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Group Doubly Coupled Designs

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Abstract: Doubly coupled designs (DCDs) have better space-filling properties between the qualitative and quantitative factors than marginally coupled designs (MCDs) which are suitable for computer experiments with both qualitative and quantitative factors. In this paper, we propose a new class of DCDs, called group doubly coupled designs (GDCDs), and provide methods for constructing two forms of GDCDs, within-group doubly coupled designs and between-group doubly coupled designs. The proposed GDCDs can accommodate more qualitative factors than DCDs, when the subdesigns for the qualitative factors are symmetric. The subdesigns of qualitative factors are not asymmetric in the existing results on DCDs, and in this paper, we construct GDCDs with symmetric and asymmetric designs for the qualitative factors, respectively. Moreover, detailed comparisons with existing MCDs show that GDCDs have better space-filling properties between qualitative and quantitative factors. Finally, the methods are particularly easy to implement.

Keywords: computer experiment; doubly coupled design; orthogonal array; sliced latin hypercube

MSC: 62K05; 62K99

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1. Introduction

Computer experiments are an effective method for exploring complex systems and scientific problems [1,2].The space-filling properties, which measure the uniformity of the design points in the experimental space, are critical for effectively exploring the experimental region of computer experiments [2]. Latin hypercube designs (LHDs), proposed by [3], are widely used space-filling designs for computer experiments. Such designs are often used in computer experiments with quantitative factors because they achieve optimal univariate uniformity. Computer experiments involving only quantitative factors have received considerable attention [1,2]. However, researchers usually encounter computer experiments involving both qualitative and quantitative factors; see [1,4–13].

Sliced Latin hypercube designs (SLHDs) proposed by [14] are LHDs that can be partitioned into some LHD slices, which not only maintain the optimal univariate uniformity but for each slice as well. SLHDs are popular for computer experiments with both qualitative and quantitative factors; see [9,10,15] and the references therein. Each slice of an SLHD can be used at one level combination of the qualitative factors. However, its number of runs increases dramatically with the number of level combinations of the qualitative factors. This is thus suitable for situations where there are few level combinations of the qualitative factors or where the cost of runs is low.

Inspired by the notion of SLHD, [16] proposed the marginally coupled designs (MCDs). Their key feature is that for each level of any qualitative factor, the design points for the quantitative factors can form a small LHD, and they have fewer runs than SLHDs. In recent years, improvements for MCDs include, but are not limited to, its quantitative factors design with column orthogonality and multi-dimensional stratifications; for more details, refer to [17–20]. MCD, however, appears to be inapplicable when it is necessary to study the stratification between multiple qualitative factors and quantitative factors, whereas a design with such properties can be useful for studying the interaction between two qualitative factors and quantitative factors.

To this end, [21] proposed the doubly coupled designs (DCDs). It not only maintains the properties of MCDs, but also ensures that the design points for the quantitative factors can form an LHD corresponding to any level combination of any two qualitative factors. In a DCD, the subdesign for qualitative factors is an orthogonal array (OA). Equal-level and mixed-level orthogonal arrays are called symmetric and asymmetric orthogonal arrays, respectively. In the DCDs constructed by [21], the subdesign for qualitative factors is a symmetric orthogonal array. However, in real-world problems, there exist qualitative factors with mixed levels, and the design of the qualitative factors is usually an asymmetric OA. At present, there are no studies of DCDs with qualitative factors being asymmetric OAs. The latter construction cannot be a simple extension of the former. Moreover, the existing DCDs have an upper bound on the number of qualitative factors, namely, no more than the number of levels of qualitative factors. Therefore, existing DCDs are inapplicable when the qualitative factors are mixed-level or when the number of qualitative factors exceeds the number of their levels.

For a computer experiment with q s-level qualitative factors and p quantitative factors, an MCD is appropriate if there is no interaction effect between any two qualitative factors and all quantitative factors; if $q \leq s$ and there is the interaction effect between any two qualitative factors and all quantitative factors, a DCD is applicable. However, neither an MCD nor a DCD is suitable, when $q > s$, some qualitative factors and all quantitative factors have such interaction effects, and some do not. Suppose that in an experiment there are four qualitative factors, the type of concentration of cell lysis reagent (A1, A2), the type of stain (Blue, Red, Pink), the shape of the cell slides (Thick, Moderate, Thin), and the cells' activity (Dead, Alive) as well as other quantitative factors. We know that only the two qualitative factors, the type of concentration of cell lysis reagent and the shape of the cell slides, have the interaction effect with all quantitative factors. Obviously, both an MCD and a DCD are not suitable for such an experiment. Thus, we need to adopt a design that satisfies two properties: (i) the whole design is an MCD; and (ii) the columns of some qualitative factors and the columns of all quantitative factors form a DCD. In this paper, we focus on such designs and call them group DCD (GDCD).

In addition, not only can the GDCDs contain more qualitative factors, but the designs for the qualitative factors can be asymmetric OAs. Therefore, the level types of GDCDs are more flexible than those of DCDs. Our methods construct two forms of GDCDs, within-group DCDs and between-group DCDs. In a within-group DCD (WGDCD), the design of the qualitative factors can be divided into several groups, and the design of any two qualitative factors from the same group coupled with the design of the quantitative factors is a DCD. Thus, columns in the same group have excellent stratification properties between qualitative and quantitative factors. In a between-group DCD (BGDCD), the design of the qualitative factors can also be split into several groups, and the design of any two qualitative factors from different level groups combined with the design of quantitative factors is a DCD. The methods for constructing WGDCDs and BGDCDs are similar and easy to implement and are given in Section 3.1 and Section 3.2, respectively. Since the space-filling property of GDCDS is similar to that of DCDs, the space-filling property of GDCDs is better than that of MCDs.

The article is organized as follows: Section 2 introduces the basic notation and definitions. Methods for constructing GDCDs and the corresponding examples are given in Section 3. Comparison was made in Section 4. Section 5 provides conclusions and discussion. All proofs are deferred to Appendix A.

2. Definitions and Notation

Let $GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$ denote the Galois field of order s, where $\alpha_0 = 0$ and $\alpha_1 = 1$. An $r \times c$ matrix D is called a difference scheme over $GF(s)$, denoted by $D(r, c, s)$, if it has the property that every element of $GF(s)$ in the vector difference between any two distinct columns in D occurs r/s times equally. For details of the difference schemes, refer to Section 6.1 of [22]. An $n \times m$ matrix is called an asymmetric orthogonal array of strength t, denoted by $OA(n, s_1^{m_1} s_2^{m_2} \cdots s_c^{m_c}, t)$, $m_1 + m_2 + \cdots + m_c = m$, if any of its $n \times t$ submatrix satisfies all possible t-tuples occur equally often, where the level of the first m_1 columns is taken from $\{0, \ldots, s_1 - 1\}$, the level of the next m_2 columns is taken from $\{0, \ldots, s_2 - 1\}$, and so on. When all the s_j's are equal to s, the orthogonal array is symmetric, denoted by $OA(n, s^m, t)$.

We now review the Rao-Hamming construction in Section 3.4 of [22]. For a prime power s, let z_1 and z_2 be two s-level columns of length s^2 with entries from $GF(s)$, $GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}$, where $\alpha_0 = 0$ and $\alpha_1 = 1$. Suppose that z_1 and z_2 are independent. We apply the Rao-Hamming construction in [22] to obtain an $OA(s^2, s^{s+1}, 2)$ Y, i.e., $Y = (z_1, z_2)\Lambda$ over $GF(s)$, where $\Lambda = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & \alpha_2 & \cdots & \alpha_s & 1 \end{pmatrix}$.

An $n \times p$ matrix that each column is a permutation of integers $\{0, 1, \ldots, n-1\}$ is called an LHD, denoted by $LHD(n, p)$. $\mathbf{0}_n$ and $\mathbf{1}_n$ are two n-dimensional column vectors with all entries being zeros and ones, respectively. Let A^T represent the transposition of matrix A. For an $n \times m$ matrix A and an $f \times g$ matrix B, $A \oplus B = (a_{ij} + B)$ and $A \otimes B = (a_{ij} B)$ represent the Kronecker sum and Kronecker product, respectively, where a_{ij} is the (i, j)th entry of A.

Suppose there is an $OA(n, s_1^{q_1} s_2^{q_2}, 2)$, if its rows can be divided into $\frac{n}{\alpha_j s_j}$ $OA(\alpha_j s_j, s_1^{q_1} s_2^{q_2}, 1)$'s, and for j = 1, 2, $\alpha_j s_j$ remains the same, then called the array as $(\alpha_1 \times \alpha_2)$-resolvable orthogonal array, denoted by $(\alpha_1 \times \alpha_2)$-$ROA(n, s_1^{q_1} s_2^{q_2}, 2)$. Especially, when $s_1 = s_2 = s$, $\alpha_1 = \alpha_2 = \alpha$, then the array reduces to α-$ROA(n, s^{q_1+q_2}, 2)$. If $\alpha = 1$, the array is called a completely resolvable orthogonal array (CROA).

Let $D = (A, L)$ be an n-run design with q qualitative factors and p quantitative factors, where the subdesigns A and L are qualitative factors and quantitative factors, respectively. The design D is called a marginally coupled design, denoted by $MCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, $q_1 + q_2 + \cdots + q_c = q$, if it satisfies: (i) A is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$; (ii) L is an $LHD(n, p)$; and (iii) the rows in L, corresponding to each level of any factor in A, form a small LHD. When $s_i = s$, $i = 1, 2, \ldots, c$, the MCD is denoted as $MCD(n, s^q, p)$.

Let $D^0 = (D_1, D_2)$ be an $MCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$. The design D^0 is called a doubly coupled design, denoted by $DCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, if it satisfies that the rows in D_2, corresponding to each level combination of any two factors in D_1, form a small LHD. When $s_i = s$, $i = 1, 2, \ldots, c$, the DCD is denoted as $DCD(n, s^q, p)$. Obviously, D_1 is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$, D_2 is an $LHD(n, p)$, and the rows in D_2, corresponding to each level combination of any t factors in D_1, form a small LHD for $t = 1, 2$.

Definition 1. *Let $D = (A, L)$ be an $MCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, $q_1 + q_2 + \cdots + q_c = q$, where $A = (A_1, A_2, \ldots, A_c)$ is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$, L is an $LHD(n, p)$, A_i is an $OA(n, s_i^{q_i}, 2)$, $i = 1, 2, \ldots, c$.*

(i) *The D is called a within-group DCD (WGDCD), denoted by $WGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, if (A_i, L) is a DCD for $i = 1, 2, \ldots, c$. When $s_1 = s_2 = \ldots = s_c = s$ and $q_1 = q_2 = \ldots = q_c = u$, then we denote such D by $WGDCD(n, s^{u \cdot c}, p)$.*

(ii) *The D is called a between-group DCD (BGDCD), denoted by $BGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, if $((A_i^j, A_m^n), L)$ is a DCD, where A_i^j and A_m^n are the jth column in A_i and the nth column in A_m, respectively, for $1 \leq i \neq m \leq c$, $j = 1, 2, \ldots, q_i$, $n = 1, 2, \ldots, q_m$. When $s_1 = s_2 = \ldots = s_c = s$ and $q_1 = q_2 = \ldots = q_c = u$, then we denote such D by $BGDCD(n, s^{u \cdot c}, p)$.*

From Definition 1, it is easy to see that (A_i, L) is a $DCD(n, s_i^{q_i}, p)$ in a $WGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$, for $i = 1, 2, \ldots, c$. For any $DCD(n, s^q, p)$, Corollary 1 of [21] shows that $q \leq s$. Similarly, we have the following Corollary 1.

Corollary 1. *If a WGDCD $D = (A, L)$ with A being an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$ exists, then $q_j \leq s_j$, $j = 1, 2, \ldots, c$.*

Corollary 1 above tells us that the $WGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$ can accommodate up to $s_1 + s_2 + \cdots + s_c$ qualitative factors.

Here we provide some results on the existence of GDCDs. Recall the definition of GDCDs, a GDCD of n runs has two subdesigns, A and L, which are for q qualitative factors and p quantitative factors, respectively. Theorems 1 and 2 below establish the necessary and sufficient conditions of the existence of WGDCDs and BGDCDs, respectively. For ease of expression, for an $n \times 1$ column vector d, define $d^{(s_i)}$ and $d^{(s_i \cdot s_m)}$ based on d. Let $d^{(v)}$ be the vth entry of d, $1 \leq v \leq n$ and $d^{(v, s_i)} = \left\lfloor \frac{d^{(v)}}{s_i} \right\rfloor$, $d^{(v, s_i \cdot s_m)} = \left\lfloor \frac{d^{(v)}}{s_i s_m} \right\rfloor$, where $d^{(v, s_i)}$ and $d^{(v, s_i \cdot s_m)}$ are the vth entries in $d^{(s_i)}$ and $d^{(s_i \cdot s_m)}$, respectively, and $\lfloor a \rfloor$ represents the largest integer not exceeding a.

