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Novel Formulas for *B*-Splines, Bernstein Basis Functions, and Special Numbers: Approach to Derivative and Functional Equations of Generating Functions

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Abstract: The purpose of this article is to give relations among the uniform B-splines, the Bernstein basis functions, and certain families of special numbers and polynomials with the aid of the generating functions method. We derive a relation between generating functions for the uniform B-splines and generating functions for the Bernstein basis functions. We derive some functional equations for these generating functions. Using the higher-order partial derivative equations of these generating functions, we derive both the generalized de Boor recursion relation and the higher-order derivative formula of uniform B-splines in terms of Bernstein basis functions. Using the functional equations of these generating functions, we derive the relations among the Bernstein basis functions, the uniform B-splines, the Apostol-Bernoulli numbers and polynomials, the Aposto–Euler numbers and polynomials, the Eulerian numbers and polynomials, and the Stirling numbers. Applying the p-adic integrals to these polynomials, we derive many novel formulas. Furthermore, by applying the Laplace transformation to these generating functions, we derive infinite series representations for the uniform B-splines and the Bernstein basis functions.

Keywords: generating functions; uniform B-splines; Bernstein basis functions; Apostol–Bernoulli numbers and polynomials; Apostol–Euler numbers and polynomials; Eulerian numbers and polynomials; Laplace transforms; p-adic integral

MSC: 05A15; 41A15; 11B68; 11S80; 12D10; 26C05; 44A10

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1. Introduction

This article covers both generating functions and B-spline curves, which are the most important members of spline theory. These have very important applications both in mathematics and other branches of science. By giving brief definitions and properties of these, the motivation for the new results of the article will be examined in detail below. We first give the generating function and some of its properties.

It is known that generating functions were first defined in 1730 by the French mathematician Abraham de Moivre (26 May 1667–27 November 1754) in order to solve the general linear recursion problem. In general, a generating function is briefly defined as follows: let $G(y,z)$ be a two-variable function which possesses a formal (not necessarily convergent for $z \neq 0$) power series expansion in z such that

$$G(y,z) = \sum_{m=0}^{\infty} P_m(y) z^m, \tag{1}$$

where each element of the coefficient set $(P_m(y))_{m=0}^{\infty}$ is independent of z. Then, $G(y,z)$ is said to have generated the set $(P_m(y))_{m=0}^{\infty}$ and $G(y,z)$ is called a generating function (generating series) for the set $(P_m(y))_{m=0}^{\infty}$. A generating function represents a number sequence as a series in powers of the formal variable. Suppose that $G(y,z)$ is an analytic function at $z = 0$. The power series of this function in Equation (1) converges in some domain of the complex z-plane that involves the origin. However, convergence is not necessary for the series expansion in Equation (1) to generate the coefficient set $(P_m(y))_{m=0}^{\infty}$. Therefore, using Equation (1), many useful properties of the coefficient set $(P_m(y))_{m=0}^{\infty}$ can be obtained. It is well known that generating functions can be studied in discrete mathematics, in complex variable theory, in linear algebra, in probability theory, in quantum physics, etc., ([1]; see also the references cited in each of these earlier works). Hence, it is not necessary to determine the radius of convergence for the power series representation of each generating function. On the other hand, if any generating function $H(y,z)$ has a power series expansion which is divergent for $t \neq 0$, then the following notation can indicate divergence:

$$H(y,z) \cong \sum_{m=0}^{\infty} f_m(y) t^m$$

(for detail, see [1]; see also the references cited in each of these earlier works). There are many types of generating functions such as linear generating functions, bilinear generating functions, bilateral generating functions, multivariable generating functions, multiple generating functions, generating functions involving Lambert series, Laurent series, Bell series, Dirichlet series, probability generating functions, etc. If we can formally establish the sum of functions $G(y,z)$ or $H(y,z)$ in terms of special functions, we shall say that generating functions $G(y,z)$ or $H(y,z)$ are found ([1,2]; see also the references cited in each of these earlier works). As a result, the generating functions and their applications cover many branches of mathematics and applied sciences. Therefore, many important different applications of the generating functions method are not even touched upon in this article, including combinatorial analysis, number theory, probability theory, the theory of differential equations, their p-adic q-analogs, and many other important topics.

Spline was first defined by Isaac Jacob Schoenberg [3], a Romanian-American mathematician. Schoenberg is also known as spline father. It is well known that spline is a special function that is defined piecewise by polynomials. It is used in many real-world problems involving interpolating problems. Thus, spline is the most popular areas of mathematics and other applied sciences. There are many famous families of splines, such as the uniform B-splines (also known as basis splines) and the Bezier curves, which are not only expressed in terms of the Bernstein basis functions, but also used for curve fitting and the numerical differentiation of experimental data. The B-splines have especially important applications in numerical analysis, computer-aided design and computer graphics, computer physics communication, engineering, etc. This is because the B-splines can be considered as spline functions with a certain degree, smoothness and corresponding minimum support, or as polynomials. This means that any spline functions of a given degree can be written as a linear combination of the B-splines ([4–10]; see also the references cited in each of these earlier works).

The motivation of this article was to give the relations among the generating functions of the Bernstein basis functions, Apostol-type numbers and polynomials, and the uniform B-splines. With the help of these relations, generating functions, and their functional equations, we present some new formulas involving the uniform B-splines and the above numbers and polynomials. Therefore, in order to give the results of this article, generating functions of some special numbers and polynomials will be given below. In addition, some of their properties will be also presented. Another important topic of this article is to use a certain family of generating functions to the uniform B-splines. Then, we derive many new formulas and relations for uniform B-splines. These splines are well established tools not only in computer-aided geometric design, but also in discovering the fundamental properties of the sequences of special numbers and polynomials. Thus, using this method, Goldman [5] gave various formulas for the uniform B-splines. This method was not previously available from classical methods, such as blossoming or the de Boor recurrence, including formulas for sums and alternating sums, for reciprocal moments, etc.

As we mentioned above, Goldman [5] constructed the generating functions for the uniform B-splines for the first time and examined their properties by them. The motivation of this article is to give new formulas and finite sums that include the uniform B-splines, the Bernstein basis functions, and certain families of special polynomials by blending the generating functions with their functional equations. These formulas are also related to the Apostol–Bernoulli numbers and polynomials, the Apostol–Euler numbers and polynomials, the Eulerian numbers and polynomials, and the Stirling numbers.

The method used in this article is different from that of almost all studies in this field. The generating function technique was not used to obtain the identities and formulas related to the uniform B-splines and the Bezier curves, especially the Bernstein polynomials. Recently, only Goldman [5] used generating functions method for the uniform B-splines. In all our studies [6,11], we gave different proofs of many new and known formulas using the generating function with their functional equations method. In this article, very useful new formulas with different and simple proof techniques will be given for the uniform B-splines.

In the remaining parts of this article, the following notations are used:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ demonstrate the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers, respectively. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{C}_p$ be the field of the p-adic completion of algebraic closure of $\mathbb{Q}_p$, which is the set of p-adic rational numbers. Let $\mathbb{Z}_p$ be the set of p-adic integers.

