mathematics

Review

Going Next after "A Guide to Special Functions in Fractional Calculus": A Discussion Survey

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Abstract: In the survey Kiryakova: "A Guide to Special Functions in Fractional Calculus" (published in this same journal in 2021) we proposed an overview of this huge class of special functions, including the Fox H-functions, the Fox–Wright generalized hypergeometric functions ${}_p\Psi_q$ and a large number of their representatives. Among these, the Mittag-Leffler-type functions are the most popular and frequently used in fractional calculus. Naturally, these also include all "Classical Special Functions" of the class of the Meijer's G- and ${}_pF_q$-functions, orthogonal polynomials and many elementary functions. However, it so happened that almost simultaneously with the appearance of the Mittag-Leffler function, another "fractionalized" variant of the exponential function was introduced by Le Roy, and in recent years, several authors have extended this special function and mentioned its applications. Then, we introduced a general class of so-called (multi-index) Le Roy-type functions, and observed that they fall in an "Extended Class of SF of FC". This includes the I-functions of Rathie and, in particular, the $\overline{H}$-functions of Inayat-Hussain, studied also by Buschman and Srivastava and by other authors. These functions initially arose in the theory of the Feynman integrals in statistical physics, but also include some important special functions that are well known in math, like the polylogarithms, Riemann Zeta functions, some famous polynomials and number sequences, etc. The I- and $\overline{H}$-functions are introduced by Mellin–Barnes-type integral representations involving multi-valued fractional order powers of Γ-functions with a lot of singularities that are branch points. Here, we present briefly some preliminaries on the theory of these functions, and then our ideas and results as to how the considered Le Roy-type functions can be presented in their terms. Next, we also introduce Gelfond–Leontiev generalized operators of differentiation and integration for which the Le Roy-type functions are eigenfunctions. As shown, these "generalized integrations" can be extended as kinds of generalized operators of fractional integration, and are also compositions of "Le Roy type" Erdélyi–Kober integrals. A close analogy appears with the Generalized Fractional Calculus with H- and G-kernel functions, thus leading the way to its further development. Since the theory of the I- and $\overline{H}$-functions still needs clarification of some details, we consider this work as a "Discussion Survey" and also provide a list of open problems.

Keywords: special functions; Le Roy function; Mittag-Leffler function; extensions of H-functions; fractional calculus; eigenfunctions

MSC: 30D20; 33E20; 33E12; 30D15; 26A33; 34L10

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Citation: Kiryakova, V.; Paneva-Konovska, J. Going Next after "A Guide to Special Functions in Fractional Calculus": A Discussion Survey. *Mathematics* **2024**, *12*, 319. https://doi.org/10.3390/math12020319

Academic Editors: Jose Luis Lopez and Dmitrii Karp

Received: 17 December 2023
Revised: 8 January 2024
Accepted: 16 January 2024
Published: 18 January 2024

1. Introduction

"Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications. *The term is defined by consensus, and thus lacks a general formal definition*, but the list of mathematical functions contains functions that are commonly accepted as special ..." (see more, in https://en.wikipedia.org/wiki/

Special_functions, accessed on 16 December 2023). Almost by the end of the 20th century, under terms such as "*Higher Transcendental Functions*", "*Special Functions of Mathematical Physics*" or "*Named Special functions*", scientists had in mind these appearing as solutions of differential equations of *integer order* (usually 2nd order) or as integrals of elementary functions. One can briefly refer to the variety of all these, like the Bessel and cylindrical functions; the Gauss, Kummer, Tricomi, confluent and generalized hypergeometric functions; the classical orthogonal polynomials (such as Laguerre, Jacobi, Gegenbauer, Legendre, Tchebisheff, Hermite, etc.); the incomplete Gamma and Beta functions; the Error functions; and the Airy, Whittaker, etc., functions. Their definitions, properties and tables have appeared in many basic handbooks on the topic of that era, but we just give a quick mention here to the three-volume Bateman project [1] (planned by Bateman as "A Guide to the Mathematical Functions" but edited posthumously by his collaborators).

In the 1970–1980s, *fractional calculus* (FC: calculus where the operators of differentiation and integration can be of arbitrary, i.e., "fractional", order) enjoyed a revival and fast development as not only an exotic mathematical theory but also due to its acknowledged useful applications in mathematical models of various processes and phenomena of the real physical and social world. Thus, the solutions of differential and integral equations of (arbitrary) fractional order became unavoidable tools. Incorporating the "classical" special functions but also much more general ones, the class of so-called "*Special Functions of Fractional Calculus*" (*SF of FC*) has been extensively investigated, and is a topic of many modern books (such as [2–8], and so on) and a huge number of surveys and articles. One of these works is our survey, Kiryakova: "*A Guide to Special Functions in Fractional Calculus*" (published in *this same journal* in 2021) whose title is borrowed by Bateman's plans for the project, appearing as [1]. As the main representatives of such functions, we consider the *Fox H-functions* and the *Fox–Wright generalized hypergeometric functions* ${}_p\Psi_q$, which are extensions of Meijer's G- and ${}_pF_q$-functions and of other "classical" ones. There is a very long list of such functions, among which *those named after Mittag-Leffler* are the most popular and often used in FC. In the mentioned survey [9], we present an attempt to overview the definitions, some properties and applications, and the works dedicated to these SF of FC.

However, many other special functions that have not up to now been mentioned under the scheme of "SF of FC" also appear to be important in the analysis, number theory, statistics and various branches of applied mathematics and physics. Here, we pay attention to the so-called *I-functions* of Rathie [10], and to their specifications, the $\overline{H}$*-functions* of Inayat-Hussain [11], see also Buschman and Srivastava [12]. These functions were introduced long ago (at the end of the 20th century) but were somehow neglected since they looked too complicated or artificial. However, they have appeared in the theory of the Feynman integrals in statistical physics, the partition of the Gaussian model, etc., and it so happens that some well-known SF are their particular cases, like the polylogarithms, Riemann Zeta functions and others.

In recent years, a new direction in the theory of SF has appeared to extend the Le Roy functions. While studying a general class of the so-called *Le Roy-type functions*, the authors of this paper observed that these functions can fall in an "*Extended Class of SF of FC*", and after our publications on the analytical properties of the Le Roy-type functions as important entire functions ([13–16]), here, we present some ideas about their relations to the I- and $\overline{H}$-functions and their role for the further development of Generalized Fractional Calculus.

As the theory of the I- and $\overline{H}$-functions still needs some clarifications and further development, and in view of the appearing multi-valued fractional order powers of Γ-functions with a lot of singularities that are branch points, we consider this work to be a "*Discussion Survey*" and also discuss a list of *Open Problems*.

2. Preliminary Definitions

From the *Bateman Project* [1] (Vol.1, p.49): "...Of all integrals which contain Gamma functions in their integrands the most important ones are the so-called *Mellin-Barnes integrals*. Such integrals were first introduced by S. Pincherle ([17], 1888); their theory has been developed in 1910 by H. Mellin ... and they were used for a complete integration of the hypergeometric differential equation by E.W. Barnes, 1908." See Mainardi–Pagnini [18].

Definition 1 (Ch. Fox [19], 1961; see books such as [3,4,6–8,20], etc.)**.** *The generalized hypergeometric function, defined by means of the Mellin–Barnes-type contour integral in the complex plane:*

$$H_{p,q}^{m,n}\left[z\left|\begin{array}{c}(a_i,A_i)_1^p\\(b_j,B_j)_1^q\end{array}\right.\right]=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\mathcal{H}_{p,q}^{m,n}(s)\,z^{-s}ds,\ z\neq 0,$$

$$\text{with}\quad \mathcal{H}_{p,q}^{m,n}(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma(a_i+A_is)},\tag{1}$$

is known as the Fox H-function. The orders (m,n,p,q) are non-negative integers so that $0\le m\le q$, $0\le n\le p$, the parameters $A_i>0, B_j>0$ are positive, and a_i, b_j, $i=1,\dots,p; j=1,\dots,q$ are arbitrary complexes such that $A_i(b_j+l)\neq B_j(a_i-l'-1)$, $l,l'=0,1,2,\dots; i=1,\dots,n; j=1,\dots,m$, and so, the poles of the Gamma functions in the numerator do not coincide. The single-valued branch of z^{-s} is chosen as $z^{-s}=\exp[-s\{\log|z|+i\arg z\}]$, $|\arg z|<\pi$. Note that $\mathcal{L}$ is an infinite contour that separates all the poles of $\Gamma(b_j+B_js)$ to the left and all the poles of $\Gamma(1-a_i-A_is)$ to the right of $\mathcal{L}$, and can be one of the three following variants, as described, e.g., in the book by Kilbas–Saigo [3]:

(i) $\mathcal{L}=\mathcal{L}_{-\infty}$ is a left loop placed in the horizontal strip starting at the point $-\infty+i\varphi_1$ and terminating at the point $-\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$.

(ii) $\mathcal{L}=\mathcal{L}_{+\infty}$ is a right loop placed in the horizontal strip starting at the point $+\infty+i\varphi_1$ and terminating at the point $+\infty+i\varphi_2$ with $-\infty<\varphi_1<\varphi_2<+\infty$.

(iii) $\mathcal{L}=\mathcal{L}_{ic\infty}$ is a contour that starts at the point $c-i\infty$ and terminates at the point $c+i\infty$, with a suitable abscissa $c\in(-\infty,+\infty)$. Often, it is also denoted as $\mathcal{L}=(c-i\infty,c+i\infty)$.

Note that when the Mellin transform of the H-function exists (see Th.2.2 in [3]), it is equal to the integrand $\mathcal{H}_{p,q}^{m,n}(s)$. Moreover, in some books and other works, the term z^{-s} in (1) is taken to be z^s (then the orientation of the contours is changed).

The following parameters are used to characterize the behavior and the properties of the H-function:

$$\begin{gathered}R=\prod_{i=1}^{p}A_i^{-A_i}\prod_{j=1}^{q}B_j^{B_j}\ ;\ \Delta=\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i;\ \ \delta=m+n-\tfrac{p+q}{2},\\ \mu=\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i+\tfrac{p-q}{2}\ ;\ a^*=\sum_{i=1}^{n}A_i-\sum_{i=n+1}^{p}A_i+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j.\end{gathered}\tag{2}$$

Depending on these, the H-function can be a function analytic in the disks $|z|<R$ or $|z|>R$, in some angular sectors or in the whole complex plane. For more details on the properties of the Fox H-functions, the reader can consult the above-mentioned and other contemporary handbooks on special functions and their applications.

It is important to know about the behavior of the H-function around the singular points. This matter is rather complicated and, especially when there is a third singular point on the circle of convergence ($|z|=R$), there are known results only in some particular cases. For more information, the reader can see the work of Karp [21], which comments on and revisits the old, basic results of Braaksma [22].

The so-called *Meijer G-function* (C.S. Meijer [23], 1936–1941, etc.) is a simpler and more popular case of the H-function (1) when $A_i=B_j=1$, $i=1,\dots,p;\ j=1,\dots,q$, and details on its definition, properties and examples appeared in [1] (Vol.1, Ch.5), as well as in the above-mentioned books:

$$G^{m,n}_{p,q}\left[z\middle|\begin{array}{c}(a_i)_1^p\\(b_j)_1^q\end{array}\right]=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\mathcal{G}^{m,n}_{p,q}(s)\,z^s ds$$

$$=\frac{1}{2\pi i}\int\limits_{\mathcal{L}}\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{i=1}^{n}\Gamma(1-a_i+s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+s)\prod\limits_{i=n+1}^{p}\Gamma(a_i-s)}z^s ds,\quad z\neq 0. \tag{3}$$

(Hereafter, we prefer denotations by z^s instead of z^{-s}, as this is more conventional, which only yields a change in the orientation of contours.) The values of the characteristic parameters in (2) for the G-function reduce as follows: $R=1$, $\Delta=q-p$, $\delta=m+n-\frac{p+q}{2}$, and its behavior depends on them. The G-function is simpler than the H-function but yet is general enough and incorporates most of the *Classical Special Functions* (known also as *Named SF*, *SF of Mathematical Physics*), many orthogonal polynomials and elementary functions. Long lists of examples can be found, e.g., in [1] (Vol.1), [20] (Appendix C) and other recent books.

The following generalized hypergeometric function was introduced and studied by Sir Edward Maitland (E.-M.) Wright in series of his works such as [24,25], etc., 1933–1940, see also Fox [26]. It is an example of a H-function that in general (unless $\forall A_j, B_k$ are integer or rational) *does not reduce to a G-function*.

Definition 2 (see, e.g., [2,7], [20] (Appendix E), etc.)**.** *The Wright generalized hypergeometric function ${}_p\Psi_q(z)$, also called the Fox–Wright function, is defined by the power series:*

$${}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{array}\middle|z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\,\frac{z^k}{k!}, \tag{4}$$

but is representable also as a H-function:

$${}_p\Psi_q\left[\begin{array}{c}(a_i,A_i)_1^p\\(b_j,B_j)_1^q\end{array}\middle|z\right]=H^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{array}\right]. \tag{5}$$

According to the denotations for the corresponding parameters (2), the power series (4) defines an entire function of z if $\Delta>-1$; it is absolutely convergent in the disk $\{|z|<R\}$ for $\Delta=-1$, and the same holds for $|z|=R$ if $Re\,(\mu)>1/2$. Aside from the numerous handbooks on SF, the details are summarized in Gorenflo–Luchko–Mainardi [27].

The Wright g.h.f. reduces to the more popular *generalized hypergeometric ${}_pF_q$-function*, and thus to a Meijer G-function (3), when $A_1=\dots=A_p=1, B_1=\dots=B_q=1$:

$${}_p\Psi_q\left[\begin{array}{c}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,1)\end{array}\middle|z\right]=c\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\,\frac{z^k}{k!}$$

$$=c\cdot{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;\,z)=G^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}1-a_1,\dots,1-a_p\\0,1-b_1,\dots,1-b_q\end{array}\right]; \tag{6}$$

where $c=\left[\prod\limits_{i=1}^{p}\Gamma(a_i)\,/\prod\limits_{j=1}^{q}\Gamma(b_j)\right]$, $(a)_0=1,(a)_k=a(a+1)\dots(a+k-1)=\Gamma(a+k)/\Gamma(a)$.

Here, it is pertinent to mention that the identity $(a)_k=\Gamma(a+k)/\Gamma(a)$ holds true only if $(a+k)\neq-1,-2,\dots$.

Details on the theory and examples of the ${}_pF_q$-functions can be found in the old sources such as [1] (Vol.1, Ch.4). Series (6) converges for all finite z if $p\leq q$; for $|z|<1$ if $p=q+1$; and diverges for $z\neq0$ if $p>q+1$. The simplest (lowest indices) particular cases are the Gauss hypergeometric function ${}_2F_1$, the Kummer (confluent hypergeometric) function ${}_1F_1$ and the Bessel type functions ${}_0F_1$. Corresponding to these three possible cases for the orders p and q, in Kiryakova [20] (Ch.4), see also [9] (Sect.8, Th.7–Th8), we suggested a classification of the ${}_pF_q$ and ${}_p\Psi_q$ functions in three basic groups of SF.

Since in the next sections we often explore the Mittag-Leffler function E_α (introduced in [28]) and its extensions, let us briefly give a reminder of some definitions.

The *Mittag-Leffler (M-L) functions* $E_{\alpha,\beta}$ (with two indices) are defined by the power series (details in [2,29–31], etc.)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \ \alpha > 0,\ \beta > 0; \text{ for } \beta = 1: \ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}. \tag{7}$$

These are entire functions of order $\rho = 1/\alpha$ and type 1, and are also representable as

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\begin{array}{c} (1,1) \\ (\beta,\alpha) \end{array} \middle| z \right] = H^{1,1}_{1,2}\left[-z \middle| \begin{array}{c} (0,1) \\ (1-\beta,\alpha) \end{array} \right].$$

A three-parameter variant, usually called a *Prabhakar function* ([32]; see detailed studies, e.g., in Garra–Garrappa [33], Giusti et al. [34]), has an additional parameter $\tau > 0$, namely:

$$\begin{aligned} E^{\tau}_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{(\tau)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} = \frac{1}{\Gamma(\tau)} \cdot {}_1\Psi_1\left[\begin{array}{c} (\tau,1) \\ (\beta,\alpha) \end{array} \middle| z \right] \\ &= \frac{1}{\Gamma(\tau)} H^{1,1}_{1,2}\left[-z \middle| \begin{array}{c} (1-\tau,1) \\ (0,1),(1-\beta,\alpha) \end{array} \right], \end{aligned} \tag{8}$$

where $(\tau)_0 = 1, (\tau)_k = \Gamma(\tau+k)/\Gamma(\tau), k = 1,2,3,\ldots$. denotes the Pochhammer symbol. For $\tau = 1$, we have the M-L function $E_{\alpha,\beta}$, and if, additionally, $\beta = 1$, it is E_α.

The following multi-index extensions of the M-L function (7) (also called the vector index extension) were introduced by Luchko et al. [35,36] and Kiryakova [37], and studied extensively in our works such as [38–40], etc.

Let $m \geq 1$ be an integer, then by means of the sets of the real "multi-indices": $(\alpha_1 > 0, \alpha_2 > 0, \ldots, \alpha_m > 0)$ and $(\beta_1, \beta_2, \ldots, \beta_m)$, we define the *multi-index Mittag-Leffler function* (multi-M-L f.) as follows:

$$\begin{aligned} E_{(\alpha_i),(\beta_i)}(z) &:= E^{(m)}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)} \\ &= {}_1\Psi_m\left[\begin{array}{c} (1,1) \\ (\beta_i,\alpha_i)_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1}\left[-z \middle| \begin{array}{c} (0,1) \\ (0,1),(1-\beta_i,\alpha_i)_1^m \end{array} \right]. \end{aligned} \tag{9}$$

Later, Kilbas–Koroleva–Rogosin [41] studied these functions without the restrictions for all of the α_i (or their real parts) to be obligatorily non-negative.

In [42,43], etc., Paneva-Konovska introduced and studied the $(3m)$*-parametric (multi-index) Mittag-Leffler–Prabhakar functions*, similar to (9) but with an additional set of Prabhakar parameters $(\tau_1 > 0, \ldots, \tau_m > 0)$:

$$\begin{aligned} E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z) &= \sum_{k=0}^{\infty} \frac{(\tau_1)_k \ldots (\tau_m)_k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)} \frac{z^k}{(k!)^m} \\ &= T \cdot {}_m\Psi_{2m-1}\left[\begin{array}{c} (\tau_1,1),\ldots,(\tau_m,1) \\ (\beta_1,\alpha_1),\ldots,(\beta_m,\alpha_m); \underbrace{(1,1),\ldots,(1,1)}_{(m-1)-times} \end{array} \middle| z \right] \\ &= T \cdot H^{1,m}_{m,2m}\left[-z \middle| \begin{array}{c} (1-\tau_i,1)_1^m \\ (0,1)_1^m; (1-\beta_i,\alpha_i)_1^m \end{array} \right], \text{ where } T = \left[\prod_{i=1}^{m} \Gamma(\tau_i) \right]^{-1}. \end{aligned} \tag{10}$$

These are extensions of both the Prabhakar function (8) and the $(2m)$-multi-index M-L functions (9). The Prabhakar function appears for $m = 1$, while for $\tau_1 = \ldots = \tau_m = 1$, these are shown in (9). It is proved that the multi-index M-L-type functions (9) and (10) are entire functions, and their Mellin transforms are proposed in our joint paper [44].