Theorem 1. *Suppose $A = (A_1, A_2, \ldots, A_c)$ is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$, $L = (l_1, l_2, \ldots, l_p)$ is an $LHD(n, p)$, where l_k is the kth column of L, $k = 1, 2, \ldots, p$. Let A_i^j be the jth column of A_i for $i = 1, 2, \ldots, c$, $j = 1, 2, \ldots, q_i$. Then design $D = (A, L)$ is a $WGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$ if and only if:*
(i) $(A_i^j, l_k^{(s_i)})$ *is an $OA(n, 2, s_i(n/s_i), 2)$, for any $i = 1, 2, \ldots, c$, $j = 1, 2, \ldots, q_i$, $k = 1, 2, \ldots, p$; and*
(ii) $(A_i^j, A_i^f, l_k^{(s_i \cdot s_i)})$ *is an $OA(n, 3, s_i^2(n/s_i^2), 3)$, for any $i = 1, 2, \ldots, c$, $1 \leq j \neq f \leq q_i$, $k = 1, 2, \ldots, p$.*

Theorem 2. *Suppose $A = (A_1, A_2, \ldots, A_c)$ is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$, $L = (l_1, l_2, \ldots, l_p)$ is an $LHD(n, p)$, where l_k is the kth column of L, $k = 1, 2, \ldots, p$. Let A_i^j be the jth column of A_i for $i = 1, 2, \ldots, c$, j=$1, 2, \ldots, q_i$. Then design $D = (A, L)$ is a $BGDCD(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, p)$ if and only if:*
(i) $(A_i^j, l_k^{(s_i)})$ *is an $OA(n, 2, s_i(n/s_i), 2)$, for any $i = 1, 2, \ldots, c$, $j = 1, 2, \ldots, q_i$, $k = 1, 2, \ldots, p$; and*
(ii) $(A_i^j, A_m^n, l_k^{(s_i \cdot s_m)})$ *is an $OA(n, 3, s_i s_m(n/s_i s_m), 3)$, for any $1 \leq i \neq m \leq c$, $j = 1, 2, \ldots, q_i$, $n = 1, 2, \ldots, q_m$, $k = 1, 2, \ldots, p$.*

Theorems 1 and 2 establish the existence of GDCDs in terms of the relations between the individual columns in A and $l_k^{(s_i)}$, and between any pair of columns in A and $l_k^{(s_i \cdot s_i)}$ or $l_k^{(s_i \cdot s_m)}$.

3. Construction of GDCDs

Since the existing DCDs in [21] have an upper bound on the number of qualitative factors, that is not exceeding the number of levels. The subdesigns for qualitative factors are all symmetric OAs, in the DCDs constructed by [21]. This section, therefore, describes four main construction algorithms to produce different GDCDs that can contain more qualitative factors. Two forms of GDCDs, WGDCDs and BGDCDs, are provided for different needs. In Section 3.1, two algorithms were proposed for constructing WGDCDs with equal-level and mixed-level qualitative factors, respectively. In Section 3.2, two algorithms were proposed for generating different BGDCDs with mixed-level qualitative factors. These newly constructed designs with qualitative factors can be either symmetric or asymmetric. Different initial DCDs are used in the construction Algorithms 1, 3 and 5. The above constructions lead the resulting designs to entertain more qualitative factors and mixed-level qualitative factors, compared with the existing DCDs.

3.1. Construction of WGDCDs

Existing DCDs have few columns and do not work if the problem under study has $s+1$ or more factors (s is the number of levels of qualitative factors). To be able to study the relationship between more qualitative and quantitative factors, this section presents WGDCDs, which can accommodate more qualitative factors with equal or mixed levels than DCDs. The construction of WGDCDs is presented in the next two subsections.

3.1.1. Construction of WGDCDs with Symmetric Qualitative Factors

Suppose there exists a difference scheme $D(r,c,s)$ of strength 2, denoted by $D(1)$, where r is a multiple of s, $c \leq r$, and an initial $DCD(n,s^q,p)$ $D^0 = (D_1, D_2)$. Such difference scheme and D^0 are used in the following algorithm to construct a $WGDCD(rn, s^{q\cdot c}, pf)$ $D = (A, L)$, where A is an $OA(rn, s^{q\cdot c}, 2)$, each group of the subdesign A is a symmetric $OA(rn, s^q, 2)$. For clarity, let Kronecker sum $\oplus$ in Algorithm 1 be defined over the Galois field of order s ($GF(s)$).

Algorithm 1 Construction of WGDCDs with symmetric qualitative factors

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Step 1. Given that $D(1)$ is a difference scheme $D(r,c,s)$, D_1 is an $OA(n,s^q,2)$, D_2 is an $LHD(n,p)$ and $D^0 = (D_1, D_2)$ is a $DCD(n,s^q,p)$, then construct a $(rn) \times (qc)$ matrix, as $A = (D(1) \oplus D_1)$ over $GF(s)$.

Step 2. Let C be an $r \times f$ matrix with all elements being ones, H be an $LHD(r, pf)$, construct an $(rn) \times (pf)$ matrix, as $L = C \otimes D_2 + nH \otimes \mathbf{1}_n$.

Step 3. The resulting design is $D = (A, L)$.

Proposition 1. *The design $D = (A, L)$ obtained by Algorithm 1 is a $WGDCD(rn, s^{q\cdot c}, pf)$.*

The initial design D^0 in Step 1 is a DCD, which can be obtained from [21]. The subdesign A of the design D constructed by Algorithm 1 is an $OA(rn, s^{cq}, 2)$ and can be divided into c groups of q columns each, forming an $OA(rn, s^q, 2)$. There are excellent stratification properties between the columns in the same group and L. DCDs with s-level qualitative factors can accommodate up to s qualitative factors according to Corollary 1 of [21]. However, the WGDCDs constructed by Algorithm 1 can accommodate up to cs qualitative factors. The following example gives an illustration of Algorithm 1, where the initial DCD D^0 is taken from the first four columns of Table 1 in [21].

Example 1. *Consider a design $WGDCD(32, 2^{2\cdot 4}, 4)$ with the initial DCD D^0 as in Table 1. By using the following difference scheme $D(1)$, 4×2 matrix C and any $LHD(4,4)$ H, we can obtain A and L in Algorithm 1, respectively. The resulting design $D = (A, L)$ is listed in Table 2. It is easy to check that the D is a $WGDCD(32, 2^{2\cdot 4}, 4)$. Obviously, the subdesign A can be divided into 4 groups of 2 columns each, forming an OA. Let $A = (A_1, A_2, A_3, A_4)$, where A_i is an $OA(16, 2^2, 2)$, then $(A_i, L)$is a $DCD(16, 2^2, 4)$, $i = 1,2,3,4$. The $WGDCD(32, 2^{2\cdot 4}, 4)$ outperforms $DCD(32, 2^2, 4)$ obtained by [21] in terms of the number of qualitative factors. Figure 1 shows that the maximum one-dimensional projection uniformity of L with respect to each level combination of any one or two factors in A_1.*

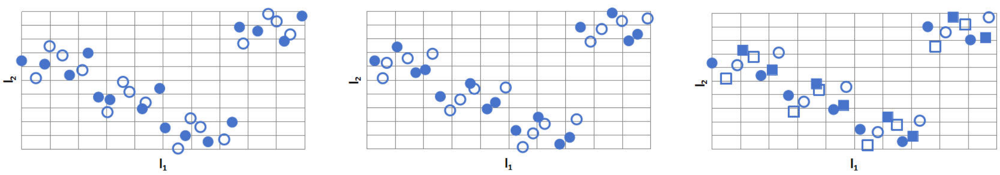
$$D(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 3 & 2 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix}.$$

Table 1. The $DCD(8, 2^2, 2)$ used in Example 1.

D_1^T	0 0	1 1	0 1	1 0	0 0	1 1	0 1	1 0
D_2^T	1 0	0 4	6 2	7 6	4 5	5 1	3 7	2 3

Table 2. $WGDCD(32, 2^{2\cdot 4}, 4)$ in Example 1.

Run	***A***								***L***				**Run**	***A***								***L***			
1	0	0	0	0	0	0	0	0	1	16	25	8	17	0	0	0	0	1	1	1	1	25	24	17	0
2	1	1	1	1	1	1	1	1	0	20	24	12	18	1	1	1	1	0	0	0	0	24	28	16	4
3	0	1	0	1	0	1	0	1	6	18	30	10	19	0	1	0	1	1	0	1	0	30	26	22	2
4	1	0	1	0	1	0	1	0	7	22	31	14	20	1	0	1	0	0	1	0	1	31	30	23	6
5	0	0	0	0	0	0	0	0	4	21	28	13	21	0	0	0	0	1	1	1	1	28	29	20	5
6	1	1	1	1	1	1	1	1	5	17	29	9	22	1	1	1	1	0	0	0	0	29	25	21	1
7	0	1	0	1	0	1	0	1	3	23	27	15	23	0	1	0	1	1	0	1	0	27	31	19	7
8	1	0	1	0	1	0	1	0	2	19	26	11	24	1	0	1	0	0	1	0	1	26	27	28	3
9	0	0	1	1	0	0	1	1	9	8	9	16	25	0	0	1	1	1	1	0	0	17	0	1	24
10	1	1	0	0	1	1	0	0	8	12	8	20	26	1	1	0	0	0	0	1	1	16	4	0	28
11	0	1	1	0	0	1	1	0	14	10	14	18	27	0	1	1	0	1	0	0	1	22	2	6	26
12	1	0	0	1	1	0	0	1	15	14	15	22	28	1	0	0	1	0	1	1	0	23	6	7	30
13	0	0	1	1	0	0	1	1	12	13	12	21	29	0	0	1	1	1	1	0	0	20	5	4	29
14	1	1	0	0	1	1	0	0	13	9	13	17	30	1	1	0	0	0	0	1	1	21	1	5	25
15	0	1	1	0	0	1	1	0	11	15	11	23	31	0	1	1	0	1	0	0	1	19	7	3	31
16	1	0	0	1	1	0	0	1	10	11	10	19	32	1	0	0	1	0	1	1	0	18	3	2	27



(**a**) (**b**) (**c**)

Figure 1. Projection of L in Example 1. Scatter plots of l_1 versus l_2 in Example 1: (**a**) points represented by $\circ$ and $\bullet$ correspond to the levels 0, 1 of A_1^1, respectively; (**b**) points represented by $\circ$ and $\bullet$ correspond to the levels 0, 1 of A_1^2, respectively; (**c**) points marked by $\square$, solid $\square$, $\circ$, and $\bullet$ correspond to the level combinations (0, 0), (0, 1), (1, 0), and (1, 1) of (A_1^1, A_1^2).

3.1.2. Construction of WGDCDs with Asymmetric Qualitative Factors

Next, we provide another method to construct a WGDCD $D = (A, L)$ with the subdesign A for qualitative factors being an asymmetric OA. Without loss of generality, we consider the case of $c = 2$, i.e., $A = (A_1, A_2)$ is an $OA(n, s_1^{q_1} s_2^{q_2}, 2)$ in this paper. Here we assume that $s_2 = s_1^2$ and n must be a multiple of s_2^2. The following algorithm is to construct a $WGDCD(\lambda s_2^2, s_1^{s_1} s_2^{s_2-1}, p)$ $D = (A, L)$, with $A = (A_1, A_2)$ being an $OA(\lambda s_2^2, s_1^{s_1} s_2^{s_2-1}, 2)$, where A_1 is an $OA(\lambda s_2^2, s_1^{s_1}, 2)$ and A_2 is an $OA(\lambda s_2^2, s_2^{s_2-1}, 2)$, L is an $LHD(\lambda s_2^2, p)$.

