\pi}. \tag{45}$$

Due to the relations $x_0, \widetilde{x}_0 \in (0,1)$, $x_0 \neq \widetilde{x}_0$ and $x_0 + \widetilde{x}_0 \neq 1$ we have

$$\widetilde{x}_0 - \theta x_0 \notin \mathbb{Z}, \quad \theta \in \{-1;1\}. \tag{46}$$

Since $\lim\limits_{k\to\infty} \varepsilon_k = 0$, due to (46), there exists $K \in \mathbb{N}$ such that for any $k > K$ it holds

$$|\varepsilon_k| < \frac{\pi}{2} \min_{\substack{\theta\in\{-1;1\}\\ m\in\mathbb{Z}}} |m - (\widetilde{x}_0 - \theta x_0)|. \tag{47}$$

We also note that due to $\theta_k \in \{-1;1\}$, we can find an integer $k_* > K$ such that either 1° $\theta_{k_*+1} = \theta_{k_*}$ or 2° $\theta_{k_*+2} = \theta_{k_*}$. In case 1° from (45) we obtain

$$k_*\frac{\widetilde{x}_0 - \theta_{k_*} x_0}{2} = n_{k_*} + \frac{\varepsilon_{k_*}}{2\pi} \quad \text{and} \quad (k_*+1)\frac{\widetilde{x}_0 - \theta_{k_*} x_0}{2} = n_{k_*+1} + \frac{\varepsilon_{k_*+1}}{2\pi}.$$

This implies $\widetilde{x}_0 - \theta_{k_*} x_0 - \frac{\varepsilon_{k_*+1} - \varepsilon_{k_*}}{\pi} = 2n$ for some $n \in \mathbb{Z}$. In view of (46) and (47), the left-hand side of this equality does not belong to $\mathbb{Z}$. This is a contradiction. Thus, we have $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$. In case 2° from (45) we deduce

$$k_*\frac{\widetilde{x}_0 - \theta_{k_*} x_0}{2} = n_{k_*} + \frac{\varepsilon_{k_*}}{2\pi} \quad \text{and} \quad (k_*+2)\frac{\widetilde{x}_0 - \theta_{k_*} x_0}{2} = n_{k_*+2} + \frac{\varepsilon_{k_*+2}}{2\pi}.$$

This yields $\widetilde{x}_0 - \theta_{k_*} x_0 - \frac{\varepsilon_{k_*+2} - \varepsilon_{k_*}}{2\pi} = n$ for some $n \in \mathbb{Z}$. Again, by (46) and (47), the left-hand side of this equality is not an element of $\mathbb{Z}$. We reached a contradiction. Therefore, $(f,\widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$. $\square$

6. Conclusions

We have proved the uniqueness for two inverse problems for FDWE with final overdetermination. In the first problem, the aim is to reconstruct two time-dependent functions g and $\widetilde{g}$ in a source term of the form $F(t) = g(t)f + \widetilde{g}(t)\widetilde{f}$, where f and $\widetilde{f}$ are given distributions in a space domain Ω. The uniqueness holds provided f and $\widetilde{f}$ satisfy the condition $(f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$. In the second inverse problem, the time factor g and point x_0 in the source function of the form $F(t) = g(t)\delta(\cdot - x_0)$ has to be found. The uniqueness for this problem in the one-dimensional case follows from the uniqueness of the first problem.

The generalization of the uniqueness result for the second problem to the multi-dimensional case is an open question. If one could prove that the implication

$$x_0, \widetilde{x}_0 \in \Omega, \quad x_0 \neq \widetilde{x}_0 \quad \Longrightarrow \quad (f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}} \tag{48}$$

is valid for $f = \delta(\cdot - x_0)$ and $\widetilde{f} = \delta(\cdot - \widetilde{x}_0)$ then the desired uniqueness follows by means of arguments similar to the proof of Theorem 4.

The relation $(f, \widetilde{f}) \in (\mathcal{D} \setminus \mathcal{F}_0)^2 \setminus \mathcal{P}_{\mathrm{sim}}$ in case $f = \delta(\cdot - x_0)$, $\widetilde{f} = \delta(\cdot - \widetilde{x}_0)$ means that the sequences $(v_k(x_0))_{k\in\mathbb{N}}$ and $(v_k(\widetilde{x}_0))_{k\in\mathbb{N}}$ have nonzero elements for arbitrarily large k, and either there exists a subsequence $(k_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $v_{k_l}(x_0) \neq 0$, $v_{k_l}(\widetilde{x}_0) = 0$ or there exists a subsequence $(p_l)_{l\in\mathbb{N}} \subset \mathbb{N}$ such that $v_{p_l}(x_0) = 0$, $v_{p_l}(\widetilde{x}_0) \neq 0$ or the sequence $\left(\lambda_k^m \frac{v_k(x_0)}{v_k(\widetilde{x}_0)}\right)_{\{k\in\mathbb{N}:\, |v_k(x_0)|+|v_k(\widetilde{x}_0)|\neq 0\}}$ has not a limit in $\mathbb{C} \setminus \{0\}$ for any $m \in \mathbb{Z}$.

Another open question is the possibility of removal of the requirement that the unknown time-dependent functions are a priori known in an interval $(T - \varepsilon, T)$ for some $\epsilon > 0$. Without such a condition, our theory does not work. In case $\varepsilon = 0$ Lemma 2 fails, because the integrals at the right-hand side of (21) become singular.

