mathematics

MDPI

Article

Single and Multi-Valued Ordered-Theoretic Perov Fixed-Point Results for θ-Contraction with Application to Nonlinear System of Matrix Equations

Fahim Ud Din [1], Salha Alshaikey [2], Umar Ishtiaq [3,*], Muhammad Din [1] and Salvatore Sessa [4,*]

1 Abdus Salam School of Mathematical Sciences, Governement College University, Lahore 54600, Pakistan; fahim@sms.edu.pk (F.U.D.); muhammaddin@sms.edu.pk (M.D.)
2 Mathematics Department, Al-Qunfudah University College, Umm Al-Qura University, Mecca 21421, Saudi Arabia; sashaikey@uqu.edu.sa
3 Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan
4 Dipartimento di Architettura, Università Dinapoli Federico II, Via Toledo 403, 80121 Napoli, Italy
* Correspondence: umar.ishtiaq@umt.edu.pk (U.I.); salvasessa@gmail.com (S.S.)

Abstract: This paper combines the concept of an arbitrary binary connection with the widely recognized principle of θ-contraction to investigate the innovative features of vector-valued metric spaces. This methodology demonstrates the existence of fixed points for both single- and multi-valued mappings within complete vector-valued metric spaces. Through the utilization of binary relations and θ-contraction, this study advances and refines the Perov-type fixed-point results in the literature. Furthermore, this article furnishes examples to substantiate the validity of the presented results. Additionally, we establish an application for finding the existence of solutions to a system of matrix equations.

Keywords: fixed point; vector-valued metric space; existence; uniqueness; contraction mappings

MSC: 47H10; 54H25

check for updates

Citation: Din, F.U.; Alshaikey, S.; Ishtiaq, U.; Din, M.; Sessa, S. Single and Multi-Valued Ordered-Theoretic Perov Fixed-Point Results for θ-Contraction with Application to Nonlinear System of Matrix Equations. *Mathematics* **2024**, *12*, 1302. https://doi.org/10.3390/math12091302

Academic Editors: Ravi P. Agarwal and Michele Nichelatti

Received: 26 March 2024
Revised: 12 April 2024
Accepted: 24 April 2024
Published: 25 April 2024

1. Introduction

The Banach fixed-point theorem [1], also known as the contraction mapping principle, is a fundamental result in functional analysis. It states that if a self-map on a complete metric space satisfies the contraction condition, then it possesses a unique fixed point within that space.

A mapping $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ on a metric space $(\mathring{\mathcal{K}}_s, \mathtt{e})$ is called a Banach contraction if there exists a constant $0 \leq k < 1$ such that for all $x, y \in \mathring{\mathcal{K}}_s$, we have $\mathtt{e}(T(x), T(y)) \leq k \cdot \mathtt{e}(x, y)$.

The Banach fixed-point theorem, named after the Polish mathematician Stefan Banach, has broad applications across various mathematical domains. It plays a crucial role in solving ordinary and partial differential equations, optimization problems, integral equations, and variational analysis. Moreover, its significance extends beyond mathematics into fields like game theory, economics, biology, and more.

The theorem ensures the existence and uniqueness of fixed points for self-maps that satisfy contraction criteria within a complete metric space. Over the years, the Banach fixed-point theorem has been extensively studied using a variety of methodological techniques, contributing significantly to the advancement of mathematical theory and its applications. As part of these studies, the mapping's scope has been expanded; Nadler [2] was the first to propose the generalization of the Banach contraction theorem for multi-valued contractions. Kikkawa and Suzuki [3] obtained three fixed-point approaches to generalized contractions, while Din et al. [4] investigated single- and multi-valued F-contractions under binary relations. A new kind of contractive multi-valued operator was presented by Moţ and Petruşel [5], whereas Sintunavarat and Kumam [6] provided a common fixed-point theorem for cyclic generalized multi-valued mappings. Multi-valued contractions on b-metric spaces were explained by Petre and Bota [7], and multi-valued fixed-point theorems in dislocated b-metric spaces were discussed by Rasham et al. [8] with applications to nonlinear integral equations.

The criteria for contraction mappings have been broadened in various ways, thereby increasing the versatility of the Banach contraction theorem. Browder [9] examined non-expansive nonlinear operators on Banach spaces and derived fixed-point results. Kannan introduced the concept of a contraction that does not require the continuity of the self-operator but still ensures the existence of a fixed point [10]. Wardowski and Dung discussed weak F-contractions and related fixed-point theorems [11]. Karapinar defined interpolative Kannan contractions and presented corresponding fixed-point results [12]. Another notable contribution was made by Jleli and Samet [13], who introduced the concept of θ-contraction to establish fixed-point theorems, representing a fascinating and profound extension of the Banach fixed-point theorem. This achievement was further advanced by Altun et al. [14], who expanded upon it to explore fixed-point results for Perov-type θ-contractions. Additionally, Alam and Imdad [15] initiated the idea of binary relations on metric spaces for single-valued Banach contraction, Lipschutz [16] and Agarwal et al. [17] discussed the ordered fixed-point results for Banach spaces with applications in nonlinear integral equations, Hussain et al. [18] studied the Krasnoselskii and Ky Fan-type ordered fixed points over Banach spaces, and Ran and Reurings [19] explored fundamental concepts related to binary relations and partial ordered theoretic fixed-point theorems. While Berzig [20] dealt with a class of matrix equations using the Bhaskar–Lakshmikantham coupled fixed-point theorem, Berzig and Samet [21] explored the systems of nonlinear matrix equations involving Lipshitzian mappings, and Long et al. [22] focused on determining the conditions for the existence of a solution of the nonlinear matrix equation $X + AX^{-1}A + BX^{-1}B = I$. Vetro and Radenović [23] discussed some Perov-type results in rectangular cone metric spaces, while Guran et al. [24] explored some multi-valued results in the metric spaces of Perov's type.

The notion of an improved vector-valued metric space using a binary relation was recently presented by Almalki et al. [25]. They used the F-contraction in the context of a generalized metric space that has a binary connection to extend the Perov and Filip–Petrusel fixed-point theorems [26,27] to single-valued and multi-valued mappings. Compared to the typical contractive inequality, the contractive inequality in this case is relatively weaker. In this case, rather than throughout the entire space, the contractive inequality must only be satisfied among components that are related to one another according to the binary relation.

In this work, we will explore fixed points for single- and multi-valued mappings by employing the notion of a binary relation and stressing θ-contraction. Additionally, we will offer examples to show the validity of our findings and an existence condition for the solutions of a system of matrix equations as well.

2. Preliminaries

We provide a summary of the key concepts required to establish our findings in this section. Let $\mathring{\mathcal{K}}_s$ be a non-empty set. The set of all non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. The set of all $\mathtt{m} \times 1$ real matrices is represented by $\mathbb{R}_{\mathtt{m}>\mathtt{v}}$, which is the set of all $\mathtt{m} \times 1$ matrices with entries greater than $\mathtt{v} \in \mathbb{R}$. In the event that $\underline{\mathtt{v}}, \underline{\mathtt{u}} \in \mathbb{R}_{\mathtt{m}}$, they have the following forms: $\underline{\mathtt{v}} = (\mathtt{v}_1, \mathtt{v}_2, \ldots, \mathtt{v}_{\mathtt{m}})^T$ and $\underline{\mathtt{u}} = (\mathtt{u}_1, \mathtt{u}_2, \ldots, \mathtt{u}_{\mathtt{m}})^T$, where T stands for matrix transposition. We denote $\underline{\mathtt{v}} \leq \underline{\mathtt{u}}$ (or $\underline{\mathtt{v}} < \underline{\mathtt{u}}$) to indicate that $\mathtt{v}_i \leq \mathtt{u}_i$ (or $\mathtt{v}_i < \mathtt{u}_i$), $\forall i = 1, 2, \ldots, \mathtt{m}$.

Definition 1 ([26])**.** *A function* $\mathtt{e} : \mathring{\mathcal{K}}_s \times \mathring{\mathcal{K}}_s \to \mathbb{R}_m$ *is defined as a vector-valued metric on* $\mathring{\mathcal{K}}_s$ *if* $\forall \mathtt{z}, \mathtt{w}, \mathtt{c} \in \mathring{\mathcal{K}}_s$*, the following hold:*

1*:* $\mathtt{e}(\mathtt{z}, \mathtt{w}) \geq \overline{0}$*;*

2: $\mathtt{e}(\mathtt{z},\mathtt{w}) = \overline{0}$ *if and only if* $\mathtt{z} = \mathtt{w}$;
3: $\mathtt{e}(\mathtt{z},\mathtt{w}) = \mathtt{e}(\mathtt{w},\mathtt{z})$;
4: $\mathtt{e}(\mathtt{z},\mathtt{c}) \leq \mathtt{e}(\mathtt{z},\mathtt{w}) + \mathtt{e}(\mathtt{w},\mathtt{c})$.
In this case, the zero matrix of order $\mathtt{m} \times 1$ *is denoted as* $\overline{0}$. *Consequently, a generalized metric space or vector-valued metric space is defined as the pair* $(\mathring{\mathcal{K}}_s, \mathtt{e})$.

It is important to know that the concepts of completeness, Cauchy sequences, and convergent sequences are similar to those in a usual metric space. The zero matrix of order $\mathtt{m} \times \mathtt{m}$ is represented as $\overline{0}_{\mathtt{m}}$, the identity matrix as $I_{\mathtt{m}}$, and the set of all square matrices of order $\mathtt{m}$ with non-negative entries as $\mathbb{M}_{\mathtt{m}}(\mathbb{R}_{\geq 0})$. Notably, we have $N^0 = I_{\mathtt{m}}$ for every $N \in \mathbb{M}_{\mathtt{m}}(\mathbb{R}_{\geq 0})$.