We apply the Rao-Hamming construction in Section 3.4 of [22] to obtain the orthogonal arrays E and F in Algorithm 2. For example, if $s_1 = 2$, then an $OA(4, 2^3, 2)$ E can be obtain by the Rao-Hamming construction over $GF(2)$, as

$$E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

After row permutation, E can be transformed into

$$E=\left(\begin{array}{ccc}0&0&0\\0&1&1\\1&0&1\\1&1&0\end{array}\right).$$

It is easy to see that

$$E_0=(e_2,E^*)=\left(\begin{array}{cccc}0&1&0&1\\0&1&1&0\end{array}\right)^T.$$

Algorithm 2 Construction of WGDCDs with asymmetric qualitative factors

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Step 1. Given an $OA(s_1^2,s_1^{s_1+1},2)$ E and an $OA(s_2^2,s_2^{s_2+1},2)$ F, $s_2=s_1^2$. After row permutation, E and F can be transformed into $E=(e_1,e_2,E^*)$ and $F=(f_1,f_2,F^*)$, respectively, so that $e_1=\left(\mathbf{0}_{s_1}^T,\mathbf{1}_{s_1}^T,\ldots,(\mathbf{s_1-1})_{s_1}^T\right)^T$, $e_2=\mathbf{1}_{s_1}\otimes(0,1,\ldots,s_1-1)^T$, $f_1=\left(\mathbf{0}_{s_2}^T,\mathbf{1}_{s_2}^T,\ldots,(\mathbf{s_2-1})_{s_2}^T\right)^T$, $f_2=\mathbf{1}_{s_2}\otimes(0,1,\ldots,s_2-1)^T$. Delete the columns e_1 and f_1 of E and F, respectively. Denote the remaining columns as E_0 and F_0, respectively, i.e., $E_0=(e_2,E^*)$ and $F_0=(f_2,F^*)$.

Step 2. On base of F_0, the first to last rows of E_0 are used to replace the s_2 levels of f_2, respectively, and denote the replaced design as G, G is an asymmetric $OA(s_2^2,s_1^{s_1}s_2^{s_2-1},2)$. Let $A=\mathbf{1}_\lambda\otimes G$.

Step 3. For a given λ and p, let π_k be a random permutation of $(0,1,\ldots,\lambda-1)^T$, g_k be a random permutation of $(0,1,\ldots,s_2-1)^T$, and $v_k=(c_1^T,c_2^T,\ldots,c_{s_1}^T)^T$, $k=1,2,\ldots,p$, where $c_i=(i-1)s_1\oplus z$, z is a random permutation of $(0,1,\ldots,s_1-1)^T$, $i=1,2,\ldots,s_1$. Construct an LHD $L=(l_1,l_2,\ldots,l_p)$, where $l_k=s_2^2(\pi_k\otimes\mathbf{1}_{s_2^2})+s_2(\mathbf{1}_\lambda\otimes(g_k\otimes\mathbf{1}_{s_2}))+\mathbf{1}_{\lambda s_2}\otimes v_k$, for $k=1,2,\ldots,p$.

Step 4. The resulting design is $D=(A,L)$.

Theorem 3. *The design $D=(A,L)$ obtained by Algorithm 2 is a $WGDCD(\lambda s_2^2,s_1^{s_1}s_2^{s_2-1},p)$ with $s_2=s_1^2$.*

Clearly, in Theorem 3, A can be divided into two groups, denoted as $A=(A_1,A_2)$, where A_1 and A_2 are $OA(\lambda s_2^2,s_1^{s_1},2)$ and $OA(\lambda s_2^2,s_2^{s_2-1},2)$, respectively. Obviously, (A_1,L) and (A_2,L) are a $DCD(\lambda s_2^2,s_1^{s_1},p)$ and a $DCD(\lambda s_2^2,s_2^{s_2-1},p)$, respectively. It is not difficult to find that the number of columns in each group of the subdesign A in the design D constructed by Algorithm 2 almost reaches its number of levels. The total number of qualitative factors is $s_1+(s_2-1)$. The designs constructed by Algorithm 2 cannot be constructed in [21], because the subdesigns for qualitative factors, constructed by [21], are all symmetric OAs. An illustration of Algorithm 2 is given in the following example.

Example 2. *Construct a $WGDCD(32,2^24^3,2)$, when $\lambda=2$. In Step 1, E and F (after rows permuting) are obtained using the Rao-Hamming construction as follows,*

$$E^T=\begin{bmatrix}0&0&1&1\\0&1&1&0\\0&1&0&1\end{bmatrix},\ F^T=\begin{bmatrix}0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3\\0&1&2&3&0&1&2&3&0&1&2&3&0&1&2&3\\0&2&3&1&1&3&2&0&2&0&1&3&3&1&0&2\\0&3&1&2&1&2&0&3&2&1&3&0&3&0&2&1\\0&1&2&3&1&0&3&2&2&3&0&1&3&2&1&0\end{bmatrix}.$$

In Step 2, obtain

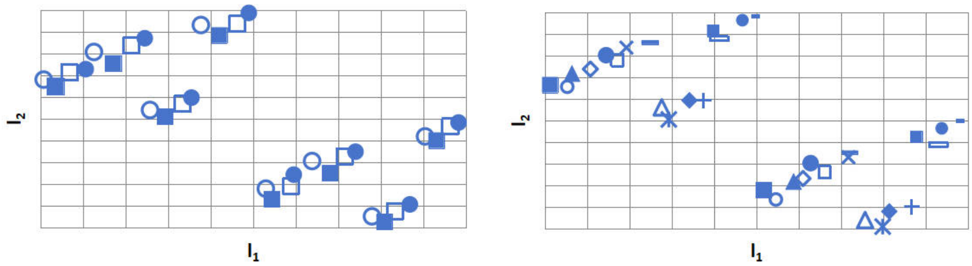
$$G^T=\begin{bmatrix}0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0\\0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\0&2&3&1&1&3&2&0&2&0&1&3&3&1&0&2\\0&3&1&2&1&2&0&3&2&1&3&0&3&0&2&1\\0&1&2&3&1&0&3&2&2&3&0&1&3&2&1&0\end{bmatrix}.$$

Let $A=\mathbf{1}_2\otimes G$. In Step 3, let $\pi_1=(1,0)^T,\pi_2=(0,1)^T,g_1=(0,3,2,1)^T,g_2=(1,3,0,2)^T,v_1=$

$(1,0,3,2)^T, v_2 = (0,1,3,2)^T$. We can get L is an $LHD(32,2)$. In Step 5, the resulting design $D = (A, L)$ is a $WGDCD(32, 2^2 4^3, 2)$ and shown in Table 3. A visualization of this example is shown in Figure 2. Clearly, Figure 2 shows the design points of quantitative factors enjoy the maximum one-dimensional stratification corresponding to each level combination of any two qualitative factors in A_1 and A_2, respectively.

Table 3. $WGDCD(32, 2^2 4^3, 2)$ in Example 2.

Run	A					L		Run	A					L	
1	0	1	0	0	0	17	4	17	0	1	0	0	0	1	20
2	1	0	2	3	1	16	5	18	1	0	2	3	1	0	21
3	1	1	3	1	2	19	7	19	1	1	3	1	2	3	23
4	0	0	1	2	3	18	6	20	0	0	1	2	3	2	22
5	0	1	1	1	1	29	12	21	0	1	1	1	1	13	28
6	1	0	3	2	0	28	13	22	1	0	3	2	0	12	29
7	1	1	2	0	3	31	15	23	1	1	2	0	3	15	31
8	0	0	0	3	2	30	14	24	0	0	0	3	2	14	30
9	0	1	2	2	2	25	0	25	0	1	2	2	2	9	16
10	1	0	0	1	3	24	1	26	1	0	0	1	3	8	17
11	1	1	1	3	0	27	3	27	1	1	1	3	0	11	19
12	0	0	3	0	1	26	2	28	0	0	3	0	1	10	18
13	0	1	3	3	3	21	8	29	0	1	3	3	3	5	24
14	1	0	1	0	2	20	9	30	1	0	1	0	2	4	25
15	1	1	0	2	1	23	11	31	1	1	0	2	1	7	27
16	0	0	2	1	0	22	10	32	0	0	2	1	0	6	26



(**a**) (**b**)

Figure 2. Projection of L in Example 2. Scatter plots of l_1 versus l_2 in Example 2: (**a**) points marked by $\square$, solid $\square$, $\circ$, and $\bullet$ correspond to the level combinations (0,0), (0,1), (1,0), and (1,1) of (A_1^1, A_1^2), respectively; (**b**) points marked by $\circ$, $\triangle$, long -, small $\bullet$, and large $\bullet$, rectangle, solid $\triangle$, + correspond to the level combinations (0,0), (0,1), (0,2), (0,3) and (1,0), (1,1), (1,2), (1,3) of (A_2^1, A_2^2), respectively; and points marked by short -, $\times$, *, large solid $\square$, and solid $\Diamond$, $\Diamond$, small solid $\square$, $\square$ correspond to the level combinations (2,0), (2,1), (2,2), (2,3), and (3,0), (3,1), (3,2), (3,3) of (A_2^1, A_2^2), respectively.

If a small MCD can be constructed, then a large MCD with more columns can be constructed following Construction 3 of [16]. Similar to Construction 3 of [16], based on the *WGDCDs* obtained by Algorithm 2, a series of new WGDCDs with more columns can be constructed by Corollary 2 as follows.

Corollary 2. *Let $A = (A_1, A_2)$ be an $OA(\lambda s_2^2, s_1^{s_1} s_2^{s_2-1}, 2)$, where A_j is the orthogonal array with s_j levels, $j = 1,2$. If for some u, there are difference schemes $D(u, c_j, s_j)$ of strength two, denoted by $D(j)$, for j = 1,2, then the design A_{new}=$(D(1) \oplus A_1, D(2) \oplus A_2)$ over $GF(s_1)$ and $GF(s_2)$ is an $OA(\lambda u s_2^2, s_1^{s_1 \cdot c_1} s_2^{(s_2-1) \cdot c_2}, 2)$. Let C be an $u \times f$ matrix with all elements being ones, H be an any $LHD(u, pf)$, obtain an LHD L_{new}=$C \oplus L + \lambda s_2^2 H \oplus \mathbf{1}_{\lambda s_2^2}$, then $(A_{new}$,$L_{new})$ is a $WGDCD(\lambda u s_2^2, s_1^{s_1 \cdot c_1} s_2^{(s_2-1) \cdot c_2}, pf)$.*

3.2. Construction of BGDCDs

Section 3.1 is devoted to constructing WGDCDs with either equal-level or mixed-level qualitative factors, where each group of A in a WGDCD achieves excellent stratification between any two qualitative factors and quantitative factors. We now construct the second type of GDCDs, the BGDCDs with mixed-level qualitative factors. In a $WGDCD(n, s_1^{q_1}s_2^{q_2}, p)$, a multiple relation between s_1 and s_2 is required, i.e., $s_2 = s_1^2$. But, in a $BGDCD(n, s_1^{q_1}s_2^{q_2}, p)$, this relation may or may not be present. Compared to WGDCDs, the BGDCDs have better stratification properties between any two qualitative factors from different groups of A and quantitative factors. Since the qualitative factor designs in DCDs constructed by [21] are symmetric OAs, we focus on the case that the qualitative factor designs are asymmetric OAs in Section 3.2.

3.2.1. Construction of BGDCDs Based on $OA(s_1s_2, s_1^1s_2^1, 2)$

In this section, we also construct BGDCDs using an initial DCD D^0. Here we only discuss the case of BGDCDs with asymmetric qualitative factors. Since D_1 in an initial DCD D^0 is a symmetric OA, then WGDCDs like the one in Section 3.1.1 can be obtained by Algorithm 3 below. In light of Corollary 2, we propose Algorithm 3 below. In Step 2 of Algorithm 3, the $\oplus$ operator is based on $GF(s_1)$ and $GF(s_2)$.

Algorithm 3 Construction of BGDCDs based on $OA(s_1s_2, s_1^1s_2^1, 2)$

Step 1. Let (M, B) be an $MCD(s_1s_2, s_1^1s_2^1, p)$, where M is an $OA(s_1s_2, s_1^1s_2^1, 2)$ and B is an $LHD(s_1s_2, p)$. Let $D_1 = \mathbf{1}_\lambda \otimes M$ and $D_2 = (B^T, (s_1s_2 \oplus B)^T, (2s_1s_2 \oplus B)^T, \ldots, ((\lambda - 1)s_1s_2 \oplus B)^T)^T$, then construct an initial $DCD(\lambda s_1s_2, s_1^1s_2^1, p)$ D^0, as $D^0 = (D_1, D_2)$, where $D_1 = (D_{11}, D_{12})$, D_{1j} is an $OA(\lambda s_1s_2, s_j^1, 1)$, j=1,2.

