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Bohr's Phenomenon for the Solution of Second-Order Differential Equations

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Abstract: The aim of this work is to establish a connection between Bohr's radius and the analytic and normalized solutions of two differential second-order differential equations, namely $y''(z) + a(z)y'(z) + b(z)y(z) = 0$ and $z^2y''(z) + a(z)y'(z) + b(z)y(z) = d(z)$. Using differential subordination, we find the upper bound of the Bohr and Rogosinski radii of the normalized solution $F(z)$ of the above differential equations. We construct several examples by judicious choice of $a(z)$, $b(z)$ and $d(z)$. The examples include several special functions like Airy functions, classical and generalized Bessel functions, error functions, confluent hypergeometric functions and associate Laguerre polynomials.

Keywords: Bohr's phenomenon; second-order differential equation; subordination; Bessel functions; Airy functions; error function; confluent hypergeometric functions

MSC: 30C10; 30C45; 30C62; 34A05

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1. Introduction

The aim of this work is to establish a connection between various special functions with one of the classical results known as Bohr's theorem for the class $\mathcal{A}$ of analytic functions of the form $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$ defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. The connection is based on subordination, an important concept in geometric functions theory. Before starting the main results, let us recall some basic information on geometric functions theory and Bohr's phenomena.

1.1. Basic Requirements from Geometric Functions Theory

The class of functions f in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized by the constraints $f(0) = 0 = f'(0) - 1$, shall be denoted by the symbol $\mathcal{A}_0$. We also require the class $\mathcal{A}_1$, which consists of functions normalized by $f(0) = 1$.

If f and g are analytic in $\mathbb{D}$, then f is subordinate [1] to g, written $f \prec g$, or $f(z) \prec g(z)$, $z \in \mathbb{D}$, if there is an analytic self-map w on $\mathbb{D}$ satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathbb{D}$. In particular, $f(D) \subset g(D)$ if g is univalent and $g(0) = f(0)$.

One of the important subclasses of $\mathcal{A}$ consisting of univalent starlike functions is denoted by $\mathcal{S}^*$. Related to this subclass is the Cárathèodory class $\mathcal{P}$ consisting of analytic functions p satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ in $\mathbb{D}$. Analytically, $f \in \mathcal{S}^*$ if $zf'(z)/f(z) \in \mathcal{P}$.

A function $f \in \mathcal{A}_0$ is lemniscate starlike if $zf'(z)/f(z) \prec \sqrt{1+z}$. On the other hand, the function $f \in \mathcal{A}_0$ is lemniscate Carathéodory if $f'(z)) \prec \sqrt{1+z}$. Clearly, a lemniscate Carathéodory function is a Carathéodory function and hence is univalent.

1.2. On Bohr's Phenomena of Analytic Functions

Bohr's result states that:

Theorem 1 ([2]). *If $f(z) \in \mathcal{A}$ satisfies $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, then $\sum_{n=0}^{\infty} |a_n||z^n| \leq 1$ for $|z| \leq 1/3$, and the constant $1/3$ cannot be improved.*

In the initial result by Bohr, the constant was $1/6$, which later was improved independently by M. Riesz, I. Schur and F. Wiener. One can find Bohr and Wiener's proof in [2], and other proof can be found in [3,4]. Recently, an easy proof of Theorem 1 was established in [5]. The constant $1/3$ in Theorem 1 is called the Bohr radius for analytic bounded functions in $\mathbb{D}$.

Another concept that is closely linked to the Bohr radius is the Rogosinski radius which can be found in the following result of Rogosinski [6]:

Theorem 2 ([6]). *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}$ and $|f(z)| \le 1$ for all $|z| \le r$, then for every $k = \{0, 1, 2, \ldots\}$ and $0 < r \le 1$, each section $s_k(f) : s_k(z; f) = \sum_{n=0}^{k} a_n z^n$ of f satisfies the inequality*

$$|s_k(f)| \le 1$$

for $|z| < r/2$. The constant $r/2$ cannot be improved.

The constant $r/2$ in Theorem 2 is called the Rogosinski radius.

For fixed $|z| = r$ and $f \in \mathcal{A}$, the Bohr operator on f is defined as

$$\mathcal{M}_r(f) := \sum_{n=0}^{\infty} |a_n||z^n| = \sum_{n=0}^{\infty} |a_n| r^n, \quad 0 \le r \le 1/3. \tag{1}$$

Clearly, if f is a polynomial of degree k, namely,

$$P_k(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_k z^k,$$

then

$$\mathcal{M}_r(P_k) = |a_0| + |a_1| r + |a_2| r^2 + \ldots + |a_k| r^k.$$

Theorem 3 ([7,8]). *If $f \prec h$ in $\mathbb{D}$, then $\mathcal{M}_r(f) \le \mathcal{M}_r(h)$ for $0 \le r \le 1/3$.*

Theorem 3 is the main result in [8], while it is also stated and proved in [7] as a lemma and applied to prove other results. Theorem 3 is refined in [9] as follows:

Theorem 4 ([9]). *Suppose that $f \prec h$ in $\mathbb{D}$ and*

$$r_1(x) = \begin{cases} \sqrt{\frac{1-x}{2}} & \text{for} \quad x \in [0, \frac{1}{2}) \\ \frac{1}{1+2x} & \text{for} \quad x \in [\frac{1}{2}, 1). \end{cases} \tag{2}$$

Then, we have

1. $\mathcal{M}_r(f) \le \mathcal{M}_r(h)$ *for* $r \le r_1(|f'(0)/h'(0)|)$, *when* $h'(0) \neq 0$.
2. $\mathcal{M}_r(f) \le \mathcal{M}_r(h)$ *for* $r \le 1/3$, *when* $h'(0) = 0$.

Moreover, $r_1(|f'(0)/h'(0)|)$ cannot be improved if $|f'(0)/h'(0)| \in [1/2, 1) \cup \{0\}$, and the constant $1/3$ in (b) cannot be improved.

Note here that $r_1(x) \ge 1/3$ for $x \in [0, 1]$.

1.3. On Subordination by $\sqrt{1+z}$ and $1 + z - z^3/3$

The following three functions are important for this study.

$$\mathcal{P}_{\mathcal{L}}(z) = \sqrt{1+z}, \qquad \phi_A(z) = 1 + Az \quad \text{and} \quad \phi_{N_e}(z) = 1 + z - \frac{z^3}{3}.$$

The function $\mathcal{P}_{\mathcal{L}}$ maps $\mathbb{D}$ to a leminscate, ϕ_A shifts $\mathbb{D}$ to a disc center at $(1, 0)$ with radius $A \in [0, 1)$ and ϕ_{N_e} maps $\mathbb{D}$ to the neuphroid domain.

Lemma 1 ([10]). *Let $p \in \mathcal{H}[1,n]$ with $p(z) \not\equiv 1$ and $n \geq 1$. Let $\Omega \subset \mathbb{C}$, and $\Psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy*

$$\Psi(r,s,t;z) \notin \Omega$$

whenever $z \in \mathbb{D}$,. For $m \geq n \geq 1$, $-\pi/4 \leq \theta \leq \pi/4$,

$$r = \sqrt{2\cos(2\theta)}e^{i\theta}, \quad s = \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}} \quad \text{and} \quad \operatorname{Re}\Big((t+s)e^{-3i\theta}\Big) \geq \frac{3m^2}{8\sqrt{2\cos(2\theta)}}. \tag{3}$$

If $\Psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $p(z) \prec \mathcal{P}_{\mathcal{L}}(z)$ in $\mathbb{D}$.

