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Gauss' Second Theorem for ${}_2F_1(1/2)$-Series and Novel Harmonic Series Identities

Chunli Li [1] **and Wenchang Chu** [2,*]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; lichunlizk@outlook.com
[2] Department of Mathematics and Physics, University of Salento, 73100 Lecce, Italy
* Correspondence: wenchang.chu@unisalento.it

Abstract: Two summation theorems concerning the ${}_2F_1(1/2)$-series due to Gauss and Bailey will be examined by employing the "coefficient extraction method". Forty infinite series concerning harmonic numbers and binomial/multinomial coefficients will be evaluated in closed form, including eight conjectured ones made by Z.-W. Sun. The presented comprehensive coverage for the harmonic series of convergence rate "1/2" may serve as a reference source for readers.

Keywords: harmonic number; gamma function; binomial/multinomial coefficient

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1. Introduction and Outline

For an indeterminate x and $n \in \mathbb{N}_0$, the Pochhammer symbol is defined by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}.$$

It can be expressed in terms of the Γ-function

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{with} \quad \Gamma(x) = \int_0^\infty \tau^{x-1} e^{-\tau} d\tau \quad \text{for} \quad \Re(x) > 0.$$

For the sake of brevity, the Γ-function quotient will be abbreviated to

$$\Gamma\left[\begin{matrix}\alpha, \beta, \cdots, \gamma \\ A, B, \cdots, C\end{matrix}\right] = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

Denote the Euler constant by $\gamma = \lim_{n\to\infty}\left(\mathbf{H}_n - \ln n\right)$. Then, the logarithmic differentiation of the Γ-function results in the digamma function (cf. Rainville [1], §9)

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty}\frac{z-1}{(n+1)(n+z)}.$$

Let $[x^m]\varphi(x)$ stand for the coefficient of x^m in the formal power series $\varphi(x)$. For a real number $\lambda \notin \mathbb{Z}\backslash\mathbb{N}$, we can extract the coefficients

$$[x]\frac{\Gamma(\lambda-x)}{\Gamma(\lambda)} = -\psi(\lambda) \quad \text{and} \quad [x^2]\frac{\Gamma(\lambda-x)}{\Gamma(\lambda)} = \frac{\psi^2(\lambda)+\psi'(\lambda)}{2}$$

from the exponential expression

$$\frac{\Gamma(\lambda-x)}{\Gamma(\lambda)} = \exp\left\{-x\psi(\lambda) + \sum_{k=2}^{\infty}\frac{x^k}{k}\zeta_k(\lambda)\right\}, \tag{1}$$

where the Riemann and Hurwitz zeta functions are defined, respectively, by

$$\zeta(m)=\sum_{n=1}^{\infty}\frac{1}{n^m}\quad\text{and}\quad\zeta_m(z)=\frac{(-1)^m}{(m-1)!}\mathcal{D}_z^{m-1}\psi(z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^m}.$$

For $\lambda\in\mathbb{R}$ and $n\in\mathbb{N}_0$, define the parametric harmonic numbers by

$$\mathbf{H}_n^{\langle m\rangle}(\lambda):=\sum_{k=0}^{n-1}\frac{1}{(\lambda+k)^m}\quad\text{and}\quad\bar{\mathbf{H}}_n^{\langle m\rangle}(\lambda):=\sum_{k=0}^{n-1}\frac{(-1)^k}{(\lambda+k)^m}.$$

When $\lambda=1$ and $\lambda=\frac{1}{2}$, they reduce to the usual harmonic numbers

$$\begin{aligned}\mathbf{H}_n^{\langle m\rangle}&:=\mathbf{H}_n^{\langle m\rangle}(1), &\quad \mathbf{O}_n^{\langle m\rangle}&:=2^{-m}\mathbf{H}_n^{\langle m\rangle}(\tfrac{1}{2});\\ \bar{\mathbf{H}}_n^{\langle m\rangle}&:=\bar{\mathbf{H}}_n^{\langle m\rangle}(1), &\quad \bar{\mathbf{O}}_n^{\langle m\rangle}&:=2^{-m}\bar{\mathbf{H}}_n^{\langle m\rangle}(\tfrac{1}{2}).\end{aligned}$$

In case $m=1$, it will be suppressed from these notations. We record also the following simple, but useful, relations:

$$\begin{aligned}\mathbf{H}_{2n}^{\langle m\rangle}&=\mathbf{O}_n^{\langle m\rangle}+2^{-m}\mathbf{H}_n^{\langle m\rangle}, &\quad \mathbf{H}_n^{\langle m\rangle}(\tfrac{1}{2})&=2^m\mathbf{O}_n^{\langle m\rangle};\\ \bar{\mathbf{H}}_{2n}^{\langle m\rangle}&=\mathbf{O}_n^{\langle m\rangle}-2^{-m}\mathbf{H}_n^{\langle m\rangle}, &\quad \bar{\mathbf{H}}_n^{\langle m\rangle}(\tfrac{1}{2})&=2^m\bar{\mathbf{O}}_n^{\langle m\rangle}.\end{aligned}$$