Preliminaries

The Stirling numbers of the second kind are defined by

$$\left(e^t - 1\right)^c = \sum_{n=0}^{\infty} c! S_2(n,c) \frac{t^n}{n!}, \tag{2}$$

where $t \in \mathbb{C}$ and $c \in \mathbb{N}_0$ (see [2,12–14]). The Stirling number is the number of ways to partition a set of n objects into c non-empty subsets. These numbers are also mentioned with different notations in the literature.

The array polynomials are defined by

$$e^{t\omega}\left(e^t - 1\right)^c = \sum_{n=0}^{\infty} c! S_c^n(\omega) \frac{t^n}{n!}, \tag{3}$$

where $t \in \mathbb{C}$ and $c \in \mathbb{N}_0$ (see [2,12–14]). The array polynomials have often appeared in as the partitioning of a set of n objects into c non-empty subset in combinatoric applications, and in separate occasions (see [15]). These numbers are also mentioned with different notations in the literature.

The Apostol–Bernoulli numbers $\mathcal{B}_n(\rho)$ are defined by

$$K_n(t;\rho) = \frac{t}{\rho e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho) \frac{t^n}{n!}, \tag{4}$$

where ρ is an arbitrary (real or complex) parameter and $|t| < 2\pi$ when $\rho = 1$ and $|t| < |\log \rho|$ when $\rho \neq 1$ (see [2,13,16–18]).

Putting $\rho = 1$ in (4), the Apostol–Bernoulli numbers $\mathcal{B}_n(\rho)$ reduce to the Bernoulli numbers B_n. Thus, we assume that $\rho \neq 1$.

Lemma 1. *Let $n \in \mathbb{N}$ with $n > 1$. Then, we have*

$$\mathcal{B}_n(\rho) = \frac{n\rho}{(\rho - 1)^n} \sum_{s=1}^{n-1} (-1)^s s! \rho^{s-1} (\rho - 1)^{n-1-s} S_2(n-1, s). \tag{5}$$

Here, we note that proof of Equation (5) was given by Apostol [16].

The Apostol–Bernoulli polynomials $\mathcal{B}_n(\omega;\rho)$ are defined by

$$K_p(\omega,t;\rho)=K_n(t;\rho)e^{t\omega}=\sum_{n=0}^{\infty}\mathcal{B}_n(\omega;\rho)\frac{t^n}{n!},\tag{6}$$

where $\rho\neq 1$ (see [16]). Applying the Mellin transformation to the generating function (6), the integral representation of the Hurwitz–Lerch zeta function can be found. This function interpolates with the Apostol–Bernoulli polynomials at negative integers ([2] Equation (4), p. 194, see also [16]).

Substituting $\rho=1$ into (6), we have the Bernoulli polynomials:

$$B_n(\omega)=\mathcal{B}_n(\omega;1).$$

When $\omega=0$, we also have the Bernoulli numbers

$$B_n=B_n(0)$$

([13,16,19,20]; see also the references cited in each of these earlier works).

By using (6) and (4), some important properties of Apostol–Bernoulli polynomials and numbers, especially those mentioned for the Apostol [16], are briefly given below:

Equation (4) yields

$$\mathcal{B}_0(\alpha)=0.$$

$$\alpha\mathcal{B}_1(1;\alpha)-\mathcal{B}_1(\alpha)=1$$

and for $n\geq 2$,

$$\alpha\mathcal{B}_n(1;\alpha)-\mathcal{B}_n(\alpha)=0.$$

Combining (6) with (4) yields

$$\mathcal{B}_n(\omega;\alpha)=\sum_{j=0}^{n}\binom{n}{j}\omega^{n-j}\mathcal{B}_j(\alpha),\tag{7}$$

and

$$\alpha\mathcal{B}_n(\omega+1;\alpha)=\mathcal{B}_n(\omega;\alpha)+n\omega^{n-1}\tag{8}$$

(see [2,13,16–18]).

Let ϕ be an arbitrary (real or complex) parameter with $\phi\neq 1$. The Euler–Frobenius polynomials $H_n(\omega;\phi)$ are defined by

$$F_P(t,\omega,\phi)=\frac{1-\phi}{e^t-\phi}e^{t\omega}=\sum_{n=0}^{\infty}H_n(\omega;\phi)\frac{t^n}{n!},\tag{9}$$

where $|t|<2\pi$ when $\frac{1}{\phi}=1$ and $|t|<\left|\log\left(\frac{1}{\phi}\right)\right|$ when $\frac{1}{\phi}\neq 1$ (see [2,3,13]).

Substituting $\omega=0$ into (9), we have the Euler–Frobenius numbers, which are defined by means of the following generating function:

$$F_N(t,\phi)=\frac{1-\phi}{e^t-\phi}=\sum_{n=0}^{\infty}H_n(\phi)\frac{t^n}{n!}.\tag{10}$$

By using (10), we have

$$H_n(\phi)=\begin{cases}1 & \text{for } n=0\\ \frac{1}{\phi}\sum\limits_{j=0}^{n}\binom{n}{j}H_j(\phi) & \text{for } n>0.\end{cases}$$

Combining (4) and (10) for $n > 1$ yields

$$nH_{n-1}(\omega;\phi) = \frac{1-\phi}{\phi}\mathcal{B}_n\left(\frac{1}{\phi}\right)$$

and substituting $\phi = -1$ into(10), we have the Euler numbers of the first kind

$$E_n = H_n(-1)$$

(see [2,14]).

The Apostol–Euler polynomials of the first kind $\mathcal{E}_n(\omega;\rho)$ and the Apostol–Euler numbers of the first kind $\mathcal{E}_n(\rho)$ are defined by means of the following generating functions:

$$F_{P1}(t,\omega;\rho) = \frac{e^{t\omega}}{\rho e^t + 1} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n(\omega;\rho)}{2}\frac{t^n}{n!} \tag{11}$$

and

$$\frac{1}{\rho e^t + 1} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n(\rho)}{2}\frac{t^n}{n!}, \tag{12}$$

where ρ is an arbitrary (real or complex) parameter and $|t| < \pi$ when $\rho = 1$ and $|t| < |\log(-\rho)|$ when $\rho \neq 1$. By combining (11) with (12), we have

$$\mathcal{E}_n(\omega;\rho) = \sum_{j=0}^{n} \binom{n}{j} \omega^{n-j}\mathcal{E}_j(\rho) \tag{13}$$

(see [2,13,17,18]).

Substituting $\rho = 1$ into (11), we have the Euler polynomials of the first kind:

$$E_n(\omega) = \mathcal{E}_n(\omega;1),$$

when $\omega = 0$, we also have $E_n = E_n(0)$ (see [2,13,14,16–18]).

The Euler–Frobenius-type polynomials (the Eulerian polynomials) $A_n(\rho)$ are defined by

$$F_a(t,\rho) = \frac{1}{1-\rho e^{t(1-\rho)}} = \sum_{n=0}^{\infty} \frac{A_n(\rho)}{1-\rho}\frac{t^n}{n!}, \tag{14}$$

where ρ is an arbitrary (real or complex) parameter and $|t| < \pi$ when $\rho = -1$ and $|t| < |\log(\rho)|$ when $\rho \neq -1$.