In the case of two dimensions, if $\Psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ satisfies $\Psi(r,s;z) \notin \Omega$ whenever $z \in \mathbb{D}$, and for $m \geq n \geq 1$, $-\pi/4 \leq \theta \leq \pi/4$,

$$r = \sqrt{2\cos(2\theta)}e^{i\theta}, \quad s = \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}}.$$

If $\Psi(p(z), zp'(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $p(z) \prec \mathcal{P}_{\mathcal{L}}(z)$ in $\mathbb{D}$.
We need the following results in sequence.

Lemma 2 ([11]). *Let $p : \mathbb{D} \to \mathbb{C}$ be analytic such that $p(0) = 1$. Then, the following subordination implies $p(z) \prec \phi_{N_e}(z)$:*

(i) $1 + \beta zp'(z) \prec \mathcal{P}_{\mathcal{L}}(z)$ *for* $\beta \geq \beta_1 := 3(1 - \log(2) \approx 0.920558$;

(ii) $1 + \beta\frac{zp'(z)}{p(z)} \prec \mathcal{P}_{\mathcal{L}}(z)$ *for* $\beta \geq \beta_2 := \frac{2\left(\sqrt{2}+\log(2)-1-\log(1+\sqrt{2}\right)}{\log(5/3)} \approx 0.884792$;

(iii) $1 + \beta\frac{zp'(z)}{(p(z))^2} \prec \mathcal{P}_{\mathcal{L}}(z)$ *for* $\beta \geq \beta_3 := 5\left(\sqrt{2} + \log(2) - 1 - \log(1+\sqrt{2})\right) \approx 1.12994$.

Note here that $\mathcal{M}_r(\phi_{N_e}) = 1 + r + r^3/3$ and we denote it as $\Phi_{N_e}(r)$ for further use in the next section.

1.4. Problems and Arrangement of This Article

Two subsections constitute Section 2, containing the major findings. The effects of Bohr's operator on leminiscate and nephroid domain functions are discussed in Section 2.1. We aim to find the solution of the following problem in Section 2.1.

Problem 1. *If a function* `f` *maps $\mathbb{D}$ inside the leminiscate $\sqrt{1+z}$ or $\mathtt{f}(z) \prec \sqrt{1+z}$, then find the upper bound of $\mathcal{M}_r(.)$ when Bohr's operator applies on any function generated by* `f`*, namely $\mathtt{f}^2$, or a specific integration of* `f`*.*

For our next problem, we consider a differential equation DE_1 as follows

$$z^2y''(z) + a(z)y'(z) + b(z)y(z) = d(z), \tag{4}$$

where $a(z)$, $b(z)$ and $d(z)$ are analytic functions for which (4) has a solution, say $F(z)$, with normalization $F(0) = 1$. The existence of a solution to (4) is a completely separate problem, but we demonstrate through examples that there is a solution $F(z)$ with normalization $F(0) = 1$ for some judicious choice of $a(z)$, $b(z)$ and $d(z)$.

We also continue the work of [12] in the context of Bohr's operator. In [12], the lemniscate starlikeness of the solution of the differential equations

$$DE_2 : y''(z) + a(z)y'(z) + b(z)y(z) = 0; \tag{5}$$

and

$$DE_3 : z^2y''(z) + a(z)zy'(z) + b(z)y(z) = 0, \tag{6}$$

has been studied. Note here that DE_3 is a special case of DE_1 with $d(z) = 0$.

We aim to find the solution of the below problem.

Problem 2. *What are the effects of Bohr's operator on the solution of a second-order differential equation? In particular, what are the conditions on $a(z)$, $b(z)$ and $d(z)$ for which we can implement the findings of Problem 1 on the solution of DE_1, DE_2 and DE_3?*

The effects of Bohr's operator on the solution of a second-order differential equation as stated in Problem 2 are covered in Section 2.2. We construct a number of examples that incorporate different special functions in Section 3.

2. Main Results

2.1. Bohr's Operator on Functions Associated with the Leminiscate and Nephroid Domain

Theorem 5. *Suppose that $f \in \mathcal{A}_1$ such that $f(0) = 1$ and $f(z) \prec \sqrt{1+z}$. Define*

$$g(z) := 1 + \frac{1}{\beta}\int_0^z \frac{f(t)-1}{t}dt, \tag{7}$$

and assume that the integration on the right-hand side is convergent. Then, the following inequalities are true

(a) $\mathcal{M}_r(f^2(z)) \le 1 + r$;

(b) $\mathcal{M}_r(g(z)) \le \Phi_{N_e}(r)$ *for* $\beta \ge \beta_1$;

(c) $\mathcal{M}_r(e^{g(z)-1}) \le \Phi_{N_e}(r)$ *for* $\beta \ge \beta_2$.

For (a),

$$r \le \begin{cases} r_1(|2f'(0)|) & \textit{if} \quad |f'(0)| < 1/2, \\ \frac{1}{3} & \textit{Otherwise}, \end{cases}$$

while for (b) and (c),

$$r \le \begin{cases} r_1\left(\left|\frac{f'(0)}{\beta}\right|\right) & \textit{if} \quad |f'(0)| < \beta, \\ \frac{1}{3} & \textit{Otherwise}. \end{cases}$$

Proof. Suppose that $f(z) \prec \sqrt{1+z}$. Then, from the definition of subordination, there is a Schawarz function $\phi(z)$ such that

$$f(z) = \sqrt{1+\phi(z)} \implies (f(z))^2 = 1 + \phi(z) \implies (f(z))^2 \prec 1 + z := \zeta_1(z).$$

Clearly, $\zeta_1'(0) = 1$. Finally, for the bound of r, let us denote $\eta(z) = (f(z))^2$. Then,

$$\eta'(0) = 2f(0)f'(0) = 2f'(0).$$

By Theorem 4, we have $r \le r_1(|\eta'(0)/\zeta_1'(0)|) = r_1(|2f'(0)|)$. Since r_1 is defined in $[0,1)$, the inequality holds if $|f_1'(0)| < 1/2$. In all other cases, the result follows from Theorem 3.

From (7), it follows by the fundamental theorem of calculus that

$$g'(z) = \frac{f(z)-1}{\beta z} \implies 1 + \beta z g'(z) = f(z) \prec \sqrt{1+z}. \tag{8}$$

From Lemma 2 (i), it follows that $g(z) \prec \phi_{N_e}(z)$, for $\beta \ge \beta_1$.

To prove the third case, denote $h(z) = e^{g(z)-1}$. Logarithmic differentiation of h yields

$$\frac{h'(z)}{h(z)} = \frac{f(z)-1}{\beta z} \implies 1 + \beta \frac{zh'(z)}{h(z)} = f(z) \prec \sqrt{1+z}.$$

Again, by Lemma 2 (ii), it follows $h(z) \prec \phi_{N_e}(z)$ for $\beta \geq \beta_2$.