Definition 2 ([26])**.** *Consider* $N \in \mathbb{M}_{\mathtt{m}}(\mathbb{R}_{\geq 0})$. *We say that* N *is a matrix converging to zero if* N^n *approaches* $\overline{0}_{\mathtt{m}}$ *as* n *tends to infinity.*

Example 1. $\mathtt{N}_1 := \begin{pmatrix} \mathtt{w}_1 & \mathtt{w}_2 \\ 0 & \mathtt{w}_3 \end{pmatrix}$, *where* $\mathtt{w}_1, \mathtt{w}_2, \mathtt{w}_3 \in \mathbb{R}_{\geq 0}$ *and* $\max\{\mathtt{w}_1, \mathtt{w}_3\} < 1$, *is a matrix convergent to zero in* $\mathbb{M}_2(\mathbb{R}_{\geq 0})$.

Example 2. *If* $\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_{\mathtt{m}} < 1$, *then*

$$N := \begin{pmatrix} \mathtt{z}_1 & 0 & \ldots & 0 \\ 0 & \mathtt{z}_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathtt{z}_{\mathtt{m}} \end{pmatrix}$$

converges to zero.

From Petrusel–Filip [27], some other comparable conditions for Definition 2 are as follows.

Proposition 1 ([27])**.** *Let* $N \in \mathbb{M}_m(\mathbb{R}_{\geq 0})$; *then, the following statements are comparable:*
(*a*) *The matrix* $N^m \to \mathbf{0}_t$ *as* $m \to \infty$;
(*b*) *Every complex number* λ *with* $\det(N - \lambda I_t) = 0$ *is contained in disc* $|\lambda| < 1$;
(*c*) $Det(I_t - N) \neq 0$ *and* $(I_t - N)^{-1} = I_t + N + \cdots + N^n + \cdots$;
(*d*) $N^n \underline{u} \to \mathbf{0}$ *as* $n \to \infty$, *for all* $\underline{u} \in \mathbb{R}_t$.

Example 3. *If* $\mathtt{z}_1 + \mathtt{z}_2 \geq 1$ *and* $\mathtt{z}_3 + \mathtt{z}_4 \geq 1$, *then*

$$N^* = \begin{pmatrix} \mathtt{z}_1 & \mathtt{z}_2 \\ \mathtt{z}_3 & \mathtt{z}_4 \end{pmatrix}$$

is not convergent to zero.

Definition 3. *Assume that the set* $\mathring{\mathcal{K}}_s$ *is not empty. The Cartesian product on* $\mathring{\mathcal{K}}_s$ *is then given by the following definition:*

$$\mathring{\mathcal{K}}_s^2 = \mathring{\mathcal{K}}_s \times \mathring{\mathcal{K}}_s = \{(\mathtt{z}_1, \mathtt{z}_2) : \mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s\}.$$

A binary relation on $\mathring{\mathcal{K}}_s$ *is defined as any subset* $\mathcal{R}$ *of* $\mathring{\mathcal{K}}_s^2$.

Observe that one of the following two circumstances must apply to each pair $(\mathtt{z}_1, \mathtt{z}_2) \in \mathring{\mathcal{K}}_s^2$:

(1) $(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$ indicates that $\mathtt{z}_1$ and $\mathtt{z}_2$ are related under $\mathcal{R}$ or that $\mathtt{z}_1$ is related to $\mathtt{z}_2$ under $\mathcal{R}$. Additionally, we can write $(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$ as $\mathtt{z}_1 \mathcal{R} \mathtt{z}_2$.

(2) $(z_1, z_2) \notin \mathcal{R}$ indicates that either z_1 does not relate to z_2 under $\mathcal{R}$ or z_1 is not $\mathcal{R}$-related to z_2. As $z_1 \not\mathcal{R} z_2$, we can alternatively write $(z_1, z_2) \notin \mathcal{R}$.

Trivial binary relations on $\mathring{\mathcal{K}}_s$ are defined as $\mathring{\mathcal{K}}_s^2$ and ϕ, which are two trivial subsets of $\mathring{\mathcal{K}}_s^2$.

Definition 4 ([15])**.** *Consider a binary relation, denoted as $\mathcal{R}$, defined over the non-empty set $\mathring{\mathcal{K}}_s$. Then, if either $(z_1, z_2) \in \mathcal{R}$ or $(z_2, z_1) \in \mathcal{R}$, then any two elements $z_1, z_2 \in \mathring{\mathcal{K}}_s$ are $\mathcal{R}$-Comparative. We set $[z_1, z_2] \in \mathcal{R}$ if $z_1, z_2 \in \mathring{\mathcal{K}}_s$ are $\mathcal{R}$-Comparative.*

By establishing appropriate conditions, it is possible to categorize a binary relation into various types. Various widely recognized binary relations, along with their significant properties, are detailed in [15,16]. The following is a well-known proposition in binary relations.

Proposition 2 ([16])**.** *Assume that $\mathcal{R}$ represents the universal relation established on a non-empty set $\mathring{\mathcal{K}}_s$. In this case, $\mathcal{R}$ is a full equivalence relation.*

Definition 5 ([25])**.** *Assume a binary relation $\mathcal{R}$ on $\mathring{\mathcal{K}}_s \neq \phi$. A sequence $(w_n) \subseteq \mathring{\mathcal{K}}_s$ is termed $\mathcal{R}$-preserving if*

$$(w_n, w_{n+1}) \in \mathcal{R}, \quad \text{for all } n \in \mathbb{N}.$$

Alam and Imdad introduced the idea of e-self-closedness for any $\mathcal{R}$ defined on some $(\mathring{\mathcal{K}}_s, e)$, as elaborated in [15]. This concept was further elaborated by Almaliki et al. in [25] as follows.

Definition 6 ([25])**.** *In a generalized metric space $(\mathring{\mathcal{K}}_s, e)$, a sequence (w_n) that converges to $w \in \mathring{\mathcal{K}}_s$ while preserving the relation $\mathcal{R}$ is called e-self-closed with a binary relation $\mathcal{R}$ if and only if (w_{n_k}) is a subsequence of (w_n) such that, for each $k \in \mathbb{N}$, $[w_{n_k}, w] \in \mathcal{R}$.*

Definition 7 ([15])**.** *Let $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ be a mapping, where $\mathring{\mathcal{K}}_s \neq \phi$. Then, $\mathcal{R}$ is called $\mathring{\mathscr{V}}$-closed if*

$$(w_1, w_2) \in \mathcal{R} \quad \text{implies} \quad (\mathring{\mathscr{V}} w_1, \mathring{\mathscr{V}} w_2) \in \mathcal{R}, \quad \text{for all } w_1, w_2 \in \mathring{\mathcal{K}}_s.$$

Lemma 1 ([25])**.** *Let $\mathring{\mathcal{K}}_s \neq \phi$, $\mathcal{R}$ be a binary relation on $\mathring{\mathcal{K}}_s$, and $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ be a mapping. Then, $\mathcal{R}$ being $\mathring{\mathscr{V}}$-closed implies that $\mathcal{R}^s$ is $\mathring{\mathscr{V}}$-closed, where $\mathcal{R}^s = \{[z_1, z_2] : z_1, z_2 \in \mathring{\mathcal{K}}_s\}$.*

Definition 8 ([16])**.** *Given $\mathring{\mathcal{K}}_s \neq \phi$ and a binary relation $\mathcal{R}$ defined on $\mathring{\mathcal{K}}_s$, a subset $\mathcal{E}$ of $\mathring{\mathcal{K}}_s$ is considered $\mathcal{R}$-directed if for all $w_1, w_2 \in \mathcal{E}$, there exists $w_3 \in \mathring{\mathcal{K}}_s$ such that both $(w_1, w_3) \in \mathcal{R}$ and $(w_2, w_3) \in \mathcal{R}$.*

The concept of a path between two points within a set furnished with a binary relation in a vector-valued metric space was introduced by Almaliki et al. in [25] as follows.

Definition 9 ([25])**.** *Let $\mathcal{R}$ be a binary relation on $\mathring{\mathcal{K}}_s \neq \phi$. A path of length l in $\mathcal{R}$ from u to v is said to be a path of length l in $\mathring{\mathcal{K}}_s$ for $u, v \in \mathring{\mathcal{K}}_s$ iff:*
(1): $w_0 = u$ and $w_l = v$;
(2): $[w_i, w_{i+1}] \in \mathcal{R}$ holds for every $i = 0, 1, \cdots, l - 1$.

Note that while not always distinct, every path of length l contains $l + 1$ members of $\mathring{\mathcal{K}}_s$.

For our inquiry, we next need the following ideas from the work of Almaliki et al. [25].

Definition 10 ([25])**.** *A compound structure is defined as the pair $(\mathcal{R} : \mathring{\mathscr{V}})$, which consists of an arbitrary binary relation $\mathcal{R}$ and a single-valued mapping $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ over a vector-valued metric space $(\mathring{\mathcal{K}}_s, e)$ such that the following are true:*

(i): $\mathscr{Y} := \{\mathtt{w} \in \mathring{\mathcal{K}}_s : (\mathtt{w}, \mathring{\mathscr{V}}\mathtt{w}) \in \mathcal{R}\} \neq \phi$;
(ii): $\mathcal{R}$ *is* $\mathtt{e}$*-self-closed;*
(iii): $\mathcal{R}$ *is* $\mathring{\mathscr{V}}$*-closed.*

Definition 11 ([25]). *Let* $\mathring{\mathcal{K}}_s$ *and* $\mathtt{e}$ *have their usual meanings as discussed earlier. Consider a multi-valued mapping* $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to cl_{\mathtt{e}}(\mathring{\mathcal{K}}_s)$*. Then, a binary relation* $\mathcal{R}$ *over* $\mathring{\mathcal{K}}_s$ *is termed* $\mathring{\mathscr{V}}$*-closed if for every pair* $\mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s$,

$$(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R} \Rightarrow (\mathtt{z}_3, \mathtt{z}_4) \in \mathcal{R}, \quad \forall \mathtt{z}_3 \in \mathring{\mathscr{V}}\mathtt{z}_1, \mathtt{z}_4 \in \mathring{\mathscr{V}}\mathtt{z}_2.$$