Step 2. Obtain two designs $A = (A_1, A_2) = (D(1) \oplus D_{11}, D(2) \oplus D_{12})$ over $GF(s_1)$ and $GF(s_2)$, and $L = C \otimes D_2 + \lambda s_1s_2H \otimes \mathbf{1}_{\lambda s_1s_2}$, where $D(j)$ is a difference scheme $D(u, c_j, s_j)$ for j=1, 2, C is an $u \times f$ matrix with all elements being ones, and H is any $LHD(u, pf)$.

Step 3. The resulting design $D = (A, L)$.

Remark 1. *The design (M, B) is an MCD which can be obtained from [23]. The D^0 generated by Algorithm 3 has two cases:* (i) *When $s_1 = s_2$ in M, the design D^0 is a DCD with equal-level qualitative factors;* (ii) *Otherwise the obtained design D^0 is a DCD with mixed-level qualitative factors. In addition, design $A = (A_1, A_2)$ obtained from Step 2 is an $OA(u\lambda s_1s_2, s_1^{c_1}s_2^{c_2}, 2)$, where A_j is an $OA(u\lambda s_1s_2, s_j^{c_j}, 2)$, j=1,2.*

Proposition 2. *According to Algorithm 3, we have the following results:*
(i) *The design $D^0 = (D_1, D_2)$ obtained from Step 1 is a $DCD(\lambda s_1s_2, s_1^1s_2^1, p)$, where D_1 is an $OA(\lambda s_1s_2, 2, s_1s_2, 2)$, D_2 is an $LHD(\lambda s_1s_2, p)$;*
(ii) *The design $D = (A, L)$ obtained from Algorithm 3 is a $BGDCD(u\lambda s_1s_2, s_1^{c_1}s_2^{c_2}, pf)$.*

In a $WGDCD(n, s_1^{q_1}s_2^{q_2}, p)$ requires $s_2 = s_1^2$. But, for the BGDCDs constructed by Algorithm 3, this multiple relation is unnecessary. Next, we give an illustrative example of Algorithm 3.

Example 3. *First, the design (M, B) in Table 4 is an $MCD(6, 2^13^1, 2)$ of 6 runs for two qualitative factors m_1, m_2 and two quantitative factors b_1, b_2. Here M is an asymmetric $OA(6, 2^13^1, 2)$ and B is an $LHD(6, 2)$. In Step 1, let $\lambda = 2$ and juxtapose the two M's row by row to obtain D_1, which is an $OA(12, 2^13^1, 2)$ and can be partitioned into two full factorial designs, while using B to obtain D_2. Then we can obtain design $D^0 = (D_1, D_2)$ in Table 5 is a $DCD(12, 2^13^1, 2)$. It can be easily checked that the rows in D_2 corresponding to any level combination of any one or two factors in D_1 form an LHD.*

Table 4. $MCD(6, 2^1 3^1, 2)$ in Example 3.

Run	M		B	
1	0	0	0	3
2	0	1	2	5
3	0	2	4	1
4	1	0	3	0
5	1	1	5	2
6	1	2	1	4

Table 5. The $DCD(12, 2^1 3^1, 2)$ used in Example 3.

D_1^T	D_{11}^T	0	0	0	1	1	1	0	0	0	1	1	1
	D_{12}^T	0	1	2	0	1	2	0	1	2	0	1	2
D_2^T		0	2	4	3	5	1	6	8	10	9	11	7
		3	5	1	0	2	4	9	11	7	6	8	10

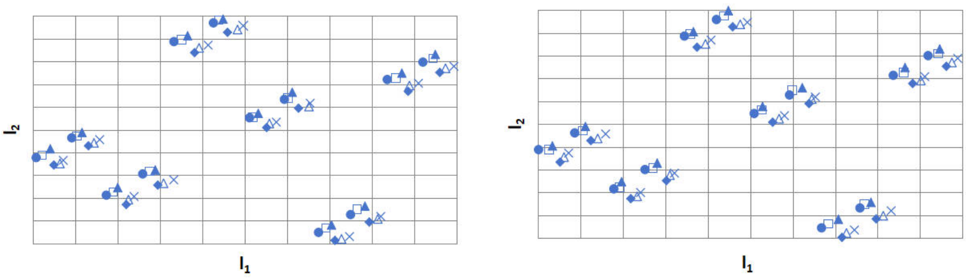
Second, take $D(1)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$, $D(2)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ *and* $H^T = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 0 & 4 \\ 3 & 4 & 2 & 5 & 1 & 0 \\ 4 & 3 & 0 & 1 & 5 & 2 \end{bmatrix}$. *Then by Step 2 and Step 3, we obtain that the design D is a $BGDCD(72, 2^2 3^3, 4)$. The final design $D = (A, L)$ is shown in Table 6. Figure 3 reflects the projection property of L, with respect to level combinations of (A_1^1, A_2^1) and (A_1^2, A_2^2), respectively.*

Table 6. $BGDCD(72, 2^2 3^3, 4)$ in Example 3.

Run	A					L				Run	A					L			
1	0	0	0	0	0	0	27	36	51	37	0	1	0	0	0	36	39	60	15
2	0	0	1	1	1	2	29	38	53	38	0	1	1	1	1	38	41	62	17
3	0	0	2	2	2	4	25	40	49	39	0	1	2	2	2	40	37	64	13
4	1	1	0	0	0	3	24	39	48	40	1	0	0	0	0	39	36	63	12
5	1	1	1	1	1	5	26	41	50	41	1	0	1	1	1	41	38	65	14
6	1	1	2	2	2	1	28	37	52	42	1	0	2	2	2	37	40	61	16
7	0	0	0	0	0	6	33	42	57	43	0	1	0	0	0	42	45	66	21
8	0	0	1	1	1	8	35	44	59	44	0	1	1	1	1	44	47	68	23
9	0	0	2	2	2	10	31	46	55	45	0	1	2	2	2	46	43	70	19
10	1	1	0	0	0	9	30	45	54	46	1	0	0	0	0	45	42	69	18
11	1	1	1	1	1	11	32	47	56	47	1	0	1	1	1	47	44	71	20
12	1	1	2	2	2	7	34	43	58	48	1	0	2	2	2	43	46	67	22
13	0	1	0	1	2	12	15	48	39	49	0	0	0	1	2	48	3	12	63
14	0	1	1	2	0	14	17	50	41	50	0	0	1	2	0	50	5	14	65
15	0	1	2	0	1	16	13	52	37	51	0	0	2	0	1	52	1	16	61
16	1	0	0	1	2	15	12	51	36	52	1	1	0	1	2	51	0	15	60
17	1	0	1	2	0	17	14	53	38	53	1	1	1	2	0	53	2	17	62
18	1	0	2	0	1	13	16	49	40	54	1	1	2	0	1	49	4	13	64
19	0	1	0	1	2	18	21	54	45	55	0	0	0	1	2	54	9	18	69
20	0	1	1	2	0	20	23	56	47	56	0	0	1	2	0	56	11	20	71
21	0	1	2	0	1	22	19	58	43	57	0	0	2	0	1	58	7	22	67
22	1	0	0	1	2	21	18	57	42	58	1	1	0	1	2	57	6	21	66
23	1	0	1	2	0	23	20	59	44	59	1	1	1	2	0	59	8	23	68
24	1	0	2	0	1	19	22	55	46	60	1	1	2	0	1	55	10	19	70
25	0	0	0	2	1	24	63	24	3	61	0	1	0	2	1	60	51	0	27
26	0	0	1	0	2	26	65	26	5	62	0	1	1	0	2	62	53	2	29
27	0	0	2	1	0	28	61	28	1	63	0	1	2	1	0	64	49	4	25

Table 6. *Cont.*

Run			A				L			Run			A				L		
28	1	1	0	2	1	27	60	27	0	64	1	0	0	2	1	63	48	3	24
29	1	1	1	0	2	29	62	29	2	65	1	0	1	0	2	65	50	5	26
30	1	1	2	1	0	25	64	25	4	66	1	0	2	1	0	61	52	1	28
31	0	0	0	2	1	30	69	30	9	67	0	1	0	2	1	66	57	6	33
32	0	0	1	0	2	32	71	32	11	68	0	1	1	0	2	68	59	8	35
33	0	0	2	1	0	34	67	34	7	69	0	1	2	1	0	70	55	10	31
34	1	1	0	2	1	33	66	33	6	70	1	0	0	2	1	69	54	9	30
35	1	1	1	0	2	35	68	35	8	71	1	0	1	0	2	71	56	11	32
36	1	1	2	1	0	31	70	31	10	72	1	0	2	1	0	67	58	7	34



(**a**) (**b**)

Figure 3. Projection of L in Example 3. Scatter plots of l_1 versus l_2 in Example 3: (**a**,**b**) points represented by •, solid △, △, solid ◊, ×, □ correspond to the level combinations (0, 0), (0, 1), (0, 2), and (1, 0), (1, 1), (1, 2) of (A_1^1, A_2^1) ((A_1^2, A_2^2)), respectively.

3.2.2. Construction of BGDCDs Based on $(s_1 \times 1)$-ROAs

For the $BGDCD$ $D = (A, L)$ constructed by Algorithm 3, the number of columns in A is determined by the number of columns in D_1, which is taken from the initial design $D^0 = (D_1, D_2)$. Since the D_1 has only two columns, the number of columns in A is very small. To solve this problem, we propose Algorithms 4 and 5. The initial design is constructed using Algorithm 4, and based on this initial design, a BGDCD with a large number of qualitative factor columns can be constructed by Algorithm 5. Before presenting Algorithm 4, we give Theorem 4 and Proposition 3, which are extremely useful for Algorithm 4.

According to Lemma 1 derived by [17] and Lemma 2 derived by [21], we present the necessary and sufficient condition for the existence of DCDs when D_1 is an asymmetric $OA(n, s_1^{q_1} s_2, 2)$ with $s_2 = s_1^2$. To drive this result, we define matrices $D_2^{'}$, $D_2^{''}$ and $D_2^{'''}$ based on D_2. For $D_2 = (d_1, d_2, \ldots, d_p)$ in a DCD $D^0 = (D_1, D_2)$, d_k is the kth column of D_2, d_{ik} is the (i,k)th entry of D_2, $1 \le i \le n, 1 \le k \le p$. Let $d_{ik}^{'} = \left\lfloor \frac{d_{ik}}{\cdot} \right\rfloor$, $d_{ik}^{''} = \left\lfloor \frac{d_{ik}}{s^2} \right\rfloor = \left\lfloor \frac{d_{ik}^{'}}{\cdot} \right\rfloor$, and $d_{ik}^{'''} = \left\lfloor \frac{d_{ik}}{s^3} \right\rfloor = \left\lfloor \frac{d_{ik}^{''}}{\cdot} \right\rfloor$, where $d_{ik}^{'}$, $d_{ik}^{''}$ and $d_{ik}^{'''}$ are the (i,k)th entries in $D_2^{'}$, $D_2^{''}$ and $D_2^{'''}$, respectively, and s is the number of levels of qualitative factors in D_1, $\lfloor a \rfloor$ represents the largest integer not exceeding a. Let $d_k^{'}$, $d_k^{''}$ and $d_k^{'''}$ be the kth columns of $D_2^{'}$, $D_2^{''}$ and $D_2^{'''}$, respectively. Refs. [17,21] derived Lemmas 1 and 2 below, respectively.

Lemma 1 ([17])**.** *Given D_1 is an $OA(n, q, s, 2)$, D_2 is an $LHD(n, p)$ and $D_2^{'}$ is defined as above, then (D_1, D_2) is an $MCD(n, s^q, p)$ if and only if for $k = 1, 2, \ldots, p$, $(D_1, d_k^{'})$ is an asymmetric $OA(n, s^m(n/s), 2)$, where $d_k^{'}$ is the kth column of $D_2^{'}$.*

Lemma 2 ([21])**.** *Suppose that $D_1 = (z_1, z_2, \ldots, z_q)$ is an $OA(n, q, s, 2)$ and $D_2 = (d_1, d_2, \ldots, d_p)$ is an $LHD(n, p)$. The design $D^0 = (D_1, D_2)$ is a $DCD(n, s^q, p)$ if and only if*

(i) $(z_i, d_k^{'})$ *is an* $OA(n, 2, s(n/s), 2)$, *for any* $1 \leq i \leq q$, $1 \leq k \leq p$; *and*
(ii) $(z_i, z_j, d_k^{''})$ *is an* $OA(n, 3, s^2(n/s^2), 3)$, *for any* $1 \leq i \neq j \leq q$, $1 \leq k \leq p$.