More details and numerous particular cases of the Mittag-Leffler-type functions (7)–(10), appearing as very important tools in FC, are provided in our recent survey "A

Guide to Special Functions of Fractional Calculus" [9]. A reminder of some of these special functions are given again in brief in Section 6.

However, it seems that many other SF, that have not been mentioned as of yet under the scheme of "SF of FC" (in the sense of Kiryakova [9]), also appear to be important in the analysis, number theory, statistics and various branches of applied mathematics. Such examples and other extensions of the Fox H-functions, Meijer G-functions, Fox–Wright ${}_p\Psi_q$- and ${}_pF_q$-functions were introduced at the end of the 20th century but were somehow neglected as they looked too complicated or artificial. Next, we pay attention to the so-called *I-functions of Rathie* [10], and to their simpler variants, the *$\overline{H}$-functions of Inayat-Hussain* [11], see also Buschman and Srivastava [12]. These are further extensions of the classes of the special functions *beyond* the cases of the Wright generalized hypergeometric functions (4) and Fox H-functions (1).

Definition 3. *The I-function is defined by a kind of Mellin–Barnes-type contour integral (Rathie [10]):*

$$I_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_j, A_j, \alpha_j)_1^p\\(b_k, B_k, \beta_k)_1^q\end{array}\right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{I}_{p,q}^{m,n}(s)\, z^s ds, \quad z \neq 0, \tag{11}$$

where $\mathcal{I}(s)$ stands for

$$\mathcal{I}_{p,q}^{m,n}(s) = \frac{\prod\limits_{k=1}^{m} \Gamma^{\beta_k}(b_k - B_k s) \prod\limits_{j=1}^{n} \Gamma^{\alpha_j}(1 - a_j + A_j s)}{\prod\limits_{k=m+1}^{q} \Gamma^{\beta_k}(1 - b_k + B_k s) \prod\limits_{j=n+1}^{p} \Gamma^{\alpha_j}(a_j - A_j s)} \tag{12}$$

with the power exponents for the Gamma functions: $\alpha_j,\ j = 1, \ldots, p$ and $\beta_k, k = 1, \ldots, q$, that in general, are not positive integers, and the contours $\mathcal{L}$ are discussed below.

Clearly, for non-integer values of these "powers" parameters, the I-function (11) is not expressible as a H-function, nor as other "classical" special functions.

Most of the denotations and some conditions on the orders and parameters in (11) and (12) are similar to those used for the H-function. Here, $a_j, j = 1, \ldots, p$ and $b_k, k = 1, \ldots, q$ can also be complex numbers such that no singularity of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, \ldots, m$ coincides with any singularity of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, \ldots, n$. Again, we prefer the definition with term z^s so as to be in agreement with the denotations in the initial works on the topic, and suppose that $z^s = \exp[s\{\log|z| + i \arg z\}]$, $0 \leq \arg z < 2\pi$.

In general, for non-integer α_j and β_k, the mentioned singularities are *no longer* poles but are converted to branching points of the multi-valued fractional powers of the Gamma functions. According to Rathie [10], the corresponding branch cuts should be chosen in a way so that the path of integration can be one of the three type of contours $\mathcal{L}$:

(a) $\mathcal{L}$ starts from $c - i\infty$ and goes to $c + i\infty$, where the real abscissa c is chosen so that the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, \ldots, m$, lie to the right of $\mathcal{L}$, and all singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, \ldots, n$, lie to the left of it;

(b) $\mathcal{L}$ is a loop that begins and ends at $+\infty$ and encircles all the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, \ldots, m$, once in the clockwise direction, but none of the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, \ldots, n$;

(c) $\mathcal{L}$ is a loop beginning and ending at $-\infty$ and encircling all the singularities of $\Gamma^{\alpha_j}(1 - a_j + A_j s), j = 1, \ldots, n$, once in the anti-clockwise direction, but none of the singularities of $\Gamma^{\beta_k}(b_k - B_k s), k = 1, \ldots, m$.

Rathie [10] noted that contour **(a)** can be considered as a particular case of contour **(c)**. In the case of contour **(b)**, the sign of the function is changed with respect to case **(c)**. When more than one of the above definitions of $\mathcal{L}$ make sense, they lead to the *same* result.

As in the theory of the Fox H-function, the following values related to the parameters (analogues of these in (2)) are important to characterize the behavior of the I-function:

$$\begin{aligned}
\Delta &= \sum_{k=1}^{m} \beta_k B_k - \sum_{k=m+1}^{q} \beta_k B_k + \sum_{j=1}^{n} \alpha_j A_j - \sum_{j=n+1}^{p} \alpha_j A_j; \\
\mu &= \sum_{k=1}^{q} \beta_k B_k - \sum_{j=1}^{p} \alpha_j A_j; \\
\nabla &= \sum_{j=1}^{p} \alpha_j [\mathrm{Re}\,(a_j) - 1/2] - \sum_{k=1}^{q} \beta_k [\mathrm{Re}\,(b_k) - 1/2]; \\
R &= \prod_{k=1}^{q} B_k^{\beta_k B_k} / \prod_{j=1}^{p} A_j^{\alpha_j A_j}.
\end{aligned} \tag{13}$$

In terms of (13), it is proved (Rathie [10]) that: the I-function (11) for contour $\mathcal{L}$ defined by **(a)**, converges when $|\arg z| < \Delta\pi/2$, if $\Delta > 0$; if $|\arg z| = \Delta\pi/2$, $\Delta \geq 0$, the integral (11) converges absolutely when: (i) $\mu = 0$ if $\nabla > 1$; (ii) $\mu \neq 0$, if with $s = \sigma + it$, σ and t real, σ chosen so that for $|t| \to \infty$, we have $\nabla + \sigma\mu > 1$. Otherwise, the integral (11) *for contour $\mathcal{L}$ defined by **(b)*** converges for $q \geq 1$ and either $\mu > 0$, or $\mu = 0$ inside the disk $|z| < R$. Alternatively, the integral (11) for contour $\mathcal{L}$ defined by **(c)** converges when $p \geq 1$ and either $\mu < 0$, or $\mu = 0$ if $|z| > R$.

To determine single-valued branches of the multi-valued Γ-functions to the fractional powers in (11), one needs to draw suitable branch cuts. The details of such a procedure were proposed in the recent paper by Rogosin and Dubatovskaya [45] (Sect.3), for a similar function denoted there as (3.1) by HG, with $\mathcal{L}$ taken as a contour of the so-called "Slater" type ([46], see Marichev [5] (Ch.4)): $\mathcal{L}_{+\infty}$ starting at $+\infty + i\varphi_2$ and finishing at $+\infty + i\varphi_1$, with suitable $\varphi_1 < \varphi_2$. However, in the case of the I-functions, the ideas like in the Slater theorem for the G-functions (see same place) would need to be expanded when we have ratios of Γ-functions to arbitrary (fractional) powers.

Definition 4. *Another generalization of the H-function, denoted by the symbol $\overline{H}$ and called the Inayat-Hussain function ([11]) or bar-H-function, appears as a slightly simpler case of the I-function:*

$$\overline{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, A_j, \alpha_j)_1^n, (a_j, A_j, 1)_{n+1}^p \\ (b_k, B_k, 1)_1^m, (b_k, B_k, \beta_k)_{m+1}^q \end{matrix} \right]$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\prod_{k=1}^{m} \Gamma(b_k - B_k s) \prod_{j=1}^{n} \Gamma^{\alpha_j}(1 - a_j + A_j s)}{\prod_{k=m+1}^{q} \Gamma^{\beta_k}(1 - b_k + B_k s) \prod_{j=n+1}^{p} \Gamma(a_j - A_j s)} z^s ds. \tag{14}$$

Compared to the definition of the I-function, part of the powers of the Γ-functions are taken to be equal to 1, namely: $\beta_k = 1$, $k = 1, \ldots, m$ and $\alpha_j = 1$, $j = n+1, \ldots, p$, and the contour $\mathcal{L}$ is chosen to be specifically as of type **(a)** (the imaginary axis, *or* if necessary shifted suitably as $(c - i\infty, +i\infty)$). In general (again like the I-function), the $\overline{H}$-function does not reduce to a H-function, unless the additional power parameters in the Gamma functions are positive integers.

For the bar-H-function (14), the values of the parameters in (13) have corresponding but similar forms.

Definition 5. *A particularly important case with $m = 1, n = p$ (see Inayat-Hussain [11] (27)) corresponds to a kind of generalized hypergeometric function, similar to the Fox–Wright function* (4). *This is called the extended Fox–Wright function:*

$$\overline{H}^{1,p}_{p,q+1}\left[-z \,\middle|\, \begin{matrix} (1 - a_j, 1, \alpha_j)_1^p \\ (0,1), (1 - b_k, 1, \beta_k)_1^q \end{matrix} \right] := {}_p\overline{\Psi}_q \left[\begin{matrix} (a_j, 1; \alpha_j)_{j=1}^p \\ (b_k, 1; \beta_k)_{k=1}^q \end{matrix} \,\middle|\, z \right]$$

$$= \frac{\prod_{j=1}^{p} \Gamma^{\alpha_j}(a_j)}{\prod_{k=1}^{q} \Gamma^{\beta_k}(b_k)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \left[(a_j)_n\right]^{\alpha_j}}{\prod_{k=1}^{q} \left[(b_k)_n\right]^{\beta_k}} \frac{z^n}{n!}, \quad with \quad (a)_0 = 1, (a)_n := a(a+1)\ldots(a+n-1). \tag{15}$$

In this case, there are substantial simplifications for the parameters (13) characterizing the behavior of the function (15):

$$\Delta = 1 - \sum_{k=1}^{q} \beta_k + \sum_{j=1}^{p} \alpha_j; \quad \mu = \sum_{k=1}^{q} \beta_k - \sum_{j=1}^{p} \alpha_j; \quad R = 1,$$

that is, under some conditions (if $\mu = 0$), the function (15) is analytic inside $|z| < 1$, or outside it; compare with the discussions in Inayat-Hussain [11], Rathie [10] and Saxena [47].

However, we go further, and introduce an even more general extension of the Fox–Wright function *that we do need and use in our studies*.

Definition 6. *We define the Generalized Fox–Wright function:*

$${}_p\widetilde{\Psi}_q\left[\begin{array}{c} (a_j, A_j, \alpha_j)_{j=1}^{p} \\ (b_k, B_k, \beta_k)_{k=1}^{q} \end{array}\middle| z\right] := \overline{H}^{1,p}_{p,q+1}\left[-z\middle| \begin{array}{c} (1-a_j, A_j, \alpha_j)_1^p \\ (0,1), (1-b_k, B_k, \beta_k)_1^q \end{array}\right], \tag{16}$$

with arbitrary positive parameters A_j, B_k as in the "classical" case (4) (*instead of these taken to be 1 in* (15)) *but again with "fractional" power parameters $\alpha_j > 0, \beta_k > 0$.*

Next, for this ${}_p\widetilde{\Psi}_q$-function, the characteristic parameters (13) take the form:

$$\begin{gathered} \mu = 1 + \sum_{k=1}^{q} \beta_k B_k - \sum_{j=1}^{p} \alpha_j A_j; \\ R = \prod_{k=1}^{q} B_k^{\beta_k B_k} / \prod_{j=1}^{p} A_j^{\alpha_j A_j} \text{ (the radius of analycity, if } \mu = 0). \end{gathered} \tag{17}$$

We observe (see next Section 4) that these generalized Fox–Wright functions, under suitable conditions for the singularities of the Gamma functions, sound very similar to the Le Roy-type functions, including the general multi-index case of the $\mathbb{F}_m$-function (see Section 3, Definition 7), introduced and studied in our recent works [13–15], and of course, the original Le Roy function.

To this end, we need to compare the corresponding Mellin–Barnes (M-B)-type integrals (14) for the $\overline{H}$-functions with the M-B-type representations for the Le Roy functions (from our above-mentioned works), but also to compare the corresponding power series and their behaviors.

First, we provide a reminder of a general result for the "computational" representation *of some specific cases of the I-function and $\overline{H}$-function in the form of power series*. Such a representation for an I-function with $\forall \beta_k = 1, k = 1, \ldots, m$ (named bar-I-function), and when all the poles of $\Gamma(b_k - B_k s), k = 1, \ldots, m$ are simple, can be found in Rathie [10] (6.8), and, resp., in Saxena [47] (3.1), where some techniques resembling the residue theorem are used. In the simpler case of the bar-H-function, when it is supposed additionally that $\forall \alpha_j = 1, j = n+1, \ldots, p$ (see definition (14)), we can cite the following result.

Theorem 1. *The function $\overline{H}^{m,n}_{p,q}(z)$ can be represented by means of the power series ([10,47]):*

$$\overline{H}^{m,n}_{p,q}(z) = \sum_{r=0}^{\infty} \sum_{h=1}^{m} \frac{\prod_{j=1}^{n} \Gamma^{\alpha_j}(1 - a_j + A_j \xi)}{\prod_{j=n+1}^{p} \Gamma^1(a_j - A_j \xi)} \cdot \frac{\prod_{k=1, k\neq h}^{m} \Gamma^1(b_k - B_k \xi)}{\prod_{k=m+1}^{q} \Gamma^{\beta_k}(1 - b_k + B_k \xi)} \cdot \frac{(-1)^r z^{\xi}}{B_h\, r!}. \tag{18}$$

Here, for shortness, $\xi := b_h + r/B_h$. Series (18) exists for: $0 < |z| < \infty$ if $q \geq 1$ and $\mu > 0$; or for $0 < |z| < R$ if $\mu = 0$, supposing that $\beta_k(B_h + \nu_1) \neq \beta_h(B_k + \nu_2)$ for $k \neq h$; $k, h = 1, \ldots, m$; $\nu_1, \nu_2 = 0, 1, 2, \ldots$.

One can observe an evident analogy of (18) with the power series representation for the Fox H-function from the handbooks, such as: [7] (§8.3.2: 3. and 4.), [4], etc.

Then, as a particular case, we can derive a *series representation for the generalized Fox–Wright function* (16), introduced here. Below, we deliberately add "stars" to the denotations of parameters and change some summation indices ($r \to k$, but $k \to i$) so as to distinguish these from the other traditional notations we use in the next sections, as previously adopted in several of the oldest and some recent works on Mittag-Leffler and Le Roy-type functions.

Theorem 2. *For the generalized Fox–Wright function ${}_p\widetilde{\Psi}_q$ we have: $m = 1, n = p$, and assume parameters $b_0^* = 0, B_0^* = 1, \beta_0^* = 1$ (taken as the first ones of the $q+1$ parameters in the $\overline{H}^{1,p}_{p,q+1}$-function below). Then, $\xi := b_0^* + k/B_0^* = k$, and $(-1)^k z^\xi = (-z)^k$. Thus, from (18), we obtain the power series representation for (16):*

$$
{}_p\widetilde{\Psi}_q\left[\begin{array}{c}(a_j^*, A_j^*; \alpha_j^*)_{j=1}^p \\ (b_i^*, B_i^*; \beta_i^*)_{i=1}^q\end{array}\middle| z\right] = \overline{H}^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-a_j^*, A_j^*, \alpha_j^*)_1^p \\ (0,1), (1-b_i^*, B_i^*, \beta_i^*)_1^q\end{array}\right]
$$
$$
= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma^{\alpha_j^*}(A_j^* k + a_j^*)}{\prod_{i=1}^{q} \Gamma^{\beta_i^*}(B_i^* k + b_i^*)} \cdot \frac{z^k}{k!}. \tag{19}
$$

The characteristic parameters (13) are written now as:

$$
\mu = 1 + \sum_{i=1}^{q} \beta_i^* B_i^* - \sum_{j=1}^{p} \alpha_j^* A_j^*, \quad R = \prod_{i=1}^{q} (B_i^*)^{\beta_i^* B_i^*} / \prod_{j=1}^{p} (A_j^*)^{\alpha_j^* A_j^*}.
$$

*Series (19) defines an entire function (that is, absolutely convergent for all $0 < |z| < \infty$) **if** $\mu > 0$ (the other condition for the $\overline{H}$-function, here: $q+1 \geq 1$, is satisfied ad hoc), or analytical one in $0 < |z| < R$ if $\mu = 0$.*

Observe the close analogy with the series representation (4) for the *Fox–Wright function*. Here, following the discussions after Definitions 3 and 4, suitable cuts are to be inserted in $\mathbb{C}$ so as to fix single-valued branches for z^k and for the included multi-valued fractional powers of the Γ-functions.

3. Multi-Index M-L-P Functions of Le Roy Type as I- and $\bar{H}$-Functions

The special function $F^{(\gamma)}$, defined in the whole complex plane $\mathbb{C}$ by the power series

$$
F^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k!]^\gamma} = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^\gamma}, \quad z \in \mathbb{C}, \ \gamma > 0, \tag{20}
$$

is known as the *Le Roy function*. It was introduced by the French mathematician Le Roy [48,49] who used it to study the asymptotics of the analytic continuation of the sum of power series. His idea and goals were similar to those of the Swedish mathematician Mittag-Leffler when introducing the function (7) in [28]. The *Mittag-Leffler function* is now very popular, and often called the *Queen function of Fractional Calculus*, see [2], also [50],

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \ \alpha > 0, \ \beta \in \mathbb{C}.
$$

It is also known as the "fractional" exponential function due to the "fractional" parameter α in the Γ-function replacing the factorial $k!$ in the exponential series, while, for (20), it emphasizes that the "fractional" index $\gamma > 0$ appears as the power of the factorial $k!$ (further extended to a Γ-function or to several Γ-functions).

In recent years, several studies appeared to revitalize the interest in (20) via its extensions and useful applications in various areas. It is important to mention that even in the simple case $\gamma = 1/2$, the Le Roy function (20) appeared as $R(z)$ in a study by Kolokoltsov [51] in quantum stochastics and particle systems. As he himself commented, "the function $R(z)$ plays the same role for stochastic equations as the exponential function and Mittag-Leffler functions for deterministic equations".

Among its generalizations, as a starting point, we mention the *Mittag-Leffler function of Le Roy-type* (MLR-function) as an extension of both the Le Roy and Mittag-Leffler functions:

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}}, \quad \alpha > 0, \beta > 0, \gamma > 0. \tag{21}$$

It was introduced by Gerhold [52] and Garra and Polito [53], and studied further by Garrappa–Rogosin–Mainardi [54] and Garra–Orsingher–Polito [55], and then also by Gorska–Horzela [56], Simon [57], Mehrez–Das [58] and Mehrez [59]. In [55], it is mentioned that the Le Roy-type functions (21) are used in probability in the context of the studies of COM-Poisson distributions (in the sense of Conway and Maxwell), which are special classes of weighted Poisson distributions. It is shown that as a first probabilistic application, these functions can also be useful in the construction of a new generalization of the COM-Poisson distribution that can be interesting for statistical applications and in physics in the case of generalized coherent states.