In [21], Foata also gave the following generating function for the Eulerian polynomials $A_n(\rho)$, which is a slightly different construction to that of [3,22,23]:

$$G_a(t,\rho) = \frac{1}{\rho - e^{t(1-\rho)}} = \sum_{n=0}^{\infty} \frac{A_n(\rho)}{1-\rho}\frac{t^n}{n!}. \tag{15}$$

For $\rho \neq 1$, Equation (14) allows us to compute the Eulerian polynomials $A_n(\rho)$ by means of the following formula:

$$A_n(\rho) = \sum_{j=0}^{n} A_{n,j}\rho^j, \tag{16}$$

where $A_{n,j}$ denotes the Eulerian numbers which are given by

$$A_{n,j} = \sum_{v=0}^{j} (-1)^v \binom{n+1}{v}(j-v)^n, \tag{17}$$

$j = 1, 2, \ldots, n$, $0 \leq j < n$, $n \in \mathbb{N}$ (see [14,22]). The Worpitzky's identity for the Eulerian numbers is given as follows:

$$\omega^n = \sum_{v=0}^{n} \binom{\omega + v - 1}{n} A_{n,v}, \tag{18}$$

where $\omega \in \mathbb{R}$, $n \in \mathbb{N}_0$ (see [22,24]).

These numbers have many interesting combinatoric applications. For instance, let $A = \{1, 2, 3, \ldots, n\}$ and $P(n, j)$ denote the number of permutation of elements of the set A that show exactly j increases between adjacent elements, the first element always being counted as a jump. Thus, it is known that

$$P(n, j) = A_{n,j}$$

and for $n > 1$,

$$\sum_{j=0}^{n-1} A_{n,j} = n!$$

see [22,24]. In the literature, the numbers $A_{n,j}$ are represented by many different notations. Some of these are presented as follows: $A_{n,j}$, $A(n,k)$, $\left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle$, $E(n,k)$, $W_{n,k}$ (see [3,14,23]).

By using the umbral calculus method in (14), we also obtain the following known formula:

$$A_n(\rho) = \rho \sum_{j=0}^{n} \binom{n}{j} (1-\rho)^{n-j} A_j(\rho),$$

where $A_0(\rho) = 1$.

The generating functions for the Bernstein basis functions $B_d^k(\omega)$ are given by

$$f_{\mathbb{B},d}(t, \omega) = (\omega t)^d e^{t(1-\omega)} = \sum_{k=d}^{\infty} d! B_d^k(\omega) \frac{t^k}{k!}, \tag{19}$$

where $f_{\mathbb{B},d}(t, \omega)$ is an analytic function in the complex plane, and $B_d^k(\omega)$ denotes the Bernstein basis functions defined by means of the following expansion:

$$B_d^k(\omega) = \binom{k}{d} \omega^d (1-\omega)^{k-d}, \tag{20}$$

where $0 \leq d \leq k$, and $d, k \in \mathbb{N}_0$,

$$\binom{k}{d} = \frac{k!}{d!(k-d)!};$$

if $d < 0$ or $d > k$, then

$$B_d^k(\omega) = 0$$

(see [4,7,11]). In recent years, many articles have been published covering the generating functions of Bernstein base functions and their applications in many different fields. Some of these studies are given by [6,9,25]). With the method of the moment-generating functions and characteristic functions applied by Simsek [9] to the Bernstein basis functions as an probability distribution function, the families of the uniform B-splines can be investigated with this method, including both their geometric structures and their probability and statistical aspects. Thanks to these methods, the potential to achieve very useful and important applications can be discovered. Similarly, the Bezier-type curves constructed by the Bernstein-type basis functions and their interesting applications were also given in [6], and the same methods may also be applied to the uniform B-splines. The Bernstein polynomials and their important properties are also available in [7,8] in a very efficient and practical way.

In order to construct a generating function for any discrete sequence, its recurrence relations play a very important role. This is a well-known classical way to derive an explicit formula for the generating function by using the recurrence relation. With the help of this way, the explicit formulas of generating functions for many classical families of special numbers and polynomials were found. Goldman [5] also used this method to derive a partial derivative equation for the generating functions of the uniform B-splines. He also solved this PDE to find an explicit formula for the generating functions of the uniform B-splines over arbitrary intervals. Therefore, Goldman [5] constructed the following generating function for the uniform B-splines from $N_{0,n}(\omega;p)$:

$$\begin{aligned} G_0(\omega,t;p) &= \sum_{j=0}^{p}(-1)^j\left(\frac{(\omega-j)^j t^j}{j!}+\frac{(\omega-j)^{j-1}t^{j-1}}{(j-1)!}\right)e^{(\omega-j)t} \\ &= \sum_{n=0}^{\infty} N_{0,n}(\omega;p)t^n, \end{aligned} \tag{21}$$

$p \leq \omega \leq p+1$.

In the literature, the uniform B-splines $N_{0,n}(\omega;p)$ are represented by many different notations. Some of these are presented as follows: $N_{0,n}(\omega)$, $B_k^n(\omega)$, $B_n(\omega)$ (see [3,5,8,26]).

By using (21), with the aid of the generating function method, the Goldman ([5] Theorem 3) proved the following well-known Schoenberg's identity and the de Boor recurrence relation for the uniform B-splines, respectively:

$$N_{0,n}(\omega;p) = \frac{1}{n!}\sum_{j=0}^{p}(-1)^j\binom{n+1}{j}(\omega-j)^n, \tag{22}$$

where $p \leq \omega \leq p+1$ and

$$N_{0,n}(\omega;p) = \frac{\omega}{n}N_{0,n-1}(\omega;p) + \frac{n+1-\omega}{n}N_{1,n-1}(\omega;p)$$

(see ([5] Theorem 3)).

By using (22), a few values of the basis $N_{0,n}(\omega;p)$ are given as follows:

For some values of p, we have

$N_{0,n}(\omega;0) = \frac{1}{n!}\omega^n$, $0 \leq \omega \leq 1$

$N_{0,n}(\omega;1) = \frac{1}{n!}\left(\omega^n - \binom{n+1}{1}(\omega-1)^n\right)$, $1 \leq \omega \leq 2$

$N_{0,n}(\omega;2) = \frac{1}{n!}\left(\omega^n - \binom{n+1}{1}(\omega-1)^n + \binom{n+1}{2}(\omega-2)^n\right)$, $1 \leq \omega \leq 2$

$N_{0,n}(\omega;3) = \frac{1}{n!}\left(\omega^n - \binom{n+1}{1}(\omega-1)^n + \binom{n+1}{2}(\omega-2)^n - \binom{n+1}{3}(\omega-3)^n\right)$, $1 \leq \omega \leq 2$.

For $n=1$, we have

$N_{0,1}(\omega;0) = \omega$, $0 \leq \omega \leq 1$

$N_{0,1}(\omega;1) = -\omega+2$, $1 \leq \omega \leq 2$

$N_{0,1}(\omega;p) = 0$, $p \leq \omega \leq p+1$ if $p \geq 2$.