Next, to find the upper bound of r in (b) and (c), note that $\phi'_{N_e}(0) = 1$. Further, the right-hand side of (8) gives

$$g'(0) = \lim_{z \to 0} \frac{f(z)-1}{\beta z} = \frac{f'(0)}{\beta}.$$

Hence, the conclusion follows from Theorem 4 if $|f'(0)| < \beta$ and Theorem 3 for all other cases. □

Next, the theorem is related to leminiscate starlike functions.

Theorem 6. *Suppose that f is leminiscate starlike and $\beta > 0$. For*

$$r \leq \begin{cases} r_1\left(\frac{|f''(0)|}{2\beta}\right) & \text{if} \quad |f''(0)| < 2\beta, \\ \frac{1}{3} & \text{Otherwise}. \end{cases}$$

Then, the following assertion holds:

(i) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\ln\left(\frac{f(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{f(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta - \ln\left(\frac{f(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_3$.

Proof. Since f is lemniscate starlike, it satisfies the subordination $zf'(z)/f(z) \prec \sqrt{1+z}$. Now, let us denote all the given functions, respectively, as

$$P_1(z) = 1 + \frac{1}{\beta}\ln\left(\frac{f(z)}{z}\right);\ P_2(z) = \left(\frac{f(z)}{z}\right)^{1/\beta};\ P_3(z) = \frac{\beta}{\beta - \ln\left(\frac{f(z)}{z}\right)}.$$

The normalization of f by $f(0) = 0 = f'(0) - 1$ implies

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f'(z)}{1} = 1.$$

Thus, $P_1(0) = 1 = P_2(0) = P_3(0)$. Further, if $\beta - \ln\left(\frac{f(z)}{z}\right) = 0$, then $f(z) = ze^{\beta}$, which contradicts the fact that $f'(0) = 1$. This implies that P_3 is defined for all z.

Differentiation of the above three equations leads to

$$\begin{aligned} P_1'(z) &= \frac{1}{\beta}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) \implies 1 + \beta z P_1'(z) = \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \\ \frac{P_2'(z)}{P_2(z)} &= \frac{1}{\beta}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) \implies 1 + \beta\frac{zP_2'(z)}{P_2(z)} = \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}, \\ -\frac{P_3'(z)}{(P_3(z))^2} &= -\frac{1}{\beta}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) \implies 1 + \beta\frac{zP_3'(z)}{(P_3(z))^2} = \frac{zf'(z)}{f(z)} \prec \sqrt{1+z}. \end{aligned}$$

Clearly, each of the three cases satisfies the requirements of Lemma 2, which is equivalent to say $P_i(z) \prec \Phi_{N_e}(z)$ for $i = 1, 2, 3$. The limit

$$\lim_{z\to 0}\left(\frac{f'(z)}{f(z)} - \frac{1}{z}\right) = \lim_{z\to 0}\frac{zf'(z) - f(z)}{zf(z)}$$
$$= \lim_{z\to 0}\frac{zf''(z)}{zf'(z) + f(z)} = \lim_{z\to 0}\frac{zf'''(z) + f''(z)}{zf''(z) + 2f'(z)} = \frac{f''(0)}{2},$$

implies $P_1'(0) = P_2'(0) = P_3'(0) = f''(0)/2\beta$.

Finally, from Theorem 4, it follows that

$$\mathcal{M}_r(P_i(z)) \le \Phi_{N_e}(r),$$

with $r \le r_1(|f''(0)|/(2\beta))$ provided $|f''(0)| < 2\beta$. In all other cases, the conclusion follows from Theorem 3. □

Theorem 6 is useful in the next section where we study Bohr's operator on the solution of a second-order differential equation.

2.2. Bohr's Operator on the Solution of Differential Equations

For our next result, we consider the differential equation DE_1 from (4).

Theorem 7. *Suppose that for the analytic functions $a(z)$, $b(z)$ and $d(z)$, the differential Equation (4) has a solution $f_1(z)$ such that $f_1(0) = 1$. If, for $\mathtt{A} \in (0, 1]$,*

$$A\,\mathrm{Re}(a(z) - 1) > A|b(z) - 1| + |b(z) - d(z)|, \tag{9}$$

then $\mathcal{M}_r(f_1) \le 1 + \mathtt{A}r$ for

$$r \le \begin{cases} r_1(|f'(0)|) & \text{if} \quad |f_1'(0)| < 1, \\ \frac{1}{3} & \text{if} \quad \text{Otherwise} \end{cases},$$

Proof. Consider

$$q(z) = \sqrt{\tfrac{1}{\mathtt{A}}(f_1(z) + \mathtt{A} - 1)}. \tag{10}$$

A simplification gives

$$f_1(z) = \mathtt{A}q^2(z) - \mathtt{A} + 1, \quad f_1'(z) = 2\mathtt{A}q'(z)q(z) \quad \mathtt{f}''_{\alpha,n}(z) = 2\mathtt{A}q''(z)q(z) + 2\mathtt{A}(q'(z))^2.$$

From (4), it follows that

$$2\mathtt{A}z^2q''(z)q(z) + 2\mathtt{A}(zq'(z))^2 + 2\mathtt{A}(\alpha + 1 - z)zq'(z)q(z) + n\mathtt{A}zq^2(z) - n\mathtt{A}z + nz = 0.$$

Let $\Omega = \{0\} \subset \mathbb{C}$ and define $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ as

$$\psi(r, s, t; z) = 2\mathtt{A}tr + 2\mathtt{A}s^2 + 2\mathtt{A}(\alpha + 1 - z)sr + nz(\mathtt{A}r^2 - \mathtt{A} + 1). \tag{11}$$

It is clear from (11) that $\psi(q(z), zq'(z), z^2q''(z); z) \in \Omega$. We shall apply Lemma 1 to show $\psi(r, s, t; z) \notin \Omega$, which implies $q(z) \prec \sqrt{1 + z}$.

Now, for $-\pi/4 \le \theta \le \pi/4$, let

$$r = \sqrt{2\cos(2\theta)}e^{i\theta}, \quad s = \frac{me^{3i\theta}}{2\sqrt{2\cos(2\theta)}}.$$

Applying elementary trigonometric identities, we have

$$r^2 - 1 = 2\cos(2\theta)e^{2i\theta} - 1 = (2\cos^2(2\theta) - 1) + i2\cos(2\theta)\sin(2\theta) = e^{4i\theta}.$$