The parametric harmonic number of the first order can be obtained by extracting the coefficient from the factorial quotient

$$\mathbf{H}_n(\lambda)=[x]\frac{(\lambda+x)_n}{(\lambda)_n}=[x]\frac{(\lambda)_n}{(\lambda-x)_n}.$$

By means of the generating function method, it can be shown without difficulty that in general, there hold the following formulae:

$$[x^m]\frac{(\lambda-x)_n}{(\lambda)_n}=\Omega_m\Big\{-\mathbf{H}_n^{\langle k\rangle}(\lambda)\Big\}\quad\text{and}\quad[x^m]\frac{(\lambda)_n}{(\lambda-x)_n}=\Omega_m\Big\{\mathbf{H}_n^{\langle k\rangle}(\lambda)\Big\}.\tag{2}$$

Here, the Bell polynomials (cf. [2], §3.3) are expressed by the multiple sum

$$\Omega_m\Big\{\pm\mathbf{H}_n^{\langle k\rangle}(\lambda)\Big\}=\sum_{\sigma(m)}\prod_{k=1}^{m}\frac{\big\{\pm\mathbf{H}_n^{\langle k\rangle}(\lambda)\big\}^{i_k}}{i_k!\,k^{i_k}},\tag{3}$$

and $\sigma(m)$ is the set of m-partitions represented by m-tuples $(i_1,i_2,\cdots,i_m)\in\mathbb{N}_0^m$ subject to the condition $\sum_{k=1}^{m}ki_k=m$. We sketch proofs of (2) for integrity. According to power series expansion of the logarithm function

$$\ln\Big(1-\frac{x}{\lambda+i}\Big)=-\sum_{k=1}^{\infty}\frac{x^k}{k(\lambda+i)^k},$$

we can manipulate the two factorial quotients

$$\begin{aligned}\frac{(\lambda-x)_n}{(\lambda)_n}&=\prod_{i=0}^{n-1}\Big\{1-\frac{x}{\lambda+i}\Big\}\\&=\exp\Big\{\sum_{i=0}^{n-1}\ln\Big(1-\frac{x}{\lambda+i}\Big)\Big\}\\&=\exp\Big\{-\sum_{k=1}^{\infty}\frac{x^k}{k}\mathbf{H}_n^{\langle k\rangle}(\lambda)\Big\},\end{aligned}\qquad\begin{aligned}\frac{(\lambda)_n}{(\lambda-x)_n}&=\prod_{i=0}^{n-1}\Big\{1-\frac{x}{\lambda+i}\Big\}^{-1}\\&=\exp\Big\{-\sum_{i=0}^{n-1}\ln\Big(1-\frac{x}{\lambda+i}\Big)\Big\}\\&=\exp\Big\{\sum_{k=1}^{\infty}\frac{x^k}{k}\mathbf{H}_n^{\langle k\rangle}(\lambda)\Big\}.\end{aligned}$$

Then the formulae in (2) follow by extracting the coefficient of x^m across the above two equations.

There exist numerous infinite series representations for π and related mathematical constants in the literature (cf. [3–11]). Some of them can be shown by means of the hypergeometric series approach. Even though many summation formulae (cf. [12–14]) have been found for the hypergeometric series, these with a half argument are quite rare. The two fundamental ones are due to Gauss (called Gauss' second summation theorem) and Bailey (cf. [12], §2.4):

$$ {}_2F_1\left[\begin{matrix} a,\ b \\ \frac{1+a+b}{2} \end{matrix}\Big|\frac{1}{2}\right] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(\frac{1+a+b}{2})_n}\left(\frac{1}{2}\right)^n = \Gamma\left[\begin{matrix} \frac{1}{2},\ \frac{1+a+b}{2} \\ \frac{1+a}{2},\ \frac{1+b}{2} \end{matrix}\right], \tag{4} $$

$$ {}_2F_1\left[\begin{matrix} a, 1-a \\ c \end{matrix}\Big|\frac{1}{2}\right] = \sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{n!\ (c)_n}\left(\frac{1}{2}\right)^n = \Gamma\left[\begin{matrix} \frac{c}{2},\ \frac{1+c}{2} \\ \frac{c+a}{2},\ \frac{1+c-a}{2} \end{matrix}\right]. \tag{5} $$

Following a recent work of the second author [15], we shall investigate infinite series involving harmonic numbers, by examining primarily Gauss' Formula (4) and secondarily Bailey's Formula (5). Several remarkable identities will be established, including eight conjectured ones made experimentally by Sun [16,17]. This will be fulfilled by utilizing the "coefficient extraction method" (cf. [5,18,19]). Considering that for a harmonic series of convergence rate "1/2", there are only a few existing formulae scattered in the literature up to now, so the relatively full coverage presented in this paper may serve as a reference source for readers.