For $n=2$, we have

$N_{0,2}(\omega;0) = \frac{1}{2}\omega^2$, $0 \leq \omega \leq 1$

$N_{0,2}(\omega;1) = \frac{1}{2}(-2\omega^2+6\omega-3)$, $1 \leq \omega \leq 2$

$N_{0,2}(\omega;2) = \frac{1}{2}(\omega^2-6\omega+9)$, $2 \leq \omega \leq 3$

$N_{0,2}(\omega;p) = 0$, $p \leq \omega \leq p+1$ if $p \geq 3$.

Thus, we see that

$$N_{0,n}(\omega;p) = 0,$$

$p \leq \omega \leq p+1$ if $p \geq n+1$ and so on.

In the relevant sections above, we have given information based on important sources about the rough usage areas, construction, and some applications of generating functions for special numbers and polynomials. Now, in the next paragraph, we briefly mention another method for constructing generating functions for certain families of special numbers and polynomials on the set of p-adic integers. We know that, by using the Volkenborn integral (bosonic p-adic integral) and Fermionic p-adic integral methods, the generating functions for the Bernoulli-type numbers and polynomials and the Euler-type numbers and polynomials are constructed, respectively, ([13,27–29]; this can also be seen in the references cited in each of these earlier works).

A very brief introduction to the p-adic integrals are presented as follows:

Let $\Psi : \mathbb{Z}_p \to \mathbb{C}_p$ be a uniformly differential function on $\mathbb{Z}_p$. The Volkenborn integral (or the p-adic bosonic integral) is given by

$$\int\limits_{\mathbb{Z}_p} \Psi(\omega) d\mu_1(\omega) = \lim_{m\to\infty} p^{-m} \sum_{v=0}^{p^m-1} \Psi(v), \tag{23}$$

where μ_1 denotes the Haar distribution:

$$\mu_1(\omega) = p^{-m}$$

(see [13,27–29]; also see also the references cited in each of these earlier works).

The fermionic p-adic integral is given by

$$\int\limits_{\mathbb{Z}_p} \Psi(\omega) d\mu_{-1}(\omega) = \lim_{m\to\infty} \sum_{v=0}^{p^m-1} (-1)^v \Psi(v), \tag{24}$$

where μ_{-1} is defined by

$$\mu_{-1}(\omega) = (-1)^\omega$$

(for detail, see [13,27,30]).

The remaining parts of this article are also briefly summarized as follows:

In the Introduction section, we gave some properties of the certain family of generating functions of special numbers and polynomials.

In Section 2, by using both generating functions with their functional equations and the bosonic p-adic integral and Fermionic p-adic integral methods, we derive some novel computation formulas for the Apostol–Bernoulli polynomials, the Apostol–Euler polynomials, the Eulerian numbers and polynomials, and the Euler–Frobenius numbers and polynomials.

In Section 3, with the aid of generating functions with their derivative and functional equations, we give many new identities and relations for the uniform B-splines and the Bernstein basis functions. By applying the Laplace transform to generating functions for the uniform B-splines, the series representations of the uniform B-spline and the Bernstein basis functions are given.

Finally, this article is concluded with the conclusion section.

2. Computation Formulas of the Apostol–Bernoulli Polynomials, Eulerian Numbers, and Polynomials

In this section, by using generating functions, we derive some new formulas of the Apostol–Bernoulli polynomials and numbers, the Apostol–Euler polynomials and numbers, and the Eulerian numbers. These formulas are involved the Stirling numbers of the second kind, the Bernstein basis function, and the array polynomials. By applying p-adic integrals to these formulas, and using the Witt identities for the Bernoulli and Euler numbers, we also derive some identities involving finite sums. These formulas have the potential to be used in many fields, especially spline theory.

By combining (4) with (6), we have

$$\sum_{n=0}^{\infty} \mathcal{B}_n(\omega;\rho)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \mathcal{B}_j(\rho)\frac{\omega^{n-j}t^n}{j!(n-j)!}.$$

Thus, the coefficients of $\frac{t^n}{n!}$ on both sides of the previous equation are equalized yields (7).

Joining (7) with (5) and (20), for $n \in \mathbb{N}_0$, after some calculations, we obtain the following relation among the Bernstein basis functions, the Apostol–Bernoulli polynomials, and the Stirling numbers of the first kind:

$$\mathcal{B}_n(\omega;\rho) = \sum_{j=1}^{n}\sum_{s=1}^{j-1}(-1)^{j-1}\frac{\binom{n}{j}}{\binom{j-1}{s}}\frac{js!\omega^{n-j}}{(\rho-1)^j}B_s^{j-1}(\rho)S_2(j-1,s). \tag{25}$$

A relation between the Apostol–Bernoulli polynomials and the array polynomials is given by the following theorem:

Theorem 1. *Let $m \in \mathbb{N}$ with $m > 1$. Then we have*

$$\mathcal{B}_m(\omega;\rho) = \frac{m}{\rho-1}\left(\omega^{m-1} + \sum_{n=1}^{m-1}\left(\frac{\rho}{1-\rho}\right)^n n!S_n^{m-1}(\omega)\right).$$

Proof. Using (6), we also obtain

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\omega;\rho)\frac{t^m}{m!} = \frac{t}{\rho-1}e^{t\omega}\left(1 + \sum_{n=1}^{\infty}\left(\frac{\rho}{1-\rho}\right)^n \left(e^t-1\right)^n\right). \tag{26}$$

Combining (26) with (3), since the array polynomials $S_n^m(\omega) = 0$ when $m < n$, after some calculations, we obtain

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\omega;\rho)\frac{t^m}{m!} = \frac{t}{\rho-1}\left(\sum_{m=0}^{\infty}\frac{\omega^m t^m}{m!} + \sum_{m=0}^{\infty}\sum_{n=1}^{m}\left(\frac{\rho}{1-\rho}\right)^n n!S_n^m(\omega)\frac{t^m}{m!}\right).$$

Therefore

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\omega;\rho)\frac{t^m}{m!} = \frac{1}{\rho-1}\left(\sum_{m=0}^{\infty} m\left(\omega^{m-1} + \sum_{n=1}^{m-1}\left(\frac{\rho}{1-\rho}\right)^n n!S_n^{m-1}(\omega)\right)\frac{t^m}{m!}\right).$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the last equation, we arrive at the desired result. □

A relation between the Apostol–Bernoulli polynomials and the Stirling numbers of the first kind is given by the following theorem:

Theorem 2. *Let $m \in \mathbb{N}$ with $m > 1$. Then, we have*

$$\mathcal{B}_m(\omega;\rho) = \frac{m}{\rho-1}\left(\omega^{m-1} + \sum_{v=0}^{m-1}\binom{m-1}{v}\omega^{m-v-1}\sum_{n=1}^{v}\left(\frac{\rho}{1-\rho}\right)^n n!S_2(v,n)\right).$$