Substitute r, s and t in (3), and simplification leads to

$$\begin{aligned}
|\psi(r,s,t;z)| &= |2\mathtt{A}tr + 2\mathtt{A}s^2 + 2\mathtt{A}(\alpha + 1 - z)sr + nz(\mathtt{A}r^2 - \mathtt{A} + 1)| \\
&= |2\mathtt{A}r(t+s) + 2\mathtt{A}s^2 + 2\mathtt{A}(\alpha - z)sr + \mathtt{A}nz(r^2 - 1) + nz| \\
&> |e^{4i\theta}|\left(2\mathtt{A}\sqrt{2\cos(2\theta)}\operatorname{Re}(t+s)e^{-3i\theta} + 2\mathtt{A}\frac{m^2\operatorname{Re}(e^{2i\theta})}{8\cos(2\theta)} + \mathtt{A}\operatorname{Re}(\alpha - 1)m\right) \\
&\qquad\qquad - n\mathtt{A}|e^{4i\theta}| - n \\
&> 2\mathtt{A}\frac{3m^2}{8} + \frac{Am^2}{4} + \mathtt{A}\operatorname{Re}(\alpha - 1) - n(\mathtt{A} + 1) \\
&> \mathtt{A} + \mathtt{A}\operatorname{Re}(\alpha) - \mathtt{A} - n(\mathtt{A} + 1) \geq 0
\end{aligned}$$

when $\operatorname{Re}(\alpha) \geq n(\mathtt{A}+1)/\mathtt{A}$. By Lemma 1, it follows that $q(z) \prec \sqrt{1+z}$, which is equivalent to

$$\sqrt{\tfrac{1}{\mathtt{A}}(f_1(z) + \mathtt{A} - 1)} = \sqrt{1 + w(z)}, \tag{12}$$

for some analytic function $w(z)$ such that $|w(z)| < 1$. A simplification of (12) gives

$$\tfrac{1}{\mathtt{A}}(f_1(z) + \mathtt{A} - 1) = 1 + w(z) \implies f_1(z) = 1 + \mathtt{A}w(z) \implies f_1(z) \prec 1 + \mathtt{A}z.$$

This completes the proof. □

The next few results are a continuation of the work in [12]. It is proven in [12] (Theorem 2.1) that the solution of the differential equation DE_2 in (5), with the normalization $\mathtt{F}(0) = 0$, $\mathtt{F}'(0) = 1$, is associated with $\phi_{N_e}(z)$ when

$$4(\sqrt{2} - 1)|za(z) - 1| + 4|zb(z) + a(z)| < 4 + \sqrt{2}.$$

A similar result was also obtained in [12] (Theorem 2.2), which is associated with the solution of the differential equation DE_3 in (6), with normalization $\mathtt{G}(0) = 0$, $\mathtt{G}'(0) = 1$.

Using the results [12] (Theorems 2.1 and 2.2) mentioned above along with Theorem 6 in this article, we have the following two results. We omit a detailed proof.

Theorem 8. *Suppose that* $\mathtt{F}$ *is the solution of the differential equation* DE_2 *in* (5), *with the normalization* $\mathtt{F}(0) = 0$, $\mathtt{F}'(0) = 1$. *Suppose that* $F(z) \neq 0$ *for all* $z \in \mathbb{D} \setminus \{0\}$. *Assume that the analytic functions a and b satisfy the inequality*

$$4(\sqrt{2} - 1)|za(z) - 1| + 4|zb(z) + a(z)| < 4 + \sqrt{2}. \tag{13}$$

Then, the following assertion are true:

(i) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\ln\left(\frac{F(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{F(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta - \ln\left(\frac{F(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_3$.

Theorem 9. *Suppose that $G(z) \neq 0$ for any $z \in \mathbb{D} \setminus \{0\}$ is the solution of the differential equation DE_3 in (6), with the normalization $\mathtt{G}(0) = 0$, $\mathtt{G}'(0) = 1$. Assume that the analytic functions a and b satisfy the inequality*

$$4(\sqrt{2}-1)|a(z)-1| + 4|b(z)+a(z)| < 4+\sqrt{2}. \tag{14}$$

Then, the following assertions are true:

(i) $\mathcal{M}_r\left(1+\frac{1}{\beta}\ln\left(\frac{G(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{G(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta-\ln\left(\frac{G(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ *for* $\beta \geq \beta_3$.

3. Examples Involving Special Functions

This section is devoted to establishing examples based on the results obtained in Sections 2.1 and 2.2. Some of the examples are based on article [12] but this article emphasizes Bohr's radius of special functions, which is a normalized solution of some second-order differential equations.

3.1. Example Involving the Associated Laguerre Polynomial

The generalized [13] or associated Laguerre polynomial (ALP) $\mathtt{L}_n^\alpha(z)$, defined by the series

$$\mathtt{L}_n^\alpha(z) = \sum_{i=0}^{n}(-1)^i \begin{pmatrix} n+\alpha \\ n-i \end{pmatrix} \frac{z^i}{i!} = \frac{(1+\alpha)_n}{n!}{}_1F_1(-n;1+\alpha;z), \tag{15}$$

is a solution of the differential equation

$$zy''(z) + (\alpha+1-z)y'(z) + ny(z) = 0, \quad \alpha \in \mathbb{R}. \tag{16}$$

Here, ${}_1F_1$ is a well-known confluent hypergeometric function and $(a)_n$ is the Pochhammer symbol defined as

$$(a)_0 = 1, \quad (a)_n = a(a+1)\ldots(a+n-1), \quad n \in \mathbb{N}.$$

The monographs by Szegó [14] and Andrews, Askey, and Roy [15] include a wealth of information about the ALP and other orthogonal polynomial families. A short summary of various applications of the ALP is given in [12]. The normalized form

$$\mathtt{f}_{n,\alpha}(z) = \frac{n!}{(\alpha+1)_n}\mathtt{L}_n^\alpha(z), \quad z \in \mathbb{D}, \tag{17}$$

which is a solution of the differential equation

$$z^2y''(z) + (\alpha+1-z)zy'(z) + nzy(z) = 0, \tag{18}$$

is studied in [12]. It is proven that $f_{n,\alpha}(z) \prec \mathcal{P}_{\mathcal{L}}(z)$ when $4\,\mathrm{Re}(\alpha) > 16n+1$.

From (17), it follows that $\mathtt{f}_{n,\alpha}(0) = 1$ and $\mathtt{f}'_{n,\alpha}(0) = -n/(1+\alpha)$. Now, applying Theorem 5, we have the following results:

(a) $\mathcal{M}_r(f^2_{n,\alpha}(z)) \leq 1+r$, where

$$r \leq r_1\left(|2f'_{n,\alpha}(0)|\right) = r_1\left(\left|\tfrac{2n}{1+\alpha}\right|\right).$$

Since, $4\,\mathrm{Re}(\alpha) > 16n+1$, it follows that

$$|1+\alpha| > 1+\mathrm{Re}(\alpha) > 4n+\frac{5}{4},$$

which further gives

$$\left|\tfrac{2n}{1+\alpha}\right| < \tfrac{8n}{16n+5} \le \tfrac{8}{21} < \tfrac{1}{2}, \quad \forall n \ge 1.$$

Finally, from the definition of r_1 in (2), it follows that

$$r \le r_1\left(\left|\tfrac{2n}{1+\alpha}\right|\right) = \sqrt{\tfrac{|1+\alpha|-2n}{2|1+\alpha|}}.$$

(*b*) Define

$$g_{n,\alpha}(z) := 1+\frac{1}{\beta}\int_0^z \frac{f_{n,\alpha}(t)-1}{t}dt. \tag{19}$$

Now,

$$\tfrac{f_{n,\alpha}(t)-1}{t} = \tfrac{{}_1F_1(-n;1+\alpha;t)-1}{t} = \sum_{j=1}^{\infty}\tfrac{(-n)_j}{(1+\alpha)_j}\tfrac{t^j}{j!} \implies g_{n,\alpha}(z) = 1+\tfrac{1}{\beta}\sum_{j=1}^{\infty}\tfrac{(-n)_j}{j(1+\alpha)_j}\tfrac{t^j}{j!}.$$