Definition 12 ([25]). *Denote any vector-valued metric space with a binary relation* $\mathcal{R}$ *by denoting* $(\mathring{\mathcal{K}}_s, \mathtt{e})$*. Let* $cl_{\mathtt{e}}(\mathring{\mathcal{K}}_s) = \{H \in N_{\mathtt{e}}(\mathring{\mathcal{K}}_s) : G \text{ is closed}\}$*. The class of all non-empty subsets of* $(\mathring{\mathcal{K}}_s)$ *is denoted by* $N_{\mathtt{e}}(\mathring{\mathcal{K}}_s)$*. Then, for multi-valued mappings, the pair* $(\mathcal{R} : \mathring{\mathscr{V}})$ *is considered to be a compound structure if the following requirements are satisfied:*

1. $\mathcal{R}$ *is* $\mathring{\mathscr{V}}$*-closed;*
2. $\mathscr{Y} := \{\mathtt{z} \in \mathring{\mathcal{K}}_s : \exists \mathtt{w} \in \mathring{\mathscr{V}}(\mathtt{z}) \quad \text{such} \quad \text{that} \quad (\mathtt{z}, \mathtt{w}) \in \mathcal{R}\} \neq \phi$;
3. $\mathcal{R}$ *is strongly* $\mathtt{e}$*-self-closed; that is, for every sequence* $(\mathtt{z_n})$ *in* $\mathring{\mathcal{K}}_s$ *with* $(\mathtt{z_n}, \mathtt{z_{n+1}}) \in \mathcal{R}$ *for all natural numbers* $\mathtt{n}$ *and* $\mathtt{z_n} \xrightarrow{\mathtt{e}} \mathtt{z}$*, we obtain* $(\mathtt{z_n}, \mathtt{z}) \in \mathcal{R}$ *for all* $\mathtt{n} \geq \mathtt{k}$*, where a positive integer* $\mathtt{k}$ *is used.*

Theorem 1 ([15]). *Consider a self mapping* $\mathring{\mathscr{V}}$ *and a binary relation* $\mathcal{R}$ *on a complete vector-valued metric space* $(\mathring{\mathcal{K}}_s, \mathtt{e})$ *such that the following are satisfied:*
(i) The pair $(\mathcal{R} : \mathring{\mathscr{V}})$ *forms a compound structure.*
(ii) For all $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s$ *with* $(\mathtt{w}_1, \mathtt{w}_2) \in \mathcal{R}$*, the condition*

$$\mathtt{e}\left(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2\right) \leq N\mathtt{e}(\mathtt{w}_1, \mathtt{w}_2)$$

holds, where $N \in \mathbb{M}_t(\mathbb{R}_{\geq 0})$ *converges to zero. Then,* $\mathring{\mathscr{V}}$ *possesses a fixed point.*
(iii) Moreover, if $C_R(\mathtt{w}_1, \mathtt{w}_2) \neq \phi, \forall \mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s$*, then the fixed point of* $\mathring{\mathscr{V}}$ *is unique.*

The term θ-contraction was first used by Altun et al. [14], who defined it as follows.

Definition 13 ([14]). *Let* $\theta : \mathbb{R}_{m_{>0}} \to \mathbb{R}_{m_{>1}}$ *be a function, where* $\mathbb{R}_{m_{>j}}$ *denotes the set of all* $m \times 1$ *real matrices with entries exceeding* j*. The function satisfies the following properties:*

Θ_1 *For any* $\mathtt{w} = (\mathtt{w}_i)$, $\mathtt{v} = (\mathtt{v}_i) \in \mathbb{R}_{m_{>0}}$*, if* $\mathtt{w} \leq \mathtt{v}$*, then* $\theta(\mathtt{w}) \leq \theta(\mathtt{v})$.

Θ_2 *For each sequence* $\{\mathtt{w}_n\} = \left(\mathtt{w}_1^{(n)}, \mathtt{w}_2^{(n)}, \ldots, \mathtt{w}_m^{(n)}\right)$ *of* $\mathbb{R}_{m_{>0}}$,

$$\lim_{n \to +\infty} \mathtt{w}_i^{(n)} = 0^+ \quad \textit{if and only if} \quad \lim_{n \to +\infty} \mathtt{v}_i^{(n)} = 1, \quad \forall i,$$

where $\theta\left[\left(\mathtt{w}_1^{(n)}, \mathtt{w}_2^{(n)}, \ldots, \mathtt{w}_m^{(n)}\right)\right] = \left(\mathtt{v}_1^{(n)}, \mathtt{v}_2^{(n)}, \ldots, \mathtt{v}_m^{(n)}\right)$.

Θ_3 *There exist* $\zeta \in (0,1)$ *and* $\xi \in (0, +\infty]$*, such that* $\lim_{\mathtt{w}_i \to 0^+} \frac{\mathtt{v}_i - 1}{[\mathtt{w}_i]^\zeta} = \xi, \forall i$*, where*

$$\theta[(\mathtt{w}_1, \mathtt{w}_2, \ldots, \mathtt{w}_m)] = (\mathtt{v}_1, \mathtt{v}_2, \ldots, \mathtt{v}_m).$$

The set of all functions θ *that satisfy* Θ_1 *to* Θ_3 *is denoted as* Ξ^m.

Example 4. *Let* $\theta^1 : \mathbb{R}_{t_{>0}} \to \mathbb{R}_{t_{>1}}$ *be given by*

$$\theta^1(\mathtt{z}_1, \mathtt{z}_2, \cdots, \mathtt{z}_t) = (e^{\sqrt{\mathtt{z}_1}}, e^{\sqrt{\mathtt{z}_2}}, \cdots, e^{\sqrt{\mathtt{z}_t}}).$$

Then, $\theta^1 \in \Xi^t$.

Example 5. *Let $\theta^2 : \mathbb{R}_{2_{>0}} \to \mathbb{R}_{2_{>1}}$ be given by*

$$\theta^2(\mathtt{z}_1, \mathtt{z}_2) = (e^{\sqrt{\mathtt{z}_1}}, e^{\sqrt{\mathtt{z}_2 e^{\mathtt{z}_2}}}).$$

Then, $\theta^2 \in \Xi^2$.

Altun et al. [14] initiated the concept of Perov-type θ-contraction by employing the family Ξ^t and defining the notion $\mathtt{z}^{[\mathtt{g}]} := (\mathtt{z}_i^{\mathtt{g}_i})$, where $\mathtt{z} = (\mathtt{z}_i), \mathtt{g} = (\mathtt{g}_i) \in \mathbb{R}_{t_{\geq 0}}$.

Definition 14 ([14]). *A mapping $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ on a vector-valued metric space $(\mathring{\mathcal{K}}_s, \mathtt{e})$ is regarded as a Perov-type θ-contraction if $\theta \in \Xi^t$ and $\mathtt{g} = (\mathtt{g}_i) \in \mathbb{R}_{t_{\geq 0}}$, with each $\mathtt{g}_i < 1$, such that*

$$\forall \mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s, \mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2) > \mathbf{0} \Rightarrow \theta(\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2)) \leq [\theta(\mathtt{e}(\mathtt{w}_1, \mathtt{w}_2))]^{[\mathtt{g}]} \tag{1}$$

For a Perov-type θ-contraction in vector-valued metric spaces, the fixed-point theorem proved by Altun et al. [14] is as follows.

Theorem 2. *Let $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ be a Perov-type θ-contraction and $(\mathring{\mathcal{K}}_s, \mathtt{e})$ be any complete vector-valued metric space. Then, in $\mathring{\mathcal{K}}_s$, the fixed point of $\mathring{\mathscr{V}}$ is unique.*

3. Main Results

Before discussing our findings, we define a Perov-type θ-contraction enriched with a binary relation and state a lemma that will be useful in the proof. This section presents a fixed-point theorem for single-valued theoretic-order Perov-type θ-contraction. We also generalize this theorem to multi-valued mappings.

Definition 15. *Assume any vector-valued metric space $(\mathring{\mathcal{K}}_s, \mathtt{e})$ and any arbitrary binary relation $\mathcal{R}$. Then, if there exists $\mathtt{g} = \left(\mathtt{g}^{(i)}\right) \in \mathbb{R}_{t_{\geq 0}}$ with each $\mathtt{g}_i < 1$, and $\theta \in \Theta^t$, then this single-valued self-mapping of $\mathring{\mathscr{V}}$ on $\mathring{\mathcal{K}}_s$ is referred to as a Perov-type θ-contraction enriched with binary relation $\mathcal{R}$ if*

$$\theta(\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2)) \leq [\theta(\mathtt{e}(\mathtt{w}_1, \mathtt{w}_2)]^{[\mathtt{g}]}, \tag{2}$$

$\forall (\mathtt{w}_1, \mathtt{w}_2) \in \mathcal{R}$ *with* $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2) > \mathbf{0}$.

Lemma 2. *Let $(\mathring{\mathcal{K}}_s, \mathtt{e})$ represent any vector-valued metric space with a binary relation $\mathcal{R}$. If $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ is a Perov-type θ-contraction enriched with the binary relation $\mathcal{R}$, where $\mathtt{g} = (\mathtt{g}_i) \in \mathbb{R}_{t_{>0}}$ with each $\mathtt{g}_i < 1$ and $\theta \in \Theta^t$, then the following statements are equivalent (given $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1^*, \mathring{\mathscr{V}}\mathtt{w}_2^*) > \mathbf{0}$):*

1. $\theta(\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2)) \leq [\theta(\mathtt{e}(\mathtt{w}_1, \mathtt{w}_2))]^{[\mathtt{g}]}$, *with* $(\mathtt{w}_1, \mathtt{w}_2) \in \mathcal{R}$;
2. $\theta(\mathtt{e}(\mathring{\mathscr{V}}\mathtt{w}_1, \mathring{\mathscr{V}}\mathtt{w}_2)) \leq [\theta(\mathtt{e}(\mathtt{w}_1, \mathtt{w}_2))]^{[\mathtt{g}]}$, *with* $[\mathtt{w}_1, \mathtt{w}_2] \in \mathcal{R}$.

Proof. This lemma's proof is straightforward and limited to the use of the metric's symmetric condition. □

We now present the following initial result for a single-valued Perov-type θ-contraction.