Proof. Combining (26) with (2), since the array polynomials $S_2(v,n) = 0$ when $v < n$, after some calculations, we obtain

$$\sum_{m=0}^{\infty} \mathcal{B}_m(\omega;\rho)\frac{t^m}{m!} = \frac{t}{\rho-1}\left(\sum_{m=0}^{\infty}\frac{\omega^m t^m}{m!} + \sum_{m=0}^{\infty}\sum_{v=0}^{m}\binom{m}{v}\omega^{m-v}\sum_{n=1}^{v}\left(\frac{\rho}{1-\rho}\right)^n n!S_2(v,n)\frac{t^m}{m!}\right).$$

Therefore

$$\sum_{m=0}^{\infty}\mathcal{B}_m(\omega;\rho)\frac{t^m}{m!}$$
$$= \frac{1}{\rho-1}\left(\sum_{m=0}^{\infty} m\left(\omega^{m-1}+\sum_{v=0}^{m-1}\binom{m-1}{v}\omega^{m-v-1}\sum_{n=1}^{v}\left(\frac{\rho}{1-\rho}\right)^n n!S_2(v,n)\right)\frac{t^m}{m!}\right).$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the last equation, we arrive at the desired result. □

Theorem 3. *Let $m \in \mathbb{N}$. Then, we have*

$$\mathcal{E}_m(\rho) = \frac{2}{1-\rho}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n n! y_1(m,n;1), \tag{27}$$

where $\left|\frac{\rho}{1-\rho}\right| < 1$.

Proof. By using (12), we obtain

$$\sum_{m=0}^{\infty}\mathcal{E}_m(\rho)\frac{t^m}{m!} = \frac{2}{1-\rho}\left(1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n\left(e^t+1\right)^n\right).$$

Combining the above equation with the following generating function

$$\left(e^t+1\right)^n = n!\sum_{m=0}^{\infty} y_1(m,n;1)\frac{t^m}{m!}$$

(see ([12] Equation (8))), we obtain

$$\begin{aligned}\mathcal{E}_0(\rho)+\sum_{m=1}^{\infty}\mathcal{E}_m(\rho)\frac{t^m}{m!} &= \frac{2}{1-\rho}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n n! y_n(0,n;1)+\\ &\quad +\frac{2}{1-\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n n! y_n(m,n;1)\frac{t^m}{m!},\end{aligned}$$

where $\left|\frac{\rho}{1-\rho}\right| < 1$, we obtain

$$\mathcal{E}_0(\rho)+\sum_{m=1}^{\infty}\mathcal{E}_m(\rho)\frac{t^m}{m!} = \frac{2}{1+\rho}+\frac{2}{1-\rho}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n n! y_1(m,n;1)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the desired result. □

Since

$$y_1(m,n;1) = \frac{1}{n!}\sum_{h=0}^{n}\binom{n}{h}h^m$$

(see [12]) Equation (27) reduces to the following corollary:

Corollary 1. *Let $m \in \mathbb{N}$. Then, we have*

$$\mathcal{E}_m(\rho) = \frac{2}{1-\rho}\sum_{n=1}^{\infty}\sum_{h=0}^{n}(-1)^n\binom{n}{h}h^m\left(\frac{\rho}{1-\rho}\right)^n, \tag{28}$$

where $\left|\frac{\rho}{1-\rho}\right| < 1$.

Joining the following formula

$$\sum_{h=0}^{n}\binom{n}{h}h^m=\frac{d^m}{dt^m}\left\{\left(e^t+1\right)^n\right\}|_{t=0},$$

(see [12,31]) with (28) yields the following result:

Corollary 2. *Let $m\in\mathbb{N}$. Then we have*

$$\mathcal{E}_m(\rho)=\frac{2}{1-\rho}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\rho}{1-\rho}\right)^n\frac{d^m}{dt^m}\left\{\left(e^t+1\right)^n\right\}|_{t=0},$$

where $\left|\frac{\rho}{1-\rho}\right|<1$.

Theorem 4. *Let $m\in\mathbb{N}$. Then, we have*

$$A_m(\rho)=\sum_{n=1}^{m}\frac{n!}{\binom{m}{n}}B_n^m(\rho)S_2(m,n).$$

Proof. Using (14), we obtain

$$A_0(\rho)+\sum_{m=1}^{\infty}A_m(\rho)\frac{t^m}{m!}=1+\sum_{n=1}^{\infty}\left(\frac{\rho}{1-\rho}\right)^n\left(e^{t(1-\rho)}-1\right)^n. \tag{29}$$

By combining (29) and (2), we obtain

$$A_0(\rho)+\sum_{m=1}^{\infty}A_m(\rho)\frac{t^m}{m!}=1+\sum_{m=1}^{\infty}\sum_{n=1}^{m}\left(\frac{\rho}{1-\rho}\right)^n n!(1-\rho)^m S_2(m,n)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation yields

$$A_m(\rho)=\sum_{n=1}^{m}\left(\frac{\rho}{1-\rho}\right)^n n!(1-\rho)^m S_2(m,n). \tag{30}$$

Combining (30) with (20), we arrive at the assertion of the theorem. □

By combining (10) and (14) yields the following functional equation:

$$F_a(t,\rho)=F_N\left((1-\rho)t,\frac{1}{\rho}\right).$$

Using this equation, we obtain a relation between the Eulerian polynomials and the Euler–Frobenius numbers with the following corollary:

Corollary 3. *Let $n\in\mathbb{N}_0$. Then, we have*

$$A_n(\rho)=\begin{cases}1 & \textit{for } n=0\\ \rho(1-\rho)^n\sum\limits_{j=0}^{n}\binom{n}{j}H_j\left(\frac{1}{\rho}\right) & \textit{for } n>0.\end{cases}$$

By combining (4) and (15), we have the following functional equation:

$$-tG_a(t,\rho)=\frac{1}{\rho}K_n((1-\rho)t,\rho).$$

Using the above equation, we obtain a known relation between the Eulerian polynomials and the Apostol–Bernoulli numbers by the following corollary:

Corollary 4. *Let $n \in \mathbb{N}$ with $(n > 1)$. Then, we have*

$$A_{n-1}(\rho) = -\frac{(1-\rho)^n}{n\rho}\mathcal{B}_n(\rho). \tag{31}$$

Remark 1. *Combining (30) with (5), we also arrive at (31). Here, we note that the formula expressed by Equation (31) has proof using different methods, for detail, as can also be seen in also [18].*

Using (5) and (31), we give a few values of the polynomials $A_n(\rho)$ as follows:

$$\begin{aligned}
A_1(\rho) &= \frac{(\rho-1)^2}{-2\rho}\mathcal{B}_2(\rho) = A_{1,0} = 1, \\
A_2(\rho) &= \frac{(\rho-1)^3}{3\rho}\mathcal{B}_3(\rho) = A_{2,0}\rho + A_{2,1} = \rho + 1, \\
A_3(\rho) &= \frac{(\rho-1)^4}{-4\rho}\mathcal{B}_4(\rho) = A_{3,0}\rho^2 + A_{3,1}\rho + A_{3,2} = \rho^2 + 4\rho + 1, \\
A_4(\rho) &= \frac{(\rho-1)^5}{5\rho}\mathcal{B}_5(\rho) = A_{4,0}\rho^3 + A_{4,1}\rho^2 + A_{4,2}\rho + A_{4,3} = \rho^3 + 11\rho^2 + 11\rho + 1, \\
A_5(\rho) &= \frac{(\rho-1)^6}{-6\rho}\mathcal{B}_6(\rho) = A_{5,0}\rho^4 + A_{5,1}\rho^3 + A_{5,2}\rho^2 + A_{5,3}\rho + A_{5,4} \\
&= \rho^4 + 26\rho^3 + 66\rho^2 + 26\rho + 1, \\
A_6(\rho) &= \frac{(\rho-1)^7}{7\rho}\mathcal{B}_7(\rho) = A_{6,0}\rho^5 + A_{6,1}\rho^4 + A_{6,2}\rho^3 + A_{6,3}\rho^2 + A_{5,4}\rho + A_{6,5} \\
&= \rho^5 + 57\rho^4 + 302\rho^3 + 302\rho^2 + 57\rho + 1,
\end{aligned}$$

and so on.