Further simplification leads to

$$g_{n,\alpha}(z) = 1+\tfrac{1}{\beta}\sum_{j=1}^{\infty}\tfrac{(-n)_j}{j(1+\alpha)_j}\tfrac{t^j}{j!} = 1-\frac{n}{\beta(1+\alpha)}{}_3F_3(-n+1,1,1;2+\alpha,2,2;z).$$

(i) $\mathcal{M}_r(g_{n,\alpha}(z)) \le \Phi_{N_e}(r)$ for $\beta \ge \beta_1$. In this case,

$$r \le r_1\left(\left|\frac{f'_{n,\alpha}(0)}{\beta}\right|\right) = r_1\left(\left|\frac{n}{\beta(1+\alpha)}\right|\right).$$

A similar calculation to the one above yields

$$\left|\tfrac{n}{\beta(1+\alpha)}\right| < \tfrac{4n}{\beta(16n+5)} \le \tfrac{4}{21\beta} < \tfrac{1}{2}, \quad \forall n \ge 1 \quad \text{and} \quad \beta \ge \beta_1.$$

Thus,

$$r \le r_1\left(\left|\tfrac{n}{\beta(1+\alpha)}\right|\right) = \sqrt{\tfrac{|1+\alpha|-n}{2\beta|1+\alpha|}}, \quad \forall n \ge 1 \quad \text{and} \quad \beta \ge \beta_1$$

(ii) $\mathcal{M}_r(e^{g_{n,\alpha}(z)-1}) \le \Phi_{N_e}(r)$ for $\beta \ge \beta_2$. In this case,

$$r \le r_1\left(\left|\tfrac{n}{\beta(1+\alpha)}\right|\right) = \sqrt{\tfrac{|1+\alpha|-n}{2\beta|1+\alpha|}}, \quad \forall n \ge 1 \quad \text{and} \quad \beta \ge \beta_2.$$

3.2. *Example Involving the Error Function*

Our next function is

$$\mathtt{h}_1(z) := \sqrt{\frac{\pi\alpha}{2}}e^{z^2/(2\alpha)}erf\left(\frac{z}{\sqrt{2\alpha}}\right), \quad \alpha > 0 \tag{20}$$

involving the error function *erf* [13], which is defined as

$$erf(z) := \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2n+1)}z^{2n+1}.$$

The error function can also be expressed by the confluent hypergeometric functions through $\sqrt{\pi}erf(z) = 2z_1F_1(1/2;3/2;-z^2)$. Functional inequalities involving the real error functions can be found in [16]. In the context of geometric functions theory, Coman [17] determined the radius of starlikeness of the error function. It is proven in [12] that $\mathtt{h}_1(z)$ is lemniscate starlike for $\alpha > 4(\sqrt{2}+1)/(8-3\sqrt{2})$. Note that $erf(0) = 0$ and hence $f_1(0) = 0$. The derivative of $erf(z)$ leads to

$$\frac{d}{dz}\mathrm{erf}\left(\frac{z}{\sqrt{2\alpha}}\right) = \frac{2}{\sqrt{\pi}\sqrt{2\alpha}}e^{-\frac{z^2}{2\alpha}}.$$

Taking the derivative of both sides of (20), it follows that

$$\mathtt{h}_1'(z) = \sqrt{\frac{\pi}{2\alpha}}ze^{\frac{z^2}{2\alpha}}\mathrm{erf}\left(\frac{z}{\sqrt{2\alpha}}\right) + 1 \tag{21}$$

and

$$\mathtt{h}_1''(z) = \sqrt{\frac{\pi}{2\alpha}}\left(e^{\frac{z^2}{2\alpha}}\mathrm{erf}\left(\frac{z}{\sqrt{2\alpha}}\right) + \frac{z^2}{\alpha}e^{\frac{z^2}{2\alpha}}\mathrm{erf}\left(\frac{z}{\sqrt{2\alpha}}\right) + \frac{2z}{\sqrt{2\pi\alpha}}\right). \tag{22}$$

It is clear from (22) that $\mathtt{h}_1''(0) = 1$. Now, we have the following assertions from Theorem 6

(i) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\ln\left(\frac{\mathtt{h}_1(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{\mathtt{h}_1(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta - \ln\left(\frac{\mathtt{h}_1(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_3$.

In the first two cases,

$$\begin{aligned} r \leq r_1\left(\frac{\mathtt{h}_1''(0)}{2\beta}\right) &= r_1\left(\frac{1}{2\beta}\right) \\ &= \begin{cases} \sqrt{\frac{2\beta-1}{4\beta}} & \text{for} \quad \beta > 1 \\ \frac{\beta}{\beta+1} & \text{for} \quad \beta \in [\beta_i, 1], \quad \text{for} \quad i = 1 \quad \text{or} \quad 2. \end{cases}, \end{aligned}$$

while in the third case, when $\beta \geq \beta_3 > 1$,

$$r \leq r_1\left(\frac{\mathtt{h}_1''(0)}{2\beta}\right) = r_1\left(\frac{1}{2\beta}\right) = \sqrt{\frac{2\beta-1}{4\beta}}.$$

We further note here that $\mathtt{h}_1$ is a solution of the differential equation

$$\alpha y''(z) - zy'(z) - y(z) = 0.$$

Thus, the above results can also be obtained by using Theorem 8.

3.3. Example Involving the Classical Bessel Function

It is proven in [12] that the function

$$\mathtt{h}_2(z) = \frac{\pi}{\sin\left(\frac{2\pi\nu}{\sqrt{\beta}}\right)}\left(J_{-\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right)J_{\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{e^z}{\beta}}\right) - J_{\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right)J_{-\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{e^z}{\beta}}\right)\right), \quad \nu \notin \mathbb{Z}, \beta > 0.$$

is leminiscate starlike when for a fixed $\beta > 0$, there exists ν for which

$$4(\nu^4 - 2\nu^2 e\cos(1) + e^2) \leq (8 - 3\sqrt{2})\beta. \tag{23}$$

Here, J_ν is the well-known classical Bessel function of order ν, which is a solution of the differential equation:

$$z^2 y''(z) + zy'(z) + (z^2 - \nu^2)y(z) = 0. \tag{24}$$

Several results related to the geometric properties of the Bessel function and its generalizations can be found in [18,19] and the references therein.

Clearly,

$$\mathtt{h}_2(0) = \frac{\pi}{\sin\left(\frac{2\pi\nu}{\sqrt{\beta}}\right)}\left(J_{-\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right)J_{\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right) - J_{\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right)J_{-\frac{2\nu}{\sqrt{\beta}}}\left(\frac{2}{\sqrt{\beta}}\right)\right) = 0.$$

After a careful computation as in [20] (Example 3, Page 561), it follows that $\mathtt{h}_2'(0) = 1$ and $\mathtt{h}_2$ is the solution of the differential equation

$$\beta F''(z) + (e^z - \nu^2)F(z) = 0.$$

The second-order derivative of $\mathtt{h}_2$ yields

$$\mathtt{h}_2''(z) = \frac{\pi\left(e^z - \nu^2\right)\csc\left(\frac{2\pi\nu}{\sqrt{\beta}}\right)\left(J_{\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{1}{\beta}}\right)J_{-\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{e^z}{\beta}}\right) - J_{-\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{1}{\beta}}\right)J_{\frac{2\nu}{\sqrt{\beta}}}\left(2\sqrt{\frac{e^z}{\beta}}\right)\right)}{\beta},$$

and $\mathtt{h}_2''(0) = 0$. Finally, we have from Theorem 6 (as well as Theorem 8) that

(i) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\ln\left(\frac{\mathtt{h}_2(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{\mathtt{h}_2(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta - \ln\left(\frac{\mathtt{h}_2(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_3$.