Theorem 3. *In any complete vector-valued metric space $(\mathring{\mathcal{K}}_s, \mathtt{e})$ with an arbitrary binary relation $\mathcal{R}$, let $\mathring{\mathscr{V}}$ be a Perov-type θ-contraction. Assuming that $\mathcal{R}$ on $\mathring{\mathcal{K}}_s$ forms a compound structure with $\mathring{\mathscr{V}}$, then $fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}}) \neq \phi$.*

Also, if $C_{\mathcal{R}}(\mathtt{z}_1, \mathtt{z}_2) \neq \phi, \forall \mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s$, then $fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}})$ is a singleton set.

Proof. Let $\mathrm{z}_0 \in \mathscr{Y} \subseteq \mathring{\mathcal{K}}_s$ be any element. We define an iterative sequence $(\mathrm{z}_n) = (\mathring{\mathscr{V}}^n \mathrm{z}_0)$. According to the definition of $\mathscr{Y}$, we have $(\mathrm{z}_0, \mathring{\mathscr{V}}\mathrm{z}_0) = (\mathrm{z}_0, \mathrm{z}_1) \in \mathcal{R}$. Since $\mathcal{R}$ is $\mathring{\mathscr{V}}$-closed, we can observe the following chain of relations:

$$(\mathring{\mathscr{V}}\mathrm{z}_0, \mathring{\mathscr{V}}^2\mathrm{z}_0), (\mathring{\mathscr{V}}^2\mathrm{z}_0, \mathring{\mathscr{V}}^3\mathrm{z}_0), \ldots, (\mathring{\mathscr{V}}^n\mathrm{z}_0, \mathring{\mathscr{V}}^{n+1}\mathrm{z}_0), \cdots \in \mathcal{R}.$$

This demonstrates that

$$(\mathrm{z}_n, \mathrm{z}_{n+1}) \in \mathcal{R}, \quad \forall n \in \mathbb{N}. \tag{3}$$

Therefore, the sequence (z_n) preserves the binary relation $\mathcal{R}$. If for some $n_0 \in \mathbb{N}$, we have $\mathrm{z}_{n_0+1} = \mathrm{z}_{n_0}$, then $\mathrm{z}_{n_0+1} = \mathring{\mathscr{V}}\mathrm{z}_{n_0} = \mathrm{z}_{n_0}$, showing that z_{n_0} is a fixed point of $\mathring{\mathscr{V}}$ in $\mathring{\mathcal{K}}_s$. Otherwise, if $\mathrm{z}_{n+1} \neq \mathrm{z}_n$ for all natural numbers n, then $\mathtt{e}(\mathring{\mathscr{V}}\mathrm{z}_n, \mathring{\mathscr{V}}\mathrm{z}_{n+1}) > \mathbf{0}$ for all $n \in \mathbb{N}$.

Consider

$$\mathtt{e}(\mathrm{z}_{n+1}, \mathrm{z}_n) = \left(\mathring{\mathrm{v}}_n^{(1)}, \mathring{\mathrm{v}}_n^{(2)}, \cdots, \mathring{\mathrm{v}}_n^{(t)}\right), \quad \forall n \in \mathbb{N}.$$

Thus, we obtain

$$\theta\left[\left(\mathring{\mathrm{v}}_n^{(1)}, \mathring{\mathrm{v}}_n^{(2)}, \cdots, \mathring{\mathrm{v}}_n^{(t)}\right)\right] = \left(\mathring{\mathrm{u}}_n^{(1)}, \mathring{\mathrm{u}}_n^{(2)}, \cdots, \mathring{\mathrm{u}}_n^{(t)}\right),$$

and using Equation (2), we obtain

$$\begin{aligned}
\left(\mathring{\mathrm{u}}_n^{(1)}, \mathring{\mathrm{u}}_n^{(2)}, \cdots, \mathring{\mathrm{u}}_n^{(t)}\right) &= \theta\left[\left(\mathring{\mathrm{v}}_n^{(1)}, \mathring{\mathrm{v}}_n^{(2)}, \cdots, \mathring{\mathrm{v}}_n^{(t)}\right)\right] \\
&= \theta(\mathtt{e}(\mathrm{z}_{n+1}, \mathrm{z}_n)) \\
&= \theta(\mathtt{e}(\mathring{\mathscr{V}}\mathrm{z}_n, \mathring{\mathscr{V}}\mathrm{z}_{n-1})) \\
&\leq [\theta(\mathtt{e}(\mathrm{z}_n, \mathrm{z}_{n-1}))]^{[\mathrm{g}]} \\
&= [\theta(\mathring{\mathrm{v}}_{n-1}^{(1)}, \mathring{\mathrm{v}}_{n-1}^{(2)}, \cdots, \mathring{\mathrm{v}}_{n-1}^{(t)})]^{[\mathrm{g}]} \\
&= [(\mathring{\mathrm{u}}_{n-1}^{(1)}, \mathring{\mathrm{u}}_{n-1}^{(2)}, \cdots, \mathring{\mathrm{u}}_{n-1}^{(t)})]^{[\mathrm{g}]} \\
&= ([\mathring{\mathrm{u}}_{n-1}^{(1)}]^{\mathrm{g}_1}, [\mathring{\mathrm{u}}_{n-1}^{(2)}]^{\mathrm{g}_2}, \cdots, [\mathring{\mathrm{u}}_{n-1}^{(t)}]^{\mathrm{g}_t}).
\end{aligned}$$

So, we have for all i,

$$\mathring{\mathrm{u}}_n^{(i)} \leq [\mathring{\mathrm{u}}_{n-1}^{(i)}]^{\mathrm{g}_i}.$$

Using this information consistently, we obtain

$$\mathring{\mathrm{u}}_n^{(i)} \leq [\mathring{\mathrm{u}}_{n-1}^{(i)}]^{\mathrm{g}_i} \leq [\mathring{\mathrm{u}}_{n-2}^{(i)}]^{\mathrm{g}_i^2} \leq \cdots \leq [\mathring{\mathrm{u}}_0^{(i)}]^{\mathrm{g}_i^n}. \tag{4}$$

Since each g_i is less than 1,

$$\lim_{n \to +\infty} \mathring{\mathrm{u}}_n^{(i)} = 1.$$

Therefore, based on Θ_2, we can deduce that for each $i \in \{1, 2, \cdots, t\}$,

$$\lim_{n \to +\infty} \mathring{\mathrm{v}}_n^{(i)} = 0^+.$$

According to Θ_3, there exist ζ in the interval $(0, 1)$ and ξ in the interval $(0, +\infty]$ such that for each $i \in \{1, 2, \cdots, t\}$,

$$\lim_{n \to +\infty} \frac{\mathring{\mathrm{u}}_n^{(i)} - 1}{[\mathring{\mathrm{v}}_n^{(i)}]^{\zeta}} = \xi.$$

In the case where ξ is finite, if we set $\curlywedge = \frac{\xi}{2} > 0$ and using the definition of a limit, we can find an $n_0 \in \mathbb{N}$ for which this yields that for all $n \geq n_0$ and $i \in \{1, 2, \cdots, t\}$,

$$\left|\frac{\mathring{u}_n^{(i)}-1}{[\mathring{v}_n^{(i)}]^{\zeta}}-\xi\right| \leq \curlywedge.$$

or

$$\frac{\mathring{u}_n^{(i)}-1}{[\mathring{v}_n^{(i)}]^{\zeta}} \geq \xi-\curlywedge=\curlywedge.$$

After rearranging the expression, we obtain that $\forall n \geq n_0$ and $i \in \{1,2,\cdots,t\}$,

$$\curlywedge[\mathring{v}_n^{(i)}]^{\zeta} \leq n[\mathring{u}_n^{(i)}-1]. \tag{5}$$

Now, if $\xi=+\infty$, for $\curlywedge>0$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $i \in \{1,2,\cdots,t\}$,

$$\frac{\mathring{u}_n^{(i)}-1}{[\mathring{v}_n^{(i)}]^{\zeta}} \geq \curlywedge.$$

Rearrangement yields

$$\curlywedge[\mathring{v}_n^{(i)}]^{\zeta} \leq n[\mathring{u}_n^{(i)}-1]. \tag{6}$$

Using (5), (6), and (4), we obtain

$$\curlywedge[\mathring{v}_n^{(i)}]^{\zeta} \leq n\Big[[\mathring{u}_n^{(i)}]^{g_i^n}-1\Big]. \tag{7}$$

By considering the limit as n approaches infinity in Equation (7), we have for each $i \in \{1,2,\cdots,t\}$,

$$\lim_{n\to+\infty} n\left[\mathring{v}_n^{(i)}\right]^{\zeta}=0.$$

So, for $\curlywedge=1$, there exists $n_i \in \mathbb{N}$ such that for all $n>n_i$,

$$n\left[\mathring{v}_n^{(i)}\right]^{\zeta} \leq 1.$$

Consequently, for any $n \geq n_0 := \max\{n_i : i=1,2,\cdots,t\}$, we derive the following expression: for each $i \in \{1,2,\cdots,t\}$,

$$n\left[\mathring{v}_n^{(i)}\right]^{\zeta} \leq 1.$$

or

$$\mathring{v}_n^{(i)} \leq \frac{1}{n^{\frac{1}{\zeta}}}. \tag{8}$$

For $\mathring{\mathcal{K}}_s$, we now claim that $\{z_n\}$ is a Cauchy sequence. To demonstrate this, we use the triangular inequality with inequality (8) and $m>k\geq n_0$ to obtain

$$\begin{aligned}
e(z_k,z_m) &\leq e(z_k,z_{k+1})+e(z_{k+1},z_{k+2})+\cdots+e(z_{m-1},z_m)\\
&=\left(\mathring{v}_k^{(i)}\right)+\left(\mathring{v}_{k+1}^{(i)}\right)+\cdots+\left(\mathring{v}_{m-1}^{(i)}\right)\\
&=\left(\sum_{j=k}^{m-1}\mathring{v}_j^{(i)}\right)\\
&\leq\left(\sum_{j=1}^{+\infty}\mathring{v}_j^{(i)}\right)\\
&\leq\left(\sum_{j=1}^{+\infty}\frac{1}{j^{\frac{1}{g}}}\right)\longrightarrow \mathbf{0}.
\end{aligned}$$

This reveals that $\{\mathtt{z}_n)\}$ in $\mathring{\mathcal{K}}_s$ is Cauchy. It follows that $\exists \mathtt{z}^* \in \mathring{\mathcal{K}}_s$ such that for every $n \to +\infty$, $\mathtt{z}_n \to \mathtt{z}^*$.