In the next section, 17 infinite series with central binomial coefficients in numerators will be evaluated in closed form by exploring four cases of Gauss' second summation Formula (4). Then, a further 23 infinite series identities with central binomial coefficients in denominators will be derived, in Section 3, by examining two cases of (4) and one case of Bailey's Formula (5).

In order to ensure the accuracy of our computations, numerical tests for all the equations have been made by appropriately devised *Mathematica* commands.

2. Series with Binomial/Multinomial Coefficients in Numerators

By examining Gauss' second summation formula (4), we shall evaluate, in closed form, several infinite series involving harmonic numbers, and binomial/multinomial coefficients in numerators, including five conjectured series made recently by Sun [17].

2.1. $\boxed{a \to \frac{1}{2} + 2ax \ \&\ b \to \frac{1}{2} + 2bx}$

The corresponding series becomes

$$ \Gamma\left[\begin{matrix} \frac{1}{2}, 1+ax+bx \\ \frac{3}{4}+ax, \frac{3}{4}+bx \end{matrix}\right] = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n \frac{(\frac{1}{2}+2ax)_n(\frac{1}{2}+2bx)_n}{n!\ (1+ax+bx)_n}. $$

Both members of the above equation are analytic function of x in the neighborhood of $x=0$ and can be expanded into power series in x. Denoting by $\mathrm{A}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities.

First, letting $x=0$, the resulting coefficient $\mathrm{A}_0(a,b)$ evaluates an easier series:

$$ \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})} = \sum_{n=0}^{\infty}\binom{2n}{n}^2\left(\frac{1}{32}\right)^n. $$

The next coefficient $\mathrm{A}_1(a,b)$ gives rise to the identities below.

Theorem 1.

$$\text{(a)}\quad \frac{\sqrt{\pi}(\pi-4\ln 2)}{2\Gamma^2(\frac{3}{4})}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{H}_n}{32^n},$$
$$\text{(b)}\quad \frac{\sqrt{\pi}\ln 2}{4\Gamma^2(\frac{3}{4})}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n}{32^n}.$$

Proof. The first identity (a) can be found in [15] (§3.1), By comparing the coefficient $\mathrm{A}_1(a,b)$, we deduce the identity

$$\frac{\sqrt{\pi}(\pi-6\ln 2)}{2\Gamma^2(\frac{3}{4})}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{H}_n-4\mathbf{O}_n}{32^n}.$$

Then, by putting the above equality in conjunction with (a), we confirm the second identity (b), which is equivalent to the following one:

$$\frac{\Gamma^2(\frac{1}{4})\ln 2}{8\sqrt{\pi^3}}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n}{32^n}$$

conjectured recently by Sun [17] (Equation (2.18)). □

By examining the two coefficients $\mathrm{A}_2(1,-1)$ and $\mathrm{A}_2(1,\sqrt{-1})$, we can evaluate the following two series on quadratic harmonic numbers, where the first one was conjectured by Sun [17] (Equation (2.19)).

Theorem 2.

$$\text{(a)}\quad \frac{\sqrt{\pi}(\pi^2-8G)}{16\Gamma^2(\frac{3}{4})}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n^{\langle 2\rangle}}{32^n},$$
$$\text{(b)}\quad \frac{\sqrt{\pi}(5\pi^2-36\pi\ln 2+108\ln^2 2)}{12\Gamma^2(\frac{3}{4})}=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{H}_n^{\langle 2\rangle}+(\mathbf{H}_n-4\mathbf{O}_n)^2}{32^n}.$$

In general, the series corresponding to coefficients of higher powers of x become more complicated. For instance, from $\mathrm{A}_3(1,-1)$ and $\mathrm{A}_3(1,\sqrt{-1})$, we have two further formulae.

Proposition 1.

$$\text{(a)}\quad \frac{\sqrt{\pi}}{32\Gamma^2(\frac{3}{4})}\Big\{56\zeta(3)+48G\ln 2-\pi^3-6\pi^2\ln 2-8\pi G\Big\}$$
$$=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{\mathbf{O}_n^{\langle 2\rangle}(\mathbf{H}_n-4\mathbf{O}_n)+4\mathbf{O}_n^{\langle 3\rangle}}{32^n},$$
$$\text{(b)}\quad \frac{\sqrt{\pi}}{8\Gamma^2(\frac{3}{4})}\Big\{240\zeta(3)-5\pi^3-30\pi^2\ln 2+108\pi\ln^2 2-216\ln^3 2\Big\}$$
$$=\sum_{n=1}^{\infty}\binom{2n}{n}^2\frac{(\mathbf{H}_n-4\mathbf{O}_n)^3+3\mathbf{O}_n^{\langle 2\rangle}(\mathbf{H}_n-4\mathbf{O}_n)+2\mathbf{H}_n^{\langle 3\rangle}+64\mathbf{O}_n^{\langle 3\rangle}}{32^n}.$$

2.2. $\boxed{a\to\frac{1}{3}+2ax\ \&\ b\to\frac{2}{3}+2bx}$

The corresponding series becomes

$$\Gamma\left[\begin{matrix}\frac{1}{2},1+ax+bx\\ \frac{2}{3}+ax,\frac{5}{6}+bx\end{matrix}\right]=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{(\frac{1}{3}+2ax)_n(\frac{2}{3}+2bx)_n}{n!\,(1+ax+bx)_n}.$$

Both members of the above equation are analytic function of x in the neighborhood of $x = 0$ and can be expanded into power series in x. Denoting by $\mathrm{B}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities.