Next, *a multi-index analogue* of (21), inspired by the multi-index Mittag-Leffler functions (9) (Kiryakova [38–40]; Luchko et al. [35,60]), was considered by Rogosin and Dubatovskaya [45,61] with $\alpha_i > 0$, $\beta_i > 0$, $\gamma_i > 0$ (and also under more general conditions):

$$F_{(\alpha,\beta)_m}^{(\gamma)_m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1} \dots [\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}}. \tag{22}$$

The following Le Roy-type analogue of the Prabhakar (three-parameter Mittag-Leffler) function (8), that we call the *Prabhakar function of Le Roy type*, was introduced by Tomovski and Mehrez [62], and studied by Paneva-Konovska [15]:

$$F_{\alpha,\,\beta;\,\tau}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{[\Gamma(\alpha k+\beta)]^{\gamma}} \frac{z^k}{k!}, \text{ where } (\tau)_k = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}, \ \tau > 0. \tag{23}$$

Further, in our recent works [13,14,16], we introduced and studied a very general Le Roy-type special function, as a multi-index analogue of the previously mentioned functions.

Definition 7. *The function, called the multi-index Mittag-Leffler–Prabhakar function of Le Roy-type (abbrev. as multi-MLPR), is defined by taking* $4m$ *parameters* $\alpha_i,\ \beta_i,\ \tau_i,\ \gamma_i,\ i = 1,\dots,m$:

$$\begin{aligned} \mathbb{F}_m(z) &:= \mathbb{F}_{\alpha_i;\beta_i;\tau_i}^{\gamma_i;m}(z) \\ &= \sum_{k=0}^{\infty} \frac{(\tau_1)_k \dots (\tau_m)_k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1} \dots [\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}} \cdot \frac{z^k}{(k!)^m} \qquad (24) \\ &= \frac{1}{\prod\limits_{i=1}^{m} \Gamma(\tau_i)} \cdot \sum_{k=0}^{\infty} \frac{\prod\limits_{i=1}^{m} \Gamma(1.k+\tau_i)}{\Gamma^{m-1}(1.k+1) \prod\limits_{i=1}^{m} \Gamma^{\gamma_i}(\alpha_i.k+\beta_i)} \cdot \frac{z^k}{k!} \qquad (25) \\ &= \sum_{k=0}^{\infty} c_k\, z^k, \ \ \textit{with} \ c_k = \prod_{i=1}^{m} \left\{ \frac{\Gamma(k+\tau_i)}{\Gamma(k+1)} \cdot \frac{1}{\Gamma(\tau_i)} \cdot \frac{1}{[\Gamma(\alpha_i k+\beta_i)]^{\gamma_i}} \right\}. \end{aligned}$$

Note that in (25)*, we replaced part of the m factorials from* (24) *with* $(k!)^{m-1} = \Gamma^{m-1}(1.k+1)$.

In view of the applications of such kinds of functions and of their particular cases, here we consider *only real positive parameters*:

$$\alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0, \ \forall i = 1,\dots,m \ \Rightarrow \ \textit{condition} \ \sum_{i=1}^{m} \alpha_i \gamma_i > 0. \tag{26}$$

In [13] (Th.2), we proved that under these conditions, the $\mathbb{F}_m$-function (24) is an entire function of order ρ and type σ, with, respectively:

$$\rho = \frac{1}{\alpha_1\gamma_1 + \ldots + \alpha_m\gamma_m}, \quad \sigma = \frac{1}{\rho}\left(\prod_{i=1}^{m}(\alpha_i)^{-\alpha_i\gamma_i}\right)^{\rho}. \tag{27}$$

The values of ρ and σ were also specified for more general requirements on the parameters, like

$$\alpha_i,\ \beta_i,\ \gamma_i,\ \tau_i \in \mathbb{C},\ \mathrm{Re}\,(\alpha_i) > 0,\ \mathrm{Re}\,(\gamma_i) > 0,\ \text{and}\ \sum_{i=1}^{m}\mathrm{Re}\,(\alpha_i\gamma_i) > 0.$$

Results for the analytical properties of the multi-MLPR functions (24) are published in our recent works [13,14,16], including Mellin–Barnes-type contour integral representations, images under Laplace transform and Erdélyi–Kober operators of fractional integration, etc. For the case of the functions (22), see the corresponding results of Rogosin–Dubatovskaya [45,61].

Next, we develop the idea that the $\mathbb{F}_m$-function, and the other mentioned Le Roy-type functions, can be represented in terms of some special functions generalizing the Fox H-function and the Fox–Wright function ${}_p\Psi_q$, namely, as some I-, $\overline{H}$- and ${}_p\widetilde{\Psi}_q$-functions. The following result is derived with a short sketch of the details. For shortness, we further denote $T := \prod\limits_{i=1}^{m} 1/\Gamma(\tau_i)$.

Theorem 3. *Under the assumptions* (26)*, the multi-index Mittag-Leffler–Prabhakar function of Le Roy type* (24) *can be represented in terms of the generalized Fox–Wright function* (16)*, and thus, also as I-function* (11) *and $\overline{H}$-function* (14)*. Namely,*

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;m}_{\alpha_i;\beta_i;\tau_i}(z)$$

$$= T \cdot {}_m\widetilde{\Psi}_{2m-1}\left[\begin{array}{c} (\tau_i,1,1)_1^m \\ (1,1,1)_{(m-1)-\text{times}},\,(\beta_i,\alpha_i,\gamma_i)_1^m \end{array}\middle|\, z\right] \tag{28}$$

$$= T \cdot \overline{H}^{1,m}_{m,2m}\left[-z\,\middle|\, \begin{array}{c} (1-\tau_i,1,1)_1^m \\ (0,1)_{m-\text{times}},\,(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{array}\right] \tag{29}$$

$$= T \cdot I^{1,m}_{m,2m}\left[-z\,\middle|\, \begin{array}{c} (1-\tau_i,1,1)_1^m \\ (0,1,1)_{m-\text{times}},\,(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{array}\right]. \tag{30}$$

Proof. For the power series representation (24) of the function $\mathbb{F}_m$, we apply Theorem 2. Here, $p = m$, $q = 2m-1$ and we use the sets of parameters denoted as follows:

$$a_j^* = \tau_j,\ A_j^* = 1,\ \alpha_j^* = 1,\ \ j = 1,\ldots,m;\ \ b_i* = 1,\ B_i^* = 1,\ \beta_i^* = 1,\ \ i = 1,\ldots,m-1;$$
$$b_{i+m-1}* = \beta_i,\ B^*_{i+m-1} = \alpha_i,\ \beta^*_{i+m-1} = \gamma_i,\ \ i = 1,\ldots,m.$$

In the case of the $\mathbb{F}_m$-function, for the related $\overline{H}$- and I-functions, we have the conditions of existence (13) satisfied, namely: $q+1 = 2m \geq 1$, and the other characteristic parameter is: $\mu = 1 + (m-1) + \sum\limits_{i=1}^{m}\alpha_i\gamma_i - m = \sum\limits_{i=1}^{m}\alpha_i\gamma_i > 0$, according to the assumed conditions (26). This allows us to conclude from Theorem 2 that (28)–(30) are entire functions of z in the complex plane.

For the definition of the $\overline{H}$-function (29), and, resp., of the I-function (30), as was mentioned before, the path of integration can be either a shifted imaginary axis $\mathcal{L} = (c - i\infty, c + i\infty)$ or a "Slater"-type contour $\mathcal{L}_{+\infty}$ starting at $+\infty + i\varphi_2$ and finishing at $+\infty + i\varphi_1$, with suitable $\varphi_1 < \varphi_2$ (following the ideas from Marichev [5] (Ch.4), Rogosin–Dubatovskaya [45]). It is supposed that these contours are chosen not to cross the branch cuts and to separate the singularities of the Gamma functions in the numerator of the corresponding integrals.

Indeed, let us analyze the singularities of the involved Gamma functions in the representation of (29):

$$\mathbb{F}_m(z) = T \cdot \overline{H}^{1,m}_{m,2m}\left[-z \,\middle|\, \begin{matrix} (1-\tau_i,1,1)_1^m \\ (0,1)_{m\text{–times}},(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{matrix} \right]$$

$$= \frac{T}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma^1(1-s)\cdot \prod_{i=1}^{m}\Gamma^1(\tau_i+s)}{\Gamma^{m-1}(1+s)\cdot \prod_{i=1}^{m}\Gamma^{\gamma_i}(\beta_i+\alpha_i s)} z^s ds.$$

The poles of $\Gamma(1-s)$ are $s_l = 1+l$, $l = 0,1,2,\ldots$; therefore, it has no singularities for $s < 1$. The functions $\Gamma(\tau_i + s)$ have poles at $s_{ik} = -\tau_i - k$, $k = 0,1,2,\ldots$, and have no singularities for $s > -m_0$, where $m_0 := \min\{\tau_1,\ldots,\tau_m\}$. Then, the Gamma functions in the numerator have no singularities for $s \in (-m_0, 1)$.

The poles of the $(m-1)$ functions $\Gamma(1+s)$ are $s_l = -1-l$, $l = 0,1,2,\ldots$, i.e., $s = -1,-2,-3,\ldots$. Finally, for the $\Gamma^{\gamma_i}(\beta_i+\alpha_i s)$, $i = 1,\ldots,m$ in the denominator, the singularities appear at $s_{in} = -\frac{\beta_i}{\alpha_i} - \frac{n}{\alpha_i} \leq -\frac{\beta_i}{\alpha_i}$, $n = 0,1,2,\ldots$ and so, if we denote $\widetilde{m_0} := \min\{\beta_1/\alpha_1,\ldots,\beta_m/\alpha_m\}$, there will be no their singularities for $s > -\widetilde{m_0}$. When the γ_i are not integers, to avoid multi-valueness, we need to make a branch cut to the left of $s = -\widetilde{m_0} < 0$.

It is now evident that the vertical strip $(c - i\infty, c + \infty)$ with $c \in (-\min(m_0, \widetilde{m_0}), 1)$ is free of any singularities of the involved Gamma functions. However, to have a contour that does not intersect the branch cuts to the left of $s = -\widetilde{m_0} < 0$ and also to the left of $s < 0$ (for single values of z^s), we can take a contour $\mathcal{L} = (c^* - i\infty, c^* + i\infty)$ with $c^* \in (0,1)$ to ensure the existence of the bar-H-function in (29), and therefore also for (30). □

4. Related "Eigen"-Operators for Some Classes of SF: Gelfond–Leontiev Operators and Operators of FC

In analysis, linear algebra, physics, etc., the notions related to the prefix "Eigen" (coming from the German word meaning "self" or "own") play important roles. In short, an eigenfunction is a function that, when acted on by an operator, yields a scalar multiple of the function itself. The scalar value is called the eigenvalue. The eigenfunctions and eigenvalues are also important, for example, in quantum mechanics, because they allow us to describe the behavior of operators in a way that is easy to understand and calculate.

Aside from many other analytical properties of the special functions, and the evaluation of their images under integral transforms and operators of fractional calculus, *it is an important but often still open problem* to determine corresponding linear integral L and differential operators D that transform a function f into itself just multiplied by a scalar, e.g., $Df = \lambda f$. For shortness, here, *we call such operators "eigen"-operators (eigenoperators) for the function f.*

It so happens that a useful tool to resolve this (generally open) problem for some classes of special functions is the notion of *Gelfond–Leontiev operators* (G-L operators) for generalized integration and differentiation, introduced in these authors' work [63] of 1951. Up to this point, we have developed the theory of the G-L operators to propose corresponding integral and differential operators for which the Mittag-Leffler functions (7) and their multi-index analogues (9) are eigenfunctions. It appears that these are also operators of Generalized Fractional Calculus (in the sense of Kiryakova [20]). See, for example, [20] (Ch.5), [37,38], etc., and survey [64].

Next, our goal is to propose eigenoperators corresponding to the considered Le Roy-type functions.

4.1. Preliminaries for the Gelfond–Leontiev Operators

We first provide a reminder of some of the basics of the theory of generalized integration and differentiation, and specifically, the notion of such Gelfond–Leontiev operators.

Consider a function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{31}$$

analytic in a disk $\Delta_R = \{|z| < R\}$ in the complex plane (in particular, it can be the unit disk with $R = 1$). Let $\{b_k\}_{k=0}^{\infty}$ be an arbitrary sequence (*multipliers' sequence*) satisfying suitable conditions and defining another analytic function in the same disk Δ_R: $b(z) = \sum\limits_{k=0}^{\infty} b_k z^k$. Then, let us define the operator

$$D\{b; f\}(z) = (b \circ f)(z) := \sum_{k=0}^{\infty} a_k b_k z^k,$$

where $\circ$ denotes the Hadamard product (convolution). If $b_k \to \infty$ for $k \to \infty$, we can consider the operator $D\{b; f\}$ as a *generalized differentiation*. For $b_k \neq 0, k = 1, 2, \ldots$, the inverse operation, or the convolution with the "reciprocal" function $b_*(z) = \sum\limits_{k=0}^{\infty} z^k / b_k$,

$$I\{b; f\}(z) = \sum_{k=0}^{\infty} \frac{a_k}{b_k} z^k = D\{b_*; f\}(z)$$

can be called a *generalized integration*, see, e.g., Samko–Kilbas–Marichev [65] (§22.3).

Following this idea, the so-called Gelfond–Leontiev operators of generalized integration and differentiation can be considered, as introduced in [63]. There, the multipliers' sequences $\{b_k\}_{k=0}^{\infty}$ and $\{1/b_k\}_{k=0}^{\infty}$ are constructed *by means of the coefficients of an entire function*.

Definition 8. *Let us take an entire function*

$$\phi(\lambda) = \sum_{k=0}^{\infty} \phi_k \lambda^k$$

with a growth (order $\rho > 0$ and type $\sigma \neq 0$) such that $\lim\limits_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\phi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$. In [63], the following operator of generalized differentiation is introduced:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \overset{D_\phi}{\longmapsto} D_\phi f(z) = \sum_{k=1}^{\infty} a_k \frac{\phi_{k-1}}{\phi_k} z^{k-1}, \tag{32}$$

together with its n-th powers D_ϕ^n, $n = 0, 1, 2, \ldots$. We call the operation (32) a Gelfond–Leontiev (G-L) operator of generalized differentiation with respect to the function $\phi(\lambda)$, and then can introduce the corresponding G-L operator of generalized integration, defined by

$$L_\phi f(z) = \sum_{k=0}^{\infty} a_k \frac{\phi_{k+1}}{\phi_k} z^{k+1}. \tag{33}$$

Evidently, $D_\phi L_\phi f(z) = f(z)$.

That is, we take $b_k = \dfrac{\phi_{k-1}}{\phi_k}$ and $1/b_{k+1} = \dfrac{\phi_{k+1}}{\phi_k} \neq 0$. The conditions required for $\phi(\lambda)$ always hold for $\limsup_{k\to\infty}$. We suppose that there exists $\lim\limits_{k\to\infty} \sqrt[k]{\left|\frac{\phi_{k-1}}{\phi_k}\right|} = 1$, and therefore, based on the Cauchy–Hadamard formula, the above-defined series (31)–(33) for $f(z), D_\phi f(z), L_\phi f(z)$ have the same radius of convergence (in particular, this can be $R = 1$).

Example 1. *For the simplest example we take $\phi(\lambda) = \exp \lambda$, with $\phi_k = 1/k! = 1/\Gamma(k+1)$, that is $b_k = k$, $k = 0, 1, 2, \ldots$. Then, the operators (32) and (33) are the differentiation and integration of 1st order:*

$$Df(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \frac{d}{dz} f(z) \ , \ Lf(z) = D^{-1} f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1} = \int_0^z f(\xi) d\xi.$$

4.2. G-L Operators Related to M-L Functions and to FC Operators

In classical fractional calculus (FC), the most often-used definition of the integration of an arbitrary (fractional) order $\delta \geq 0$ is the *Riemann–Liouville (R-L) operator of fractional integration* (left-hand sided), defined by the formula

$$I^{\delta} f(z) = I^{\delta}_{0+} f(z) = \frac{1}{\Gamma(\delta)} \int\limits_0^z (z-\xi)^{\delta-1} f(\xi) d\xi \tag{34}$$

$$= \frac{z^{\delta}}{\Gamma(\delta)} \int\limits_0^1 (1-\sigma)^{\delta-1} f(z\sigma) d\sigma, \quad I^0 f(z) := f(z).$$

Then, the corresponding *R-L fractional derivative*, and, resp., the *Caputo fractional derivative* of order $\delta > 0$, are defined by compositions of the n-th order casual derivative ($n-1<\delta\leq n$, $n \in \mathbb{N}$) and fractional order integral of form (34), taken in the two cases in reverse order:

$$D^{\delta} f(z) = {}_{RL}D^{\delta} f(z)$$

$$:= D^n I^{n-\delta} f(z) = \left(\frac{d}{dz}\right)^n \left\{ \frac{1}{\Gamma(n-\delta)} \int\limits_0^z (z-\xi)^{n-\delta-1} f(\xi) d\xi \right\}, \tag{35}$$

$$ {}_C D^{\delta} f(z) := I^{n-\delta} D^n f(z) = \frac{1}{\Gamma(n-\delta)} \int\limits_0^z (z-\xi)^{n-\delta-1} f^{(n)}(\xi) d\xi, \ \ D^0 f(z) := f(z). \tag{36}$$

The theory of *fractional calculus* (FC) based on these (classical) operators is widely presented in many basic handbooks such as [4,65,66], etc. For its historical development, see survey [67].

Under these operators of FC, the images of analytic functions $f(z)$ of the form (31) are as follows:

$$I^{\delta} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} z^{k+\delta}, \quad D^{\delta} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} z^{k-\delta}. \tag{37}$$

This suggests that by replacing the multiplier sequences from Example 1 with more general ones, one can define G-L-type operators corresponding to some special functions.

Example 2. *The G-L operator of generalized integration generated by the Mittag-Leffler entire function $\phi(\lambda) := E_{\alpha,1}(\lambda) = E_{\alpha}(\lambda)$ has the form*

$$J^{\alpha} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} z^{k+1}. \tag{38}$$

The following result for (38) is given in Samko–Kilbas–Marichev [65] (§22.3):

The G-L operator of generalized integration J^{α} generated by the entire function $E_{\alpha}(\lambda)$ is the following modification of the R-L integral of order α:

$$J^{\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int\limits_0^1 (1-t)^{\alpha-1} f(zt^{\alpha}) dt = \frac{z^{-1}}{\Gamma(\alpha)} \int\limits_0^z \left(z^{1/\alpha} - \xi^{1/\alpha}\right)^{\alpha-1} f(\xi)\, d(\xi^{1/\alpha}). \tag{39}$$

The *proof* uses the known definition of the Beta-function

$$\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} = \frac{B(\alpha, \alpha k+1)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int\limits_0^1 (1-t)^{\alpha-1} t^{\alpha k} dt.$$

This integral representation allows us to extend the definition of the integration operator (38) from analytic functions $f(z)$ in a disk to continuous functions (and even to integrable ones), given in a (complex) domain starlike with respect to the origin $z = 0$.

A more general "classical" operator of fractional integration is the *Erdélyi–Kober (E-K) fractional integral* of order $\delta > 0$, with two additional parameters: real "weight" γ and some $\beta > 0$:

$$\begin{aligned} I_\beta^{\gamma,\delta} f(z) &= \frac{z^{-\beta(\gamma+\delta)}}{\Gamma(\delta)} \int_0^z (z^\beta - \xi^\beta)^{\delta-1} \xi^{\beta\gamma} f(\xi)\, d(\xi^\beta) \\ &= \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^\gamma f(zt^{1/\beta})\, dt = \int_0^1 \frac{\beta(1-\sigma^\beta)^{\delta-1}\, \sigma^{\beta\gamma+\beta-1}}{\Gamma(\delta)} f(z\sigma) d\sigma \\ &= \int_0^1 H_{1,1}^{1,0}\left[\sigma \left| \begin{array}{c} (\gamma+\delta+1-\frac{1}{\beta}, \frac{1}{\beta}) \\ (\gamma+1-\frac{1}{\beta}, \frac{1}{\beta}) \end{array} \right.\right] f(z\sigma) d\sigma. \end{aligned} \tag{40}$$

For $\gamma = 0, \beta = 1$, it reduces to the R-L fractional integral, namely: $I_1^{0,\delta} f(z) = z^{-\delta}\, I^\delta f(z)$.