In all three cases,

$$r \leq r_1\left(\frac{\mathtt{h}_2''(0)}{2\beta}\right) = r_1(0) = \frac{1}{\sqrt{2}}.$$

3.4. Example Involving Airy Functions

For next example, consider the following function considered in [12]:

$$\mathtt{h}_3(z) = \frac{\Gamma\left(\frac{1}{3}\right)\left(3^{5/6}Bi(\sqrt[3]{a}z) - 3^{4/3}Ai(\sqrt[3]{a}\,z)\right)}{6\sqrt[3]{a}}, a \neq 0.$$

Here, Ai and Bi are well-known Airy functions [13] which are independent solutions of the differential equation $y''(z) - zy(z) = 0$ with initial value

$$Ai(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}, \quad Ai'(0) = -\frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}, \quad Bi(0) = \frac{1}{3^{1/6}\Gamma\left(\frac{2}{3}\right)}, \quad Bi'(0) = \frac{3^{1/6}}{\Gamma\left(\frac{1}{3}\right)}.$$

Thus,

$$\mathtt{h}_3(0) = \frac{\Gamma\left(\frac{1}{3}\right)\left(3^{5/6}Bi(0) - 3^{4/3}Ai(0)\right)}{6\sqrt[3]{a}} = \frac{\Gamma\left(\frac{1}{3}\right)}{6\sqrt[3]{a}}\left(\frac{3^{5/6}}{3^{1/6}\Gamma\left(\frac{2}{3}\right)} - \frac{3^{4/3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\right) = 0$$

and

$$\begin{aligned}\mathtt{h}_3'(0) &= \left.\frac{\Gamma\left(\frac{1}{3}\right)\left(3^{5/6}\sqrt[3]{a}\mathrm{Bi}'\left(\sqrt[3]{a}z\right) - 3\sqrt[3]{3}\sqrt[3]{a}\mathrm{Ai}'\left(\sqrt[3]{a}z\right)\right)}{6\sqrt[3]{a}}\right|_{z=0} \\ &= \frac{\Gamma\left(\frac{1}{3}\right)\left(3^{5/6}\sqrt[3]{a}Bi'(0) - 3^{4/3}\sqrt[3]{a}Ai'(0)\right)}{6\sqrt[3]{a}} = \frac{\Gamma\left(\frac{1}{3}\right)}{6}\left(\frac{3}{\Gamma\left(\frac{1}{3}\right)} + \frac{3^{4/3}}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}\right) = 1.\end{aligned}$$

Further computation yields that $\mathtt{h}_3$ is a solution of the differential equation

$$\mathtt{F}''(z) - a\,z\,\mathtt{F}(z) = 0.$$

Then, it is shown in [12] that $\mathtt{h}_3$ is lemniscate starlike for $|a| < (8 - 3\sqrt{2})/4 \approx 0.93934$. Now, the second-order derivative of $\mathtt{h}_3$ at $z = 0$ gives

$$\mathtt{h}_3''(0) = \left.\frac{\Gamma\left(\frac{1}{3}\right)^2 z\left(3^{5/6}a\mathrm{Bi}\left(\sqrt[3]{a}z\right) - 3\sqrt[3]{3}az\mathrm{Ai}\left(\sqrt[3]{a}\right)\right)}{36a^{2/3}}\right|_{z=0} = 0.$$

Similar to the earlier example, we now have from Theorem 6 (as well as Theorem 8) that

(i) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\ln\left(\frac{\mathtt{h}_3(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{\mathtt{h}_3(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta - \ln\left(\frac{\mathtt{h}_3(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_3$,

and in all three cases,

$$r \leq r_1\left(\frac{\mathtt{h}_3''(0)}{2\beta}\right) = r_1(0) = \frac{1}{\sqrt{2}}.$$

3.5. Example Involving Generalized Bessel Functions

One of the most significant functions included in the literature of geometric functions theory is the generalized and normalized Bessel functions of the form

$$\mathtt{U}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{4^n(\kappa)_n}\,\frac{z^n}{n!}, \quad 2\kappa = 2p + b + 1 \neq 0, -2, -4, -6, \dots;$$

which are the solutions of

$$4z^2\mathtt{U}''(z) + 4\kappa z\mathtt{U}'(z) + cz\mathtt{U}(z) = 0. \tag{25}$$

For $b = c = 1$, the function $\mathtt{U}_p$ represents the normalized Bessel function of order p, while for $b = -c = 1$, the function $\mathtt{U}_p$ represents the normalized modified Bessel function of order p. The spherical Bessel function can also be obtained by using $b = 2$, $c = 1$.

The inclusion of $\mathtt{U}_p$ in various subclasses of univalent functions theory has been extensively studied by many authors [18,21–23] and some references therein. Recently, the lemniscate convexity and other properties of $\mathtt{U}_p$ have been studied in [21].

Now, consider

$$\mathtt{h}_4(z) = z\mathtt{U}_p(z).$$

It is proved in [12] that the function f_4 is lemniscate starlike if

$$16(\sqrt{2}-1)|\kappa - 3| + |c| < 4 + \sqrt{2}.$$

A simple computation yields

$$\mathtt{h}_4'(z) = z\mathtt{U}_p'(z) + \mathtt{U}_p(z), \quad \mathtt{h}_4''(z) = z\mathtt{U}_p''(z) + 2\mathtt{U}_p'(z).$$

Then, it follows that $\mathtt{h}_4''(0) = 2\mathtt{U}_p'(0) = -c/2\kappa$.

We now have from Theorem 6 (as well as Theorem 8) that

(i) $\mathcal{M}_r\left(1+\frac{1}{\beta}\ln\left(\frac{\mathtt{h}_4(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{\mathtt{h}_4(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta-\ln\left(\frac{\mathtt{h}_4(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_3$,

and in all three cases, with respect to β, we have

$$r \leq r_1\left(\left|\frac{\mathtt{h}_4''(0)}{2\beta}\right|\right) = r_1\left(\frac{|c|}{4|\kappa\beta}\right) = \begin{cases} \sqrt{\frac{4\beta|\kappa|-|c|}{8|\kappa|\beta}} & \text{for} \quad 2\beta|\kappa| > |c| \\ \frac{2|\kappa|\beta}{2|\kappa|\beta+|c|} & \text{for} \quad |c| < 4\beta|\kappa| \leq 2|c| \end{cases}.$$

3.6. Example Involving Confluent Hypergeometric Functions

Geometric functions theory has a close association with the hypergeometric functions ${}_2F_1$ and the confluent hypergeometric functions ${}_1F_1$ (refer to the articles [20,24–30]).