Boyadzhiev [18] gave a relation between the Apostol–Bernoulli numbers and the geometric polynomials $W_n(w)$ is given as follows:

$$\mathcal{B}_n(\rho) = \frac{n}{\rho - 1}W_{n-1}\left(\frac{\rho}{1-\rho}\right) \tag{32}$$

and

$$\mathcal{B}_n\left(\frac{\rho}{1+\rho}\right) = -n(\rho+1)W_{n-1}(\rho),$$

where $n \in \mathbb{N}$ and

$$W_n(\rho) = \sum_{j=0}^{n} j! S_2(n,j)\rho^j.$$

By using the Euler (derivative) operator

$$\rho\frac{d}{d\rho},$$

Boyadzhiev [18] also showed that

$$\left(\rho\frac{d}{d\rho}\right)^k\left\{\frac{1}{1-\rho}\right\} = \frac{1}{1-\rho}\omega_k\left(\frac{\rho}{1-\rho}\right) = \sum_{v=0}^{\infty} v^k\rho^v,$$

where $|\rho| < 1$. Joining the above equation with (16), we obtain the following result:

$$A_k(\rho) = (1-\rho)^{k+1}\left(\rho\frac{d}{d\rho}\right)^k\left\{\frac{1}{1-\rho}\right\}.$$

Combining the above equation with (31) yields

$$\mathcal{B}_{k+1}(\rho) = -(k+1)\left(\rho\frac{d}{d\rho}\right)^{k}\left\{\frac{1}{1-\rho}\right\}. \tag{33}$$

Joining the above equation with the following well-known equation which is combined (10) with (6):

$$\mathcal{E}_n(\omega;\lambda) = -\frac{2}{n+1}\mathcal{B}_{n+1}(\omega;-\lambda) \tag{34}$$

(see [2]) we also obtain

$$\mathcal{E}_n(\omega;\lambda) = \left(\rho\frac{d}{d\rho}\right)^{k}\left\{\frac{1}{1-\rho}\right\}. \tag{35}$$

There are many different proofs of the Equations (33) and (35). Some of them were also given by (see [2,17–20]).

Theorem 5. *Let $n \in \mathbb{N}$. Then, we have*

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}j(\rho-1)^{-j}A_{j-1}(\rho)\left(\rho\sum_{k=0}^{n-j}\binom{n-j}{k}B_k - B_{n-j}\right) = nB_{n-1}.$$

Proof. Combining (31) with (7) and (8) yields

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}j(\rho-1)^{-j}A_{j-1}(\rho)\left(\rho(\omega+1)^{n-j} - \omega^{n-j}\right) = n\omega^{n-1}, \tag{36}$$

where $n \in \mathbb{N}$. By applying the Volkenborn integral, given by Equations (23)–(36) and using

$$\int_{\mathbb{Z}_p}\omega^k d\mu_1(\omega) = B_k, \tag{37}$$

which is known as the Witt identity for the Bernoulli numbers, for $k > 0$,

$$B_{2k+1} = 0$$

which yields

$$\int_{\mathbb{Z}_p}\omega^{2k+1} d\mu_1(\omega) = 0,$$

where $k > 0$; thus, we obtain

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}j(\rho-1)^{-j}A_{j-1}(\rho)\left(\rho B_{n-j}(1) - B_{n-j}\right) = nB_{n-1}.$$

Combining the above equation with the following formula

$$B_{n-j}(1) = B_{n-j} + \sum_{k=0}^{n-j-1}\binom{n-j}{k}B_k,$$

we arrive at the desired result. □

Theorem 6. *Let $n \in \mathbb{N}$. Then, we have*

$$\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}j(\rho-1)^{-j}A_{j-1}(\rho)\left(\rho\sum_{k=0}^{n-j}\binom{n-j}{k}E_k - E_{n-j}\right) = nE_{n-1}.$$

Proof. By applying the Ferminoic p-adic integral, given by Equations (24)–(36) and using

$$\int_{\mathbb{Z}_p} \omega^k d\mu_{-1}(\omega) = E_k,$$

which is known as the Witt identity for the Euler numbers, for $k > 0$,

$$E_{2k} = 0$$

which yields

$$\int_{\mathbb{Z}_p} \omega^{2k} d\mu_{-1}(\omega) = 0,$$

where $k > 0$, thus we obtain

$$\sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} j(\rho-1)^{-j} A_{j-1}(\rho) \left(\rho E_{n-j}(1) - E_{n-j}\right) = nE_{n-1}.$$

Comparing the above equation with following formula

$$E_{n-j}(1) = E_{n-j} + \sum_{k=0}^{n-j-1} \binom{n-j}{k} E_k,$$

we arrive at the desired result. □

3. Identities, Relations, and Series Representations for the Uniform *B*-Splines and the Bernstein Basis Functions

In this section, we give many new formulas involving the uniform B-spline, the Apostol–Bernoulli numbers, and the Eulerian numbers. We also give some functional equations and partial derivative equations for generating the functions of the uniform B-splines and the Bernstein basis functions. Using these equations, we derive generalized formula for the de Boor recurrence relation, a higher-order derivative formula for the uniform B-spline, and a relation between the uniform B-spline and the Bernstein basis function. By applying the Laplace transform to generating functions, we give the series representations for the uniform B-splines.

3.1. Relations among the Uniform B-Spline, Apostol–Bernoulli Numbers and Eulerian Numbers

Here, by using p-adic integrals, we introduce very interesting results among the B-Spline, the Apostol–Bernoulli numbers, and the Eulerian numbers.

Let us briefly give these interesting relationships as follows:

Theorem 7. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{B}_{n+1}(\rho) = (-1)^n (\rho-1)^{n+1} (n+1)! \sum_{j=0}^{n} N_{0,n}(j;j)\rho^j.$$

Proof. By using (14) and (4), we obtain

$$\sum_{n=0}^{\infty} A_n(\rho) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} \mathcal{B}_n(\rho) \frac{t^n}{n!}.$$

Joining (31) and (22) with (16) and (17), after some calculations, we arrive at the desired result. □

Substituting $\omega = p$ into Equation (22), a relation between the uniform B-spline and the Eulerian numbers is also given by

$$N_{0,n}(p;p) = \frac{1}{n!}A_n(p). \tag{38}$$

Here, we note that there are another proof of the formula in Equation (38). For instance, for detail, also see [4,5,32].