The involved additional parameters give more freedom and thus, much wider application, see, for example, Sneddon [68] and Kiryakova [20] (Ch.2). The corresponding *Erdélyi–Kober (E-K) fractional derivative* is introduced in [20] (Ch.2), (1.6.9) in the form

$$D_\beta^{\gamma,\delta} f(z) = D_\eta\, I_\beta^{\gamma+\delta,\eta-\delta} f(z),$$

$$\text{with } \eta - 1 < \delta \leq \eta,\ \eta \in \mathbb{N}, \text{ and } D_\eta := \prod_{j=1}^{\eta} (\frac{1}{\beta} z \frac{d}{dz} + \gamma + j), \text{ a polynomial } P(\frac{d}{dz}).$$

Let Ω be a complex domain, starlike with respect to the origin $z = 0$ (in particular, it can be a disk $|z| < R$). We can consider the R-L and E-K fractional integrals, the corresponding R-L and E-K fractional derivative, and the operators of Generalized FC that are defined in Section 4.3 in *functional spaces* of the form

$$\begin{aligned} &\mathcal{H}_\mu(\Omega) = \left\{ f(z) = z^\mu \widetilde{f}(z);\ \mu \geq 0;\ \widetilde{f} \in \mathcal{H}(\Omega)\text{—the space of functions analytic in } \Omega \right\}, \\ &\text{where } 0 \leq \arg z < 2\pi, \text{ i.e., with a cut along the positive half-line } \{\mathrm{Re}\, z \geq 0, \mathrm{Im}\, z = 0\}, \\ &\text{or } -\pi \leq \arg z < \pi, \text{ i.e., with a cut along the negative half-line } \{\mathrm{Re}\, z \leq 0, \mathrm{Im}\, z = 0\}, \end{aligned} \tag{41}$$

so as to avoid multiplicities in members like $z^{\mu+k}$ in $f(z)$.

In Kiryakova [20] (Ch.4), [69], etc., it is shown that the E-K operators transform the ${}_pF_q$- and ${}_p\Psi_q$-functions into the same kind of functions with increased orders p and q: resp., into ${}_{p+1}F_{q+1}$- and ${}_{p+1}\Psi_{q+1}$-functions. For the Le Roy-type functions (24), the images under the Erdélyi–Kober (E-K) fractional integration operator (40) are provided in our recent paper [14], and in particular cases, for the Riemann–Liouville fractional integrals (34) of the function (22), by Rogosin–Dubatovskaya [45]. Note also that R-L fractional integrals of the $\overline{H}$-function are presented in Srivastava et al. [70].

4.3. G-L Operators for Multi-Index M-L Functions and Generalized FC

To explain the construction of the G-L operators generated by the *multi-index M-L functions* (9), we need to introduce briefly the notions of *Generalized Fractional Calculus* (GFC) from Kiryakova [20] and other works. Fractional calculus, as an exotic theory of the differentiation and integration of arbitrary (not-integer) order, arose in the 19th century, and has nowadays become a hot topic with a very wide scope of application. To read about its development, the reader can consult, e.g., the survey by Machado–Kiryakova [67]. We need to pay honor to Kalla [71] who introduced the notion of *generalized operators of fractional integration* by using an arbitrary special function as a kernel. Then, he and many other authors explored kernels such as the Gauss hypergeometric function, the Bessel functions, the Appell F_3-function, and in general, the Fox H-function; for details, see [72]. However, the choice of suitable kernels as $G^{m,0}_{m,m}$- and $H^{m,0}_{m,m}$-functions was crucial for the possibility to develop a detailed and comparatively full theory of GFC [20].

Definition 9. *Take an integer (multiplicity) $m \geq 1$. By means of the sets of parameters: $\boldsymbol{\delta} = (\delta_1 \geq 0, \ldots, \delta_m \geq 0)$ as a multi-order of fractional integration, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ as a real multi-weight and additional multi-parameter $\boldsymbol{\beta} = (\beta_1 > 0, \ldots, \beta_m > 0)$, we define the integral operator:*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \int\limits_0^1 H^{m,0}_{m,m}\left[t \left| \begin{array}{c} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_{i=1}^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_{i=1}^m \end{array} \right. \right] f(zt)dt, \text{ if } \sum_{i=1}^m \delta_i > 0, \tag{42}$$

and $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = f(z)$ if $\delta_1 = \delta_2 = \cdots = \delta_m = 0$. We call it a multiple (m-tuple) Erdélyi–Kober fractional integration operator. More generally, all operators of the form

$$If(z) = z^{\delta_0}\, I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) \quad \text{with} \quad \delta_0 \geq 0, \tag{43}$$

are our generalized $(m$-tuple$)$ fractional integrals of multi-order $(\delta_1 \geq 0, \ldots, \delta_m \geq 0)$.

The kernel function in (42) is a suitably chosen case $H^{m,0}_{m,m}$ of the Fox H-function (1), and when $\forall \beta_i = \beta > 0, i = 1, \ldots, m$, we have slightly simpler generalized fractional integrals with the kernel as Meijer $G^{m,0}_{m,m}$-function (3), see [20] (Ch.1).

The generalized fractional derivatives $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ of multi-order $(\delta_1, \ldots, \delta_m)$, corresponding to (42) and (43), are also introduced explicitly, in a way similar to that for D^δ and $D^{\gamma,\delta}_\beta$ but with more complicated differ-integral representations. See [20] and also [73,74].

Here, the operators of GFC are considered in the spaces $\mathcal{H}_\mu(\Omega)$ of analytic functions with power weights of the form (41). In [20] (Ch.1, Ch.5) and in subsequent works, see, e.g., survey [73], the properties of these operators of GFC in various other functional spaces are studied and a full chain of operational rules is proposed, with a wide range of applications and many particular examples. Here, we mention only the following basic result for functions analytic in a disk that are useful for our next considerations.

Let the following conditions on the parameters be satisfied: $\beta_k(\gamma_k + 1) > -\mu$, $\delta_k \geq 0$, $k = 1, \ldots, m$. Then, the multiple E-K integral (42) maps the class $\mathcal{H}_\mu(\Omega)$ of weighted analytic functions $f(z)$ into itself. In particular, the images of the functions $f(z)$ analytic in a disk $\Delta_R = \{|z| < R\}$:

$$f(z) = z^\mu \sum_{k=0}^\infty a_k z^k = z^\mu (a_0 + a_1 z + \ldots) \in \mathcal{H}_\mu(\Delta_R), \tag{44}$$

have the same form:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = z^\mu \sum_{k=0}^\infty a_k \left\{ \prod_{i=1}^m \frac{\Gamma(\gamma_i + \frac{k+\mu}{\beta_i} + 1)}{\Gamma(\gamma_i + \delta_i + \frac{k+\mu}{\beta_i} + 1)} \right\} z^k \;\in\; \mathcal{H}_\mu(\Delta_R), \tag{45}$$

with the same radius of convergence $R = \left\{ \limsup_{k\to\infty} \sqrt[k]{|a_k|} \right\}^{-1} > 0$.

For $m = 1$, (42) is the "classical" E-K integral (40). A very important property of the single-integral operators (42) with H-functions kernels (or with Meijer's G-functions in the simpler case of equal $\beta_i = \beta > 0, i = 1, \ldots, m$) is the so-called *composition/decomposition rule*: they can also be represented by means of commutative compositions of classical E-K integrals ($m = 1$):

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \left[\prod_{i=1}^m I^{\gamma_i,\delta_i}_{\beta_i} \right] f(z)$$

$$= \int\limits_0^1 \ldots \int\limits_0^1 \left[\prod_{i=1}^m \frac{(1-t_i)^{\delta_i - 1} t_i^{\gamma_i}}{\Gamma(\delta_i)} \right] f\left(z t_1^{\frac{1}{\beta_1}} \ldots \sigma_m^{\frac{1}{\beta_m}} \right) dt_1 \ldots dt_m\,, \tag{46}$$

that is, by iterated integrals without special functions involved in the kernel. This explains the wide and efficient use of the operators of the GFC from [20] because compositions like (46) often appear in problems related to applications, but their study and evaluation is based on the simple but effective tools of the theory of the kernel special functions (H- and G-functions) in definition (42).

Next, let us consider the G-L-type integration and differentiation operators we introduced as generated by the M-L function (7) and by the multi-index M-L function (9). The series representations for these operators can be analytically continued in the functional spaces $\mathcal{H}_\mu(\Omega)$ as particular cases of the generalized fractional integrals $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ and derivatives $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ of fractional multi-order. We called these operators *multiple Dzrbashjan–Gelfond–Leontiev operators*, so as to honor the Armenian mathematician Dzrbashjan, author of the book [75], one of the most detailed studies of the 20th century on the M-L function (7) in the complex plane. In [29], he also introduced a 2×2 ($m = 2$) M-L-type function as the first representative of the multi-index M-L functions (9). Now, we refer to these operators as *G-L operators for the multi-index M-L functions*, defined as follows.

Let $f(z)$ be an analytic function in a disk $\Delta_R = \{|z| < R\}$ and $\alpha_i > 0, \beta_i \in \mathbb{R}, i = 1, \ldots, m$ be arbitrary parameters. The operators:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \longmapsto \quad D_{(\alpha_i),(\beta_i)} f(z) \ \text{ and } \ L_{(\alpha_i),(\beta_i)} f(z),$$

constructed as

$$\begin{aligned} D_{(\alpha_i),(\beta_i)} f(z) &= \sum_{k=1}^{\infty} a_k \frac{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)}{\Gamma(\alpha_1 (k-1) + \beta_1) \ldots \Gamma(\alpha_m (k-1) + \beta_m)} \, z^{k-1}, \\ L_{(\alpha_i),(\beta_i)} f(z) &= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)}{\Gamma(\alpha_1 (k+1) + \beta_1) \ldots \Gamma(\alpha_m (k+1) + \beta_m)} \, z^{k+1}, \end{aligned} \tag{47}$$

are called *G-L operators generated by the multi-index M-L function* (9).

Evidently, $D_{(\alpha_i),(\beta_i)} L_{(\alpha_i),(\beta_i)} f(z) = f(z)$ for $f(z)$ analytic in Δ_R, and the coincidence of the radii of convergence of $f(z)$ and of the series in (47) easily follows by the Cauchy–Hadamard formula and the asymptotic estimation of the Γ-function multipliers (details are, e.g., in [20] (Th.5.5.2)). In Kiryakova [20] (§5.4.ii) and subsequent works such as [37,39], etc., we provided the following integral representation of the G-L integration $L_{(\alpha_i),(\beta_i)}$ as its analytical extension in terms of the GFC operators of multi-order $(\alpha_1, \ldots, \alpha_m)$.

Theorem 4. *Let $\Omega \supset \Delta_R$ be a domain in $\mathbb{C}$ starlike with respect to the origin $z = 0$. Then, the multi-index G-L integration operator in* (47) *can be analytically continued from $\mathcal{H}(\Delta_R)$ into $\mathcal{H}_\mu(\Omega)$ by means of the single-integral operator*

$$L_{(\alpha_i),(\beta_i)} f(z) = z \int_0^1 H^{m,0}_{m,m} \left[\sigma \left| \begin{matrix} (\beta_i, \alpha_i)_1^m \\ (\beta_i - \alpha_i, \alpha_i)_1^m \end{matrix} \right. \right] f(z\sigma) d\sigma = z \, I^{(\beta_i - 1),(\alpha_i)}_{(1/\alpha_i),m} f(z), \tag{48}$$

that is, by a generalized fractional integral of form (43).

Here, we concentrate on the G-L operators of generalized integration and their representations by generalized fractional integrals, and skip the details on the corresponding differentiation analogues. However, let us mention that the multi-index G-L derivative in (47) can be represented by generalized fractional derivatives $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ ([20,73]), and for analytic functions in $\mathcal{H}(\Omega) \supset \mathcal{H}(\Delta_R)$, and under restrictions $\beta_i - \alpha_i \neq 0, -1, -2, \ldots$; $z \neq 0$,

$$D_{(\alpha_i),(\beta_i)} f(z) = z^{-1} \, D^{(\beta_i - \alpha_i - 1),(\alpha_i)}_{(1/\alpha_i),m} f(z) - \left[\prod_{i=1}^{m} \frac{\Gamma(\beta_i)}{\Gamma(\beta_i - \alpha_i)} \right] \frac{f(0)}{z}. \tag{49}$$

By the rules of the GFC one can check that $D_{(\alpha_i),(\beta_i)} L_{(\alpha_i),(\beta_i)} f(z) = f(z)$.

The following result from [38–40] shows that the multi-index M-L functions (9) are "*eigenfunctions*" of the multi-index G-L operators (47)–(49). Namely:

The multi-index Mittag-Leffler functions (9) *satisfy the integral relation*

$$L_{(\alpha_i),(\beta_i)}\, E_{(\alpha_i),(\beta_i)}(\lambda z) = \frac{1}{\lambda}\, E_{(\alpha_i),(\beta_i)}(\lambda z) - \frac{1}{\lambda \prod\limits_i \Gamma(\beta_i)}, \;\; \lambda \neq 0, \tag{50}$$

and the differential equation of fractional multi-order $(\alpha_1, \dots, \alpha_m)$*:*

$$D_{(\alpha_i),(\beta_i)}\, E_{(\alpha_i),(\beta_i)}(\lambda z) = \lambda\, E_{(\alpha_i),(\beta_i)}(\lambda z), \;\; \lambda \neq 0. \tag{51}$$

Then, we may say that the G-L operators (47)–(49) are *"eigen"-operators for the multi-index Mittag-Leffler functions* (9).

Example 3. *For* $m = 1$*, the operators* (47) *are the Dzrbashjan–Gelfond–Leontiev differentiation and integration operators, generated by the Mittag-Leffler function* (7)*, as introduced in Kiryakova* [20] *(Ch.2):*

$$\begin{aligned} D_{\alpha,\beta} f(z) &= \sum_{k=1}^{\infty} a_k \frac{\Gamma(\alpha k + \beta)}{\Gamma(\alpha(k-1) + \beta)} z^{k-1}, \\ L_{\alpha,\beta} f(z) &= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha k + \beta)}{\Gamma(\alpha(k+1) + \beta)} z^{k+1}. \end{aligned} \tag{52}$$

And their analytical continuations appear as (single, $m = 1$*) E-K fractional integrals* (40) *and corresponding E-K derivatives:*

$$\begin{aligned} L_{\alpha,\beta} f(z) &= z^1\, I_{1/\alpha}^{\beta-1,\alpha} f(z) = \frac{z}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} f(z t^{\alpha})\, dt, \\ D_{\alpha,\beta} f(z) &= z^{-1} D_{1/\alpha}^{\beta-\alpha-1,\alpha} f(z) - \frac{f(0)\Gamma(\beta)}{\Gamma(\beta-\alpha)}\, z^{-1}, \;\; z \neq 0. \end{aligned} \tag{53}$$

For $\beta = 1$*, one has the simpler G-L operators* (38) *and* (39) *from Samko–Kilbas–Marichev* [65]*, as in the previous Example* 2*.*

4.4. G-L Operators Generated by the Le Roy-Type Functions $F_{(\alpha,\beta)_m}^{(\gamma)_m}$

We consider *only* the case of the multi-index Mittag-Leffler functions (22) of Le Roy type $F_{(\alpha,\beta)_m}^{(\gamma)_m}$, as the simpler cases introduced in Rogosin and Dubatovskaya's articles [45,61]. It seems that the more general case of special functions (24) with additional Prabhakar parameters $\tau_i \neq 1$, $i = 1, \dots, m$, still cannot be treated with respect to corresponding eigenoperators (in the sense of Gelfond–Leontiev operators). At this point, we consider it to be an *open problem*.

First, let us present the corresponding (slightly simpler) representations of these Le Roy-type functions in terms of the generalizations of the Fox H-function and Fox–Wright function ${}_p\Psi_q$. As a corollary of Theorem 3, bearing in mind that $\forall \tau_i = 1$, $T = 1$, and that some equal sets of parameters in the upper and lower rows in the $\widetilde{\Psi}$- and I-functions cancel each other out (Property (7.1) in Rathie [10], quite similar to that for the H-functions, e.g., [7], etc.), we have:

$$F_{(\alpha,\beta)_m}^{(\gamma)_m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod\limits_{i=1}^{m} \Gamma^{\gamma_i}(\alpha_i k + \beta_i)}$$

$$= {}_m\widetilde{\Psi}_{2m-1}\left[\begin{matrix} (1,1,1)_1^m \\ (1,1,1)_{(m-1)\text{–times}}, (\beta_i, \alpha_i, \gamma_i)_1^m \end{matrix} \middle| z \right] \tag{54}$$

$$= {}_1\widetilde{\Psi}_m\left[\begin{matrix} (1,1,1) \\ (\beta_i, \alpha_i, \gamma_i)_1^m \end{matrix} \middle| z \right]$$

$$= \overline{H}_{m,2m}^{1,m}\left[-z \middle| \begin{matrix} (0,1,1)_1^m \\ (0,1,1)_{m\text{–times}}, (1-\beta_i, \alpha_i, \gamma_i)_1^m \end{matrix} \right] \tag{55}$$

$$
\begin{aligned}
&= \overline{H}^{1,1}_{1,m+1}\left[-z\left|\begin{array}{c}(0,1,1)\\(0,1,1),(1-\beta_i,\alpha_i,\gamma_i)_1^m\end{array}\right.\right]\\
&= I^{1,m}_{m,2m}\left[-z\left|\begin{array}{c}(0,1,1)_1^m\\(0,1,1)_{m\text{–times}},(1-\beta_i,\alpha_i,\gamma_i)_1^m\end{array}\right.\right]\\
&= I^{1,1}_{1,m+1}\left[-z\left|\begin{array}{c}(0,1,1)\\(0,1,1),(1-\beta_i,\alpha_i,\gamma_i)_1^m\end{array}\right.\right].
\end{aligned}
\tag{56}
$$

Now, let us construct the Gelfond–Leontiev generalized operators of integration and differentiation (32) and (33), generated by the coefficients of the mMLR-function $F^{(\gamma)_m}_{(\alpha,\beta)_m}$:

$$\phi_k = 1/\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i).$$

For an entire function $f(z)=\sum\limits_{k=0}^{\infty}a_k z^k$ (or analytic in a disk), these have the following form of power series, to be a *generalized G-L differentiation*:

$$\mathbf{D}\, f(z) := D_{mMLR}\, f(z) = \sum_{k=1}^{\infty} a_k z^{k-1}\cdot\prod_{i=1}^{m}\frac{\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\Gamma^{\gamma_i}(\alpha_i k+\beta_i-\alpha_i)}, \tag{57}$$

and, respectively, a *generalized G-L integration*:

$$\mathbf{L}\, f(z) := L_{mMLR}\, f(z) = \sum_{k=0}^{\infty} a_k z^{k+1}\cdot\prod_{i=1}^{m}\frac{\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\Gamma^{\gamma_i}(\alpha_i k+\beta_i+\alpha_i)}. \tag{58}$$

Then, $\mathbf{D}\,\mathbf{L}\, f(z) = f(z)$.