The initial coefficient $\mathrm{B}_0(a,b)$ yields the following formula:

$$\frac{\sqrt{\pi}}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = \sum_{n=0}^{\infty} \binom{3n}{n,n,n} \left(\frac{1}{54}\right)^n.$$

Theorem 3.

$$\text{(a)} \quad \frac{\sqrt{\pi}(2\pi\sqrt{3} - 9\ln 3)}{6\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = \sum_{n=1}^{\infty} \binom{3n}{n,n,n} \frac{\mathbf{H}_n}{54^n},$$

$$\text{(b)} \quad \frac{\sqrt{\pi}(2\pi\sqrt{3} + 6\ln 2 - 9\ln 3)}{18\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = \sum_{n=1}^{\infty} \binom{3n}{n,n,n} \frac{\mathbf{H}_{3n}}{54^n}.$$

Proof. The identity (a) can be found in [15] (§4.1). By considering the coefficient $\mathrm{B}_1(1,1)$, we have the next formula

$$\frac{\sqrt{\pi}(2\pi\sqrt{3} - 6\ln 2 - 9\ln 3)}{6\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} = \sum_{n=1}^{\infty} \binom{3n}{n,n,n} \frac{2\mathbf{H}_n - 3\mathbf{H}_{3n}}{54^n}.$$

Then, combining this one with (a) leads us to identity (b). The two formulae in this theorem can be considered as refinements of the following conjectured identity made by Sun [17] (Equation (2.20)):

$$\frac{3\ln 2\,\Gamma^3(\frac{1}{3})}{4\pi^2\sqrt[3]{2}} = \sum_{n=1}^{\infty} \binom{3n}{n,n,n} \frac{3\mathbf{H}_{3n} - \mathbf{H}_n}{54^n}.$$

□

Two further series can be evaluated by examining the coefficients $\mathrm{B}_2(1,1)$ and $\mathrm{B}_3(1,1)$ as in the following proposition.

Proposition 2.

$$\text{(a)} \quad \frac{\sqrt{\pi}}{12\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} \left\{6\pi^2 + 3\ln^2(108) - 4\pi\sqrt{3}\ln(108) - 3\zeta_2(\tfrac{2}{3}) - 3\zeta_2(\tfrac{5}{6})\right\}$$
$$= \sum_{n=1}^{\infty} \binom{3n}{n,n,n} \frac{(2\mathbf{H}_n - 3\mathbf{H}_{3n})^2 + (2\mathbf{H}_n^{\langle 2\rangle} - 9\mathbf{H}_{3n}^{\langle 2\rangle})}{54^n},$$

$$\text{(b)} \quad \frac{\sqrt{\pi}}{48\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})} \left\{ \begin{array}{c} 40\pi^3\sqrt{3} - 36\pi^2\ln(108) + 12\pi\sqrt{3}\ln^2(108) - 6\ln^3(108) \\ -1152\zeta(3) - 3[4\pi\sqrt{3} - 3\ln(11664)][\zeta_2(\frac{2}{3}) + \zeta_2(\frac{5}{6})] \end{array} \right\}$$
$$= \sum_{n=1}^{\infty} \frac{1}{54^n} \binom{3n}{n,n,n} \left\{ \begin{array}{c} (2\mathbf{H}_n - 3\mathbf{H}_{3n})^3 + 2(2\mathbf{H}_n^{\langle 3\rangle} - 27\mathbf{H}_{3n}^{\langle 3\rangle}) \\ +3(2\mathbf{H}_n - 3\mathbf{H}_{3n})(2\mathbf{H}_n^{\langle 2\rangle} - 9\mathbf{H}_{3n}^{\langle 2\rangle}) \end{array} \right\}.$$

2.3. $\boxed{a \to \frac{1}{4} + 2ax \;\&\; b \to \frac{3}{4} + 2bx}$

The corresponding series becomes

$$\Gamma\left[\begin{array}{c} \frac{1}{2}, 1 + ax + bx \\ \frac{5}{8} + ax, \frac{7}{8} + bx \end{array}\right] = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{(\frac{1}{4} + 2ax)_n (\frac{3}{4} + 2bx)_n}{n!\,(1 + ax + bx)_n}.$$

Both members of the above equation are analytic functions of x in the neighborhood of $x = 0$ and can be expanded into power series in x. Denoting by $\mathrm{C}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities.