Applying the definitions of $\mathring{\mathscr{V}}$ and Θ, we find that for all $(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$ where $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2) > \mathbf{0}$,

$$\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2) \leq \mathtt{e}(\mathtt{z}_1, \mathtt{z}_2). \tag{9}$$

According to the definition of $(\mathcal{R} : \mathring{\mathscr{V}})$, $\mathcal{R}$ is $\mathtt{e}$-self-closed. Then, for a sequence $\{\mathtt{z}_n\}$ that preserves $\mathcal{R}$ and converges to $\mathtt{z}^*$, there must exist a subsequence $\{\mathtt{z}_{n_k}\}$ of $\{\mathtt{z}_n\}$ such that $[\mathtt{z}_{n_k}, \mathtt{z}^*] \in \mathcal{R}$ and $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_{n_k}, \mathring{\mathscr{V}}\mathtt{z}^*) > \mathbf{0}$ for all $k \in \mathbb{N}$. Lemma 2 and inequality (9) therefore allow us to obtain, for $[\mathtt{z}_{n_k}, \mathtt{z}^*] \in \mathcal{R}$,

$$\mathtt{e}(\mathtt{z}_{n_k+1}, \mathring{\mathscr{V}}\mathtt{z}^*) = \mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_{n_k}, \mathring{\mathscr{V}}\mathtt{z}^*) \leq \mathtt{e}(\mathtt{z}_{n_k}, \mathtt{z}^*) \to \mathbf{0} \quad as \quad k \to +\infty.$$

which yields

$$\lim_{k \to +\infty} \mathtt{z}_{n_k+1} = \mathring{\mathscr{V}}(\mathtt{z}^*).$$

Hence,

$$\mathring{\mathscr{V}}(\mathtt{z}^*) = \lim_{k \to +\infty} \mathtt{z}_{n_k+1} = \lim_{n \to +\infty} \mathtt{z}_n = \mathtt{z}^*.$$

which shows that $fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}}) \neq \phi$.

To demonstrate that $fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}})$ has a cardinality of one, we begin by assuming that $C_{\mathcal{R}}(\mathtt{z}_1, \mathtt{z}_2) \neq \emptyset$ for all $\mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s$. This implies that there exists a path between every pair of points in $\mathring{\mathcal{K}}_s$. Now, suppose to the contrary that $\mathtt{z}_1, \mathtt{z}_2 \in fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}})$ with $\mathring{\mathscr{V}}\mathtt{z}_1 = \mathtt{z}_1 \neq \mathtt{z}_2 = \mathring{\mathscr{V}}\mathtt{z}_2$ (i.e., $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2) > \mathbf{0}$). Thus, for $\mathtt{z}_1, \mathtt{z}_2 \in fix_{\mathring{\mathcal{K}}_s}(\mathring{\mathscr{V}}) \subseteq \mathring{\mathcal{K}}_s$, there exists $\{\mathtt{y}_1, \mathtt{y}_2, \cdots, \mathtt{y}_{o+1}\}$ satisfying the following:

1. $\mathtt{y}_1 = \mathtt{z}_1$ and $\mathtt{y}_{o+1} = \mathtt{z}_2$;
2. $[\mathtt{y}_i, \mathtt{y}_{i+1}] \in \mathcal{R}, \forall i = 1, 2, \cdots, o$.

So, by letting $\mathtt{e}(\mathtt{y}_k, \mathtt{y}_{k+1}) = (\wedge_k^{(i)})_{i=1}^t$, we obtain

$$\begin{aligned}
\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2) &= \mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2) = \mathtt{e}(\mathring{\mathscr{V}}\mathtt{y}_1, \mathring{\mathscr{V}}\mathtt{y}_{o+1}) \\
&\leq \sum_{\mathfrak{k}=1}^{o} \mathtt{e}(\mathring{\mathscr{V}}\mathtt{y}_{\mathfrak{k}}, \mathring{\mathscr{V}}\mathtt{y}_{\mathfrak{k}+1}) \leq \sum_{\mathfrak{k}=1}^{o} \mathtt{e}(\mathtt{y}_{\mathfrak{k}}, \mathtt{y}_{\mathfrak{k}+1}) \\
&= \sum_{\mathfrak{k}=1}^{o} \left(\wedge_{\mathfrak{k}}^{(i)}\right) = \left(\sum_{\mathfrak{k}=1}^{o} \wedge_{\mathfrak{k}}^{(i)}\right) \\
&\leq \left(\sum_{\mathfrak{k}=1}^{+\infty} \wedge_{\mathfrak{k}}^{(i)}\right)_{i=1}^{t} \leq \left(\sum_{\mathfrak{k}=1}^{+\infty} \frac{1}{\mathfrak{k}^{\frac{1}{\mathtt{g}}}}\right) \quad \longrightarrow \mathbf{0}.
\end{aligned}$$

Thus, $\mathtt{z}_1 = \mathtt{z}_2$. □

Theorem 4. *Let $(\mathring{\mathcal{K}}_s, \mathtt{e})$ represent a vector-valued metric space that is complete, along with a binary relation $\mathcal{R}$, and a multi-valued mapping $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to cl_{\mathtt{e}}(\mathring{\mathcal{K}}_s)$. Additionally, suppose that the following are satisfied:*

1. *The pair $(\mathcal{R} : \mathring{\mathscr{V}})$ forms a compound structure;*
2. *$\forall(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$ and $\mathtt{z}_3 \in \mathring{\mathscr{V}}\mathtt{z}_1, \exists \mathtt{z}_4 \in \mathring{\mathscr{V}}\mathtt{z}_2$ s.t.*

$$\theta(\mathtt{e}(\mathtt{z}_3, \mathtt{z}_4)) \leq [\theta(\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2))]^{[\mathtt{g}]}, \tag{10}$$

where $\mathtt{g} = \left(\mathtt{g}^{(i)}\right) \in \mathbb{R}_{t_{\geq 0}}$ with each $\mathtt{g}_i < 1$ and $\theta \in \Theta^t$.

Then, $\mathring{\mathscr{V}}$ has a fixed point.

Proof. Given any element $\mathtt{z}_0 \in \mathscr{Y} \subseteq \mathring{\mathcal{K}}_s$, there exists a $\mathtt{z}_1 \in \mathring{\mathscr{V}}\mathtt{z}_0$ such that $(\mathtt{z}_0, \mathtt{z}_1) \in \mathcal{R}$. From the definition of a compound structure, we can say that for $(\mathtt{z}_0, \mathtt{z}_1) \in \mathcal{R}$ and $\mathtt{z}_1 \in \mathring{\mathscr{V}}\mathtt{z}_0$, there exists a $\mathtt{z}_2 \in \mathring{\mathscr{V}}\mathtt{z}_1$ such that

$$\theta(\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2)) \leq [\theta(\mathtt{e}(\mathtt{z}_0, \mathtt{z}_1))]^{[\mathtt{g}]}, \tag{11}$$

Since $\mathcal{R}$ is $\mathring{\mathscr{V}}$-closed, $(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$. Moreover, by assumption, for $(\mathtt{z}_1, \mathtt{z}_2) \in \mathcal{R}$ and $\mathtt{z}_2 \in \mathring{\mathscr{V}}\mathtt{z}_1$, $\exists \mathtt{z}_3 \in \mathring{\mathscr{V}}\mathtt{z}_2$ for which

$$\theta(\mathtt{e}(\mathtt{z}_2, \mathtt{z}_3)) \leq [\theta(\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2))]^{[\mathtt{g}]},$$

and $(\mathtt{z}_2, \mathtt{z}_3)$, which implies, by inequality (11), that

$$\theta(\mathtt{e}(\mathtt{z}_2, \mathtt{z}_3)) \leq [\theta(\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2))]^{[\mathtt{g}]} \leq [\theta(\mathtt{e}(\mathtt{z}_0, \mathtt{z}_1))]^{[\mathtt{g}^2]},$$

where $[\mathtt{g}^2] = \left([\mathtt{g}^{(i)}]^2\right)_{i=1}^{t}$. We obtain a sequence $\{\mathtt{z}_n\}$ by repeating the same procedure. This sequence is defined by $\mathtt{z}_n \in \mathring{\mathscr{V}}\mathtt{z}_{n-1}$, $\forall n \in \mathbb{N}$, with the property that $(\mathtt{z}_n, \mathtt{z}_{n+1}) \in \mathcal{R}$ (i.e., $\{\mathtt{z}_n\}$ is $\mathcal{R}$-preserving) and

$$\theta(\mathtt{e}(\mathtt{z}_n, \mathtt{z}_{n+1})) \leq [\theta(\mathtt{e}(\mathtt{z}_0, \mathtt{z}_1))]^{[\mathtt{g}^n]}, \tag{12}$$

Let $\forall n \in \mathbb{N}$,

$$\mathtt{e}(\mathtt{z}_n, \mathtt{z}_{n+1}) = \left(\mathring{\mathtt{v}}_n^{(1)}, \mathring{\mathtt{v}}_n^{(2)}, \cdots, \mathring{\mathtt{v}}_n^{(t)}\right)$$

and

$$\theta[\mathtt{e}(\mathtt{z}_n, \mathtt{z}_{n+1})] = \left(\mathring{\mathtt{u}}_n^{(1)}, \mathring{\mathtt{u}}_n^{(2)}, \cdots, \mathring{\mathtt{u}}_n^{(t)}\right).$$