Theorem 5. *The generalized G-L operator of differentiation (32) is an eigenoperator for the function $F^{(\gamma)_m}_{(\alpha,\beta)_m}$, that is, $F^{(\gamma)_m}_{(\alpha,\beta)_m}$ is an eigenfunction for this "differentiation" operator:*

$$\boldsymbol{D}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = D_{mMLR}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \lambda\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z),\ \ \lambda\neq 0. \tag{59}$$

For the G-L integration, we obtain the corresponding relation

$$\boldsymbol{L}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = L_{mMLR}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \frac{1}{\lambda}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) - \frac{1}{\lambda\prod\limits_{i=1}^{m}\Gamma^{\gamma_i}(\beta_i)},\ \ \lambda\neq 0. \tag{60}$$

Proof. Indeed, using the power series representation (22),

$$
\begin{aligned}
D_{mMLR}\left(F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z)\right) &= D_{mMLR}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{\prod\limits_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}\, z^k\right)\\
&= \sum_{k=1}^{\infty}\frac{\lambda^k}{\prod\limits_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}\, z^{k-1}\,\frac{\prod\limits_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\prod\limits_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i-\alpha_i)}
\end{aligned}
$$

(where the groups of the Gamma functions $\Gamma^{\gamma_i}(\alpha_i k+\beta_i)$ cancel each other out, and we change the index of summation as $k^* := k-1$, to have)

$$= \lambda\sum_{k^*=0}^{\infty}(\lambda z)^{k^*}\prod_{i=1}^{m}\frac{1}{\Gamma^{\gamma_i}(\alpha_i k^*+\beta_i)} = \lambda\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z).$$

Analogously,

$$\mathbf{L}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = L_{mMLR}\left(\sum_{k=0}^{\infty}\frac{\lambda^k}{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_1)}z^k\right)$$

$$= \sum_{k=0}^{\infty}\frac{\lambda^k}{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}z^{k+1}\frac{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k+\beta_i+\alpha_i)}$$

$$= \frac{1}{\lambda}\sum_{k^*=1}^{\infty}\frac{(\lambda z)^{k^*}}{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\alpha_i k^*+\beta_i)} = \frac{1}{\lambda}F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) - \frac{1}{\lambda\prod_{i=1}^{m}\Gamma^{\gamma_i}(\beta_i)}. \qquad \square$$

Both relations (59) and (60) are exactly the expected *analogues of those for the multi-index M-L functions*, (51) and (50).

Now, we aim to find an integral representation of the G-L integral (58), as an analogue of the generalized fractional integration operator (48) in Theorem 4. Instead of the H-function $H^{m,0}_{m,m}$, we have now an I-function as a kernel. Then, we can admit that *such an integral operator can also be somehow thought of as an operator of "fractional multi-order"*.

Theorem 6. *The G-L integration operator* (58) *of an entire function $f(z)$, generated by means of the Le Roy-type function* (22)*, can also be represented by means of the integral operator*

$$\mathbb{I}\, f(z) = L_{mMLR}\, f(z) = z\int_0^1 I^{m,0}_{m,m}\left[\sigma\,\middle|\,\begin{matrix}(\beta_i,\alpha_i,\gamma_i)_1^m\\(\beta_i-\alpha_i,\alpha_i,\gamma_i)_1^m\end{matrix}\right]f(z\sigma)d\sigma. \qquad (61)$$

This can be interpreted as a kind of a generalized fractional integration of multi-order $(\alpha_1,\ldots,\alpha_m)$.

We call the operators (58) and (61) the *Gelfond–Leontiev–Le Roy (G-L-Le Roy) operators of generalized integration*. To emphasize when these concern the general case of $m = 1,2,3,\ldots$, we sometimes also use the notation $\mathbb{I}^m$ and, for example, when $m = 1$, the notation is $\mathbb{I}^1$. If necessary, sub-indices i are also used, $\mathbb{I}^1_i$, so as to specify the relation to the particular parameters $\alpha_i > 0, \beta_i > 0, \gamma_i > 0$.

Proof. First, let us analyse the nature of the kernel function $I^{m,0}_{m,m}$ with the parameters as in (61). As an I-function, its definition (11) and (12) reads as

$$I^{m,0}_{m,m}(\sigma) = \frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\beta_i-\alpha_i-\alpha_i s)}{\prod_{i=1}^{m}\Gamma^{\gamma_i}(\beta_i-\alpha_i s)}\sigma^s ds. \qquad (62)$$

The singular points for $\Gamma^{\gamma_i}(\beta_i-\alpha_i-\alpha_i s)$ are: $s_{ik} = \frac{\beta_i}{\alpha_i}+\frac{k}{\alpha_i}-1$, $k = 0,1,2,\ldots$, that is $\forall i = 1,\ldots,m$: $s_{ik} > -1$, and there are no singularities for $s < -1$. For $\Gamma^{\gamma_i}(\beta_i-\alpha_i s)$, the singular points appear at $s_{il} = \frac{\beta_i}{\alpha_i}+\frac{l}{\alpha_i}$, $l = 0,1,2,\ldots$ and all are $s_{il} > 0$. Then, it seems that for $s < -1$, none of the Gamma functions have singularities. The cut to ensure single values of the involved members can now be taken along the real half-line $s \geq 0$ (see assumptions (41)). Therefore, we can have a contour $\mathcal{L} = (c-i\infty, c+i\infty)$ with some $c < -1$ for which all singularities are to the right and which does not intersect the branch cut.

The parameters (13) for the $I^{m,0}_{m,m}$-function in (61) are as follows:

$$\mu = \sum_{i=1}^{m}\alpha_i\gamma_i - \sum_{i=1}^{m}\alpha_i\gamma_i = 0;\quad \Delta = \sum_{i=1}^{m}\alpha_i\gamma_i - \sum_{i=1}^{m}\alpha_i\gamma_i = 0;$$

$$\nabla = \sum_{i=1}^{m}\gamma_i(\beta_i-1/2) - \sum_{i=1}^{m}\gamma_i(\beta_i-\alpha_i-1/2) = \sum_{i=1}^{m}\alpha_i\gamma_i > 0;$$

$$R = \prod_{i=1}^{m} \alpha_i^{\alpha_i \gamma_i} / \prod_{i=1}^{m} \alpha_i^{\alpha_i \gamma_i} = 1.$$

This means that the kernel function $I_{m,m}^{m,0}$ is an analytic function in the unit disk $0 < |z| < 1$ and it vanishes for $|z| > 1$. Then, the integral in (61) from 0 to 1, when necessary (for the evaluations below), can also be taken with limits from 0 to ∞, having the same value.

Let us evaluate this integral operator of an entire function with power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$:

$$\begin{aligned}\mathbb{I} f(z) &= z \int_0^1 I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\beta_i, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i)_1^m \end{matrix} \right.\right] \left\{ \sum_{k=0}^{\infty} a_k \, z^k \sigma^k \right\} d\sigma \\ &= z \sum_{k=0}^{\infty} a_k z^k \left\{ \int_0^1 \sigma^k \, I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\beta_i, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i)_1^m \end{matrix} \right.\right] d\sigma \right\} \\ &= z \sum_{k=0}^{\infty} a_k z^k \left\{ \int_0^1 I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\beta_i + \alpha_i k, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i + \alpha_i k, \alpha_i, \gamma_i)_1^m \end{matrix} \right.\right] d\sigma \right\},\end{aligned}$$

where the exchanged order of integration and summation is admissible, and we also use the shifting property from Rathie [10] (7.3) for $\sigma^k I_{m,m}^{m,0}(\sigma)$.

Now, we need to use an *auxiliary result* that in general will read as follows:

$$\begin{aligned}\int_0^1 I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (a_i, A_i, \alpha_i)_1^m \\ (b_i, B_i, \beta_i)_1^m \end{matrix} \right.\right] d\sigma &= \int_0^{\infty} I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (a_i, A_i, \alpha_i)_1^m \\ (b_i, B_i, \beta_i)_1^m \end{matrix} \right.\right] d\sigma \\ &= \prod_{i=1}^{m} \frac{\Gamma^{\beta_i}(b_i + B_i)}{\Gamma^{\alpha_i}(a_i + A_i)}, \quad \text{for } \forall a_i > b_i > 0. \end{aligned} \tag{63}$$

This is an analogue of our "Auxiliary integral" (Kiryakova [20] (App.E, (E.21))) for the similar $H_{m,m}^{m,0}$-function with $a_i > b_i > 0, \ i = 1, \ldots, m$:

$$\int_0^1 H_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (a_i, A_i)_1^m \\ (b_i, B_i)_1^m \end{matrix} \right.\right] d\sigma = \prod_{i=1}^{m} \frac{\Gamma(b_i + B_i)}{\Gamma(a_i + A_i)}. \tag{64}$$

The simpler analogous formula for the case of $G_{m,m}^{m,0}$ is proved in Kiryakova [20] (Lemma B.2), and the same one is repeated in Karp–Lopez [76].

For the $H_{m,m}^{m,0}$-function, Formula (64) follows by also taking $s = 1$ in the result for the Mellin transform of $H_{p,m}^{m,0}$-function (it can also be $p = m$) from Karp–Prilepkina [77] (Th.6):

$$\int_0^{R\,(\text{or }\infty)} \sigma^{s-1} \, H_{p,m}^{m,0}\left[\sigma \left| \begin{matrix} (a_i, A_i) \\ (b_j, B_j) \end{matrix} \right.\right] d\sigma = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s)}{\prod_{i=1}^{p} \Gamma(a_i + A_i s)}.$$

Analogously, the result for $I_{m,m}^{m,0}$ follows from the Mellin transform image of the I-function ([78] (3.9)) taken for $s = 1$. As in the case of the H-functions, the same result can be derived as a corollary by a more general integral formula, see Vellaisamy–Kataria [78] (Prop.3.1), and Lemma 1 in Section 4.5, involving a weighted product of two I-functions. For the corresponding formula in the case of the product of two H-functions, we refer to [3] (2.8.4) and [20] (E.21′).

For the particular parameters of the $I_{m,m}^{m,0}$-function here, from (63), we obtain

$$\begin{aligned}&\int_0^1 I_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\beta_i + \alpha_i k, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i + \alpha_i k, \alpha_i, \gamma_i)_1^m \end{matrix} \right.\right] d\sigma \ \left(= \int_0^{\infty} \ldots\right) \\ &= \prod_{i=1}^{m} \frac{\Gamma^{\gamma_i}(\beta_i - \alpha_i + \alpha_i k + \alpha_i)}{\Gamma^{\gamma_i}(\beta_i + \alpha_i k + \alpha_i)} = \prod_{i=1}^{m} \frac{\Gamma^{\gamma_i}(\alpha_i k + \beta_i)}{\Gamma^{\gamma_i}(\alpha_i k + \alpha_i + \beta_i)},\end{aligned}$$

which we need to show that

$$\mathbb{I}\,f(z)=\mathbb{I}\left(\sum_{k=0}^{\infty}a_k z^k\right)=\sum_{k=0}^{\infty}a_k\,z^{k+1}\prod_{i=1}^{m}\frac{\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\Gamma^{\gamma_i}(\alpha_i k+\alpha_i+\beta_i)}=L_{mMLR}\,f(z).\qquad\square$$

Remark 1. *Along with the check above that the two operators $L_{mMLR}\,f(z)$ and $\mathbb{I}\,f(z)$ are equivalent for entire functions $f(z)$ given by power series, we can discuss the behavior of the kernel $I^{m,0}_{m,m}$ around the singularities 0 and 1 in the improper integral (61). First, it is known (see, e.g., Rathie [10] (6.9)) that for small values of σ, i.e., for $\sigma\to+0$:*

$$I^{m,0}_{m,m}\left[\sigma\middle|\begin{array}{c}(\beta_i,\alpha_i,\gamma_i)_1^m\\(\beta_i-\alpha_i,\alpha_i,\gamma_i)_1^m\end{array}\right]\sim\sigma^c,\ \ \textit{where}$$

$$c=\min_{1\le i\le m}\{(\beta_i-\alpha_i)/\alpha_i\}=\min_{1\le i\le m}\{\frac{\beta_i}{\alpha_i}-1\}>-1,\ \textit{for}\ \forall\,\alpha_i>0,\beta_i>0.$$

The asymptotic behavior of the I-functions near the singular point $\sigma=1$ (or in more general cases, near $\sigma=R$ on the circle of convergence) is not yet well studied. The same situation stands for the Fox H-functions' behavior at the third singular point when these are analytic in disks with finite radius $|z|<R$, or outside, in $|z|>R$. A particular result for the case of $H^{m,0}_{p,m}$-functions $(n=0,q=m)$ with $\mu=\sum_{j=1}^{m}B_j-\sum_{i=1}^{p}A_i=0$ is contained in Karp [21] (12.24). This can be paraphrased briefly as

$$H^{m,0}_{p,m}\left[R\cdot\sigma\middle|\begin{array}{c}(a_i,A_i)_1^p\\(b_j,B_j)_1^m\end{array}\right]=(1-\sigma)^{\eta-1}\sum_{k=0}^{\infty}\frac{h_k(1-\sigma)^k}{\Gamma(\eta+k)},$$

where in our case $(m=p=q)$:

$$R=1\ \textit{and}\ \eta=\sum_{i=1}^{p}a_i-\sum_{j=1}^{m}b_j-\frac{m-p}{2}=\sum_{i=1}^{p}a_i-\sum_{j=1}^{m}b_j>0,\ \textit{that is},\ \eta-1>-1.$$

In the case of $\mu=\sum_{j=1}^{m}B_j-\sum_{i=1}^{p}A_i=0$ (which is also true, in particular, if $\forall A_i=B_j$), the main term in the representation above depends only on the value of η given above.

The result for the simpler case of the Meijer $G^{m,0}_{m,m}$ function (known also as the Meijer–Nørlund function) by Marichev (1981) is analogous, and was used by Kiryakova [20] (1.1.14):

$$G^{m,0}_{m,m}\left[\sigma\middle|\begin{array}{c}(a_k)_1^m\\(b_k)_1^m\end{array}\right]\sim\frac{(1-\sigma)^{\eta_m^*}}{\Gamma(\eta_m^*+1)}\ \ \textit{as}\ \ \sigma\to1,\sigma<1,$$

with $\eta_m^*=\sum_{k=1}^{m}(a_k-b_k)-1>-1$ in the case when (see [20] (Ch.1)) $a_k-b_k=(\gamma_k+\delta_k)-\gamma_k=\delta_k>0,\ \ \forall k=1,\dots,m$ and $\sum_{k=1}^{m}\delta_k\neq1,2,3,\dots$ (to avoid the logarithmic case, which separately considered leads to similar asymptotics, ensuring $\eta_m^*>-1$). The same behavior follows from the result found by Karp–López [76] (8), for $G^{m,0}_{m,m}(\sigma)$ in the disk $|1-\sigma|<1$ under the same conditions, showing that the behavior near the singular point $\sigma=1$ depends only on this $\eta>-1$.

In the case of I-function (62) of orders $n=0$, $m=p=q$, with parameters $a_i:=\beta_i>\beta_i-\alpha_i:=b_i$, one has quite a similar result because of all of the equal second $(\alpha_i>0)$ and third $(\gamma_i>0)$ parameters, $i=1,\dots,m$. The main term of the asymptotics depends only on η:

$$I^{m,0}_{m,m}\left[\sigma\middle|\begin{array}{c}(\beta_i,\alpha_i,\gamma_i)\\(\beta_i-\alpha_i,\alpha_i,\gamma_i)\end{array}\right]\sim(1-\sigma)^{\eta-1},$$
$$\textit{with}\ \eta=\sum_{i=1}^{m}[\beta_i-(\beta_i-\alpha_i)]\gamma_i=\sum_{i=1}^{m}\alpha_i\gamma_i>0\ \Rightarrow\ \eta-1>-1,\ \textit{as}\ \sigma\to1,\ \sigma<1.\qquad(65)$$

We clarify this situation in more detail for $m=1$ in Section 4.5.

Alternatively, for the $I^{m,0}_{m,m}$*-functions* (62)*, one may interpret and specify the results (6.4) and (6.5) from Rathie* [10] *(Procedure 1) when* $\mu = 0$ *and* $\nabla = \sum_{i=1}^{m} \alpha_i \gamma_i > 0$ *but is not integer. The technique used employs a known expansion for large s, involving the Bernoulli polynomials. Then,* $I(\sigma)$ *can be expressed in a series of Beta functions in the unit disk that leads to similar asymptotics as those in* (65)*. Otherwise, if* $\nabla = \sum_{i=1}^{m} \alpha_i \gamma_i$ *is a positive integer, one falls in a logarithmic case, a result that is considered in* [10] *(6.6). In our situation (for shortness, we mention the result for* $m = 1$, $\nabla = \alpha\gamma = 1, 2, 3, \ldots$*), one can derive it in a form:*

$$I^{m,0}_{m,m}(\sigma) \sim \frac{(1-\sigma^{1/\alpha})^{\alpha\gamma-1}}{(\alpha\gamma-1)!} + \ldots \ \ \textit{for } \sigma \to 1,\, \sigma < 1 \textit{ with } (\alpha\gamma - 1) > -1.$$

4.5. Composition/Decomposition Property of the Gelfond–Leontiev–Le Roy Integrations

For the generalized fractional integrals (42) with $H^{m,0}_{m,m}$- (resp. $G^{m,0}_{m,m}$-) kernels in the Generalized Fractional Calculus (Kiryakova [20]), we have already mentioned the important composition/decomposition property (46). In the case of the G-L generalized integration in (53) for $m = 1$, generated by the M-L function $E_{\alpha,\beta}$, we provide a reminder that its integral representation has the form

$$L_{\alpha,\beta} f(z) = z\, I^{\beta-1,\alpha}_{1/\alpha} f(z) = \frac{z}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\beta-1} f(zt^{\alpha})\, dt$$

$$= z \int_0^1 \frac{(1-\sigma^{1/\alpha})^{\alpha-1}\, \sigma^{\beta/\alpha-1}}{\alpha\, \Gamma(\alpha)} f(z\sigma)\, d\sigma = z \int_0^1 H^{1,0}_{1,1}\left[\sigma \,\middle|\, \begin{matrix} (\beta,\alpha) \\ (\beta-\alpha,\alpha) \end{matrix} \right] f(z\sigma) d\sigma,$$

where the kernel $H^{1,0}_{1,1}$ analytic in $|\sigma| < 1$ can also be represented by the *generalized binomial series*

$$\frac{(1-\sigma^{1/\alpha})^{\alpha-1}\, \sigma^{\beta/\alpha-1}}{\alpha\, \Gamma(\alpha)} = \frac{\Gamma(\alpha)\, \sigma^{\beta/\alpha-1}}{\alpha\, \Gamma(\alpha)} \sum_{k=0}^{\infty} \binom{\alpha-1}{k} \left(-\sigma^{1/\alpha}\right)^k = \frac{\sigma^{\beta/\alpha-1}}{\alpha} \sum_{k=0}^{\infty} \frac{\left(-\sigma^{1/\alpha}\right)^k}{\Gamma(\alpha-k)\, k!}.$$

Note that if $\alpha - 1 = n$, a non-negative integer, the $k = n + 1$-term and all the next ones in the series are 0, i.e., the series is finite.