The differential equation

$$z^2y''(z) + (b-z)y'(z) - azy(z) = 0$$

has the solution ${}_1F_1(a,b;z)$.

Now, consider the function $\mathtt{h}_5(z) := z{}_1F_1(a,b;z)$. Then, $\mathtt{h}_5$ is lemniscate starlike if

$$4(\sqrt{2}-1)|\beta - 3| + |\alpha| < 8 - 3\sqrt{2} \tag{26}$$

which is proven in [12]. The second derivative of $\mathtt{h}_5$ leads to

$$\mathtt{h}_5''(0) = 2{}_1F_1'(a,b;0) = a/b.$$

Similar to earlier examples, we have

(i) $\mathcal{M}_r\left(1+\frac{1}{\beta}\ln\left(\frac{\mathtt{h}_5(z)}{z}\right)\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$;

(ii) $\mathcal{M}_r\left(\left(\frac{\mathtt{h}_5(z)}{z}\right)^{\frac{1}{\beta}}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_2$;

(iii) $\mathcal{M}_r\left(\frac{\beta}{\beta-\ln\left(\frac{\mathtt{h}_5(z)}{z}\right)}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_3$, and in all three cases, with respect to β, we have

$$r \leq r_1\left(\left|\frac{\mathtt{h}_5''(0)}{2\beta}\right|\right) = r_1\left(\frac{|a|}{2\beta|b|}\right) = \begin{cases} \sqrt{\frac{2\beta|b|-|a|}{4\beta|b|}} & \text{for} \quad \beta|b| > |a| \\ \frac{2|b|\beta}{|b|\beta+|a|} & \text{for} \quad |a| < 2\beta|b| \leq 2|a| \end{cases}.$$

3.7. Example Involving Some General Functions

For $\alpha \in (-\infty, 1/2]$, define the functions

$$\mathtt{h}_6(z) = \sqrt{1-z}(1+\alpha z) \quad \text{and} \quad \mathtt{w}(z) = (2\alpha-1)z + \left(\alpha^2-\alpha\right)z^2 - \alpha^2 z^3. \tag{27}$$

Clearly, $\mathtt{h}_6(0) = 1$, $\mathtt{h}_6'(0) = \alpha - 1/2$ and $w(0) = 0$.
Now, a calculation yields

$$1 + w(z) = 1 + (2\alpha-1)z + \left(\alpha^2 - 2\alpha\right)z^2 - \alpha^2 z^3 = (1-z)(1+\alpha z)^2.$$

This implies $\mathtt{h}_6(z) = \sqrt{1+w(z)}$. Further, for $|z| < 1$,

$$|w(z)| < |2\alpha - 1| + |\alpha^2 - 2\alpha| + |\alpha|^2 = 1$$

when $\alpha \leq \frac{1}{2}$. Thus, $\mathtt{h}_6(z) \prec \sqrt{1+z}$ for $\alpha \in (-\infty, 1/2]$.
For $\beta > 0$, let us define g_6 as

$$\mathrm{g}_6(z) := 1 + \frac{1}{\beta}\int_0^z \frac{\mathtt{h}_6(t)-1}{t}dt. \tag{28}$$

Next, we aim to find a closed form of g_6. The solution of the integration in (28) can be easily established using computational software, but here we solve the problem to achieve the completeness of the result. First, we consider the following indefinite integration:

$$I = \int \frac{\sqrt{1-t}-1}{t}dt = \int \frac{\left(\sqrt{1-t}-1\right)\left(\sqrt{1-t}+1\right)}{t\left(\sqrt{1-t}+1\right)}dt = -\int \frac{1}{\sqrt{1-t}+1}dt.$$

Next, substitute $r = \sqrt{1-t}$. Then,

$$dr = -\frac{1}{2\sqrt{1-t}}dt \Rightarrow dt = -2r\ dr.$$

The integration I reduces to

$$\begin{aligned} I = 2\int \frac{r}{r+1}dr &= 2\int\left(1 - \frac{1}{r+1}\right)dr \\ &= 2r - 2\ln|r+1| + c_1 \\ &= 2\sqrt{1-t} - 2\ln\left(1+\sqrt{1-t}\right) + c_1. \end{aligned}$$

By a routine calculation, the second integration in I_1 leads to

$$\int \sqrt{1-t}\, dt = -\frac{2}{3}(1-t)^{3/2} + c_2.$$

This finally leads to the closed form of g_6 as follows:

$$\begin{aligned}
g_6(z) &= 1 + \frac{1}{\beta}\int_0^z \frac{\sqrt{1-t}(1+\alpha t)-1}{t}dt \\
&= 1 + \frac{1}{\beta}\int_0^z \frac{\sqrt{1-t}-1}{t}dt + \frac{\alpha}{\beta}\int_0^z \sqrt{1-t}dt \\
&= 1 + \frac{1}{\beta}\left(2\sqrt{1+t} - 2\ln\left(1+\sqrt{1+t}\right)\right)_0^z + \frac{\alpha}{\beta}\left(-\frac{2}{3}(1-t)^{3/2}\right)_0^z \\
&= 1 + \frac{1}{\beta}\left(2\sqrt{1+z} - 2\ln\left(1+\sqrt{1+z}\right) - 2 + 2\ln(2)\right) - \frac{2\alpha}{3\beta}(1-z)^{3/2} + \frac{2\alpha}{3\beta}
\end{aligned}$$

Finally, from Theorem 5, we have the following conclusions:

(*a*) $\mathcal{M}_r\left((1-z)(1+\alpha z)^2\right) \leq 1+r$, with $r \leq r_1(\mathtt{h}_6'(0))$. Since $\mathtt{h}_6'(0) = \alpha - 1/2$ and $\alpha \leq 1/2$, we can say that $r_1(\alpha - 1/2)$ is defined only for $\alpha = 1/2$. Hence,

$$r \leq r_1(\mathtt{h}_6'(0)) = r_1(0) = 1/\sqrt{2}.$$

(*b*) $\mathcal{M}_r\left(1 + \frac{1}{\beta}\left(2\sqrt{1+z} - 2\ln\left(1+\sqrt{1+z}\right) - 2 + \ln(4)\right) - \frac{2\alpha}{3\beta}(1-z)^{3/2} + \frac{2\alpha}{3\beta}\right) \leq \Phi_{N_e}(r)$ for $\beta \geq \beta_1$, and $r \leq r_1\left(\left|\frac{\mathtt{h}_6'(0)}{\beta}\right|\right)$. Now,

$$r_1\left(\left|\frac{\mathtt{h}_6'(0)}{\beta}\right|\right) = r_1\left(\left|\frac{2\alpha-1}{2\beta}\right|\right) = \begin{cases} \sqrt{\frac{2\beta-|2\alpha-1|}{4\beta}} & \text{for} \quad 2|2\alpha-1| \leq \beta \\ \frac{\beta}{\beta+|2\alpha-1|} & \text{for} \quad \beta < 2|2\alpha-1| \leq 2\beta. \end{cases}$$

4. Conclusions

Bohr's operator has been the subject of numerous investigations. To the best of our knowledge, no research has addressed Bohr's operator in relation to the solution to a second-order differential equation. This article offers a novel approach to research in this area of study. It is also interesting to incorporate special functions into the investigation of Bohr's operator. We established the upper bound of Bohr's radius involving special functions such as:

1. Associated Laguerre polynomials (Section 3.1);
2. Error functions (Section 3.2);
3. Classical Bessel functions (Section 3.3);
4. Airy functions (Section 3.4);
5. Generalized Bessel functions (Section 3.5);
6. Confluent hypergeometric functions (Section 3.6).