Then, using methodology similar to that used in Theorem 3, yields that for all $n \geq n_0$ and $i \in \{1, 2, \cdots, t\}$,

$$\mathring{\mathtt{v}}_n^{(i)} \leq \frac{1}{n^{\frac{1}{\mathtt{g}}}}. \tag{13}$$

Now, in accordance with inequality (13) and the triangular inequality, for $m > p \geq n_0$,

$$\begin{aligned}
\mathtt{e}(\mathtt{z}_p, \mathtt{z}_m) &\leq \mathtt{e}(\mathtt{z}_p, \mathtt{z}_{p+1}) + \mathtt{e}(\mathtt{z}_{p+1}, \mathtt{z}_{p+2}) + \cdots + \mathtt{e}(\mathtt{z}_{m-1}, \mathtt{z}_m) \\
&= \left(\mathring{\mathtt{v}}_p^{(i)}\right)_{i=1}^{t} + \left(\mathring{\mathtt{v}}_{p+1}^{(i)}\right)_{i=1}^{t} + \cdots + \left(\mathring{\mathtt{v}}_{m-1}^{(i)}\right)_{i=1}^{t} \\
&= \left(\sum_{\mathfrak{k}=p}^{m-1} \mathring{\mathtt{v}}_{\mathfrak{k}}^{(i)}\right)_{i=1}^{t} \\
&\leq \left(\sum_{\mathfrak{k}=1}^{+\infty} \mathring{\mathtt{v}}_{\mathfrak{k}}^{(i)}\right)_{i=1}^{t} \\
&\leq \left(\sum_{\mathfrak{k}=1}^{+\infty} \frac{1}{\mathfrak{k}^{\frac{1}{\mathtt{g}}}}\right)_{i=1}^{t} \longrightarrow \mathbf{0}.
\end{aligned}$$

Consequently, $\{\mathtt{z}_n\}$ in $\mathring{\mathcal{K}}_s$ is a Cauchy sequence. We obtain $\mathtt{z} \in \mathring{\mathcal{K}}_s$ such that $\mathtt{z}_n \to \mathtt{z}$ using the completeness property of $\mathring{\mathcal{K}}_s$. Using Θ_1 and given assumption, we determine that for each $(\mathtt{z}_1, \mathtt{z}_2) \in R$ implies that

$$\mathtt{e}(\mathtt{z}_3, \mathtt{z}_4) \leq \mathtt{e}(\mathtt{z}_1, \mathtt{z}_2), \quad \textit{where} \quad \mathtt{z}_3 \in \mathring{\mathscr{V}}\mathtt{z}_1, \mathtt{z}_4 \in \mathring{\mathscr{V}}\mathtt{z}_2. \tag{14}$$

We obtain $(\mathtt{z}_n, \mathtt{z}) \in \mathcal{R}$, since $\mathcal{R}$ is strongly $\mathtt{e}$-self-closed, where N_0 is any natural number and $\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_{n_k}, \mathring{\mathscr{V}}\mathtt{z}) > \mathbf{0}$ for all $n \geq N_0$. In the context of a given assumption and inequality (14), $\exists \mathtt{z}^* \in \mathring{\mathscr{V}}\mathtt{z}$ such that for all $(\mathtt{z}_n, \mathtt{z}) \in \mathcal{R}$, for all $n \geq N_0$, and $\mathtt{z}_{n+1} \in \mathring{\mathscr{V}}\mathtt{z}_n$,

$$\mathtt{e}(\mathtt{z}_{n+1}, \mathtt{z}^*) \leq \mathtt{e}(\mathtt{z}_n, \mathtt{z}) \to \mathbf{0} \quad as \quad n \to +\infty.$$

But

$$\mathtt{z}^* = \lim_{n \to +\infty} \mathtt{z}_{n+1} = \lim_{n \to +\infty} \mathtt{z}_n = \mathtt{z}.$$

Hence, $\mathtt{z} \in \mathring{\mathscr{V}}\mathtt{z}$. □

Remark 1. *It is crucial to note that $C_{\mathcal{R}}(\mathtt{z}_1, \mathtt{z}_2) \neq \phi$ holds true for all $\mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s$ if $\mathcal{R}$ represents a full order or if $\mathring{\mathcal{K}}_s$ is $\mathcal{R}$-directed.*

Proof. If $\mathcal{R}$ forms a complete order, then every pair $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s$ is $\mathcal{R}$-comparative, implying that $[\mathtt{w}_1, \mathtt{w}_2] \in \mathcal{R}$ for all $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathscr{V}}_s$. This indicates that $\{\mathtt{w}_1, \mathtt{w}_2\}$ constitutes a path from $\mathtt{w}_1$ to $\mathtt{w}_2$ with a length of 1 in $\mathcal{R}$. Consequently, $C_{\mathcal{R}}(\mathtt{w}_1, \mathtt{w}_2) \neq \phi$ for all $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s$.

If $\mathring{\mathcal{K}}_s$ is an $\mathcal{R}$-directed set, then for every $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathscr{V}}\mathtt{w}$, there exists $\mathtt{w}_3 \in \mathring{\mathcal{K}}_s\mathtt{w}$ such that $(\mathtt{w}_1, \mathtt{w}_3) \in \mathcal{R}$ and $(\mathtt{w}_2, \mathtt{w}_3) \in \mathcal{R}$. This demonstrates that for each $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathscr{V}}\mathtt{w}$, we have a path $\{\mathtt{w}_1, \mathtt{w}_3, \mathtt{w}_2\}$ from $\mathtt{w}_1$ to $\mathtt{w}_2$ with a length of 2 in $\mathcal{R}$. Thus, $C_{\mathcal{R}}(\mathtt{w}_1, \mathtt{w}_2)$ is non-empty for every $\mathtt{w}_1, \mathtt{w}_2 \in \mathring{\mathcal{K}}_s$. □

Corollary 1. *If $\mathring{\mathcal{K}}_s$ is an $\mathcal{R}$-directed set or $\mathcal{R}$ represents a full order, then all the hypotheses of Theorem 3 hold. Then, there exists a unique fixed point for $\mathring{\mathscr{V}}$.*

By setting the full relation, that is, taking $\mathcal{R} = \mathring{\mathcal{K}}_s \times \mathring{\mathcal{K}}_s$ in Theorem 3, we obtain the main result of Jleli and Samet [13] as follows.

Corollary 2. *In any complete vector-valued metric space $(\mathring{\mathcal{K}}_s, \mathtt{e})$, let $\mathring{\mathscr{V}}$ be a Perov-type θ-contraction. Then, $\mathring{\mathscr{V}}$ has a unique fixed point in $\mathring{\mathcal{K}}_s$.*

Example 6. *For $\mathtt{m} = 2$, let $\mathring{\mathcal{K}}_s = [0, +\infty)$ be equipped with the standard vector-valued metric, that is, $\mathtt{e}(\mathtt{u}, \mathtt{v}) = (|\mathtt{u} - \mathtt{v}|, |\mathtt{u} - \mathtt{v}|)$. Consider the sequence $\{\mathtt{r_n}\}$ in $\mathring{\mathcal{K}}_s$ defined by*

$$\mathtt{r_n} = \frac{(\mathtt{n}+1)(\mathtt{n}+2)}{2}, \textit{ for all } \mathtt{n} \in \mathbb{N}.$$

Define the binary relation $\mathcal{R}$ as follows:

$$\mathcal{R} = \{(\mathtt{r}_1, \mathtt{r}_1)\} \cup \{(\mathtt{r_n}, \mathtt{r_{n+1}}) : \mathtt{n} \in \mathbb{N}\}.$$

Now, define the mapping $\mathring{\mathscr{V}} : \mathring{\mathcal{K}}_s \to \mathring{\mathcal{K}}_s$ by

$$\mathring{\mathscr{V}}\mathtt{z} = \begin{cases} \mathtt{z} & \textit{if } 0 < \mathtt{z} \leq \mathtt{r}_1 \\ \mathtt{r}_1 & \textit{if } \mathtt{r}_1 \leq \mathtt{z} \leq \mathtt{r}_2 \\ \mathtt{r}_i + \left(\frac{\mathtt{r}_{i+1} - \mathtt{r}_i}{\mathtt{r}_{i+2} - \mathtt{r}_{i+1}}\right)(\mathtt{z} - \mathtt{r}_{i+1}) & \textit{if } \mathtt{r}_{i+1} \leq \mathtt{z} \leq \mathtt{r}_{i+2}, i = 1, 2, \ldots \end{cases}$$

Then, $\mathring{\mathscr{V}}$ is continuous, $\mathtt{r}_1 \mathcal{R} \mathring{\mathscr{V}} \mathtt{r}_1$. Next, we will show that $\mathring{\mathscr{V}}$ satisfies the contraction condition for $\theta(\mathtt{b}) = e^{\sqrt{\mathtt{b}e^{\mathtt{b}}}}$ for $\mathtt{b} > 0$. Now, let $\mathtt{z}_1, \mathtt{z}_2 \in \mathring{\mathcal{K}}_s$ be such that $\mathtt{z}_1 \mathcal{R} \mathtt{z}_2$ and $\mathring{\mathscr{V}}\mathtt{z}_1 \neq \mathring{\mathscr{V}}\mathtt{z}_2$. It must be the case that $\mathtt{z}_1 = \mathtt{r_n}$ and $\mathtt{z}_2 = \mathtt{r_{n+1}}$ for some $\mathtt{n} \in \mathbb{N}$. To prove contraction condition (2), it is enough to show that

$$\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2) e^{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1, \mathring{\mathscr{V}}\mathtt{z}_2)} \leq h^2 \mathtt{e}(\mathtt{z}_1, \mathtt{z}_2) e^{\mathtt{e}(\mathtt{z}_1, \mathtt{z}_2)}$$

for some $h \in (0, 1)$; that is, we have to show that

$$\frac{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1,\mathring{\mathscr{V}}\mathtt{z}_2)}{\mathtt{e}(\mathtt{z}_1,\mathtt{z}_2)}e^{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1,\mathring{\mathscr{V}}\mathtt{z}_2)-\mathtt{e}(\mathtt{z}_1,\mathtt{z}_2)} \le h^2$$

for some $h \in (0,1)$. Now, observe that

$$\frac{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1,\mathring{\mathscr{V}}\mathtt{z}_2)}{\mathtt{e}(\mathtt{z}_1,\mathtt{z}_2)}e^{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{z}_1,\mathring{\mathscr{V}}\mathtt{z}_2)-\mathtt{e}(\mathtt{z}_1,\mathtt{z}_2)} = \frac{\mathtt{r_n}-\mathtt{r_{n-1}}}{\mathtt{r}_{n+1}-\mathtt{r_n}}e^{(\mathtt{r_n}-\mathtt{r_{n-1}})-(\mathtt{r_{n+1}}-\mathtt{r_n})} < e^{-1}.$$

Hence, inequality (2) holds for $h = e^{\frac{-1}{2}}$. Therefore, all the hypotheses of Theorem 3 are satisfied. Therefore, $\mathring{\mathscr{V}}$ has a fixed point in $\mathring{\mathcal{K}}_s$.