Next, we prove a similar result in that the Gelfond–Leontiev–Le Roy integrals (61) of "multi-order" $(\alpha_1, \ldots, \alpha_m)$ *can be represented as commutable compositions of m operators* $\mathbb{I}^1$ *of the same kind* for $m = 1$, each one with different parameters. As mentioned before in the proof of Theorem 6, we use the notation $\mathbb{I}^m$ for the case of arbitrary (multiplicity) $m = 1, 2, 3, \ldots$, while a particular operator with $m = 1$ is denoted by $\mathbb{I}^1$, and we add a sub-index i for $\mathbb{I}^1_i$ to keep in mind that it is the case with parameters $(\alpha_i, \beta_i, \gamma_i)$. These operators $\mathbb{I}^1_i$ *play the role of "Le Roy type" extensions of the Erdélyi–Kober operators* (40) for the integration of "order" α_i. In short, these can be called *Erdélyi–Kober–Le Roy integrations* (E-K-Le Roy), with each of them having a representation as for (61) with $m = 1$:

$$\mathbb{I}^1_i f(z) = z \int_0^1 I^{1,0}_{1,1}\left[\sigma \,\middle|\, \begin{matrix} (\beta_i, \alpha_i, \gamma_i) \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i) \end{matrix} \right] f(z\sigma) d\sigma \tag{66}$$

$$= \int_0^z I^{1,0}_{1,1}\left[\frac{\zeta}{z} \,\middle|\, \begin{matrix} (\beta_i, \alpha_i, \gamma_i) \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i) \end{matrix} \right] f(\zeta) d\zeta.$$

It is interesting to represent this kernel function $I^{1,0}_{1,1}$ in the way we did above for the $H^{1,0}_{1,1}$, as a kind of similar "generalized" binomial series. Namely, for $|\sigma| < 1$, $\gamma > 0$, one has

$$I_{1,1}^{1,0}\left[\sigma\left|\begin{array}{c}(\beta,\alpha,\gamma)\\(\beta-\alpha,\alpha,\gamma)\end{array}\right.\right]=\frac{\sigma^{\beta/\alpha-1}}{\alpha}\sum_{k=0}^{\infty}\frac{\left(-\sigma^{1/\alpha}\right)^k}{\Gamma^\gamma(\alpha\gamma-k)\,k!}.$$

The main term in such series gives an asymptotics as

$$I_{1,1}^{1,0}\left[\sigma\left|\begin{array}{c}(\beta,\alpha,\gamma)\\(\beta-\alpha,\alpha,\gamma)\end{array}\right.\right]\sim\frac{\sigma^{\beta/\alpha-1}\,(1-\sigma^{1/\alpha})^{\alpha\gamma-1}}{\alpha\Gamma^\gamma(\alpha\gamma)},$$

$$\text{with}\quad \beta/\alpha-1>-1 \text{ for } \sigma\to 0,\ \sigma>0,\quad \text{and}\quad \alpha\gamma-1>-1 \text{ for } \sigma\to 1,\ \sigma<1.$$

This is to confirm again, for the simplest case of $m=1$, the described behavior of the kernel $I_{m,m}^{m,0}$ near the singular points 0 and 1 in the improper integral (66).

Before proving the decomposition of the operator $\mathbb{I}^m$, we need the following *auxiliary result* that appears analogous to the formula for the Mellin transform, and thus, for an integral from 0 to ∞ of the product of two different H-functions. This can be found, for example, in some handbooks such as [3] (2.8.4) and (E.21′) in Kiryakova [20].

Here, it is derived from Proposition 3.1 in Vellaisamy–Kataria [78] for the *Mellin transform of the product of two different I-functions*. For our purposes, we reproduce it as an integral formula in a simpler form.

Lemma 1. *For $\lambda\neq 0$, $\nu\neq 0$, $s>0$,*

$$\int_0^\infty \sigma^{s-1}\cdot I_{p,q}^{m,n}\left[\lambda\sigma\left|\begin{array}{c}(a_i,A_i,\alpha_i)_1^p\\(b_j,B_j,\beta_j)_1^q\end{array}\right.\right]\cdot I_{u,v}^{k,l}\left[\nu\sigma\left|\begin{array}{c}(c_i,C_i,\varphi_i)_1^u\\(d_j,D_j,\psi_j)_1^v\end{array}\right.\right]d\sigma$$
$$=\frac{1}{s}\cdot I_{q+u,p+v}^{n+k,m+l}\left[\frac{\nu}{\lambda}\left|\begin{array}{c}(c_i,C_i,\varphi_i)_1^l,\ (1-b_j-B_j,B_j,\beta_j)_1^q,\ (c_i,C_i,\varphi_i)_{l+1}^u\\(d_j,D_i,\psi_j)_1^k,\ (1-a_i-A_i,A_i,\alpha_i)_1^p,\ (d_j,D_j,\psi_i)_{k+1}^v\end{array}\right.\right].\tag{67}$$

A set of necessary operational properties for the I-functions that are rather similar to those for the Fox H-functions can be found in Rathie [10] (Sect. 7). We mention only a few of them that are needed below, such as: symmetry in some sets of parameters; shifting property (7.3) for $\sigma^c I_{p,q}^{m,n}(\sigma)$; and (7.5) for $I_{p,q}^{m,n}(\sigma)\mapsto I_{q,p}^{n,m}(1/\sigma)$.

Now, we are ready to state and prove the *composition/decomposition property of the Gelfond–Leontiev–Le Roy generalized integration*.

Theorem 7. *For entire functions $f(z)$, the equivalent representations hold:*

$$\mathbb{I}^m f(z)=\left[\prod_{i=1}^m \mathbb{I}_i^1\right]f(z)=\mathbb{I}_m^1\left\{\mathbb{I}_{m-1}^1\cdots\left[\mathbb{I}_1^1 f(z)\right]\right\}.\tag{68}$$

This composition of operators is commutative.

Proof. Let us check the statement first for $m=2$. We consider the composition of two Le Roy-type Erdélyi–Kober operators of the form (66):

$$\mathbb{I}_2^1\,\mathbb{I}_1^1 f(z)=\int_0^z I_{1,1}^{1,0}\left[\frac{\zeta}{z}\left|\begin{array}{c}(\beta_2,\alpha_2,\gamma_2)\\(\beta_2-\alpha_2,\alpha_2,\gamma_2)\end{array}\right.\right]\left\{\mathbb{I}_1^1 f(z)\right\}d\zeta$$

$$=\int_0^z I_{1,1}^{1,0}\left[\frac{\zeta}{z}\left|\begin{array}{c}(\beta_2,\alpha_2,\gamma_2)\\(\beta_2-\alpha_2,\alpha_2,\gamma_2)\end{array}\right.\right]\left\{\int_0^\zeta I_{1,1}^{1,0}\left[\frac{\tau}{\zeta}\left|\begin{array}{c}(\beta_1,\alpha_1,\gamma_1)\\(\beta_1-\alpha_1,\alpha_1,\gamma_1)\end{array}\right.\right]f(\tau)\,d\tau\right\}d\zeta$$

$$=\int_0^z f(\tau)\,d\tau\left\{\int_\tau^z I_{1,1}^{1,0}\left[\frac{\zeta}{z}\left|\begin{array}{c}(\beta_2,\alpha_2,\gamma_2)\\(\beta_2-\alpha_2,\alpha_2,\gamma_2)\end{array}\right.\right]\cdot I_{1,1}^{1,0}\left[\frac{\tau}{\zeta}\left|\begin{array}{c}(\beta_1,\alpha_1,\gamma_1)\\(\beta_1-\alpha_1,\alpha_1,\gamma_1)\end{array}\right.\right]d\zeta\right\}$$

$$:= \int_0^z f(\tau) \left\{ \widetilde{\mathcal{I}} \right\} d\tau.$$

The interchange of the order of integrals, as above, is well admissible. We have to note that both of the involved $I_{1,1}^{1,0}$-functions in the above expression $\widetilde{\mathcal{I}}$ vanish outside the unit disk. That is, the one with $(i=1)$ parameters is $\equiv 0$ for $\zeta < \tau$, and the other one for $(i=2)$ is $\equiv 0$ for $\zeta > z$. Therefore, the above inner integral denoted by $\widetilde{\mathcal{I}}$ will have the same value if its limits from τ to z are extended to an interval from 0 to ∞. Moreover, we replace $I_{1,1}^{1,0}$ with the $(i=1)$ parameters by $I_{1,1}^{0,1}$ of the reciprocal argument, according to the property (7.5) from Rathie [10] (quite similar to one for the H-functions) and then use Formula (67), taking $\lambda := z^{-1}$, $\nu := \tau^{-1}$ and $s = 1$. We have:

$$\mathbb{I}_2^1 \, \mathbb{I}_1^1 \, f(z) = \int_0^z I_{2,2}^{0,2} \left[\frac{z}{\tau} \middle| \begin{array}{c} (1-\beta_1+\alpha_1,\alpha_1,\gamma_1),(1-\beta_2+\alpha_2,\alpha_2,\gamma_2) \\ (1-\beta_2,\alpha_1,\gamma_2),(1-\beta_1,\alpha_1,\gamma_1) \end{array} \right] f(\tau)\, d\tau,$$

and using again Rathie's property (7.5) for $I_{2,2}^{0,2} \to I_{2,2}^{2,0}$, and symmetry of the $I_{2,2}^{2,0}$-function with respect to the parameters in the upper and lower rows, this gives

$$\mathbb{I}_2^1 \, \mathbb{I}_1^1 \, f(z) = \int_0^z I_{2,2}^{2,0} \left[\frac{\tau}{z} \middle| \begin{array}{c} (\beta_1,\alpha_1,\gamma_1),(\beta_2,\alpha_2,\gamma_2) \\ (\beta_1-\alpha_1,\alpha_1,\gamma_1),(\beta_2-\alpha_2,\alpha_2,\gamma_2) \end{array} \right] f(\tau)\, d\tau, \quad \text{i.e.,}$$

$$\mathbb{I}_2^1 \, \mathbb{I}_1^1 \, f(z) = z\int_0^1 I_{2,2}^{2,0} \left[\sigma \middle| \begin{array}{c} (\beta_1,\alpha_1,\gamma_1),(\beta_2,\alpha_2,\gamma_2) \\ (\beta_1-\alpha_1,\alpha_1,\gamma_1),(\beta_2-\alpha_2,\alpha_2,\gamma_2) \end{array} \right] f(z\sigma)\, d\sigma = \mathbb{I}_{2,1}^2 \, f(z).$$

The statement for the composition of the arbitrary number $m \geq 1$ of operators (66) can then be derived by mathematical induction, following the same procedure as above. In the final stage, we repeat the above evaluation for

$$\mathbb{I}^{m-1} \left\{ \mathbb{I}_m^1 \, f(z) \right\} = \mathbb{I}^m \, f(z).$$

Note that due to symmetry of the $I_{m,m}^{m,0}$-function with respect to the parameters in the upper and lower rows, the composition is *commutative*, i.e., does not depend on the order of the operators $\mathbb{I}_i^1$, $i = 1, 2, \ldots, m$ in (66). $\square$

Remark 2. *An alternative way to prove Theorem 7 is to show that the Mellin transform images $\mathcal{M}$ of the operators $\mathbb{I}^m$ and of the composition $\left[\prod_{i=1}^m \mathbb{I}_i^1\right]$ coincide for entire functions $f(z)$. This kind of proof is similar to that we used for the composition/decomposition property of the generalized fractional integrals* (43) *with $H_{m,m}^{m,0}$-kernels, see Th. 5.1.5 and Th. 5.2.1 in Kiryakova* [20] *(Ch.5). Using* (67) *again and the auxiliary formula* (63)*, we have:*

$$\mathcal{M}\{\mathbb{I}^m f(z); s\} = \left[\prod_{i=1}^m \frac{\Gamma^{\gamma_i}(\beta_i - s\alpha_i)}{\Gamma^{\gamma_i}(\beta_i + \alpha_i - s\alpha_i)} \right] \mathcal{M}\{f(z); s\} = \prod_{i=1}^m \left[\mathcal{M}\left\{ \mathbb{I}_i^1 f(z); s \right\} \right].$$

In a future study, we can look for a *semigroup property for the new generalized operators of the fractional integration of multi-order* $(\alpha_1, \ldots, \alpha_m)$ of a form more general than that appearing in (61).

5. Illustration of Results from Sections 3 and 4 for Particular Cases of the Le Roy-Type Functions

Here, we give examples as to how the discussed results work for the previously introduced and studied Le Roy-type functions $F^{(\gamma)}$, $F_{\alpha,\beta}^{(\tau)}$, $F_{(\alpha,\beta)_m}^{(\gamma)_m}$, and for some of their particular cases.

For $\underline{m=1}$, the simplest case is the original Le Roy function (20), $\gamma > 0$. According to Theorem 3, (28), etc., for $\alpha = \beta = 1$,

$$F^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma^\gamma(1.k+1)} = {}_1\widetilde{\Psi}_1 \left[\begin{array}{c} (1,1,1) \\ (1,1,\gamma) \end{array} \middle| z \right]$$
$$= \overline{H}^{1,1}_{1,2} \left[-z \middle| \begin{array}{c} (0,1,1) \\ (0,1),(0,1,\gamma) \end{array} \right] = I^{1,1}_{1,2} \left[-z \middle| \begin{array}{c} (0,1,1) \\ (0,1,1),(0,1,\gamma) \end{array} \right].$$

Naturally, for $\gamma = 1$,

$$F^{(1)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = {}_1\Psi_1 \left[\begin{array}{c} (1,1) \\ (1,1) \end{array} \middle| z \right] = {}_1F_1(1;1;z) = \exp(z),$$

as a "classical" generalized hypergeometric function, and in particular, the exponential one. Based on Theorem 4, the "eigen" integral operator generated by the Le Roy function for entire functions $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k$ has the form

$$\mathbf{L}^1_{1,1} f(z) = z \sum_{k=0}^{\infty} a_k \frac{\Gamma^\gamma(k+1)}{\Gamma^\gamma(k+2)}\, z^k = z \int\limits_0^1 I^{1,0}_{1,1} \left[\sigma \middle| \begin{array}{c} (1,1,\gamma) \\ (0,1,\gamma) \end{array} \right] f(z\sigma) d\sigma,$$

with $\mathbf{L}^1_{1,1}\, F^\gamma(\lambda z) = (1/\lambda)\, F^\gamma(\lambda z) - 1/\lambda$, and the corresponding G-L differentiation (57): $\mathbf{D}^1_{1,1}\, F^\gamma(\lambda z) = \lambda\, F^\gamma(\lambda z),\ \lambda \neq 0$.

For the Mittag-Leffler-type Le Roy function (21) (Gerhold, Garra–Polito, Garrappa–Rogosin–Mainardi, etc.) with $\alpha > 0, \beta > 0, \gamma > 0$, the same results read as follows:

$$F^{(\gamma)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma^\gamma(\alpha k + \beta)} = {}_1\widetilde{\Psi}_1 \left[\begin{array}{c} (1,1,1) \\ (\beta,\alpha,\gamma) \end{array} \middle| z \right]$$
$$= \overline{H}^{1,1}_{1,2} \left[-z \middle| \begin{array}{c} (0,1,1) \\ (0,1),(1-\beta,\alpha,\gamma) \end{array} \right] = I^{1,1}_{1,2} \left[-z \middle| \begin{array}{c} (0,1,1) \\ (0,1,1),(1-\beta,\alpha,\gamma) \end{array} \right].$$

The corresponding eigenoperators $\mathbf{L}^\gamma_{\alpha,\beta}$, resp. $\mathbf{B}^\gamma_{\alpha,\beta}$ satisfy the eigenfunction relations:

$$\mathbf{L}^\gamma_{\alpha,\beta}\, F^{(\gamma)}_{\alpha,\beta}(\lambda z) = \frac{1}{\lambda}\, F^{(\gamma)}_{\alpha,\beta}(\lambda z) - \frac{1}{\lambda\, \Gamma^\gamma(\beta)}, \quad \mathbf{D}^\gamma_{\alpha,\beta}\, F^{(\gamma)}_{\alpha,\beta}(\lambda z) = \lambda\, F^{(\gamma)}_{\alpha,\beta}(\lambda z),$$

where we have the Le Roy-type *analogue of the Erdélyi–Kober fractional integral*:

$$\mathbf{L}^\gamma_{\alpha,\beta}\, f(z) = z \sum_{k=0}^{\infty} a_k \frac{\Gamma^\gamma(\alpha k + \beta)}{\Gamma^\gamma(\alpha k + \alpha + \beta)}\, z^k = z \int\limits_0^1 I^{1,0}_{1,1} \left[\sigma \middle| \begin{array}{c} (\beta,\alpha,\gamma) \\ (\beta-\alpha,\alpha,\gamma) \end{array} \right] f(z\sigma) d\sigma.$$

For $\underline{m=2}$, in our previous papers [13,14,16], we discussed the Le Roy-type function mentioned in Pogány [79], and kept the original denotations from there:

$$F^{\alpha,\beta}_{(p,q;r,s)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma^\alpha(pk+q)\, \Gamma^\beta(rk+s)}.$$

Then, we provided the values for the order and type of this entire function. In view of the results of Sections 3 and 4, the following new representations follow:

$$F^{\alpha,\beta}_{(p,q;r,s)}(z) = {}_2\widetilde{\Psi}_3 \left[\begin{array}{c} (1,1,1) \\ (1,1,1),(q,p,\alpha),(s,r,\beta) \end{array} \middle| z \right]$$
$$= \overline{H}^{1,2}_{2,4}(-z) = I^{1,2}_{2,4} \left[-z \middle| \begin{array}{c} (0,1,1) \\ (0,1,1),(0,1,1),(1-q,p,\alpha),(1-s,r,\beta) \end{array} \right],$$

and the corresponding Gelfond–Leontiev–Le Roy integration operator is represented by (with the adapted denotations)

$$\mathbf{L}^{\alpha,\beta}_{(p,q;r,s)}\, f(z) = z \int\limits_0^1 I^{2,0}_{2,2} \left[\sigma \middle| \begin{array}{c} (q,p),(s,r) \\ (q-p,p),(s-r,r) \end{array} \right] f(z\sigma) d\sigma.$$

The above kernel function can be considered as an analogue of the Gauss hypergeometric function related to $G^{2,0}_{2,2}$, so such an integral operator can be thought as analogous to the popular *Saigo fractional integrals*. It is important to mention that a case of G-L-Le Roy-type integration with $m = 3$ and a kernel $I^{3,0}_{3,3}$ would resemble an extension of the so-called *Marichev–Saigo–Maeda operators of FC*, see more details in Kiryakova [80].

In the case of arbitrary $m \geq 1$, for the general M-L-P function of Le Roy type (24), the representations in terms of the generalized Fox–Wright function, $\overline{H}$- and I-functions are given as in Theorem 3. However, the problem of defining corresponding Gelfond–Leontiev operators for which $\mathbb{F}^m$ is an eigenfunction seems *to still be open*, in the general case.