3.2. Relations between the Uniform B-Spline and the Bernstein Basis Functions

By using generating functions and their functional equations and PDEs, relations between the uniform B-spline and the Bernstein basis functions are given.

Theorem 8. *Let $n, p \in \mathbb{N}_0$ and $p \leq \omega \leq p+1$. Then we have*

$$N_{0,n}(\omega;p) = \frac{1}{n!}\sum_{v=0}^{n}\binom{n}{v}\sum_{j=0}^{p}(-1)^{v+j}\left(B_j^v(\omega-j) + B_{j-1}^v(\omega-j)\right). \tag{39}$$

Proof. By using generating functions for the uniform B-spline $N_{0,n}(\omega;p)$ and the Bernstein basis function $B_j^v(\omega)$, we define the following functional equation:

$$G_0(\omega,t;p) = \sum_{j=0}^{p}(-1)^j\left(f_{\mathbb{B},j}(-t,\omega-j) + f_{\mathbb{B},j-1}(-t,\omega-j)\right)e^t. \tag{40}$$

Combining (40) with (19) and (21), we obtain

$$\sum_{n=0}^{\infty}N_{0,n}(\omega;p)t^n = \sum_{j=0}^{p}\sum_{n=0}^{\infty}(-1)^{n+j}\left(B_j^n(\omega-j) + B_{j-1}^n(\omega-j)\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{t^n}{n!}.$$

Using the Cauchy product rule in the above equation, we obtain

$$\sum_{n=0}^{\infty}N_{0,n}(\omega;p)t^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{v=0}^{n}(-1)^{v+j}\binom{n}{v}\sum_{j=0}^{p}\left(B_j^v(\omega-j) + B_{j-1}^v(\omega-j)\right).$$

Now, by comparing the coefficients of t^n on both sides of the above equation, we arrive at the assertion (39) of Theorem 8. □

Remark 2. *Equation* (39) *gives us the modification of the Schoenberg's identity. This identity was proved by Goldman* ([5] Theorem 3) *with the aid of the generating function method.*

Theorem 9. *Let $n, p, v \in \mathbb{N}_0$ and $p \leq \omega \leq p+1$. Then, we have*

$$\begin{aligned} \frac{d^v}{d\omega^v}\{N_{0,n}(\omega;p)\} \quad &= \quad \sum_{d=0}^{n}\binom{n}{d}\binom{d}{v}v!\sum_{m=0}^{v}(-1)^{v+n-d-m}\binom{v}{m} \\ &\quad \times\sum_{j=0}^{p}(-1)^j\left(B_{j-m}^{d-v}(\omega-j) + B_{j-m-1}^{d-v}(\omega-j)\right). \end{aligned} \tag{41}$$

Proof. Substituting $t = -z$ into (40), we have

$$G_0(\omega,-z;p) = \sum_{j=0}^{p}(-1)^j\left(f_{\mathbb{B},j}(z,\omega-j) + f_{\mathbb{B},j-1}(z,\omega-j)\right)e^{-z}. \tag{42}$$

Taking the vth derivative of Equation (42), with respect to ω, we have the following partial differential equation:

$$\frac{\partial^v}{\partial\omega^v}\{G_0(\omega,-z;p)\} = \sum_{j=0}^{p}(-1)^j\left(\frac{\partial^v}{\partial\omega^v}\{f_{\mathbb{B},j}(z,\omega-j)\} + \frac{\partial^v}{\partial\omega^v}\{f_{\mathbb{B},j-1}(z,\omega-j)\}\right)e^{-z}.$$

Combining the above equation with the following PDE for $f_{\mathbb{B},j}(t,\omega)$:

$$\frac{\partial^v}{\partial\omega^v}\{f_{\mathbb{B},j}(z,\omega)\} = \sum_{m=0}^{v}(-1)^{v-m}\binom{v}{m}z^v f_{\mathbb{B},j-m}(z,\omega)$$

(see [11] Equation (15)), we obtain the following PDE for the generating function $G_0(\omega,-z;p)$:

$$\frac{\partial^v}{\partial\omega^v}\{G_0(\omega,-z;p)\} = \sum_{j=0}^{p}\sum_{m=0}^{v}(-1)^{j+v-m}\binom{v}{m}z^v\left(f_{\mathbb{B},j-m}(z,\omega-j)+f_{\mathbb{B},j-m-1}(z,\omega-j)\right)e^{-z}.$$

Therefore,

$$\begin{aligned}\frac{\partial^v}{\partial\omega^v}\{G_0(\omega,-z;p)\} &= \sum_{j=0}^{p}\sum_{m=0}^{v}(-1)^{v-m+j}\binom{v}{m}\\ &\quad\times\left(f_{\mathbb{B},j-m}(z,\omega-j)+f_{\mathbb{B},j-m-1}(z,\omega-j)\right)z^v e^{-z}.\end{aligned}$$

Combining the right-hand side of the above equation with the following the higher-order derivative formula of the Bernstein basis function:

$$\frac{d^v}{d\omega^v}\left\{B_j^n(\omega)\right\} = \frac{n!}{(n-v)!}\sum_{m=0}^{v}(-1)^{v-m}\binom{v}{m}B_{j-m}^{n-v}(\omega)$$

(see [11] Equation (16)), we arrive at the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\frac{d^v}{d\omega^v}\{N_{0,n}(\omega;p)\}z^n &= \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}\sum_{n=0}^{\infty}\binom{n}{v}v!\sum_{m=0}^{v}(-1)^{v-m}\binom{v}{m}\\ &\quad\times\sum_{j=0}^{p}(-1)^j\left(B_{j-m}^{n-v}(\omega-j)+B_{j-m-1}^{n-v}(\omega-j)\right)\frac{z^n}{n!}.\end{aligned}$$

By using the Cauchy product rule in the right-hand side of the above equation yields

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\tfrac{d}{d\omega}\{N_{0,n}(\omega;p)\}z^n &= \sum_{n=0}^{\infty}\tfrac{z^n}{n!}\sum_{d=0}^{n}(-1)^{n-d}\tbinom{n}{d}\tbinom{d}{v}v!\\ &\quad\times\sum_{m=0}^{v}\tbinom{v}{m}\sum_{j=0}^{p}(-1)^{j+v-m}\left(B_{j-m}^{d-v}(\omega-j)+B_{j-m-1}^{d-v}(\omega-j)\right).\end{aligned}$$

Comparing the coefficients of z^n on both sides of the above equation, we arrive at the assertion (41) of the Theorem. □

Remark 3. *Substituting $v=1$ into (41), we obtain the derivative formula $\frac{d}{d\omega}\{N_{0,n}(\omega;p)\}$, which was proven by Goldman* ([5] Theorem 4).