All of the above functions are the solution of some second-order differential equations. It is well known that Gaussian hypergeometric functions are also solutions of second-order differential equations and are very important special functions. Hence, we raised following problem.

Problem 3. *How can the Gaussian hypergeometric functions* ${}_2F_1(a,b,c;z)$ *be connected with Bohr's radius problem?*

In addition, by utilizing the definition of subordination, we proved in Section 3.7 that $\sqrt{1-z}P_1(\alpha_1,z) \prec \sqrt{1+z}$ for $\alpha_1 \leq 1/2$, which further leads to the inequality

$$\mathcal{M}_r\left((1-z)(P_1(\alpha_1,z))^2\right) \leq 1+r,$$

only for $\alpha = 1/2$. Here, $P_1(\alpha_1,z) = 1 + \alpha_1 z$ is a polynomial of degree 1. In this aspect, we raised the following problem for further study:

Problem 4. *Can we define an n-th degree polynomial*

$$P_n(\alpha_1, \alpha_2, \ldots, \alpha_n, z) = 1 + \sum_{i=1}^{n} \alpha_i z^n$$

such that $P_n(\alpha_1, \alpha_2, \ldots, \alpha_n, z) \prec \sqrt{1+z}$*? What will be the range of each* α_i, $i = 1, 2, \ldots, n$ *in such cases? Further, if* $\mathcal{M}_r\big((1-z)(P_n(\alpha_1, \alpha_2, \ldots, \alpha_n, z))^2\big) \leq 1 + r$*, what is the range or exact value of each* α_i*?*

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References

1. Duren, P.L. Univalent functions. In *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*; Springer: New York, NY, USA, 1983; Volume 259, pp. xiv+382.
2. Bohr, H. A Theorem Concerning Power Series. *Proc. London Math. Soc.* **1914**, *13*, 1–5. [CrossRef]
3. Sidon, S. Über einen Satz von Herrn Bohr. *Math. Z.* **1927**, *26*, 731–732. [CrossRef]
4. Tomić, M. Sur un théorème de H. Bohr. *Math. Scand.* **1962**, *11*, 103–106. [CrossRef]
5. Paulsen, V.I.; Popescu, G.; Singh, D. On Bohr's inequality. *Proc. London Math. Soc.* **2002**, *85*, 493–512. [CrossRef]
6. Rogosinski, W. Über Bildschranken bei Potenzreihen und ihren Abschnitten. *Math. Z.* **1923**, *17*, 260–276. [CrossRef]
7. Bhowmik, B.; Das, N. Bohr phenomenon for subordinating families of certain univalent functions. *J. Math. Anal. Appl.* **2018**, *462*, 1087–1098. [CrossRef]
8. Abu Muhanna, Y.; Ali, R.M. Bohr's phenomenon for analytic functions into the exterior of a compact convex body. *J. Math. Anal. Appl.* **2011**, *379*, 512–517. [CrossRef]
9. Huang, Y.; Liu, M.S.; Ponnusamy, S. The Bohr-type operator on analytic functions and sections. *Complex Var. Elliptic Equ.* **2023**, *68*, 317–332. [CrossRef]
10. Madaan, V.; Kumar, A.; Ravichandran, V. Starlikeness associated with lemniscate of Bernoulli. *Filomat* **2019**, *33*, 1937–1955. [CrossRef]
11. Swaminathan, A.; Wani, L.A. Sufficiency for nephroid starlikeness using hypergeometric functions. *Math. Methods Appl. Sci.* **2022**, *45*, 5388–5401. [CrossRef]
12. Mondal, S.R. On Lemniscate Starlikeness of the Solution of General Differential Equations. *Mathematics* **2022**, *10*, 3254. [CrossRef]
13. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; US Government Printing Office: Washington, DC, USA, 1968; Volume 55.
14. Szegö, G. *Orthogonal Polynomials*; American Mathematical Society Colloquium Publications, American Mathematical Society: New York, NY, USA, 1939; Volume 23, pp. ix+401.
15. Andrews, G.E.; Askey, R.; Roy, R. Special Functions. In *Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1999; Volume 71, pp. xvi+664. [CrossRef]
16. Alzer, H. Error function inequalities. *Adv. Comput. Math.* **2010**, *33*, 349–379. [CrossRef]
17. Coman, D. The radius of starlikeness for the error function. *Stud. Univ. Babeş-Bolyai Math.* **1991**, *36*, 13–16.
18. Baricz, A. Geometric properties of generalized Bessel functions. *Publ. Math. Debr.* **2008**, *73*, 155–178. [CrossRef]
19. Brown, R.K. Univalence of Bessel functions. *Proc. Am. Math. Soc.* **1960**, *11*, 278–283. [CrossRef]
20. Ali, R.M.; Lee, S.K.; Mondal, S.R. Geometric features of general differential solutions. *Bull. Belg. Math. Soc. Simon Stevin* **2019**, *26*, 551–570. [CrossRef]
21. Madaan, V.; Kumar, A.; Ravichandran, V. Lemniscate convexity of generalized Bessel functions. *Stud. Sci. Math. Hungar.* **2019**, *56*, 404–419. [CrossRef]
22. Baricz, A.; Szász, R. The radius of convexity of normalized Bessel functions of the first kind. *Anal. Appl.* **2014**, *12*, 485–509. [CrossRef]
23. Baricz, A. Geometric properties of generalized Bessel functions of complex order. *Mathematica* **2006**, *48*, 13–18.
24. Ponnusamy, S.; Vuorinen, M. Univalence and convexity properties for confluent hypergeometric functions. *Complex Var. Theory Appl.* **1998**, *36*, 73–97. [CrossRef]
25. Ponnusamy, S.; Vuorinen, M. Univalence and convexity properties for Gaussian hypergeometric functions. *Rocky Mt. J. Math.* **2001**, *31*, 327–353. [CrossRef]
26. Ruscheweyh, S.; Singh, V. On the order of starlikeness of hypergeometric functions. *J. Math. Anal. Appl.* **1986**, *113*, 1–11. [CrossRef]

27. Ali, R.M.; Mondal, S.R.; Ravichandran, V. On the Janowski convexity and starlikeness of the confluent hypergeometric function. *Bull. Belg. Math. Soc. Simon Stevin* **2015**, *22*, 227–250. [CrossRef]
28. Miller, S.S.; Mocanu, P.T. Univalence of Gaussian and confluent hypergeometric functions. *Proc. Am. Math. Soc.* **1990**, *110*, 333–342. [CrossRef]
29. Naz, A.; Nagpal, S.; Ravichandran, V. Exponential starlikeness and convexity of confluent hypergeometric, Lommel, and Struve functions. *Mediterr. J. Math.* **2020**, *17*, 204. [CrossRef]
30. Kanas, S.a.; Stankiewicz, J. Univalence of confluent hypergeometric functions; *Ann. Univ. Mariae Curie-Sklodowska* **1998**, *52*, 51–56.