The initial coefficient $\mathrm{C}_0(a,b)$ results in the following identity:

$$\frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} = \sum_{n=0}^{\infty} \binom{4n}{n,n,2n} \left(\frac{1}{128}\right)^n.$$

Then, we have three independent series related to $\mathrm{C}_1(a,b)$.

Theorem 4.

$$\text{(a)} \quad \frac{\sqrt{\pi}(\pi\sqrt{2} - 6\ln 2)}{2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \frac{\mathbf{H}_n}{128^n},$$

$$\text{(b)} \quad \frac{\ln 2\sqrt{\pi}}{4\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \frac{\mathbf{O}_{2n}}{128^n},$$

$$\text{(c)} \quad \frac{\sqrt{\pi}\{\pi - \sqrt{2}\ln(3+2\sqrt{2})\}}{8\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \frac{\bar{\mathbf{O}}_{2n}}{128^n}.$$

Proof. For the first series (a), refer to Chu [15] (§4.1). By examining the coefficient $\mathrm{C}_1(1,1)$, we obtain the next formula

$$\frac{\sqrt{\pi}(\pi\sqrt{2} - 8\ln 2)}{2\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \frac{\mathbf{H}_n - 4\mathbf{O}_{2n}}{128^n}.$$

By combining this with (a), we derive (b), which was conjectured by Sun [17] (Equation (2.22)). Finally the identity (c) corresponds to the coefficient $\mathrm{C}_1(1,-1)$. □

Analogously, two further identities can be shown by considering $\mathrm{C}_2(1,1)$ and $\mathrm{C}_2(1,-1)$.

Proposition 3.

$$\begin{aligned} \text{(a)} \quad & \frac{\sqrt{\pi}}{12\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\left\{8\pi^2 + 192\ln^2 2 - 48\pi\sqrt{2}\ln 2 - 3\zeta_2(\tfrac{5}{8}) - 3\zeta_2(\tfrac{7}{8})\right\} \\ & = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \left(\frac{1}{128}\right)^n \left\{\begin{matrix} \mathbf{H}_n^{\langle 2\rangle} - 16\mathbf{O}_{2n}^{\langle 2\rangle} + 16\mathbf{O}_{2n}^2 \\ +\mathbf{H}_n(\mathbf{H}_n - 8\mathbf{O}_{2n}) \end{matrix}\right\}, \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \frac{\sqrt{\pi}}{64\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\left\{\pi^2 + 2\ln^2(3+2\sqrt{2}) - 2\pi\sqrt{2}\ln(3+2\sqrt{2}) - \zeta_2(\tfrac{5}{8}) - \zeta_2(\tfrac{7}{8})\right\} \\ & = \sum_{n=1}^{\infty} \binom{4n}{n,n,2n} \frac{\bar{\mathbf{O}}_{2n}^2 - \mathbf{O}_{2n}^{\langle 2\rangle}}{128^n}. \end{aligned}$$

2.4. $\boxed{a \to \frac{1}{6} + 2ax \ \&\ b \to \frac{5}{6} + 2bx}$

The corresponding series becomes

$$\Gamma\left[\begin{matrix} \frac{1}{2}, 1 + ax + bx \\ \frac{7}{12} + ax, \frac{11}{12} + bx \end{matrix}\right] = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \frac{(\frac{1}{6} + 2ax)_n(\frac{5}{6} + 2bx)_n}{n!\,(1 + ax + bx)_n}.$$

Both members of the above equation are analytic function of x in the neighborhood of $x = 0$ and can be expanded into power series in x. Denoting by $\mathrm{D}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities.

The initial coefficient $\mathrm{D}_0(a,b)$ gives rise to the formula below.

$$\frac{\sqrt{\pi}}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}=\sum_{n=0}^{\infty}\binom{6n}{n,2n,3n}\Big(\frac{1}{864}\Big)^n.$$

Then, we have two independent series evaluations.

Theorem 5.

$$\text{(a)}\quad \frac{\sqrt{\pi}\{2\pi-\ln(432)\}}{2\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}=\sum_{n=1}^{\infty}\binom{6n}{n,2n,3n}\frac{\mathbf{H}_n}{864^n},$$

$$\text{(b)}\quad \frac{\sqrt{\pi}\ln 2}{2\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}=\sum_{n=1}^{\infty}\binom{6n}{n,2n,3n}\frac{3\mathbf{O}_{3n}-\mathbf{O}_n}{864^n}.$$

Proof. The first series (a) is due to the second author [15] (§4.1). According to the coefficient $\mathrm{D}_1(1,1)$, we can derive another formula

$$\frac{\sqrt{\pi}\{2\pi-\ln(1728)\}}{2\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}=\sum_{n=1}^{\infty}\binom{6n}{n,2n,3n}\frac{\mathbf{H}_n+2\mathbf{O}_n-6\mathbf{O}_{3n}}{864^n}.$$

Then by combining (a) with the above identity, we find the identity (b), which was conjectured by Sun [17] (Equation (2.27)). □

3. Series with Central Binomial Coefficients in Denominators

By employing Gauss' second theorem (4) and then Bailey's theroem (5), we shall establish several summation formulae concerning harmonic numbers, and central binomial coefficients in denominators. Three of them were previously conjectured (without proofs) by Sun [16] (2014).