In the case of Prabhakar parameters $\forall \tau_i = 1$, we have the following representations:

$$F^{(\gamma)_m}_{(\alpha,\beta)_m}(z) = {}_m\widetilde{\Psi}_{2m-1}\left[\begin{array}{c} (1,1,1)_1^m \\ (1,1,1)_{(m-1)\text{–times}}, (\beta_i,\alpha_i,\gamma_i)_1^m \end{array} \middle| z \right]$$

$$= \overline{H}^{1,m}_{m,2m}\left[-z \middle| \begin{array}{c} (0,1,1)_1^m \\ (0,1)_{m\text{–times}}, (1-\beta_i,\alpha_i,\gamma_i)_1^m \end{array} \right] = I^{1,m}_{m,2m}\left[-z \middle| \begin{array}{c} (0,1,1)_1^m \\ (0,1,1)_{m\text{–times}}, (1-\beta_i,\alpha_i,\gamma_i)_1^m \end{array} \right],$$

where some of the orders of the above $\widetilde{\Psi}$, $\overline{H}$- and I-functions can be suitably reduced because of the coincidence of some parameters $(1,1,1)$ in the upper row with part of the same in the lower row. Then, these are analogous to the representations for the multi-index Mittag-Leffler functions (9) with $\forall \gamma_i = 1$:

$$F^{(1)_m}_{(\alpha,\beta)_m} := E^{(m)}_{(\alpha_i,\beta_i)}(z) = {}_1\Psi_m\left[\begin{array}{c} (1,1) \\ (\beta_i,\alpha_i)_1^m \end{array} \middle| z \right] = H^{1,1}_{1,m+1}\left[-z \middle| \begin{array}{c} (0,1) \\ (0,1), (1-\beta_i,\alpha_i) \end{array} \right].$$

For the $F^{(\gamma)_m}_{(\alpha,\beta)_m}$-functions, in Section 4 we developed the Gelfond–Leontiev generalized integrations and differentiation (58) and (57) as their eigenoperators, called Gelfond–Leontiev–Le Roy operators.

It is necessary to emphasize the evident analogy of the representations of (58) and (57), with the results for the operators (47): $L_{(\alpha_i,\beta_i)}$ and $D_{(\alpha_i,\beta_i)}$ generated by the multi-index Mittag-Leffler functions, in terms of the Generalized Fractional Calculus (GFC), see Kiryakova [20]. Compare the integral representation (48) with the new one of Le Roy-type, (61), when the kernel Fox $H^{m,0}_{m,m}$-function is replaced by a more general $I^{m,0}_{m,m}$-function. For $m = 1$, we have $L_{(\alpha,\beta)}$ in the form of an Erdélyi–Kober integral (40), and, respectively, the operator $\mathbb{I}^1$ in (66) as a kind of "*Erdélyi-Kober-Le Roy integral*" (66).

6. Some Short Reminders from "Guide to SF of FC"

Here, we provide only a brief reminder of some of the SF of FC that are considered in detail in the survey of Kiryakova [9], and partly in the monographs by Kiryakova [20] and Paneva-Konovska [43]. The reason is because these special functions are to be compared with related extensions to the Le Roy type, I- and $\overline{H}$-functions.

6.1. Classes of G- and H-Functions as Kernels of Laplace-Type Integral Transforms and of Operators of Generalized Fractional Calculus

In survey [9] (Sect.3), we attracted readers' attention to the use of *two basic classes of G- and H-functions* with specific orders: **(i)** $G^{m,0}_{0,m}$, resp., $H^{m,0}_{0,m}$ with $m = q, n = p = 0$; and **(ii)** $G^{m,0}_{m,m}$, resp., $H^{m,0}_{m,m}$ with $m = p = q, n = 0$.

Concerning the G- and H-functions of type **(i)**, let us mention that the analogues of the Laplace transform, the so-called *G- and H-transforms*

$$\mathcal{G}\{f(t);s\} = \int_0^\infty G^{m,n}_{p,q}\left[st \middle| \begin{array}{c} (a_j)_1^p \\ (b_k)_1^q \end{array} \right] f(t)dt,$$

and, respectively,

$$\mathcal{H}\{f(t);s\} = \int_0^\infty H^{m,n}_{p,q}\left[st \left| \begin{array}{c} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{array} \right.\right] f(t)dt,$$

are said to be *generalized integral transforms of Laplace type* when

$$\delta = m + n - \frac{p+q}{2} > 0, \quad \text{resp.} \quad a^* = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{k=1}^{m} B_k - \sum_{k=m+1}^{q} B_k > 0.$$

As their simplest examples, we mention the *Laplace and Meijer transforms*

$$\mathcal{L}\{f(t);s\} = \int_0^\infty \exp(-st) f(t)dt, \quad \mathcal{K}_\nu\{f(t);s\} = \int_0^\infty \sqrt{st}\, K_\nu(st) f(t)dt, \tag{69}$$

and also the *Borel transform* (considered by Dzrbasjan [75], see also Kiryakova [20] (Ch.2))

$$\mathcal{B}_{(\rho),(\mu)}\{f(t);s\} = \int_0^\infty \exp(-s^\rho t^\rho)\, t^{\mu\rho-1} f(t)dt, \quad \rho > 0,\ \mu \in \mathbb{C}.$$

In 1958, the Bulgarian mathematician Obrechkoff introduced a far-reaching generalization of both Laplace and Meijer transforms, particular cases of which were studied years later by many authors (Ditkin, Prudnikov, Krätzel, etc.). We mention, for example, the *Krätzel transform* ([81,82]),

$$\mathcal{L}^{(m)}_\nu\{f(t);s\} := \int_0^\infty \Lambda(s,t) f(t)dt, \text{ where } \Lambda(s,t) \text{ is the \textit{Krätzel function} as below.}$$

For details and studies on the latter function, see [81–83].

Note that the kernels of the above-mentioned integral transforms appear, resp., as

$$\exp(-s) = G^{1,0}_{0,1}\left[s \left| \begin{array}{c} -- \\ 0 \end{array} \right.\right], \quad K_\nu(s) = \frac{1}{2} G^{2,0}_{0,2}\left[\frac{s^2}{4} \left| \begin{array}{c} -- \\ \frac{\nu}{2}, \frac{-\nu}{2} \end{array} \right.\right],$$

$$\exp(-s^\rho t^\rho) = H^{1,0}_{0,1}\left[st \left| \begin{array}{c} -- \\ (\mu - \frac{1}{\rho}, \frac{1}{\rho}) \end{array} \right.\right], \quad \Lambda(s,t) = s^{-\nu-1+\frac{2}{m}}\, G^{m,0}_{0,m}\left[st \left| \begin{array}{c} -- \\ 0, (\nu + \frac{k-2}{m})_{k=2}^{k=m} \end{array} \right.\right].$$

These integral transforms have been used mainly for the purposes of operational calculi for different classes of differential operators, like

$$D = \frac{d}{dt}, \ B_\nu = \frac{d}{dt} t^{1-\nu} \frac{d}{dt} t^\nu, \ D^{\,1/\rho}, \ B^{(m)}_\nu = \frac{d}{dt} t^{\frac{1}{m}-\nu} \left(t^{1-\frac{1}{m}} \frac{d}{dt} \right)^{m-1} t^{\nu+1-\frac{2}{m}}, \text{ etc.}$$

The initial aims of Obrechkoff were to extend the theorem of S.N. Bernstein for absolutely monotonic functions representable by means of Laplace–Stieltjes transforms when the conditions for n-th derivatives are replaced by similar for differential operators, more general than those above. The *Obrechkoff transform*, as modified by Dimovski and studied in detail by Kiryakova (see [84], also [20,85] (Ch.3)), was defined originally in the form

$$\mathcal{O}\{f(t);s\} = \beta \int_0^\infty t^{\beta(\gamma_m+1)-1}\, K\left[(st)^\beta\right] f(t)dt = \beta \int_0^\infty \lambda(t,s) f(t)dt,$$

with the kernel function

$$K(s) = \int_0^\infty \ldots \int_0^\infty \exp\left(-u_1 - \ldots - u_{m-1} - \frac{s}{u_1 \ldots u_{m-1}}\right) \prod_{k=1}^{m} u_k^{\gamma_m - \gamma_k - 1}\, du_1 \ldots du_{m-1}. \tag{70}$$

Later, in Kiryakova [20] (Ch.3) (and in other works, see e.g., [84]), we proved that the kernel function (70) of the Obrechkoff transform is representable by a Meijer's $G^{m,0}_{0,m}$-function, namely

$$\lambda(t,s) = s^{-\beta(\gamma_m+1)+1}\, G^{m,0}_{m,m}\left[(st)^{\beta}\middle|\begin{array}{c} -- \\ (\gamma_k - \frac{1}{\beta}+1)_1^m \end{array}\right].$$

Thus, *the Obrechkoff transform appears to be a G-transform of Laplace type*, related to the *hyper-Bessel differential operators* (Dimovski [86], Kiryakova [85], Dimovski–Kiryakova [84])

$$Bf(t) = t^{\alpha_0}\frac{d}{dt}t^{\alpha_1}\frac{d}{dt}\cdots t^{\alpha_{m-1}}\frac{d}{dt}t^{\alpha_m}f(t) \tag{71}$$

$$= t^{-\beta}\,P_m\left(t\frac{d}{dt}\right)f(t) = t^{-\beta}\prod_{k=1}^{m}\left(t\frac{d}{dt}+\beta\gamma_k\right)f(t),\ \ t>0,$$

with arbitrary parameters $\alpha_0,\alpha_1,\ldots,\alpha_m,\ \beta := m-(\alpha_0+\alpha_1+\ldots+\alpha_m)>0,\ \gamma_k := (\alpha_k+\alpha_{k+1}+\ldots+\alpha_m)/\beta,\ \ k=1,\ldots,m$, and P_m a polynomial of degree m. Evidently, for particular choices of these parameters, (71) gives the above-mentioned simpler differential operators and many others frequently appearing in problems of mathematical physics.

Next, the *generalized Obrechkoff transform* (its *fractionalized* analogue) was introduced and studied by Kiryakova [20] (Ch.5) (and next in papers by Luchko, Al Mussalam and V.K. Tuan, and also in Yakubovich–Luchko [36]) with a *Fox $H^{m,0}_{m,m}$-function as a kernel*:

$$\mathcal{GO}(s) := \mathcal{B}_{(\rho_i),(\mu_i)}\{f(t);s\} = \int_0^\infty H^{m,0}_{0,m}\left[st\middle|\begin{array}{c} -- \\ (\mu_i-\frac{1}{\rho_i},\frac{1}{\rho_i})_1^m \end{array}\right] f(t)dt. \tag{72}$$

We called it *multi-index Borel–Dzrbashjan transform* because $m=1$ is reduced to the mentioned Borel transform. The "fractional" Obrechkoff transform (72) can be considered not only as a H-transform but as a tool in operational calculus for the *fractional multi-order analogues of hyper-Bessel differential operators* (71), formally written as

$$D_{(\rho_i),(\mu_i)}f(t) = t^{-1}\prod_{i=1}^{m}\left(t^{1+(1-\mu_i)\rho_i}\,D^{1/\rho_i}_{t^{\rho_i}}\,t^{(\mu_i-1)\rho_i}\right)f(t),$$

in the same way as the Laplace transform, Obrechkoff transform and its particular cases serve for the classical differentiation, for the hyper-Bessel operators (71), etc.

All the above details on the use of the functions $G^{m,0}_{0,m}$ and $H^{m,0}_{0,m}$ serve to pose an *Open Problem* (see 8.4. in the concluding Section 8) for the possibility to introduce and study a Laplace-type integral transform with $I^{m,0}_{0,m}$ and $\overline{H}^{m,0}_{0,m}$ as kernel functions.

We also have to emphasize in terms of the other class of special functions the $G^{m,0}_{m,m}$ and $H^{m,0}_{m,m}$-functions of type **(ii)**, as discussed in survey [9], because here in Section 4, the classical FC operators of integration (such as R-L and E-K, (40)) are mentioned as having such kernels for $m=1$, and also the generalized fractional integrals (42) have these kernels for arbitrary order $m\geq 1$. In addition, in Section 4.4 (Theorem 6), their $I^{m,0}_{m,m}$-analogue appears as a kernel function of the integral operator (61), being an alternative representation of the series (58) for the introduced Gelfond–Leontiev generalized integration generated by the Le Roy-type functions (22). Thus, the need for the analogy is evident.

6.2. Some Important Cases of Mittag-Leffler Functions (7) *and Multi-Index Mittag-Leffler Functions* (9)

- We mention a few cases of M-L functions ($m=1$) from §5.1 of [9]: *Generalized trigonometric functions* of higher integer orders $m\geq 1$, $\cos_m$ and, resp., sine-functions; and next mentioned their "fractalized" analogues; also the *Lorenzo–Hartley functions*, the *Rabotnov function*, etc. Extensive literature is nowadays available on the theory of the M-L functions and their cases, with a few to mention such as: [2,30,31,39,50,87], etc.
- For $m=2$, a not very popular function introduced by *Dzrbashjan* [29], in his and, respectively, in "our" denotations, is:

$$\Phi_{\rho_1,\rho_2}(z;\mu_1,\mu_2) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\Gamma(\mu_2+\frac{k}{\rho_2})} \tag{73}$$

$$:= E_{(\frac{1}{\rho_1},\frac{1}{\rho_2}),(\mu_1,\mu_2)}(z) = E_{(\alpha_1,\alpha_2),(\beta_1,\beta_2)}(z),\ \ 1/\rho_i := \alpha_i > 0, \mu_i := \beta_i.$$

In addition to the classical M-L function, the geometric series and the *Bessel functions* J_ν, we observed ([9]) that (73) also includes: the *Struve and Lommel functions* $s_{\mu,\nu}$ and H_ν, and the *(classical) Wright function* [24,25] (details in [27] and in many books)

$$\phi(\alpha,\beta;z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!} = {}_0\Psi_1\left[\begin{array}{c} - \\ (\beta,\alpha) \end{array} \middle| z \right] = E^{(2)}_{(\alpha,1),(\beta,1)}(z),$$

and thus, also the *Mainardi function* $M(z;\alpha/2) = \phi(-\alpha/2, 1-\alpha/2; -z)$, and the *Airy function* $M(z;1/3) = 3^{2/3}\mathrm{Ai}(z/3^{1/3})$, etc.

In other denotations, the Wright function $\phi(\alpha,\beta;z)$ is also known as the *generalized Bessel function*, the *Wright–Bessel function* or misnamed as the *Bessel–Maitland function*:

$$J^{\mu}_{\nu}(z) = \phi(\mu,\nu+1;-z) = {}_0\Psi_1\left[\begin{array}{c} - \\ (\nu+1,\mu) \end{array} \middle| -z \right] \tag{74}$$

$$= \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\nu+k\mu+1)\,k!} = E^{(2)}_{(1/\mu,1),(\nu+1,1)}(-z)\,,$$

again as an example of the Dzrbashjan function. We discuss its extension as an $\overline{H}$-function (81) in Section 7. In the same survey [9] (§5.2), further extensions of the Bessel function with 3, 4, etc., parameters are mentioned.

- For the case $m \geq 2$, we mention the *hyper-Bessel functions of Delerue* [88] (for details and essential use of these functions, see [20] (Ch.3)):

$$J^{(m-1)}_{\gamma_i,\ldots,\gamma_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1}\gamma_i} E^{(m)}_{(1,1,\ldots,1),(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1,1)}\left(-(\frac{z}{m})^m\right) \tag{75}$$

$$= \left[\prod_{i=1}^{m-1}\Gamma(\gamma_i+1)\right]^{-1} \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1}\gamma_i} {}_0F_{m-1}\left(\gamma_1+1,\gamma_2+1,\ldots,\gamma_{m-1}+1;-(\frac{z}{m})^m\right).$$

This representation suggests that the multi-index M-L functions (9) with arbitrary $(\alpha_1,\ldots,\alpha_m) \neq (1,\ldots,1)$ can be interpreted as *fractional-indices analogues of the hyper-Bessel functions* (75), which themselves are *multi-index analogues* of the Bessel function. Functions (75) are closely related to the *hyper-Bessel differential operators* (71) of Dimovski [86] and form a fundamental system of solutions of the differential equations of the form $By(z) = \lambda y(z)$, as proved in Kiryakova [20] (Ch.3, Th.3.4.3).

In relation to the multi-index analogues of the Mittag-Leffler functions, we can also mention two other variants that appear as solutions of some fractional multi-order (multi-term) differential equations. In Gorenflo–Kilbas–Rogosin [89], the function

$$E_{\alpha,\mu,l}(z) = \sum_{k=0}^{\infty} c^k z^k,\ \ \text{where}\ \ c_k = \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(j\mu+l)+1]}{\Gamma[\alpha(j\mu+l+1)+1]},$$

is introduced and studied, with indices $\mathrm{Re}\,(\alpha) > 0, l \in \mathbb{C}, \mu \in \mathbb{R}$. One can be curious about a case when Le Roy-type fractional power indices are attached to the Gamma functions in the coefficient c_k. Another variant is the function introduced and studied recently by Droghei [90]

$$\mathcal{W}^{(\bar{\alpha},\bar{\nu}),n}(z) = \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k}\prod_{j=1}^{n} \frac{\Gamma(\alpha_{n+1}i+a_j)}{\Gamma(\alpha_{n+1}i+b_j)} \right) \frac{z^k}{\Gamma(\alpha_{n+1}k+b_{n+1})},$$

with parameters $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{n+1})$; $\bar{\nu} = (\nu_1, \ldots, \nu_n)$ and $a_j = 1 + \sum_{m=1}^{j}(\nu_{m-1} - \alpha_m)$; and $b_j = 1 + \sum_{m=1}^{j}(\nu_{m-1} - \alpha_{m-1})$, such that $a_j = b_j - \alpha_j$ with $j = 1, \ldots, n+1$. In some particular cases, say for $n = 1$, its three-parameter variant:

$$\mathcal{W}_{\alpha,\beta,\nu}(z^\beta) = \sum_{k=0}^{\infty}\left(\prod_{i=1}^{k}\frac{\Gamma(\beta i + 1 - \alpha)}{\Gamma(\beta i + 1)}\right)\frac{z^{\beta k}}{\Gamma(\beta k + 1 - \alpha + \nu)}$$

is shown to be a solution of the fractional order variant of the Bessel equation, and reducible to the Bessel function; to the Wright function ($\alpha = 1$); to the Mittag-Leffler function ($\alpha = 0$: $\mathcal{W}_{0,\alpha,\beta-1}(z) = E_{\alpha,\beta}(z)$); and to some cases of the hyper-Bessel and 2×2 Mittag-Leffler function of Dzrbashjan: $\mathcal{W}_{\nu,\nu,\nu}(z^\nu) = \sum\limits_{k=0}^{\infty} \frac{z^{\nu k}}{\Gamma^2(\nu k+1)}$. Therefore, this can also be considered as a Le Roy-type function but with integer index $\gamma = 2$.

Other cases of the ${}_p\Psi_q$ and of multi-index M-L functions are mentioned in [9] (Ch.6), such as Virchenko and Ricci generalized hypergeometric functions, Mainardi–Massina and Paris generalized exponential integrals, generalized K-series and M-series, etc., as well as critics on the numerous so-called k-analogues defined by replacement of the Γ-functions by Γ_k-functions that bring no essential novelties at all (details in Kiryakova [91]).

A *multi-variable Mittag-Leffler multi-index function* (called also *multinomial* M-L function) was introduced by Luchko et al. (e.g., [36,60]) as well. Further, a *multinomial Prabhakar function* was introduced by Bazhlekova–Bazhlekov, see Bazhlekova [92] (2.2), where an additional Prabhakar parameter δ is involved. The complete monotonicity of this new special function is studied via the Laplace transform image, and the appearance of the above-mentioned multinomial M-L type functions in resolving multi-term time-fractional evolution equations is discussed.