Theorem 4. *Suppose that* $D_1 = (M, B)$ *is an* $OA(\lambda s_1 s_2, s_1^{q_1} s_2, 2)$, *where* $s_2 = s_1^2$, $\lambda \geq 2$, *M and B are the first* q_1 *columns and the last column of* D_1, *respectively,* D_2 *is an* $LHD(\lambda s_1^3, p)$=$(d_1, d_2, \ldots, d_p)$, *then the design* D^0=(D_1, D_2) *is a* $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)$, *if and only if, for* $1 \leq k \leq p$:
(i) $(m_i, d_k^{'})$ *is an* $OA(\lambda s_1^3, 2, s_1(\lambda s_1^2), 2)$, m_i *is the ith column of M,* $1 \leq i \leq q_1$;
(ii) $(B, d_k^{''})$ *is an* $OA(\lambda s_1^3, 2, s_2(\lambda s_1), 2)$;
(iii) $(m_i, m_j, d_k^{''})$ *is an* $OA(\lambda s_1^3, 3, s_1^2(\lambda s_1), 3)$,$1 \leq i \neq j \leq q_1$;
(iv) $(m_i, B, d_k^{'''})$ *is an* $OA(\lambda s_1^3, 3, s_1 s_2 \lambda, 3)$, $1 \leq i \leq q_1$.

Proposition 3. *When* D_1=(M, B) *is an* $OA(\lambda s_1^3, s_1^{q_1} s_2, 2)$, s_2=s_1^2, $\lambda \geq 2$, $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)$ *exists, if and only if* D_1 *can be divided into* λ $(s_1 \times 1)$-$ROA(s_1^3, s_1^{q_1} s_2, 2)'s$, *and M is a* $CROA(\lambda s_1^3, s_1^{q_1}, 2)$.

Inspired by Proposition 3, we give Algorithm 4 to construct a DCD, which can be used as an initial design for Algorithm 5.

Algorithm 4 Construction of $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)'s$

Step 1. Given $F_i = (M_i, B_i)$ is an $(s_1 \times 1)$-$ROA(s_1^3, s_1^{q_1} s_2, 2)$, where M_i is an $CROA(s_1^3, s_1^{q_1}, 2)$ and B_i is an $OA(s_1^3, s_2, 1)$, $1 \leq i \leq \lambda$, then let $D_1 = (F_1^T, F_2^T, \ldots, F_\lambda^T)^T$. Note that all the F_i's can be either the same or different.
Step 2. Let e_k be a random permutation of $(0, 1, \ldots, \lambda - 1)^T$, f_k be a random permutation of $((f_{k1} \otimes \mathbf{1}_{s_1^2})^T, (f_{k2} \otimes \mathbf{1}_{s_1^2})^T, \ldots, (f_{k\lambda} \otimes \mathbf{1}_{s_1^2})^T)^T$, where f_{ki} is a random arrangement of $(0, 1, \ldots, s_1 - 1)$, $i = 1, 2, \ldots, \lambda$.
Step 3. Let $d_k = \left(e_k \otimes \mathbf{1}_{s_1^3}\right) s_1^3 + f_k s_1^2 + \left(h_{k1}^T, h_{k2}^T, \ldots, h_{k\lambda s_1}^T\right)^T$,$1 \leq k \leq p$. Let $D_2 = (d_1, d_2, \ldots ,d_p)$. Here h_{ki} is a random arrangement of $\left(h_{ki1}^T, h_{ki2}^T, \ldots, h_{kis_1}^T\right)^T$ and h_{kij} is a random arrangement of $((j-1)s_1, \ldots, js_1 - 1)^T$, $1 \leq k \leq p$, $1 \leq i \leq \lambda s_1$, $1 \leq j \leq s_1$.
Step 4. Obtain $D^0 = (D_1, D_2)$.

Theorem 5. *The design* $D^0 = (D_1, D_2)$ *obtained by Algorithm 4 is a* $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)$, *where* D_1 *is an asymmetric* $OA(\lambda s_1^3, s_1^{q_1} s_2, 2)$, D_2 *is an* $LHD(\lambda s_1^3, p)$.

From Proposition 3, it is clear that Theorem 5 is true. According to Corollary 1 of [21], we have the following Corollary 3.

Corollary 3. *For the design* $D^0 = (D_1, D_2)$ *in Theorem 5, we have* $q_1 \leq s_1$.

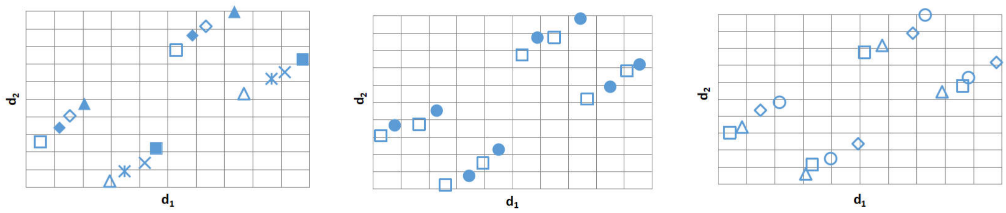
The proof of Theorem 4 is straightforward. In Algorithm 4, Step 1 is devoted to creating the D_1 satisfying the requirements in Proposition 3. The above steps produce $\lambda! \cdot (s_1!)^\lambda \cdot \lambda s_1 (s_1!)^{s_1}$ different quantitative columns. In other words, Algorithm 4 provides the DCDs with many quantitative factors. The following example gives an illustration of Algorithm 4.

Example 4. *Consider a (2×1)-resolvable orthogonal array* F_i=(M_i, B_i) *is an* $OA(8, 2^2 4, 2)$, *where* $s_1 = 2, s_2 = s_1^2 = 4$, $1 \leq i \leq \lambda$. *Let* $\lambda = 2, p = 3$, $F_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 3 & 2 & 0 & 3 & 1 \end{bmatrix}^T$, $F_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 2 & 3 & 1 & 2 & 0 \end{bmatrix}^T$. *According to Step 1, we obtain* D_1. *In Step 2 and Step 3, let* $e_1 = e_2 = e_3 = (0, 1)^T$, $f_1 = (0, 1)^T, f_2 = (1, 0)^T, f_3 = (0, 1)^T$, $h_{1i} = h_{2i} =$

$(0,1,2,3)^T, h_{3i} = (1,0,3,2)^T$, *for* $i = 1,2\ldots,4$. *Then we can obtain the corresponding* d_1, d_2, d_3 *in* D_2. *According to Step 4, a* $DCD(16, 2^2 4, 3)$ D^0 *is constructed as in Table 7. From Figure 4, we can verify that the rows in* D_2 *corresponding to each level combination of any one or two factors in* D_1 *form an LHD. Obviously, this satisfies the definition of DCD.*

Table 7. $DCD(16, 2^2 4, 3)$ $D^0 = (D_1, D_2)$ in Example 4, where $\lambda = 2$.

	D_{11}^T	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
D_1^T		0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
	D_{12}^T	0	2	1	3	2	0	3	1	1	3	0	2	3	1	2	0
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D_2^T		4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
		1	0	3	2	5	4	7	6	9	8	10	11	13	12	15	14



(**a**) (**b**) (**c**)

Figure 4. Projection of D_2 in Example 4. Scatter plots of d_1 versus d_2 in Example 4: (**a**) points represented by $\square$, $\Diamond$, $\triangle$, $\times$, * and solid $\square$, solid $\Diamond$, solid $\triangle$ correspond to the level combinations (0, 0), (0, 1), (0, 2), (0, 3) and (1, 0), (1, 1), (1, 2), (1, 3) of (m_1, B); (**b**) points marked by $\square$ and $\bullet$ correspond to the levels 0 and 1 of m_1; (**c**) points represented by $\square$, $\Diamond$, $\triangle$ and $\circ$ correspond to the levels 0, 1, 2 and 3 of B.

Next, we propose another algorithm to construct $BGDCD(u\lambda s_1^3, s_1^{q_1 \cdot c_1} s_2^{c_2}, pf)$ based on the DCD $D^0 = (D_1, D_2)$ constructed by Algorithm 4. Similar to Corollary 2, mixed-level difference schemes are used in Algorithm 5.

Algorithm 5 Construction of BGDCDs based on $(s_1 \times 1)$-ROAs

Step 1. Obtain a $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)$ $D^0 = (D_1, D_2)$ from Algorithm 4, where $D_1 = (D_{11}, D_{12})$ is an $OA(\lambda s_1^3, s_1^{q_1} s_2, 2)$, D_2 is an $LHD(\lambda s_1^3, p)$, s_2=s_1^2, D_{11} is an $OA(\lambda s_1^3, s_1^{q_1}, 2)$, and D_{12} is an $OA(\lambda s_1^3, s_2, 1)$.

Step 2. Obtain two designs $A = (D(1) \oplus D_{11}, D(2) \oplus D_{12})$ over $GF(s_1)$ and $GF(s_2)$, and $L = C \otimes D_2 + \lambda s_1^3 H \otimes \mathbf{1}_{\lambda s_1^3}$.

Step 3. Obtain a design $D = (A, L)$.

Theorem 6. *The design* $D = (A, L)$ *obtained by Algorithm 5 is a* $BGDCD(u\lambda s_1^3, s_1^{q_1 \cdot c_1} s_2^{c_2}, pf)$ *, where A is an asymmetric* $OA(u\lambda s_1^3, s_1^{q_1 \cdot c_1} s_2^{c_2}, 2)$, *L is an* $LHD(u\lambda s_1^3, pf)$.

The result of Theorem 6 just follows from the proofs of Propositions 1 and 2.

Remark 2. *In Algorithm 5, since* $D^0 = (D_1, D_2)$ *is a* $DCD(\lambda s_1^3, s_1^{q_1} s_2, p)$, (D_{11}, D_2) *is also a* $DCD(\lambda s_1^3, s_1^{q_1}, p)$. *Therefore* $(A_1, L) = ((D(1) \oplus D_{11}), L)$ *is a* $WGDCD(u\lambda s_1^3, s_1^{q_1 \cdot c_1}, pf)$, *i.e., subgroup* A_{1j} *of* A_1 *satisfies* (A_{1j}, L) *is a* $DCD(u\lambda s_1^3, s_1^{q_1}, pf)$, *for* $1 \leq j \leq c_1$.

Example 5. *Consider a design* $BGDCD(64, 2^{2 \cdot 4} 4^2, 4)$, *using the initial DCD* D^0 *as in Table 7, and delete the first column in* D_2 *(for saving space). By using the following difference schemes* $D(1)$ *and* $D(2)$, 4×2 *matrix C and any* $LHD(4,4)$ *H, we can obtain A and L in Algorithm 5, respectively. The resulting design* $D = (A, L)$ *is listed in Table 8. As we can verify that* (A_1, L) *is* *a* $WGDCD(64, 2^{2 \cdot 4}, 4)$, *where* A_1 *is an* $OA(64, 2^8, 2)$.

$$D(1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, D(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 2 & 3 & 1 \\ 1 & 1 & 1 & 2 \\ 3 & 3 & 2 & 0 \\ 2 & 0 & 0 & 3 \end{bmatrix}.$$

Table 8. $BGDCD(64, 2^{2 \cdot 4}4^2, 4)$ $D = (A, L)$ in Example 5.