3.1. $\boxed{a\to 2ax\ \&\ b\to 2bx}$

The corresponding series in (4) becomes

$$\Gamma\left[\begin{matrix}\frac{1}{2},\frac{1}{2}+ax+bx\\ \frac{1}{2}+ax,\frac{1}{2}+bx\end{matrix}\right]=\sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n\frac{(2ax)_n(2bx)_n}{n!\,(\frac{1}{2}+ax+bx)_n}.$$

Both members of the above equation are analytic function of x in the neighborhood of $x=0$ and can be expanded into power series in x. Denoting by $\mathrm{U}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities.

The summation formula corresponding to the coefficient $\mathrm{U}_2(a,b)$ reads as

$$\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{2^n}{n^2}\binom{2n}{n}^{-1}.$$

By examining the coefficients $\mathrm{U}_3(a,b)$ and $\mathrm{U}_4(1,-1)$, we derive the next two identities, where the former one was conjectured by Sun [16] (Equation (1.14)).

Theorem 6.

$$\text{(a)}\quad \frac{7}{8}\zeta(3)=\sum_{n=1}^{\infty}\frac{2^n}{n^3}\binom{2n}{n}^{-1}\Big\{1-n\mathbf{H}_n+n\mathbf{O}_n\Big\},$$

$$\text{(b)}\quad \frac{\pi^4}{384}=\sum_{n=1}^{\infty}\frac{2^n}{n^4}\binom{2n}{n}^{-1}\Big\{n^2\mathbf{H}_n^{\langle 2\rangle}-1\Big\}.$$

We have also two further summation formulae of infinite series.

Proposition 4.

$$\text{(a)}\quad \frac{3\pi^4}{128}=\sum_{n=1}^{\infty}\frac{2^n}{n^4}\binom{2n}{n}^{-1}\Big\{1+n^2\mathbf{O}_n^{\langle 2\rangle}+n(\mathbf{H}_n-\mathbf{O}_n)(n\mathbf{H}_n-n\mathbf{O}_n-2)\Big\},$$

$$\text{(b)}\quad \frac{62\zeta(5)-7\pi^2\zeta(3)}{64}=\sum_{n=1}^{\infty}\frac{2^n}{n^5}\binom{2n}{n}^{-1}\left\{\begin{matrix}2-n\mathbf{H}_n+n\mathbf{O}_n-n^2\mathbf{H}_n^{\langle 2\rangle}\\ -n^3(\mathbf{H}_n^{\langle 3\rangle}-\mathbf{H}_n\mathbf{H}_n^{\langle 2\rangle}+\mathbf{H}_n^{\langle 2\rangle}\mathbf{O}_n)\end{matrix}\right\}.$$

Proof. The first identity (a) follows directly by extracting coefficient $\mathrm{U}_4(1,\sqrt{-1})$. Instead, the identity (b) is confirmed by evaluating the limit of $\frac{\mathrm{U}_5(a,b)}{a+b}$ as $a\to 1$ and $b\to -1$. □

3.2. $\boxed{a\to 1+2ax\ \&\ b\to 1+2bx}$

The corresponding series in (4) becomes

$$\Gamma\begin{bmatrix}\frac{1}{2},\frac{3}{2}+ax+bx\\ 1+ax,1+bx\end{bmatrix}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{(1+2ax)_n(1+2bx)_n}{n!\,(\frac{3}{2}+ax+bx)_n}.$$

Letting $\mathrm{V}_m(a,b)$ be the coefficient of x^m across the equation, we can derive the following infinite series identities. The initial one related to $\mathrm{V}_0(a,b)$ is given by

$$\frac{\pi}{2}=\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)\binom{2n}{n}}.$$

Next, by considering $\mathrm{V}_1(1,1)$, $\mathrm{V}_2(1,-1)$ and $\mathrm{V}_4(1,-1)$, we can derive the following three elegant summation formulae, where (b) can be found in Sun [20].

Theorem 7.

$$\text{(a)}\quad \frac{\pi\ln 2}{2}=\sum_{n=0}^{\infty}\frac{2^n(\mathbf{O}_{n+1}-\mathbf{H}_n)}{(2n+1)\binom{2n}{n}},$$

$$\text{(b)}\quad \frac{\pi^3}{48}=\sum_{n=1}^{\infty}\frac{2^n\,\mathbf{H}_n^{\langle 2\rangle}}{(2n+1)\binom{2n}{n}},$$

$$\text{(c)}\quad \frac{\pi^5}{1920}=\sum_{n=1}^{\infty}\frac{2^n\big\{(\mathbf{H}_n^{\langle 2\rangle})^2-\mathbf{H}_n^{\langle 4\rangle}\big\}}{(2n+1)\binom{2n}{n}}.$$

Two further identities are recorded in the next proposition, that are deduced by the limiting case $a\to 1,\ b\to -1$ of $\frac{\mathrm{V}_3(a,b)}{a+b}$, and the coefficient $\mathrm{V}_6(1,-1)$, respectively.