Remark 2. *The theorem referenced in [14] is not appropriate in the circumstances of Example 6 due to the fact that $\lim_{n\to+\infty}\frac{\mathtt{e}(\mathring{\mathscr{V}}\mathtt{r_n},1)}{\mathtt{e}(\mathtt{r_n},1)} = 1$, which means that their contraction conditions are not satisfied. Therefore, our findings represent a suitable extension of the work by Altun et al. [14].*

4. Applications Associated with Nonlinear System of Matrix Equations

Fixed-point theorems have undergone thorough explorations for a variety of functions within ordered metric spaces, resulting in numerous applications spanning across different fields of science and mathematics. In particular, these theorems find significance in contexts involving differential, integral, and matrix equations. These extensive studies and their applications are well-documented in various references such as [17–19], along with additional sources referenced therein.

Let $\mathring{\mathcal{K}}_s(\mathtt{n})$ represent the set of all $\mathtt{n}\times\mathtt{n}$ complex matrices, $\mathtt{H}(\mathtt{n})$ the set of all Hermitian matrices in $\mathring{\mathcal{K}}_s(\mathtt{n})$, $\mathtt{v}(\mathtt{n})$ the collection of all positive definite matrices in $\mathring{\mathcal{K}}_s(\mathtt{n})$, and $\mathtt{H}^+(\mathtt{n})$ the class of all positive semidefinite matrices in $\mathring{\mathcal{K}}_s(\mathtt{n})$. To indicate that a matrix N in $\mathtt{v}(\mathtt{n})$ (or $\mathtt{H}^+(\mathtt{n})$) is positive definite (or positive semidefinite), we use the notation $\mathtt{N}\succ 0$ (or $\mathtt{N}\succeq 0$). Furthermore, $\mathtt{N}_1\succ\mathtt{N}_2$ (or $\mathtt{N}_1\succeq\mathtt{N}_2$) implies that $\mathtt{N}_1-\mathtt{N}_2\succ 0$ (or $\mathtt{N}_1-\mathtt{N}_2\succeq 0$). The symbol $\|\cdot\|$ represents the vector-valued spectral norm of matrix $\mathtt{Z}$, defined as $\|\mathtt{Z}\| = (\sqrt{\mathtt{g}^+(\mathtt{Z}^*\mathtt{Z})}, \frac{\sqrt{\mathtt{g}^+(\mathtt{Z}^*\mathtt{Z})}}{2})$, where $\mathtt{g}^+(\mathtt{Z}^*\mathtt{Z})$ is the largest eigenvalue of $\mathtt{Z}^*\mathtt{A}$, with $\mathtt{Z}^*$ being the conjugate transpose of $\mathtt{Z}$. Moreover, $\|\mathtt{Z}\|_{tr}$ is defined as $\left(\sum_{k=1}^{n}s_k(\mathtt{Z}), \frac{1}{2}\sum_{k=1}^{n}s_k(\mathtt{Z})\right)$, where $s_k(\mathtt{Z})$ (for $1\le k\le n$) denotes the singular values of $\mathtt{Z}\in\mathring{\mathcal{K}}_s(n)$. The pair $(\mathtt{H}(n), \|\cdot\|_{tr})$ constitutes a complete vector-valued metric space (for further details, refer to References [19–21]). Furthermore, the binary relation $\preceq$ on $\mathtt{H}(n)$ is defined as follows: $\mathtt{Z}_1\preceq\mathtt{Z}_2$ if and only if $\mathtt{Z}_2\succeq\mathtt{Z}_1$ holds for all $\mathtt{Z}_1,\mathtt{Z}_2\in\mathtt{H}(n)$.

In this section, we utilize our findings to provide a solution to the system comprising two nonlinear matrix equations as presented below.

$$\mathtt{X}_i = \mathtt{v} + \sum_{k=1}^{n}\mathtt{N}_k^*\mathcal{G}_i(\mathtt{E})\mathtt{N}_k, \quad i = 1,2. \tag{15}$$

Here, $\mathcal{G}_i$ represents a continuous, order-preserving mapping with $\mathcal{G}_i(0)=0$, $\mathtt{v}$ denotes a Hermitian positive definite matrix, and $\mathtt{N}_k$ are arbitrary $n\times n$ matrices, with $\mathtt{N}_k^*$ denoting their conjugates.

Now, we present the following lemmas, which prove to be beneficial in the subsequent results.

Lemma 3 ([19])**.** *Let $\mathtt{Z}_1,\mathtt{Z}_2\in\mathtt{H}(n)$ such that $\mathtt{Z}_1\succeq 0$ and $\mathtt{Z}_2\succeq 0$. Then,*

$$0\le \mathrm{tr}(\mathtt{Z}_1\mathtt{Z}_2)\le\|\mathtt{Z}_1\|\,\mathrm{tr}(\mathtt{Z}_2)$$

Lemma 4 ([22])**.** *If $\mathtt{Z}_1\in\mathtt{H}(n)$ such that $\mathtt{Z}_1\prec I_n$, then $\|\mathtt{Z}_1\|<1$.*

Theorem 5. *Take into account the system of Matrix Equation (15), along with the parameters L and $\mathtt{g} \in (0,1)$ being positive real numbers such that we have the following:*

1. *For $\mathtt{Z}_1, \mathtt{Z}_2 \in \mathtt{H}(n)$ with $\mathtt{Z}_1 \preceq \mathtt{Z}_2$ and $\sum_{i=1}^{n} \mathtt{N}_i^* \mathcal{G}_j(\mathtt{Z}_1)\mathtt{N}_i \neq \sum_{i=1}^{n} \mathtt{N}_i^* \mathcal{G}_j(\mathtt{Z}_2)\mathtt{N}_i$, we have*

$$\left|\operatorname{tr}\left(\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right)\right| \leq \frac{\mathtt{g}^2 |\operatorname{tr}(\mathtt{Z}_2 - \mathtt{Z}_1)|}{L}, \quad \forall j = 1,2; \tag{16}$$

2. $\sum_{i=1}^{m} \mathtt{N}_i \mathtt{N}_i^* \prec L I_n$ *and* $\sum_{i=1}^{m} \mathtt{N}_i^* \mathcal{G}_j(\mathtt{Z}_1)\mathtt{N}_i \succ 0, \quad \forall j = 1,2.$
   Then, the system of Equation (15) has a solution.

Proof. Define the operators $\mathring{\mathscr{V}}_j : \mathtt{H}(n) \times \mathtt{H}(n) \to \mathtt{H}(n)$ by

$$\mathring{\mathscr{V}}_j(\mathtt{Z}) = \mathtt{v} + \sum_{i=1}^{m} \mathtt{N}_i^* \mathcal{G}_j(\mathtt{Z})\mathtt{N}_i, \quad j = 1,2, \forall \mathtt{Z} \in \mathtt{H}(n). \tag{17}$$

In such a case, $\mathring{\mathscr{V}}_j$'s elements are properly defined, and the ordering $\preceq$ on $\mathtt{H}(n)$ is closed under both $\mathring{\mathscr{V}}_1$ and $\mathring{\mathscr{V}}_2$.

Now, define a mapping $\mathring{\mathscr{V}} : \mathtt{H}(n) \times \mathtt{H}(n) \to \mathtt{H}(n) \times \mathtt{H}(n)$ by

$$\mathring{\mathscr{V}}(\tilde{Z}) = \mathring{\mathscr{V}}(\mathtt{Z}_1, \mathtt{Z}_2) = (\mathring{\mathscr{V}}_1(\mathtt{Z}_1, \mathtt{Z}_2), \mathring{\mathscr{V}}_2(\mathtt{Z}_1, \mathtt{Z}_2)). \tag{18}$$

Then, the system of Equation (15) can be transformed to the following fixed-point problem:

$$\mathring{\mathscr{V}}(\tilde{Z}) = \tilde{Z}. \tag{19}$$

Given that both $\mathring{\mathscr{V}}_1$ and $\mathring{\mathscr{V}}_2$ maintain closure under the binary relation $\preceq$, this consequently implies that the operator $\mathring{\mathscr{V}}$ also upholds closure under $\preceq$. Furthermore, since $\sum_{i=1}^{m} \mathtt{N}_i^* \mathcal{G}_j(\mathtt{Z})\mathtt{N}_i \succ 0$ for all $j = 1,2$, it follows that $\mathtt{Z} \preceq \mathring{\mathscr{V}}_j(\mathtt{Z})$ for both $j = 1,2$.