We give the generalized de Boor recurrence relation by the following theorem:

Theorem 10. *Let $n, p, v \in \mathbb{N}_0$ and $p \leq \omega \leq p+1$. Then, we have*

$$\begin{aligned} N_{0,n+v}(\omega; p) &= \frac{1}{n!v!\binom{n+v}{v}} \sum_{d=0}^{n} (-1)^{v+d} \binom{n}{d} \sum_{m=0}^{v} (-1)^{v-m} \binom{v}{m} \\ &\quad \times \sum_{j=0}^{p} (-1)^j \sum_{c=0}^{m} B_c^m(\omega - j) \Big(B_{j-c}^d(\omega - j) + B_{j-c-1}^d(\omega - j) \Big). \end{aligned} \tag{43}$$

Proof. By applying the Leibnitz's formula for the vth derivative, with respect to z, to Equation (42), with the help of the following partial differential equation for the generating function $f_{\mathbb{B},j}(z, \omega)$:

$$\frac{\partial^m}{\partial z^m} \{ f_{\mathbb{B},j}(z, \omega) \} = \sum_{c=0}^{m} B_c^m(\omega) f_{\mathbb{B},j-c}(z, \omega)$$

(see [11] Equation (18)), we obtain

$$\begin{aligned} \frac{\partial^v}{\partial z^v} \{ G_0(\omega, -z; p) \} &= e^{-z} \sum_{j=0}^{p} \sum_{m=0}^{v} (-1)^{j+v-m} \binom{v}{m} \\ &\quad \times \sum_{c=0}^{m} B_c^m(\omega) \big(f_{\mathbb{B},j-c}(z, \omega - j) + f_{\mathbb{B},j-c-1}(z, \omega - j) \big). \end{aligned} \tag{44}$$

Combining (40) with (19) and (44), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} (-1)^{n+v} v! \binom{n+v}{v} N_{0,n+v}(\omega; p) z^n \\ = &\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{d=0}^{n} (-1)^{n-d} \binom{n}{d} \sum_{j=0}^{p} \sum_{m=0}^{v} (-1)^{j+v-m} \binom{v}{m} \\ &\times \sum_{c=0}^{m} B_c^m(\omega) \Big(B_{j-c}^d(\omega - j) + B_{j-c-1}^d(\omega - j) \Big). \end{aligned}$$

Comparing the coefficients of z^n on both sides of the above equation, we arrive at the assertion (43) of Theorem. □

Remark 4. *Substituting $v = 1$ into (43), we obtain the de Boor recurrence relation, which was proven by Goldman* ([5] Theorem 5)*.*

3.3. Series Representations for the Uniform B-Spline and the Bernstein Basis Functions

By applying the Laplace transform to the generating functions for the uniform B-spline and the Bernstein basis functions, we give some series representations. We also give some formulas for these series.

Theorem 11. *Let $p \in \mathbb{N}_0$, $p \leq \omega + y \leq p+1$ and $\omega > 0$, $y > 0$. Then, we have*

$$\sum_{n=0}^{\infty} N_{0,n}(\omega; p) \frac{n!}{y^{n+2}} = \sum_{j=0}^{p} (-1)^j \frac{(\omega + y - j)^{j-1}}{(j - \omega)^{j+1}}. \tag{45}$$

Proof. The following functional equation is derived from (21):

$$G_0(\omega + y, t; p) e^{-yt} = \sum_{j=0}^{p} (-1)^j \left(\frac{(\omega + y - j)^j t^j}{j!} + \frac{(\omega + y - j)^{j-1} t^{j-1}}{(j-1)!} \right) e^{-(j-\omega)t},$$

where $p \in \mathbb{N}_0$, $p \leq \omega + y \leq p+1$ and $\omega > 0, y > 0$. Applying the Laplace transform to the above functional equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} N_{0,n}(\omega;p) \int_0^{\infty} t^n e^{-yt} dt \\ = &\sum_{j=0}^{p} (-1)^j \left(\frac{(\omega+y-j)^j}{j!} \int_0^{\infty} t^j e^{-t(j-\omega)} dt + \frac{(\omega+y-j)^{j-1}}{(j-1)!} \int_0^{\infty} t^{j-1} e^{-t(j-\omega)} dt \right).\end{aligned}$$

Joining the above equation with the following well-known identity

$$\int_0^{\infty} t^j e^{-t} dt = j!,$$

we obtain

$$\sum_{n=0}^{\infty} N_{0,n}(\omega;p) \frac{n!}{y^{n+1}} = \sum_{j=0}^{p} (-1)^j \left(\frac{(\omega+y-j)^j}{(j-\omega)^{j+1}} + \frac{(\omega+y-j)^{j-1}}{(j-\omega)^j} \right).$$

After some elementary calculations, the proof of theorem is completed. □

Substituting $y = 0$ into (45) yields the following result:

Corollary 5. *Let $n, p \in \mathbb{N}_0$ and $p \leq \omega \leq p+1$. Then, we have*

$$\sum_{n=0}^{\infty} n! N_{0,n}(\omega;p) = \sum_{j=0}^{p} (-1)^j \frac{(\omega+1-j)^{j-1}}{(j-\omega)^{j+1}}$$

Theorem 12. *Let $n, p \in \mathbb{N}_0$ and $p \leq \omega \leq p+1$ and $y > 0$. Then we have*

$$\sum_{n=0}^{\infty} \frac{n! N_{0,n}(\omega;p)}{(y+1)^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \sum_{j=0}^{p} \frac{B_j^n(\omega-j) - B_{j-1}^n(\omega-j)}{y^{n+1}}.$$

Proof. The following functional equation is derived from (40):

$$G_0(\omega,t;p) e^{-(1+y)t} = \sum_{j=0}^{p} \left(f_{\mathbb{B},j}(-t,\omega-j) - f_{\mathbb{B},j-1}(-t,\omega-j) \right) e^{-yt},$$

where $y > 0$. By applying the Laplace transform to the above functional equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} N_{0,n}(\omega;p) \int_0^{\infty} t^n e^{-(1+y)t} dt \\ = &\sum_{j=0}^{p} \sum_{n=0}^{\infty} (-1)^n \left(\frac{B_j^n(\omega-j) - B_{j-1}^n(\omega-j)}{n!} \int_0^{\infty} t^n e^{-ty} dt \right).\end{aligned}$$

Since the rest of proof of this theorem is similar to the that of (45), we skip this proof here. □

4. Conclusions

In this article, the results and how they were obtained are discussed together with their methods. For this purpose, by applying p-adic integrals to the identities found for special polynomials, new formulas were given that maybe serve as a resource for researchers on the subject. In addition, by using the differential and functional equations of the generating functions, some novel formulas for special numbers and special polynomials were found. In addition, thanks to these methods, some formulas were given for finite sums containing these special numbers and polynomials. Derivative formulas of the B-spline curves were found using the higher-order derivative formula of Bernstein base functions. Moreover, by applying the Laplace transform to the new generating functional equations, we found infinite series containing B-spline curves.

It is planned that new mathematical models will be developed with the aid of different applications using functional equations and the differential equations of the generating functions for the B-splines. By blending these equations, we shall investigate novel connections with spline theory, special polynomials, and Bernstein basis functions.

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