Run	***A***										***L***				**Run**	***A***										***L***			
	A_1								A_2							A_1								A_2					
1	0	0	0	0	0	0	0	0	0	0	4	33	52	17	33	0	0	0	0	1	1	1	1	0	2	52	49	36	1
2	1	1	1	1	1	1	1	1	2	2	5	32	53	16	34	1	1	1	1	0	0	0	0	2	0	53	48	37	0
3	0	1	0	1	0	1	0	1	1	1	6	35	54	19	35	0	1	0	1	1	0	1	0	1	3	54	51	38	3
4	1	0	1	0	1	0	1	0	3	3	7	34	55	18	36	1	0	1	0	0	1	0	1	3	1	55	50	39	2
5	0	0	0	0	0	0	0	0	2	2	0	37	48	21	37	0	0	0	0	1	1	1	1	2	0	48	53	32	5
6	1	1	1	1	1	1	1	1	0	0	1	36	49	20	38	1	1	1	1	0	0	0	0	0	2	49	52	33	4
7	0	1	0	1	0	1	0	1	3	3	2	39	50	23	39	0	1	0	1	1	0	1	0	3	1	50	55	34	7
8	1	0	1	0	1	0	1	0	1	1	3	38	51	22	40	1	0	1	0	0	1	0	1	1	3	51	54	35	6
9	0	0	0	0	0	0	0	0	0	0	12	41	60	25	41	0	0	0	0	1	1	1	1	0	2	60	57	44	9
10	1	1	1	1	1	1	1	1	2	2	13	40	61	24	42	1	1	1	1	0	0	0	0	2	0	61	56	45	8
11	0	1	0	1	0	1	0	1	1	1	14	42	62	26	43	0	1	0	1	1	0	1	0	1	3	62	58	46	10
12	1	0	1	0	1	0	1	0	3	3	15	43	63	27	44	1	0	1	0	0	1	0	1	3	1	63	59	47	11
13	0	0	0	0	0	0	0	0	2	2	8	45	56	29	45	0	0	0	0	1	1	1	1	2	0	56	61	40	13
14	1	1	1	1	1	1	1	1	0	0	9	44	57	28	46	1	1	1	1	0	0	0	0	0	2	57	60	41	12
15	0	1	0	1	0	1	0	1	3	3	10	47	58	31	47	0	1	0	1	1	0	1	0	3	1	58	63	42	15
16	1	0	1	0	1	0	1	0	1	1	11	46	59	30	48	1	0	1	0	0	1	0	1	1	3	59	62	43	14
17	0	0	1	1	0	0	1	1	0	1	20	17	20	33	49	0	0	1	1	1	1	0	0	0	3	36	1	4	49
18	1	1	0	0	1	1	0	0	2	3	21	16	21	32	50	1	1	0	0	0	0	1	1	2	1	37	0	5	48
19	0	1	1	0	0	1	1	0	1	2	22	19	22	35	51	0	1	1	0	1	0	0	1	1	0	38	3	6	51
20	1	0	0	1	1	0	0	1	3	0	23	18	23	34	52	1	0	0	1	0	1	1	0	3	2	39	2	7	50
21	0	0	1	1	0	0	1	1	2	3	16	21	16	37	53	0	0	1	1	1	1	0	0	2	1	32	5	0	53
22	1	1	0	0	1	1	0	0	0	1	17	20	17	36	54	1	1	0	0	0	0	1	1	0	3	33	4	1	52
23	0	1	1	0	0	1	1	0	3	0	18	23	18	39	55	0	1	1	0	1	0	0	1	3	2	34	7	2	55
24	1	0	0	1	1	0	0	1	1	2	19	22	19	38	56	1	0	0	1	0	1	1	0	1	0	35	6	3	54
25	0	0	1	1	0	0	1	1	0	1	28	25	28	41	57	0	0	1	1	1	1	0	0	0	3	44	9	12	57
26	1	1	0	0	1	1	0	0	2	3	29	24	29	40	58	1	1	0	0	0	0	1	1	2	1	45	8	13	56
27	0	1	1	0	0	1	1	0	1	2	30	26	30	42	59	0	1	1	0	1	0	0	1	1	0	46	10	14	58
28	1	0	0	1	1	0	0	1	3	0	31	27	31	43	60	1	0	0	1	0	1	1	0	3	2	47	11	15	59
29	0	0	1	1	0	0	1	1	2	3	24	29	24	45	61	0	0	1	1	1	1	0	0	2	1	40	13	8	61
30	1	1	0	0	1	1	0	0	0	1	25	28	25	44	62	1	1	0	0	0	0	1	1	0	3	41	12	9	60
31	0	1	1	0	0	1	1	0	3	0	26	31	26	47	63	0	1	1	0	1	0	0	1	3	2	42	15	10	63
32	1	0	0	1	1	0	0	1	1	2	27	30	27	46	64	1	0	0	1	0	1	1	0	1	0	43	14	11	62

4. Comparison

In this section, the GDCDs presented in this paper, including WGDCDs and BGDCDs, are compared with the existing DCDs and MCDs in terms of qualitative factors.

Table 9 compares the qualitative factor designs of the WGDCDs constructed by Algorithm 1 with that of the DCDs constructed by [21]. As we can see in Table 9, as the run size of the design increases, the number of qualitative factors in WGDCDs also increases, but the number of qualitative factors in DCDs remains constant. Therefore, the WGDCDs constructed by Algorithm 1 can accommodate more equal-level qualitative factors than DCDs constructed by [21] under the same runs; details see Table 9. Moreover, the subdesigns for qualitative factors in the DCDs from [21] are all symmetric OAs. To this end, from Algorithm 4 we construct the DCDs with the qualitative factor subdesigns being asymmetric OAs.

On the other hand, we compare GDCDs, including WGDCDs and BGDCDs, with MCDs. Firstly, because the GDCDs have similar space-filling properties to DCDs, the GDCDs have better stratification properties between two qualitative factors and quantitative factors than the MCDs in [16–20]. Secondly, when the subdesigns for qualitative factors are mixed-level, Table 10 compares the qualitative factor designs of the GDCDs (WGDCDs and BGDCDs) with that of the MCDs. In [16] there is a relation in an MCD, i.e., $s_2 = \alpha s_1$, when the subdesign for qualitative factors is an $OA(n, s_1^{q_1} s_2^{q_2}, 2)$. When the constraint conditions ($s_2 = s_1^2$) are the same, the designs constructed by Algorithms 2, 4 and 5 have better space-filling properties between qualitative factors and quantitative factors than the MCDs in [16]. Especially, the relation, $s_2 = \alpha s_1$, is not required in the BGDCDs constructed by Algorithm 3. Therefore, the level types of the BGDCDs constructed by Algorithm 3 are more flexible than those of MCDs in [16].

Table 9. Comparisons between DCDs (D_1, D_2) in [21] and WGDCDs (A, L).

s	Run	D_1	A
2	16	$OA(16, 2^2, 2)$ *	$OA(16, 2^4, 2)$
	32	$OA(32, 2^2, 2)$ *	$OA(32, 2^8, 2)$
3	54	$OA(54, 3^3, 2)$ *	$OA(54, 3^9, 2)$
	162	$OA(162, 3^3, 2)$ *	$OA(162, 3^{15}, 2)$
4	128	$OA(128, 4^4, 2)$ *	$OA(128, 4^{16}, 2)$
	512	$OA(512, 4^4, 2)$ *	$OA(512, 4^{24}, 2)$
5	250	$OA(250, 5^5, 2)$ *	$OA(250, 5^{25}, 2)$
	1250	$OA(1250, 5^5, 2)$ *	$OA(1250, 5^{35}, 2)$
6	216	$OA(216, 6^2, 2)$ *	$OA(216, 6^4, 2)$
	1296	$OA(1296, 6^2, 2)$ *	$OA(1296, 6^{16}, 2)$

1 D_1: The subdesigns for qualitative factors in DCDs generated from [21]. Symbol * represents that the number of columns in D_1 reaches the upper bound. 2 A: The subdesigns for qualitative factors in WGDCDs constructed by Algorithm 1.

Table 10. Comparisons of GDCDs (WGDCDs and BGDCDs) with the existing MCDs.

s_1	s_2	$A^{(1)}$	$A^{(2)}$	$A^{(3)}$	$A^{(4)}$	$A^{(5)}$
2	3	-	-	$OA(72, 2^2 3^3, 2)$	-	-
2	4	$OA(32, 2^4 4^4, 2)$	$OA(32, 2^2 4^3, 2)$	$OA(64, 2^4 4^4, 2)$	$OA(16, 2^2 4, 2)$	$OA(64, 2^8 4^2, 2)$
2	5	-	-	$OA(200, 2^2 5^2, 2)$	-	-
2	6	$OA(72, 2^2 6^2, 2)$	-	$OA(144, 2^2 6^2, 2)$	-	-
2	7	-	-	$OA(392, 2^2 7^2, 2)$	-	-
2	8	$OA(128, 2^8 8^2, 2)$	-	$OA(256, 2^8 8^2, 2)$	-	-
3	4	-	-	$OA(288, 3^2 4^2, 2)$	-	-
3	5	-	-	$OA(450, 3^2 5^5, 2)$	-	-
3	6	$OA(108, 3^3 6^2, 2)$	-	$OA(216, 3^3 6^2, 2)$	-	-
3	7	-	-	$OA(882, 3^2 7^2, 2)$	-	-
3	8	-	-	$OA(1152, 3^2 8^2, 2)$	-	-
3	9	$OA(243, 3^5 9^2, 2)$	$OA(162, 3^3 9^8, 2)$	$OA(486, 3^5 9^2, 2)$	$OA(81, 3^3 9, 2)$	$OA(729, 3^{20} 9^9, 2)$
4	5	-	-	$OA(800, 4^2 5^2, 2)$	-	-
4	6	-	-	$OA(576, 4^2 6^2, 2)$	-	-
4	7	-	-	$OA(1568, 4^2 7^2, 2)$	-	-
4	8	$OA(256, 4^2 8^2, 2)$	-	$OA(512, 4^2 8^2, 2)$	-	-

1 $A^{(1)}$: The subdesigns for qualitative factors in MCDs generated from Construction 3 of [16]. 2 $A^{(2)}$: The subdesigns for qualitative factors in WGDCDs constructed by Algorithm 2. 3 $A^{(3)}$: The subdesigns for qualitative factors in BGDCDs constructed by Algorithm 3. 4 $A^{(4)}$: The subdesigns for qualitative factors in DCDs constructed by Algorithm 4. 5 $A^{(5)}$: The subdesigns for qualitative factors in BGDCDs constructed by Algorithm 5.

5. Conclusions and Future Research Directions

The existence of the interaction effects between any two qualitative factors and all quantitative factors in a computer experiment involving both qualitative and quantitative factors is very important for design selection. If no such effects exist, then an MCD is chosen; if the effects exist and the number of qualitative factors is no greater than the number of their levels, then a DCD is the best choice. When the number of qualitative factors exceeds the number of levels, neither an MCD nor a DCD can be used if some qualitative factors have the effects with quantitative factors and some do not. Inspired by this, we propose a new class of DCDs, namely GDCDs. A GDCD is an MCD, and the columns of some qualitative factors and all quantitative factors form a DCD. DCDs in [21] can only accommodate equal-level qualitative factors and the number of qualitative factors is also limited. Unlike DCDs, GDCDs can not only entertain more qualitative factors, but the qualitative factors can be either symmetric or asymmetric. In addition, GDCDs are equipped with better stratification properties between the qualitative and quantitative factors than the existing MCDs, whether the qualitative factors are symmetric or asymmetric.

In this paper, we propose two classes of GDCDs, namely WGDCDs and BGDCDs. While the algorithms for constructing WGDCDs and BGDCDs are similar and easy to implement, they differ in the initial DCDs used to construct the subdesign A. Four algorithms for constructing different GDCDs are provided. Algorithm 1 constructs WGDCDs based on the initial DCD $D^0 = (D_1, D_2)$, where the design of the qualitative factors D_1 is an $OA(n, s^q, 2)$. In contrast to DCDs in [21], whose number of qualitative factors is at most s, the WGDCDs obtained from Algorithm 1 can accommodate cs qualitative factors. The WGDCDs obtained from Algorithm 2 can entertain two different levels of qualitative factors, and the number of qualitative factors in each group can almost reach its bound. Similar to Algorithm 1, the designs obtained from Algorithm 2 with more qualitative factors can be further extended by using the difference schemes with mixed levels. Algorithm 3 not only constructs BGDCDs with two different levels of qualitative factors, but also constructs DCDs with mixed-level qualitative factors, and there is no multiple relation between s_1 and s_2. Algorithm 4 provides the initial DCDs that contain more mixed-level qualitative factors and are needed in Algorithm 5. The BGDCDs obtained by Algorithm 5 can also be realized as WGDCDs in the groups extended by D_{1i} being an OA. Moreover, according to the comparisons in Section 4, the GDCDs in this paper, including WGDCDs and BGDCDs, outperform both MCDs in [16–20] and DCDs in [21].

An interesting but challenging direction for future research is to construct initial DCDs with more qualitative factors possessing different levels. Another possible direction is to construct GDCDs with L having high-dimensional space-filling properties, such as 2 to 3 dimensions, or to consider adding column-orthogonality within or between groups. The construction of such GDCDs is not trivial and cannot be easily extended. We hope to investigate this and report the results in the near future.

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Appendix A. Proofs

Proof of Theorem 1. Since $D = (A, L)$ is a special MCD, then (i) follows from the Proposition 1 of [17]. For part (ii), according to the property of a WGDCD, we know that (A_i, L) is a DCD, therefore (ii) can be obtained from Theorem 1 of [21]. This completes the proof. □

Proof of Theorem 2. Since $D = (A, L)$ is a special MCD, then (i) follows from the Proposition 1 of [17]. For part (ii), when $A = (A_1, A_2, \ldots, A_c)$ is an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_c^{q_c}, 2)$, following the property of a BGDCD, i.e., the rows in L corresponding to each of $s_i s_m$ level combinations of any two factors in A_i and A_m, respectively, form an $LHD(n/s_i s_m, p)$. This means that each of all possible three-tuples in (A_i^j, A_m^n, l_k'') occurs only once. This completes the proof. □

From the Construction 3 of [16], we have the following result.