7. Other Important SF in the Scheme of the I- and $\overline{H}$-Functions

Next, it is time to mention some other well-known or not so well-known special functions, but all with important significance, that also fall in the scheme of the $\overline{H}$-functions of Inayat-Hussain [11] and of the more general I-functions of Rathie [10]. This class of generalized hypergeometric functions is introduced and studied as an extension of the Fox–Wright functions ${}_p\Psi_q$ and of the Fox H-functions, and the functions are defined by kinds of "Mellin-Barnes"-type integrals (or more exactly, their analogues). They involve quotients of Gamma functions on arbitrary (fractional) powers, and in general, present multi-valued functions with branch points, thus needing precision of the meanings and conditions of existence. Several authors have studied the details of the I- and $\overline{H}$-functions, along with the above-mentioned initiators, such as: Buschman–Srivastava [12], Saxena [47], Gupta–Soni [93], Srivastava–Lyn–Wang [70], Mathai–Saxena–Haubold [6], Vellaisamy–Kataria [78], Jolly [94], Srivastava [95], etc. Nevertheless, this theory has not yet been deeply developed and this has raised some open problems. However, we need to mention that the I- and $\overline{H}$-functions arose from the needs for mathematical tools, such as for the evaluation of certain Feynman integrals (arising in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions), partition function of the Gauss model from statistical mechanics, etc.

We skip the particular examples of I- and $\overline{H}$-functions from the initial works of Inayat-Hussain and Rathie, invoking their appearance; these are also mentioned in Appendix A.5.1 in Mathai–Saxena–Haubold [6], as (A.60)–(A.62).

Here, we provide *some examples of popular "mathematical" functions*, starting from the simplest ones.

- *The polylogarithm function.*

  Consider (see for example, [1] (Vol.1), [5], etc.)

$$\mathrm{Li}_\alpha(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^\alpha} = z + \frac{z^2}{2^\alpha} + \frac{z^3}{3^\alpha} + \cdots, \quad |z| < 1,\ \alpha \in \mathbb{C}, \tag{76}$$

which for particular choices of the index α appears as: ordinary logarithm $\mathrm{Li}_1(z) = -\ln(1-z)$; $\mathrm{Li}_0(z) = \frac{z}{1-z}$; $\mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}$; $\mathrm{Li}_n(z) = z\,{}_{n+1}F_n(1,1,\ldots;2,2,\ldots;z)$, $n = 0,1,2,\ldots$; $\mathrm{Li}_{-n}(z) = z\,{}_nF_{n-1}(2,2,\ldots;1,1,\ldots;z)$, $-n = -1,-2,\ldots$; etc. Note also that for (the singular value) $z = 1$ when $\alpha > 1$, it gives the *famous Riemann Zeta function*, $\mathrm{Li}_\alpha(1) = \zeta(\alpha) = \sum_{k=1}^{\infty} 1/(k^\alpha)$. The function (76) is also referred to as the *Jonquière function*. As particular examples, for special choice of argument and parameters, one can also mention some popular polynomials such as the Jonquière and Bernoulli polynomials, and number sequences such as the Stirling and Eulerian numbers, etc.

For the purposes here, we consider the *real positive index* $\alpha > 0$. Contour-type integral representations of "Mellin-Barnes"-type have been considered in many of the above-mentioned works, like, e.g., Jolly [94] (1.1.24). We appeal to the representation as presented by Gerhold–Tomovski [96], (1.2)–(1.3):

$$Li_\alpha(z)(z) = -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\pi}{s^\alpha \sin \pi s}(-z)^s\, ds,$$

and using [1] (Vol.1, Ch.1),

$$\frac{\pi}{\sin \pi s} = \Gamma(s)\Gamma(1-s)\,;\;\; s^\alpha = \left[\frac{\Gamma(1+s)}{\Gamma(s)}\right]^\alpha, \text{ we have:}$$

$$\begin{aligned} Li_\alpha(z) &= -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma^{\alpha+1}(s)\,\Gamma(1-s)}{\Gamma^\alpha(1+s)}(-z)^s\, ds \\ &= -\overline{H}^{1,1}_{1,2}\left[-z\,\middle|\, \begin{matrix} (1,1,\alpha+1) \\ (1,1,1),(0,1,\alpha) \end{matrix}\right], \end{aligned}$$

as a $\overline{H}^{1,1}_{1,2}$-function of the same orders as the $\overline{H}$-function (29) *for our Mittag-Leffler–Prabhakar function of Le Roy type*, $\mathbb{F}^1$ for $m = 1$ (but with different parameters). Note that the singularities of the Gamma functions in the numerator of the above contour integral for Li_α are all lying to the left of $s = 0$ and to the right of $s = 1$, and so there are none in the interval $c \in (0,1)$, which explains the choice $c = 1/2$ for the "vertical" contour $\mathcal{L} = (c - i\infty, c + i\infty)$. Meanwhile, the series (76) can be rewritten as

$$Li_\alpha(z) = \sum_{k=1}^{\infty} \frac{\Gamma^\alpha(k)}{\Gamma^\alpha(k+1)}\, z^k,$$

that is, *in the form of an "extended" Le Roy-type function*, as discussed in Section 4 of our joint work [14].

- *The generalized Riemann Zeta function (Hurwitz–Lerch Zeta function)* ([1] eq.(1), §1.11).

  As mentioned in [94] (1.1.27), see also [97] (1.1),

$$\Phi(z,\alpha,b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^\alpha} \tag{77}$$

$$= \overline{H}^{1,2}_{2,2}\left[-z\,\middle|\, \begin{matrix} (0,1,1),(1-b,1,\alpha) \\ (0,1,1),(-b,1,\alpha) \end{matrix}\right],\;\; b \neq 0,-1,-2,\ldots,\;\; |z| \leq 1,$$

and more precisely, for $\alpha \in \mathbb{C}$ when $|z| < 1$, or $\mathrm{Re}\,(\alpha) > 1$ when $|z| = 1$; with a branch cut along with the positive semi-axis from 1 to $+\infty$. Then, the function is analytic in z in the so-cut z-plane.

This contains not only the Riemann Zeta function $\zeta(\alpha)$ $(z = 1, b = 0)$ and the Hurwitz Zeta function $\zeta(\alpha, b) = \Phi(1,\alpha,b)$ with $\mathrm{Re}\,(\alpha) > 1$ but also the above-mentioned polylogarithm function (76), as $\mathrm{Li}_\alpha(z) = z\,\Phi(z,\alpha,1)$, $\alpha \in \mathbb{C}$ when $|z| < 1$, or $\mathrm{Re}\,(\alpha) > 1$ when $|z| = 1$.

The more general Hurwitz–Lerch Zeta function is:

$$\Phi_{\mu,\nu}^{\rho,\sigma}(z,\alpha,b) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \cdot \frac{z^n}{(n+b)^{\alpha}}, \text{ for } |z| < \delta = \rho^{-\rho}\sigma^{\sigma},$$

involving Pochhamer symbols, and in some cases as:

$$\Phi_{\nu,\nu}^{\sigma,\sigma}(z,\alpha,b) = \Phi(z,\alpha,b), \text{ and } \Phi_{\mu,1}^{1,1}(z,\alpha,b) = \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \cdot \frac{z^n}{(n+b)^{\alpha}}.$$

Details of the parameters and contoursfor the more general case are explained below.

- *The generalized Hurwitz–Lerch Zeta function* (Srivastava–Saxena–Pogány–Saxena [97]).

It is assumed that $\lambda, \mu \in \mathbb{C}$; $b, \nu = 0, -1, -2, \ldots$; real $\rho, \sigma, \kappa > 0$; $\kappa - \rho - \sigma > -1$ when $\alpha, z \in \mathbb{C}$, or $\kappa - \rho - \sigma = -1$ and $\alpha \in \mathbb{C}$ when $|z| < \delta^* = \rho^{-\rho}\sigma^{-\sigma}\kappa^{\kappa}$, while $\kappa - \rho - \sigma = -1$ and $\mathrm{Re}\,(\alpha + \nu - \lambda\mu) > 1$ when $|z| = \delta^*$. Then, the following generalized (extended) Hurwitz–Lerch Zeta functions is considered:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\, n!} \cdot \frac{z^n}{(n+b)^{\alpha}}. \tag{78}$$

In Srivastava–Saxena–Pogány–Saxena [97], the Riemann–Liouville fractional derivatives of the above Hurwitz–Lerch Zeta-type functions are evaluated, as expected in terms of similar kind of functions, in full agreement with the results as in Kiryakova [69,80], and the ideas for the SF of FC ([9]). Functional relations are also presented of (78) with the generalized Wright–Fox functions ${}_p\widetilde{\Psi}_q$ of the form (16) from Section 2.

According to [97] (Th.4), the generalized Hurwitz–Lerch Zeta function (78) has the following contour integral representation:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$
$$\times \int_{\mathcal{L}} \frac{\Gamma(-s)\Gamma(\lambda+\rho s)\Gamma(\mu+\sigma s)\,\Gamma^{\alpha}(s+b)}{\Gamma(\nu+\kappa s)\,\Gamma^{\alpha}(s+b+1)} (-z)^s\, ds, \tag{79}$$

for $|\arg(-z)| < \pi$, and path of integration $\mathcal{L} = (c - i\infty, c + i\infty)$ in the complex s-plane that separates the singularities of $\Gamma(-s), \Gamma(\lambda+\rho s), \Gamma(\mu+\sigma s)$ and $\Gamma^{\alpha}(s+b)$. Then, the relation with the $\overline{H}$-function is obtained, as in [97] (Th.5):

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$
$$\times \overline{H}_{3,3}^{1,3}\left[-z \,\middle|\, \begin{matrix} (1-\lambda,\rho,1),(1-\mu,\sigma,1),(1-b,1,\alpha) \\ (0,1),(1-\nu,\kappa,1),(-b,1,\alpha) \end{matrix}\right]. \tag{80}$$

Several important special cases are then considered, such as asymptotic estimations, etc.

- *Generalized Wright–Bessel function* (see Jolly [94], (1.1.27), p.13)

$$\overline{J}_{\lambda}^{\nu,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma^{\mu}(\lambda+k\nu+1)\,k!} = \overline{H}_{0,2}^{1,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),\,(-\lambda,\nu,\mu) \end{matrix}\right]. \tag{81}$$

Let us compare this with the popular *Wright–Bessel/Bessel–Maitland function* (e.g., Marichev [5]) considered often by Kiryakova at al., for example, (57) in [9], see in Section 6:

$$J_{\lambda}^{\nu}(z) = {}_0\Psi_1\left[\begin{matrix} -- \\ (\lambda+1,\nu) \end{matrix} \,\middle|\, -z\right] = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\lambda+k\nu+1)\,k!} = H_{0,2}^{1,0}\left[z \,\middle|\, \begin{matrix} -- \\ (0,1),(-\lambda,\nu) \end{matrix}\right],$$

which appears as a case of (81) if $\mu = 1$. See also Srivastava et al. [70] (3.2).

One can prolong this list with some more complicated or exotic-looking special functions that can fall in the scheme of the I-functions. We limit ourselves to the case below.

- In Bhatter et al. [98], the authors introduce the so-called *E-function*:

$$ {}_{\tau}E_k^h\left[z\left|\begin{array}{l}(\rho,a);(\gamma_i,q_i,\alpha_i)_1^h\\(\alpha,\beta);(\delta_j,p_j,r_j)_1^k\end{array}\right.\right]=\sum_{n=0}^{\infty}c_n\,\frac{(-1)^{\rho n}\,z^{an+\tau}}{\Gamma(\alpha n+\beta)} \tag{82} $$

with coefficients including the Pochhamer symbols $c_n = \dfrac{\left[(\gamma_1)_{q_1 n}\right]^{\alpha_1}\dots\left[(\gamma_h)_{q_h n}\right]^{\alpha_h}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\dots\left[(\delta_k)_{p_k n}\right]^{r_k}}$.

These lead to Γ-functions to (arbitrary) fractional powers both in the numerator and denominator of the power series (82), which is convergent for all finite values of

$$ \prod_{i=1}^{h}(q_i)^{q_i\alpha_i}\left[\alpha^{\alpha}\prod_{j=1}^{k}(p_j)^{p_j r_j}\right]^{-1}|z^a|. $$

We omit all assumptions on parameters and on the contour that are provided in the cited work. As shown in [98] (Th.5), this E-function allows a kind of Mellin–Barnes-type integral

$$ {}_{\tau}E_k^h(z)=\frac{\prod\limits_{v=1}^{k}\Gamma^{r_v}(\delta_v)}{\prod\limits_{u=1}^{h}\Gamma^{\alpha_u}(\gamma_u)}\cdot\frac{z^{\tau}}{2\pi i}\int\limits_{\mathcal{L}}\frac{\Gamma(s)\Gamma(1-s)\prod\limits_{i=1}^{h}\Gamma^{\alpha_i}(\gamma_i-q_i s)}{\Gamma(\beta-\alpha s)\prod\limits_{j=1}^{k}\Gamma^{r_j}(\delta_j-p_j s)}\left[(-1)^{\rho}(-z^a)\right]^{-s}ds. \tag{83} $$

Then, it can be considered as a case of a $\overline{H}_{2,k}^{1,1+h}$-function, and also has a form close to *the "extended" Le Roy-type functions*, discussed in Sect. 4 of our joint work [14].

8. Some Remarks and Open Problems

We now mention in brief some open or related problems that can be further discussed for the Le Roy-type functions (in the sense as discussed here), and more generally, for the I-functions.

8.1. *Open problem* to determine *G-L-type operators of generalized integration* (33) *in the case of the Prabhakar-type Le Roy functions* (24), and the corresponding "eigenoperators" as the generalized differentiation (32), for which these special functions can appear as eigenfunctions. As mentioned in Kiryakova [64], even for the simplest case $m = 1$ and $\gamma = 1$, the problem stays open if $\tau \neq 1$. In the particular case of Prabhakar parameters $\forall\tau_i = 1$, we proposed such operators in Section 4 here, see (58) as well as (61).

8.2. *Open problem* to find conditions of parameters for which the Le Roy-type functions (24), or the simpler (22), or the multi-index Mittag-Leffler functions (9) and (10), are *Completely Monotone* (CM). Over many years, several attempts have been made to study the CM of some simpler classes of special functions. As a reminder, a function $f(z)$ is called CM if it is infinitely differentiable and $(-1)^n f^{(n)}(z) \geq 0,\ \ \forall n = 0,1,2,\dots$. According to the Bernstein theorem ([99]), a function f is CM if and only if it can be uniquely represented as a Laplace transform of a non-negative (weight) function.

The results for the Mittag-Leffler function $E_{\alpha,\beta}$: $0 \leq \alpha \leq 1$, $\beta \geq \alpha$ are known, see, e.g., [2] (after the earlier results by Pollard for $\beta = 1$, and by Miller–Samko), for the classical Prabhakar function by Giusti at al. [34] and for the multinomial Prabhakar function by Bazhlekova [92]. See also Karp–Prilepkina [77], Berg–Çetinkaya–Karp [100] (Sect.3), etc., for some conditions related to the CM of some Fox H- and similar functions.

For the Le Roy-type function $F_{(\alpha,\beta)}^{(\gamma)}$, there are results given by Gorska and Horzela [56] and Simon [57].

8.3. *Open problem* for the *behavior of the I- and $\overline{H}$-functions near the third singular points* on the circle of convergence when these functions are analytic inside/outside disks with final radius R. Even for the Fox H-functions, in general, it is still an open problem, but some particular results are available in Karp [21] (12.24).

8.4. *Open problem* (as mentioned in Section 5) to define and study the properties of *integral transforms of Laplace type with kernels* $I^{m,0}_{0,m}$. These should be analogues of the Laplace; Borel-Dzrbashjan; Meijer; Obrechkoff; and "fractional" Obrechkoff transforms, where the kernel functions are, resp.: $\exp(-z)$; $z^{\rho-1}\exp(-z^{\rho}) = G^{1,0}_{0,1}(z^{\rho})$; $K_{\nu}(z) = \frac{1}{2}\, G^{2,0}_{0,2}\left[\frac{z^2}{4}\,\middle|\, \begin{array}{c} \frac{\nu}{2} \\ \frac{-\nu}{2} \end{array}\right]$; $G^{m,0}_{0,m}(z)$; $H^{m,0}_{0,m}(z)$, etc. See some details in Kiryakova [9] (Sect.3), and our works such as [20] (Ch.2, Ch.3, Ch.5); [37,38,84].

8.5. *Open problem* related to the notion of the *non-holomonicity* for the case of I-functions and Le Roy-type functions. A sequence $\{u(n)\}$ is called holonomic (p-recursive) if it satisfies a homogeneous linear recurrence $p_0(n)u(n) + p_1(n)u(n+1) + \ldots + p_d(n)u(n+d) = 0,\ \ n \geq 0$ with polynomials p_k, and p_d not identically zero. A formal power series $f(z) = \sum_{0}^{\infty} u(n)z^n$ is holonomic (d-finite) if it satisfies a homogenous linear ordinary differential equation $p_0(z)f(z) + p_1(z)f'(z) + \ldots + p_d(z)f^{(d)}(z) = 0$ with polynomial coefficients. It is well known that such a power series is holonomic if and only if its coefficient sequence is.

For definitions of the above-mentioned notions, and *non-holomonicity of the hypergeometric sequences* $\Gamma^{\alpha_1}(n-u_1)\ldots\Gamma^{\alpha_m}(n-u_m)$ with non-integer α_i, related to functions $f(z) = \Gamma^{\alpha_1}(z-u_1)\ldots\Gamma^{\alpha_m}(z-u_m)$, see Gerhold [101], Bell–Gerhold et al. [102] (Theorem 8), Flajolet–Gerhold–Salvy [103], Gerhold [52] (Sect.3), etc. Important *examples of non-holonomic functions* that are I-functions have been mentioned, such as the polylogarithms, Riemann Zeta functions, etc. More generally, I- and $\overline{H}$-functions that involve such non-holonomic sequences are also expected to fall in the case of the non-holonomic functions, and in particular, the considered Le Roy-type functions.

Author Contributions: The ideas and results in this paper resulted from the authors' recent joint research on the Le Roy-type functions. In an earlier survey paper in this journal [9], we confined the class of the special functions of fractional calculus only to the level of the Fox–Wright generalized hypergeometric functions, Fox H-functions, and especially, the Mittag-Leffler-type functions. Now, inspired by some works from the end of the 20th century by A.A. Inayat-Hussain, R.G. Buschman-H.M. Srivastava and A.K. Rathie, and by more recent ones, we observed that the Le Roy type functions as well as other important special functions, can be represented by the more general I-functions and $\overline{H}$-functions. Thus, it sounds suitable to extend the class of functions related to fractional calculus. Since the theory of the latter-mentioned functions still puts forward several open problems, we title our work as a "Discussion Survey". All authors have read and agreed to the published version of the manuscript.

Funding: This research received no financial funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors acknowledge the fruitful collaboration with Sergei Rogosin and Maryna Dubatovskaya for the joint research on the considered multi-index Mittag-Leffler–Prabhakar functions of Le Roy-type, as published in the very recent articles [14,16]. We are thankful also to Dmitrii Karp for the useful discussions on some subtle details in the theory of the Mellin–Barnes-type integral representations of the involved multi-valued functions. Thanks are due to the reviewers and editors who carefully checked the original version and made useful suggestions that improved this work.

Conflicts of Interest: The authors declare no conflicts of interest.

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