Funding: This research was funded by Estonian Research Council grant number PRG832.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Berkowitz, B.; Klafter, J.; Metzler, R.; Scher, H. Physical pictures of transport in heterogeneous media: advection- dispersion, random-walk, and fractional derivative formulations. *Water Resour. Res.* **2002**, *38*, 1–12. [CrossRef]
2. Fang, C.; Cao, J.Z.; Sun, L.F. Exact solution of fractional diffusion model with source term used in the study of the concentration of fission product in uranium dioxide particle. *Commun. Theor. Phys.* **2011**, *55*, 863–867. [CrossRef]
3. Magin, R.L. Fractional calculus models of complex dynamics in biological tissues. *Comput. Math. Appl.* **2010**, *59*, 1586–1593. [CrossRef]
4. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
5. Mohammadzadeh, A.; Castillo, O.; Band, S.S.; Mosavi, A. A novel fractional-order multiple-model type-3 fuzzy control for nonlinear systems with unmodeled dynamics. *Int. J. Fuzzy Syst.* **2021**, *23*, 1633–1651. [CrossRef]
6. Isakov, V. *Inverse Problems for Partial Differential Equations*; Springer: New York, NY, USA, 2006.
7. Lorenzi, A. *An Introduction to Identification Problems via Functional Analysis*; VSP: Utrecht, The Netherlands, 2001.
8. Janno, J.; von Wolfersdorf, L. Identification of weakly singular memory kernels in viscoelasticity. *Z. Angew. Math. Mech.* **1998**, *78*, 391–403. [CrossRef]
9. Jin, B.; Rundell, W. A tutorial on inverse problems for anomalous diffusion processes. *Inverse Probl.* **2015**, *31*, 035003. [CrossRef]
10. Ikehata, M.; Kian, Y. The enclosure method for the detection of variable order in fractional diffusion equations. *Inverse Probl. Imag.* **2023**, *17*, 180–202. [CrossRef]
11. Blåsten, E.; Päivärinta, L.; Sadique, S. Unique determination of the shape of a scattering screen from a passive measurement. *Mathematics* **2020**, *8*, 1156. [CrossRef]
12. Kernevez, J.P. *The Sentinel Method and Its Application to Environmental Pollution Problems*; CRC Press: New York, NY, USA, 1997.
13. Sun, C.; Liu, Q.; Li, G. Conditional well-posedness for an inverse source problem in the dffusion equation using the variational adjoint method. *Adv. Math. Phys.* **2017**, *2017*, 6801260. [CrossRef]
14. Ali, M.; Aziz, S.; Malik, S.A. Inverse problem for a space-time fractional diffusion equation: Application of fractional Sturm-Liouville operator. *Math. Meth. Appl. Sci.* **2018**, *41*, 2733–2747. [CrossRef]

15. Janno, J.; Kasemets, K. Uniqueness for an inverse problem for a semilinear time fractional diffusion equation. *Inverse Probl. Imag.* **2017**, *11*, 125–149. [CrossRef]
16. Kinash, N.; Janno, J. Inverse problems for a generalized subdiffusion equation with final overdetermination. *Math. Model. Anal.* **2019**, *24*, 236–262.
17. Kirane, M.; Samet, B.; Torebek, B. Determination of an unknown source term temperature distribution for the sub-diffusion equation at the initial and final data. *Electron. J. Differ. Eq.* **2017**, *2017*, 257.
18. Orlovsky, D. Parameter determination in a differential equation of fractional order with Riemann-Liouville fractional derivative in a Hilbert space. *J. Siberian Fed. Univ. Math. Phys.* **2015**, *8*, 55–63. [CrossRef]
19. Sakamoto, K.; Yamamoto, M. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **2011**, *382*, 426–447. [CrossRef]
20. Yamamoto, M.; Zhang, Y. Conditional stability in determining a zeroth-order coefficient in a half-order fractional diffusion equation by a Carleman estimate. *Inverse Probl.* **2012**, *28*, 105010. [CrossRef]
21. Janno, J. Determination of time-dependent sources and parameters of nonlocal diffusion and wave equations from final data. *Fract. Calc. Appl. Anal.* **2020**, *23*, 1678–1701. [CrossRef]
22. Janno, J. Inverse problems with unknown boundary conditions and final overdetermination for time fractional diffusion-wave equations in cylindrical domains. *Mathematics* **2021**, *9*, 2541. https:// doi.org/10.3390/math9202541. [CrossRef]
23. Pskhu, A.V. Green functions of the first boundary-value problem for a fractional diffusion - wave equation in multidimensional domains. *Mathematics* **2020**, *8*, 464. [CrossRef]
24. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
25. Chechkin, A.; Gorenflo, R.; Sokolov, I. Fractional diffusion in inhomogeneous media. *J. Phys. A Math. Gen.* **2005**, *38*, 679–684. [CrossRef]
26. Frömberg, D. Reaction Kinetics under Anomalous Diffusion. Ph.D. Thesis, Humboldt-Universität zu Berlin, Berlin, Germany, 2011. Available online: https://edoc.hu-berlin.de/bitstream/handle/18452/17026/froemberg.pdf (accessed on 23 June 2011).
27. Povstenko, Y. Thermoelasticity that uses fractional heat conduction equation. *J. Math. Sci.* **2009**, *162*, 296–305. [CrossRef]
28. Zacher, R. Quasilinear Parabolic Problems with Nonlinear Boundary Conditions. Ph.D. Thesis, Martin-Luther-Universität Halle-Wittenberg, Halle, Germany, 2003. Available online: https://www.yumpu.com/en/document/view/4926858/quasilinear-parabolic-problems-with-nonlinear-boundary-conditions (accessed on 2 December 2012).