Lemma A1 ([16])**.** *Let $B = (B_1, \ldots, B_v)$ be an $OA(n, s_1^{q_1} s_2^{q_2} \cdots s_v^{q_v}, 2)$, where B_i is an $OA(n, s_i^{q_i}, 2)$, for $i = 1, 2, \ldots, v$, $D(j)$ be a difference scheme $D(u, c_j, s_j)$ (of strength 2), for $j = 1, 2, \ldots, v$, C be an $u \times f$ matrix with all elements being ones, H be an $LHD(u, pf)$, and M be an $LHD(n, p)$. If (B, M) is an $MCD(n, s_1^{q_1} s_2^{q_2} \cdots s_v^{q_v}, p)$, then (A, L) is an $MCD(nu, s_1^{q_1 c_1} s_2^{q_2 c_2} \cdots s_v^{q_v c_v}, pf)$, where $A = (D(1) \oplus B_1, \ldots, D(v) \oplus B_v)$ over the Galois field $GF(s_i)$, $i = 1, 2, \ldots, v$, and $L = C \otimes M + nH \otimes \mathbf{1}_n$.*

When $v = 1$, Lemma A2 follows from Lemma A1.

Lemma A2. *Let B be an $OA(n, s^q, 2)$, $D(1)$ be a difference scheme $D(u, c, s)$ (of strength 2), C be an $u \times f$ matrix of all 1's, H be an $LHD(u, pf)$, and M be an $LHD(n, p)$. If (B, M) is an $MCD(n, s^q, p)$, then (A, L) is an $MCD(un, s^{qc}, pf)$, where $A = D(1) \oplus B$ over $GF(s)$ and $L = C \otimes M + nH \otimes \mathbf{1}_n$.*

Lemma A3. *Let U be an $OA(ks^2, s^2, 2)$, V be an $LHD(ks^2, 1)$, $r = \xi s$, h be an $LHD(r, 1)$. η is a permutation of $(\mathbf{0}_\xi^T, \mathbf{1}_\xi^T, \ldots, \mathbf{s\text{-}1}_\xi^T)^T$ or $\eta = \mathbf{0}_.^T$. Let $T = \eta \oplus U$ over $GF(s)$, $W = \mathbf{1}_. \otimes V + (ks^2)h \otimes \mathbf{1}_{ks^2})$. If (U, V) is a $DCD(ks^2, s^2, 1)$, then (T, W) is a $DCD(rks^2, s^2, 1)$.*

Proof of Lemma A3. First, when $k = 1$, (T, W) is a $DCD(rks^2, s^2, 1)$. When $k = 1$, U is a full factorial design, i.e., if (a, b) is one row in U, then (a, b) occurs in U only once.

(i) According to Lemma A1, (T, W) is an MCD. From Lemma 1, (t_i, w') is an $OA(rs^2, s(rs), 2)$, where t_i is the ith column of T, $i = 1, 2$.

(ii) For $\alpha \in GF(s) = \{\alpha_0, \alpha_1, \ldots, \alpha_{s-1}\}, \alpha_0 = 0$, then $\alpha \oplus U$ over $GF(s)$ can be transformed into U via row permutations, hence, T is r replicates of U. Therefore, (T, W) can be transformed by row permutations into

$$\begin{pmatrix} U & V + \mathbf{0}_n \\ U & V + n\mathbf{1}_n \\ \vdots & \vdots \\ U & V + n(r-1)\mathbf{1}_n \end{pmatrix}, \tag{A1}$$

where $n = s^2$. For $i = 1, 2, \ldots, s^2$, let (a_i, b_i) be the ith row of U, v_i be the ith row of V, then following (1) above and $\begin{pmatrix} a_i & b_i & v_i + 0 \\ a_i & b_i & v_i + s^2 \\ \vdots & \vdots & \vdots \\ a_i & b_i & v_i + s^2(r-1) \end{pmatrix}$, we know that

$$\begin{pmatrix} a_i & b_i & (v_i + 0)'' \\ a_i & b_i & (v_i + s^2)'' \\ \vdots & \vdots & \vdots \\ a_i & b_i & (v_i + s^2(r-1))'' \end{pmatrix} = \begin{pmatrix} a_i & b_i & 0 \\ a_i & b_i & 1 \\ \vdots & \vdots & \vdots \\ a_i & b_i & r-1 \end{pmatrix}.$$

Since V is an $LHD(s^2, 1)$, for $v_i \in \{0, 1, 2, \ldots, s^2 - 1\}$, $((v_i + 0)'', (v_i + s^2)'', \ldots, (v_i + s^2(r-1))'')$=$(0, 1, \ldots, r-1)$.

Since U is a full factorial design, then (T, W'') is an $OA(rs^2, s^2(r), 3)$. According to Lemma A2, we know that (T, W) is a $DCD(rs^2, s^2, 1)$.

The second, when $k > 1$, according to the case $k = 1$, we need to show that (T, W) can be transformed by row permutations into

$$\begin{pmatrix} U^* & \tilde{V} + \mathbf{0}_{s^2} \\ U^* & \tilde{V} + s^2\mathbf{1}_{s^2} \\ \vdots & \vdots \\ U^* & \tilde{V} + (rk-1)s^2\mathbf{1}_{s^2} \end{pmatrix}, \tag{A2}$$

where U^* is a full factorial design, i.e., U^* is an $OA(s^2, s^2, 2)$, $\tilde{V}$ is an $LHD(s^2, 1)$. When $k > 1$, U is k replicates of U^*, then T is rk replicates of U^*. Following the definitions of T and W, (T, W) can be transformed into $\begin{pmatrix} \mathbf{1}_k \otimes U^* & V^* + \mathbf{0}_n \\ \mathbf{1}_k \otimes U^* & V^* + n\mathbf{1}_n \\ \vdots & \vdots \\ \mathbf{1}_k \otimes U^* & V^* + n(r-1)\mathbf{1}_n \end{pmatrix}$, where $n = ks^2$, $V^* = (0, 1, \ldots, ks^2 - 1)$. Since $V^* + in\mathbf{1}_n$=$(0 + in, 1 + in, \ldots, (ks^2 - 1) + in)$= $(0 + iks^2, 1 + iks^2, \ldots, (ks^2 - 1) + iks^2)$, $i = 0, 1, \ldots, r-1$, $\left((V^* + \mathbf{0}_n)^T, (V^* + n\mathbf{1}_n)^T, \ldots, (V^* + n(r-1)\mathbf{1}_n)^T\right)^T$ = $\left((\tilde{V} + \mathbf{0}_{s^2})^T, (\tilde{V} + s^2\mathbf{1}_{s^2})^T, \ldots, (\tilde{V} + s^2(rk-1)\mathbf{1}_{s^2})^T\right)^T$. From (i) and (ii), we know that (T, W) is a $DCD(rks^2, s^2, 1)$. This completes the proof. □

Proof of Proposition 1. (i) From Lemma A2, (D, L) is an $MCD(rn, s^{qc}, pf)$.

(ii) Let (a, b) be any two columns of A_i, $i = 1, 2, \ldots, c$, l be any column of L, then (a, b) can be represented as $(a, b) = d(1)_{(a,b)} \oplus D_{1_{(a,b)}}$ over $GF(s)$, where $d(1)_{(a,b)}$ is one column of $D(1)$ corresponding to (a, b), $D_{1(a,b)}$ are two columns in D_1 corresponding to (a, b), and l can be expressed as $l = \mathbf{1}_{.} \otimes D_{2(l)} + nh \otimes \mathbf{1}_n$, where $D_{2(l)}$ is one column of D_2 corresponding to l, h is $LHD(r, 1)$. Since D_1 is an $OA(n, s^2, 2)$, then there exist a k, such that $n = ks^2$. As $D(1)$ is difference scheme $D(r, c, s)$ of strength 2, and $d(1)_{(a,b)}$ is one column in $D(1)$, therefore, there exist g, such that $r = gs$, and $d(1)_{(a,b)}$ is a permutation of $(\mathbf{0}_g^T, \mathbf{1}_g^T, \ldots, \mathbf{s\text{-}1}_g^T)^T$ or $d(1)_{(a,b)} = \mathbf{1}_{.}^T$. Since $D^0 = (D_1, D_2)$ is a $DCD(n, s^q, p)$, $(D_{1(a,b)}, D_{2(l)})$ is a $DCD(n, s^2, 1)$, where $D_{1(a,b)}$ is an $OA(n, s^2, 2)$, $D_{2(l)}$ is an $LHD(n, 1)$.

According to Lemma A3, we know that (a, b, l) is a $DCD(rn, s^2, 1)$. From Lemma 2, (i) both (a, l') and (b, l') are $OA(rn, s(rn/s), 2)$; (ii) (a, b, l'') is an $OA(rn, s^2(rn/s^2), 2)$. According to the randomness of a, b, l, it can be checked that (A_i, L) is a $DCD(rn, s^q, pf)$, $i = 1, 2, \ldots, c$. From (i) and (ii) above, we know that $D = (A, L)$ is a $WGDCD(rn, s^{q \cdot c}, pf)$. This completes the proof. □

Proof of Theorem 3. First, it is easily to check that l_k is a permutation of $\{0, 1, \ldots, \lambda s_2^2 - 1\}$, hence L constructed by Algorithm 2 is an $LHD(\lambda s_2^2, p)$. Second, to prove that the design D is an MCD, without loss of generality, we need to show that (A_j, l_k') is an $OA(\lambda s_2^2, s_j^{q_j}(\lambda s_2^2/s_j)^1, 2)$, j=1,2, k=1,2,...,p. For j=2, according to $l_k = s_2^2(\pi_k \otimes \mathbf{1}_{s_2^2}) + s_2(\mathbf{1}_\lambda \otimes (g_k \otimes \mathbf{1}_{s_2})) + \mathbf{1}_{\lambda s_2} \otimes v_k$, $s_2 = s_1^2$, then l_k'=$\lfloor l_k/s_2 \rfloor = s_2(\pi_k \otimes \mathbf{1}_{s_2^2}) + \mathbf{1}_\lambda \otimes (g_k \otimes \mathbf{1}_{s_2})$. Next, we divide l_k' into λ parts, correspondingly, A_2 can be partitioned into λ parts, and since each part is a completely resolvable orthogonal array, (A_2, l_k') is an $OA(\lambda s_2^2, s_2^{q_2}(\lambda s_2)^1, 2)$. The proof of the case when j=1 is similar to the proof for j=2, and thus omit it. Therefore, the resulting design D is an $MCD(\lambda s_2^2, s_1^{s_1} s_2^{s_2-1}, p)$. Finally, on account of replacing levels $0, 1, \ldots, s_2 - 1$ of the column with the form $\mathbf{1}_{s_2} \otimes (0, 1, \ldots, s_2 - 1)^T$ in F_0 by s_1^2 level combinations in E_0 in order, and following the above proof of the case when j=2, we can verify that for j=1, (A_1, L) is a $DCD(\lambda s_2^2, s_1^{s_1}, p)$. Finally, it is easy to check that $(A_2, \lfloor l_k/s_2^2 \rfloor)$ is an $OA(\lambda s_2^2, s_2^{s_2-1}\lambda, 2)$, then (A_2, L) is a $DCD(\lambda s_2^2, s_2^{s_2-1}, p)$. This completes the proof. □