Proposition 5.

$$\text{(a)}\quad \frac{3\pi\zeta(3)+\pi^3\ln 2}{48}=\sum_{n=1}^{\infty}\frac{2^n\big\{\mathbf{H}_n^{\langle 2\rangle}(\mathbf{O}_{n+1}-\mathbf{H}_n)+\mathbf{H}_n^{\langle 3\rangle}\big\}}{(2n+1)\binom{2n}{n}},$$

$$\text{(b)}\quad \frac{\pi^7}{107{,}520}=\sum_{n=1}^{\infty}\frac{2^n\big\{(\mathbf{H}_n^{\langle 2\rangle})^3-3\mathbf{H}_n^{\langle 2\rangle}\mathbf{H}_n^{\langle 4\rangle}+2\mathbf{H}_n^{\langle 6\rangle}\big\}}{(2n+1)\binom{2n}{n}}.$$

3.3. Bailey's ${}_2F_1(\frac{1}{2})$-Series

Under the replacements $a\to 2ax$ and $c\to\frac{1}{2}+2bx$, we can reformulate Bailey's formula (5) as

$$\Gamma\begin{bmatrix}\frac{1}{4}+bx,\frac{3}{4}+bx\\ \frac{1}{4}+ax+bx,\frac{3}{4}-ax+bx\end{bmatrix}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\frac{(2ax)_n(1-2ax)_n}{n!\,(\frac{1}{2}+2bx)_n}.$$

Both members of the above equation are analytic function of x in the neighborhood of $x = 0$ and can be expanded into power series in x. Denoting by $\mathrm{W}_m(a,b)$ the coefficient of x^m across the equation, we derive the following infinite series identities, which may serve as complementary results to those obtained recently by the second author [15]. Since there are more summation formulae in this subsection, we exhibit them in groups according to the similarity of their summands.

First, by considering the coefficients $[a^2]\mathrm{W}_1(a,b)$ and $[ab]\mathrm{W}_1(a,b)$, we immediately obtain two identities, where G is the Catalan constant as usual.

Theorem 8.

$$\text{(a)}\quad \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\binom{2n}{n}^{-1},$$

$$\text{(b)}\quad 2G = \sum_{n=1}^{\infty} \frac{2^n}{n}\binom{2n}{n}^{-1}\mathbf{O}_n.$$

Then we have closed formulae below for three independent series.

Theorem 9.

$$\text{(a)}\quad \frac{7}{2}\zeta(3) - \pi G = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\binom{2n}{n}^{-1}\mathbf{O}_n,$$

$$\text{(b)}\quad \pi G + \frac{\pi^2}{8}\ln 2 - \frac{35}{16}\zeta(3) = \sum_{n=1}^{\infty} \frac{2^n}{n^3}\binom{2n}{n}^{-1},$$

$$\text{(c)}\quad \frac{7}{16}\zeta(3) - 2\pi G = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\binom{2n}{n}^{-1}\mathbf{H}_n.$$

We remark that this theorem refines the following two identities conjectured by Sun [16] (Equations (1.15) and (1.16)):

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}\binom{2n}{n}^{-1}\Big\{4 - 4n\mathbf{H}_n + 3n\mathbf{O}_n\Big\} = \pi G,$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}\binom{2n}{n}^{-1}\Big\{3n\mathbf{H}_n - n\mathbf{O}_n - 1\Big\} = \frac{\pi^2}{4}\ln 2.$$

Proof. The first identity (a) is easiest, which follows simply by extracting the coefficient $[a^2b]\mathrm{W}_2(a,b)$. Instead, the second identity (b) is not deducible by hypergeometric series approach. We offer the following integration proof.

Recalling the power series expansion

$$2\arcsin^2 y = \sum_{n=1}^{\infty} \frac{(2y)^{2n}}{n^2\binom{2n}{n}},$$

we can manipulate the integrals

$$\sum_{n=1}^{\infty} \frac{2^n}{n^3}\binom{2n}{n}^{-1} = 4\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin^2 y}{y}dy \quad \boxed{y \to \sin x}$$

$$= 4\int_0^{\frac{\pi}{4}} x^2 \cot x dx = -\frac{\pi^2}{8}\ln 2 - 8\int_0^{\frac{\pi}{4}} x\ln(\sin x)dx.$$

By making use of the Fourier series expansion

$$\ln(\sin x) = -\ln 2 - \sum_{k=1}^{\infty} \frac{\cos(2kx)}{k}, \text{ where } 0 < x < \pi,$$

we can evaluate the integral

$$\begin{aligned}\int_0^{\frac{\pi}{4}} x\ln(\sin x)dx &= -\frac{\pi^2}{32}\ln 2 - \sum_{k=1}^{\infty}\left\{\frac{x\sin(2kx)}{2k^2} + \frac{\cos(2kx)}{4k^3}\right\}\Bigg|_0^{\frac{\pi}{4}} \\ &= \frac{35\zeta(3)}{128} - \frac{\pi^2\ln 2}{32} - \frac{\pi G}{8}.\end{aligned}$$

By making substitution and simplifications, we find the identity (b).