Next, we show that (2) holds for the mapping $\mathring{\mathscr{V}}$. For this purpose, take $\theta \in \Theta$ with $\mathtt{g} \in (0,1)$ as given by

$$\theta(t) = e^{\sqrt[\mathtt{g}]{t}}, \quad t \in (0, +\infty).$$

Consider $\mathtt{Z}_1, \mathtt{Z}_2 \in \mathtt{H}(n)$ such that $\mathtt{Z}_1 \preceq \mathtt{Z}_2$ and $\mathcal{G}_j(\mathtt{Z}_1) \neq \mathcal{G}_j(\mathtt{Z}_2)$, which implies that $\mathtt{Z}_1 \prec \mathtt{Z}_2$. Given that $\mathcal{G}_j$ is order-preserving, it follows that $\mathcal{G}_j(\mathtt{Z}_1) \prec \mathcal{G}_j(\mathtt{Z}_2)$ for $j = 1,2$. Therefore,

$$\begin{aligned}
\left\|\mathring{\mathscr{V}}(\mathtt{Z}_2) - \mathring{\mathscr{V}}(\mathtt{Z}_1)\right\|_{tr} &= \operatorname{tr}\left(\mathring{\mathscr{V}}(\mathtt{Z}_2) - \mathring{\mathscr{V}}(\mathtt{Z}_1)\right) \\
&= \operatorname{tr}\left(\sum_{i=1}^{m} \mathtt{N}_i^* \left(\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right)\mathtt{N}_i\right) \\
&= \sum_{i=1}^{m} \operatorname{tr}\left(\mathtt{N}_i^* \left(\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right) NV_i\right) \\
&= \sum_{i=1}^{m} \operatorname{tr}\left(\mathtt{N}_i \mathtt{N}_i^* \left(\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right)\right) \\
&= \operatorname{tr}\left(\left(\sum_{i=1}^{m} \mathtt{N}_i \mathtt{N}_i^*\right)\left(\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right)\right) \\
&\leq \left\|\sum_{i=1}^{m} \mathtt{N}_i \mathtt{N}_i^*\right\| \left\|\mathcal{G}_j(\mathtt{Z}_2) - \mathcal{G}_j(\mathtt{Z}_1)\right\|_{tr} \\
&\leq \frac{\left\|\sum_{i=1}^{m} \mathtt{N}_i \mathtt{N}_i^*\right\|}{L} \left(\mathtt{g}^2 \|\mathtt{Z}_2 - \mathtt{Z}_1\|_{tr}\right) \\
&< \mathtt{g}^2 \|\mathtt{Z}_2 - \mathtt{Z}_1\|_{tr},
\end{aligned}$$

which further implies that

$$e^{\sqrt{\|\mathring{\mathscr{V}}(E_2)-\mathring{\mathscr{V}}(E_1)\|_{tr}}} \leq e^{g\sqrt{\|E_2-E_1\|_{tr}}}.$$

Thus, we have

$$\theta\left(\left\|\mathring{\mathscr{V}}(Z_2)-\mathring{\mathscr{V}}(Z_1)\right\|_{tr}\right) \leq [\theta(\|Z_2-Z_1\|_{tr})]^{g}.$$

This shows that $\mathring{\mathscr{V}}$ satisfies all the assumptions of Theorem 3, so it has a fixed-point $\tilde{Z}^* \in H(n) \times H(n)$ such that

$$\mathring{\mathscr{V}}(\tilde{Z}^*) = \mathring{\mathscr{V}}((Z_1^*, Z_2^*)) = (\mathring{\mathscr{V}}_1(Z_1^*, Z_2^*), \mathring{\mathscr{V}}_2(Z_1^*, Z_2^*)) = (Z_1^*, Z_2^*).$$

This implies that $\mathring{\mathscr{V}}_1(Z_1^*, Z_2^*) = Z_1^*$ and $\mathring{\mathscr{V}}_2(Z_1^*, Z_2^*) = Z_2^*$. Thus, the system of Matrix Equation (15) has a solution in $H(n) \times H(n)$. $\square$

5. Conclusions

In this study, we presented a novel and unique adaptation of the classical fixed-point theorems proposed by Perov as well as Altun et al. This theorem applies to both single-valued and multi-valued mappings within a complete generalized metric space equipped with a binary relation. Notably, our version encompasses and extends the fixed-point theorems of Perov and Altun et al., particularly when considering trivial θ mappings or relations (universal). Furthermore, we illustrated the versatility of our findings through examples and practical applications.

Author Contributions: Conceptualization, F.U.D., S.A., U.I., M.D. and S.S.; methodology, F.U.D., S.A., U.I., M.D. and S.S.; software, F.U.D., S.A., U.I., M.D. and S.S.; validation, F.U.D., S.A., U.I., M.D. and S.S.; formal analysis, F.U.D., S.A., U.I., M.D. and S.S.; investigation, F.U.D., S.A., U.I., M.D. and S.S.; resources, F.U.D., S.A., U.I., M.D. and S.S.; data curation, F.U.D., S.A., U.I., M.D. and S.S.; writing—original draft preparation, F.U.D., S.A., U.I., M.D. and S.S.; writing—review and editing, F.U.D., S.A., U.I., M.D. and S.S.; visualization, F.U.D., S.A., U.I., M.D. and S.S.; supervision, F.U.D., S.A., U.I., M.D. and S.S.; project administration, F.U.D., S.A., U.I., M.D. and S.S.; funding acquisition, F.U.D., S.A., U.I., M.D. and S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available upon request from the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Banach, S. Sur les opêrations dans les ensembles abstraits et leur application aux quations intêgrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Nadler, S.B., Jr. Multi-valued contraction mappings. *Pac. J. Math.* **1969**, *30*, 475–488. [CrossRef]
3. Kikkawa, M.; Suzuki, T. Three fixed point theorems for generalized contractions with constants in complete metric spaces. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2942–2949. [CrossRef]
4. Din, F.U.; Din, M.; Ishtiaq, U.; Sessa, S. Perov fixed-point results on *F*-contraction mappings equipped with binary relation. *Mathematics* **2023**, *11*, 238. [CrossRef]
5. Moţ, G.; Petruşel, A. Fixed point theory for a new type of contractive multivalued operators. *Nonlinear Anal. Theory Methods Appl.* **2009**, *70*, 3371–3377. [CrossRef]
6. Sintunavarat, W.; Kumam, P. Common fixed point theorem for cyclic generalized multi-valued contraction mappings. *Appl. Math. Lett.* **2012**, *25*, 1849–1855. [CrossRef]
7. Petre, I.R.; Bota, M. Fixed point theorems on generalized b-metric spaces. *Publ. Math. Debrecen* **2013**, *83*, 139–159. [CrossRef]
8. Rasham, T.; Shoaib, A.; Hussain, N.; Alamri, B.A.; Arshad, M. Multivalued fixed point results in dislocated b-metric spaces with application to the system of nonlinear integral equations. *Symmetry* **2019**, *11*, 40. [CrossRef]
9. Browder, F.E. Nonexpansive nonlinear operators in a Banach space. *Proc. Natl. Acad. Sci. USA* **1965**, *54*, 1041. [CrossRef]
10. Kannan, R. Some results on fixed points—II. *Am. Math. Mon.* **1969**, *76*, 405–408.
11. Wardowski, D.; Dung, N.V. Fixed points of *F*-weak contractions on complete metric spaces. *Demonstratio Math.* **2014**, *47*, 146–155. [CrossRef]

12. Karapinar, E. Revisiting the Kannan type contractions via interpolation. *Adv. Theory Nonlinear Anal. Its Appl.* **2018**, *2*, 85–87. [CrossRef]
13. Jleli, M.; Samet, B. A new generalization of the Banach contraction principle. *J. Inequalities Appl.* **2014**, *2014*, 38. [CrossRef]
14. Altun, I.; Hussain, N.; Qasim, M.; Al-Sulami, H.H. A new fixed point result of Perov type and its application to a semilinear operator system. *Mathematics* **2019**, *7*, 1019. [CrossRef]
15. Alam, A.; Imdad, M. Relation-theoretic contraction principle. *J. Fixed Point Theory Appl.* **2015**, *17*, 693–702. [CrossRef]
16. Lipschutz, S. *Schaum's Outline of Theory and Problems of Set Theory and Related Topics*; McGraw-Hill: New York, NY, USA, 1964.
17. Agarwal, R.P.; Hussain, N.; Taoudi, M.A. Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations. In *Abstract and Applied Analysis*; Hindawi: London, UK, 2012; Volume 2012.
18. Hussain, N.; Khan, A.; Agarwal, R.P. Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces. *J. Nonlinear Convex Anal.* **2010**, *11*, 475–489.
19. Ran, A.C.; Reurings, M.C. A fixed point theorem in partially ordered sets and some applications to matrix equations. *Proc. Am. Math. Soc.* **2004**, *132*, 1435–1443. [CrossRef]
20. Berzig, M. Solving a class of matrix equations via the Bhaskar–Lakshmikantham coupled fixed point theorem. *Appl. Math. Lett.* **2012**, *25*, 1638–1643. [CrossRef]
21. Berzig, M.; Samet, B. Solving systems of nonlinear matrix equations involving Lipshitzian mappings. *Fixed Point Theory Appl.* **2011**, *2011*, 89. [CrossRef]
22. Long, J.H.; Hu, X.Y.; Zhang, L. On the Hermitian positive defnite solution of the nonlinear matrix equation $X + AX^{-1}A + BX^{-1}B = I$. *Bull. Braz. Math. Soc. New Ser.* **2008**, *39*, 371–386. [CrossRef]
23. Vetro, F.; Radenović, S. Some results of Perov type in rectangular cone metric spaces. *J. Fixed Point Theory Appl.* **2018**, *20*, 41. [CrossRef]
24. Guran, L.; Bota, M.F.; Naseem, A.; Mitrović, Z.D.; Sen, M.D.L.; Radenović, S. On some new multivalued results in the metric spaces of Perov's type. *Mathematics* **2020**, *8*, 438. [CrossRef]
25. Almalki, Y.; Din, F.U.; Din, M.; Ali, M.U.; Jan, N. Perov-fixed point theorems on a metric space equipped with ordered theoretic relation. *Aims Math.* **2022**, *11*, 20199–20212. [CrossRef]
26. Perov, A.I. On the Cauchy problem for a system of ordinary differential equations. *Pviblizhen. Met. Reshen. Differ. Uvavn* **1964**, *2*, 115–134.
27. Filip, A.D.; Petruşel, A. Fixed point theorems on spaces endowed with vector-valued metrics. *Fixed Point Theory Appl.* **2010**, *2010*, 281381. [CrossRef]