Proof of Proposition 2. (i) We show that $D^0 = (D_1, D_2)$ is a $DCD(\lambda s_1 s_2, s_1 s_2, p)$. The first, it can be easily checked that D_2 is an $LHD(\lambda s_1 s_2, p)$. Since (M, B) is an MCD, we denote m_1 and m_2 are the first and the second column of M, respectively, b_i is ith column of B, $1 \leq i \leq p$. Then we have $\left(m_1, \left\lfloor \frac{b_i}{s_1} \right\rfloor\right)$ and $\left(m_2, \left\lfloor \frac{b_i}{s_2} \right\rfloor\right)$ are $OA(s_1s_2, s_1^1s_2^1, 2)$ and $OA(s_1s_2, s_2^1s_1^1, 2)$, respectively. Second, to show that (D_1, D_2) is an MCD we only need to prove $\left(D_{11}^{\sim}, \left\lfloor \frac{d_i}{s_1} \right\rfloor\right)$ and $\left(D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_2} \right\rfloor\right)$ are $OA(\lambda s_1s_2, 2, s_1^1(\lambda s_2)^1, 2)$ and $OA(\lambda s_1s_2, 2, s_2^1(\lambda s_1)^1, 2)$, where $D_{11}^{\sim}$ and $D_{12}^{\sim}$ are also the first and the second column of D_1, respectively , d_i is ith column of D_2, $1 \le i \le p$. Due to $\left(D_{11}^{\sim}, \left\lfloor \frac{b_i}{s_1} \right\rfloor\right)$ can be represented as $(\alpha \otimes 1_{s_2}, 1_{s_1} \otimes \beta)$, $\alpha = (0, 1, \ldots, s_1 - 1)^T$, $\beta = (0, 1, \ldots, s_2 - 1)^T$, so that $\left(D_{11}^{\sim}, \left\lfloor \frac{d_i}{s_1} \right\rfloor\right) = \left(1_\lambda \otimes (\alpha \otimes 1_{s_2}), \gamma \oplus \left\lfloor \frac{d_i}{s_1} \right\rfloor\right)$, where $\gamma = (0, s_2, 2s_2, \ldots, (\lambda-1)s_2)^T$. After transferring $\left(D_{11}^{\sim}, \left\lfloor \frac{d_i}{s_1} \right\rfloor\right)$ into $(\alpha \otimes 1_{\lambda s_2}, 1_{s_1} \otimes \xi)$, where $\xi = (0, 1, 2, \ldots, \lambda s_2 - 1)^T$. Therefore $\left(D_{11}^{\sim}, \left\lfloor \frac{d_i}{s_1} \right\rfloor\right)$ is an $OA(\lambda s_1s_2, 2, s_1^1(\lambda s_2)^1, 2)$. Similarly, it can be checked that $\left(D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_2} \right\rfloor\right)$ is an $OA(\lambda s_1s_2, 2, s_2^1(\lambda s_1)^1, 2)$. Therefore (D_1, D_2) is an MCD. Finally, in order to prove (D_1, D_2) is a DCD, all that remains is for $\left(D_{11}^{\sim}, D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_1s_2} \right\rfloor\right)$ to be an $OA(\lambda s_1s_2, 3, s_1^1s_2^1\lambda^1, 3)$. $\left(D_{11}^{\sim}, D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_1s_2} \right\rfloor\right)$ can be represented as $(\mathbf{1}_\lambda \otimes (\alpha \otimes \mathbf{1}_{s_2}), \mathbf{1}_\lambda \otimes (\mathbf{1}_{s_1} \otimes \beta), \eta \oplus \mathbf{0}_{s_1s_2})$, where $\eta = (0, 1, \ldots, \lambda - 1)$, i.e.,

$$\left(D_{11}^{\sim}, D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_1s_2} \right\rfloor\right) = \begin{pmatrix} \alpha \otimes \mathbf{1}_{s_2} & \mathbf{1}_{s_1} \otimes \beta & \mathbf{0}_{s_1s_2} \\ \alpha \otimes \mathbf{1}_{s_2} & \mathbf{1}_{s_1} \otimes \beta & \mathbf{1}_{s_1s_2} \\ \vdots & \vdots & \vdots \\ \alpha \otimes \mathbf{1}_{s_2} & \mathbf{1}_{s_1} \otimes \beta & (\lambda-1)_{s_1s_2} \end{pmatrix}. \tag{A3}$$

It is easy to see that each of all possible three-tuples occurs exactly once in $\left(D_{11}^{\sim}, D_{12}^{\sim}, \left\lfloor \frac{d_i}{s_1s_2} \right\rfloor\right)$. Thus, (i) of Proposition 2 is true.

(ii) Since (D_1, D_2) is a $DCD(\lambda s_1s_2, s_1^1s_2^1, p)$, from Lemma A1, (A, L) is an $MCD(u\lambda s_1s_2, s_1^{c_1}s_2^{c_2}, pf)$. Let $A = (A_1, A_2)$, where A_i is an $OA(u\lambda s_1s_2, s_i^{c_i}, 2)$, for $i = 1, 2$. Let A_1^i and A_2^j be the ith and jth column of A_1 and A_2, respectively, for $i = 1, 2, \ldots, c_1$, $j = 1, 2, \ldots, c_2$. Next, we only prove that $\left(A_1^i, A_2^j, \left\lfloor \frac{l}{s_1s_2} \right\rfloor\right)$ is an $OA(u\lambda s_1s_2, s_1^1s_2^1(u\lambda)^1, 3)$, where l is any column of L, for $i = 1, 2, \ldots, c_1$, $j = 1, 2, \ldots, c_2$. Since $\frac{\lambda s_1s_2h \oplus \mathbf{1}_{\lambda s_1s_2}}{s_1s_2} = \lambda h \oplus \mathbf{1}_{\lambda s_1s_2} = h \oplus \lambda \mathbf{1}_{\lambda s_1s_2}$, where h is an $LHD(u, 1)$ in H corresponding l in L, from (3), $\left(A_1^i, A_2^j, \left\lfloor \frac{l}{s_1s_2} \right\rfloor\right)$ can be represented as $\left(A_1^i, A_2^j, \left\lfloor \frac{l}{s_1s_2} \right\rfloor\right) = (\mathbf{1}_u \otimes (\mathbf{1}_\lambda \otimes (\alpha \otimes \mathbf{1}_{s_2}, \mathbf{1}_{s_1} \otimes \beta)), \mathbf{1}_u \otimes (\eta \otimes \mathbf{1}_{s_1s_2}) + h \oplus \lambda \mathbf{1}_{\lambda s_1s_2}) = (\mathbf{1}_{u\lambda} \otimes (\alpha \otimes \mathbf{1}_{s_2}, \mathbf{1}_{s_2} \otimes \beta), \xi \oplus \mathbf{1}_{s_1s_2})$, where $\xi = (0, 1, \ldots, \lambda u - 1)^T$. Thus, $\left(A_1^i, A_2^j, \left\lfloor \frac{l}{s_1s_2} \right\rfloor\right)$ is an $OA(u\lambda s_1s_2, s_1^1s_2^1(u\lambda)^1, 3)$. This completes the proof. $\square$

Proof of Theorem 4. It is clear that the sufficiency is true. Next, we show the necessity, since the $D^0 = (D_1, D_2)$ is a $DCD(\lambda s_1^3, s_1^{q_1}s_2, p)$ and $D_1 = (M, B)$, the (M, D_2) and the (B, D_2) are a $DCD(\lambda s_1^3, s_1^{q_1}, p)$ and a $DCD(\lambda s_1^3, s_2, p)$, respectively. Thus, from Lemma 2, conditions (i), (ii), and (iii) are true. Let m_i be the ith column of M, $i = 1, 2, \ldots, q_1$. Thus, (m_i, B, D_2) is a $DCD(\lambda s_1^3, s_1^1s_2^1, p)$, for $i = 1, 2, \ldots, q_1$, since (D_1, D_2) is a $DCD(\lambda s_1^3, s_1^{q_1}s_2, p)$. For condition (iv), when (m_i, B, D_2) is a DCD, from the Definition of DCD, we have the rows in D_2 corresponding to each level combination between m_i and B form an $LHD(\lambda, p)$. This indicates that all possible three-tuples occur equally once. $\square$

Proof of Proposition 3. First, we show if a $DCD(\lambda s_1^3, s_1^{q_1}s_2, p)$ exists, then D_1=(M, B) can be partitioned into λ $(s_1 \times 1)$-$ROA(s_1^3, s_1^{q_1}s_2, 2)$'s. Since $DCD(\lambda s_1^3, s_1^{q_1}s_2, p)$ exists, so does $MCD(\lambda s^3, s_1^{q_1}s_2, p)$. Thus, M is a $CROA(\lambda s_1^3, s_1^{q_1}, 2)$ follows from Proposition 1 of [16]. Next, according to the condition (iv) in Theorem 6, the rows in D_1 can be split into λ $OA(s_1^3, s_1^{q_1}s_2, 2)$'s, say $F_1, F_2, \ldots, F_\lambda$. It remains to show that each F_i is an $(s_1 \times 1)$-$ROA(s_1^3, s_1^{q_1}s_2, 2)$, $i = 1, 2, \ldots, \lambda$,. From conditions (ii) and (iii) of Theorem 6, the rows in D_1 corresponding to each level of d_k'' is an $OA(s_1^2, s_1^{q_1}s_2, 1)$. Recall that the relationship $\left\lfloor \frac{d_k''}{s_1} \right\rfloor = d_k'''$, which implies that each level in

d_k''' corresponds to s_1 levels in d_k''. Hence, for each F_i corresponding to each level of d_k''' can be divided into s_1 $OA(s_1^2, s_1^{q_1}s_2, 1)$'s, and by definition, it is an $(s_1{\times}1)$-$ROA(s_1^3, s_1^{q_1}s_2, 2)$.

Second, we show if $D_1 = (M, B)$ can be partitioned into λ $(s_1{\times}1)$-$ROA(s_1^3, s_1^{q_1}s_2, 2)$'s, then a $DCD(\lambda s^3, s_1^{q_1}s_2, p)$ exists. Since D_1 can be represented as $\left(F_1^T, F_2^T, \ldots, F_\lambda^T\right)^T$, where $F_i = \left(F_{i1}^T, F_{i2}^T, \ldots, F_{is_1}^T\right)^T$=(M$_i$,B$_i$), F_i is an $(s_1{\times}1)$-$ROA(s_1^3, s_1^{q_1}s_2, 2)$, $s_2 = s_1^2, 1 \le i \le \lambda$. For each $F_{ij} = (M_{ij}, B_{ij})$ is an $OA(s_1^2, s_1^{q_1}s_2, 1)$, where M_{ij} is a $CROA(s_1^2, s_1^{q_1}, 2)$, B_{ij} is an $LHD(s_2, 1)$. Known that F_i is an $OA(s_1^3, s_1^{q_1}s_2, 2)$, so each level combination between any column of M_i and the column B_i occurs exactly once, and thus λ times in D_1. In order to verify the existence of a DCD, given such a D_1 as above, then construct a D_2. Denote d as a column of D_2, let $d = \left(h_1^T, h_2^T, \ldots, h_\lambda^T\right)^T$. Where $h_i = \left(h_{i1}^T, h_{i2}^T, \ldots, h_{is_1}^T\right)^T$, $h_{ij} = (i-1)s_1^3 + (j-1)s_2, \ldots, (i-1)s_1^3 + js_2 - 1$, for $i = 1, 2, \ldots, \lambda$, $j = 1, 2, \ldots, s_1$. Since $s_2 = s_1^2$, $h_{ij} = \left(h_{ij1}^T, h_{ij2}^T, \ldots, h_{ijs_1}^T\right)^T$, $h_{ijk} = (i-1)s_1^3 + (j-1)s_1^2 + (k-1)s_1, \ldots, (i-1)s_1^3 + (j-1)s_1^2 + ks_1 - 1$, for $i = 1, 2, \ldots, \lambda$, $j = 1, 2, \ldots, s_1$, $k = 1, 2, \ldots, s_1$. Finally, we show (D_1, d) is a $DCD(\lambda s_1^3, s_1^{q_1}s_2, 1)$. Obviously, d is an $LHD(\lambda s_1^3, 1)$. Since $\left\lfloor \frac{h_{ijk}}{s_1} \right\rfloor = \left[(i-1)s_1^2 + (j-1)s_1 + k - 1\right] \cdot \mathbf{1}_{s_1}$, then the rows in d' corresponding to each level of any column of M is an $LHD(\lambda s_1^2, 1)$. $\left\lfloor \frac{h_{ij}}{s_1^2} \right\rfloor = [(i-1)s_1 + (j-1)] \cdot \mathbf{1}_{s_1^2}$ reveals that corresponding to both each level combination of any two columns of M and each level of B, the rows of d'' forms an $LHD(\lambda s_1, 1)$. $\left\lfloor \frac{h_i}{s_1^3} \right\rfloor = (i-1) \cdot \mathbf{1}_{s_1^3}$ shows that for each level combination between any column of M and B with s$_1$s$_2$ level combinations, the corresponding λ entries in d''' are $\{0, 1, \ldots, \lambda - 1\}$, that is to say, they form an $LHD(\lambda, 1)$. Additionally, randomly permuting h_i in d, randomly permuting h_{ij} in h_i, and the entries in h_{ijk} means D_2 can accommodate more quantitative factors. This completes the proof. □

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