Finally, the identity (c) follows from a linear combination of (a), (b) as well as that displayed in Theorem 6 (a). □

Now, we give three pairs of similar summation formulae. The first pair of series are evaluated in closed form by extracting the coefficient $[ab^2]\mathrm{W}_2(a,b)$ and $[a^2b^2]\mathrm{W}_3(a,b)$.

Theorem 10.

$$\text{(a)} \quad \frac{\pi^3}{8} = \sum_{n=1}^{\infty} \frac{2^n}{n}\binom{2n}{n}^{-1}\left\{\mathbf{O}_n^{\langle 2\rangle} + \mathbf{O}_n^2\right\},$$

$$\text{(b)} \quad \frac{\pi^4 - 64G^2}{16} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}\binom{2n}{n}^{-1}\left\{\mathbf{O}_n^{\langle 2\rangle} + \mathbf{O}_n^2\right\}.$$

The following two similar series are evaluated in closed form by extracting the coefficients $[a^3]\mathrm{W}_2(a,b)$ and $[a^4]\mathrm{W}_3(a,b)$.

Theorem 11.

$$\text{(a)} \quad \frac{\pi^3}{48} = \sum_{n=1}^{\infty} \frac{2^n}{n^3}\binom{2n}{n}^{-1}\left\{n^2\mathbf{H}_n^{\langle 2\rangle} - 1\right\},$$

$$\text{(b)} \quad \frac{\pi^4}{384} = \sum_{n=1}^{\infty} \frac{2^n}{n^4}\binom{2n}{n}^{-1}\left\{n^2\mathbf{H}_n^{\langle 2\rangle} - 1\right\}.$$

By examining the coefficients $[a^5]\mathrm{W}_4(a,b)$ and $[a^6]\mathrm{W}_5(a,b)$, we establish two identities as in the next theorem.

Theorem 12.

$$\text{(a)} \quad \frac{\pi^5}{1920} = \sum_{n=1}^{\infty} \frac{2^n}{n^5}\binom{2n}{n}^{-1}\left\{2 - 2n^2\mathbf{H}_n^{\langle 2\rangle} + n^4(\mathbf{H}_n^{\langle 2\rangle})^2 - n^4\mathbf{H}_n^{\langle 4\rangle}\right\},$$

$$\text{(b)} \quad \frac{\pi^6}{23{,}040} = \sum_{n=1}^{\infty} \frac{2^n}{n^6}\binom{2n}{n}^{-1}\left\{2 - 2n^2\mathbf{H}_n^{\langle 2\rangle} + n^4(\mathbf{H}_n^{\langle 2\rangle})^2 - n^4\mathbf{H}_n^{\langle 4\rangle}\right\}.$$

Finally we record three seemingly unrelated series, which correspond to the coefficients $[ab^3]\mathrm{W}_3(a,b)$, $[a^3b]\mathrm{W}_3(a,b)$ and $[a^3b^2]\mathrm{W}_4(a,b)$, respectively.

Proposition 6.

(a) $$\frac{3\zeta_4(\frac{1}{4})-3\zeta_4(\frac{3}{4})}{64}=\sum_{n=1}^{\infty}\frac{2^n}{n}\binom{2n}{n}^{-1}\Big\{\mathbf{O}_n^3+2\mathbf{O}_n^{\langle 3\rangle}+3\mathbf{O}_n\mathbf{O}_n^{\langle 2\rangle}\Big\},$$

(b) $$\frac{\zeta_4(\frac{1}{4})-\zeta_4(\frac{3}{4})-8\pi^2G-56\pi\zeta(3)}{32}=\sum_{n=1}^{\infty}\frac{2^n}{n^3}\binom{2n}{n}^{-1}\Big\{\mathbf{O}_n(1-n^2\mathbf{H}_n^{\langle 2\rangle})\Big\},$$

(c) $$\frac{5\pi^5+384\pi G^2-2688G\zeta(3)}{192}=\sum_{n=1}^{\infty}\frac{2^n}{n^3}\binom{2n}{n}^{-1}\Big\{(1-n^2\mathbf{H}_n^{\langle 2\rangle})(\mathbf{O}_n^{\langle 2\rangle}+\mathbf{O}_n^2)\Big\}.$$

4. Concluding Comments

In this paper, we have presented forty closed-form evaluations for harmonic series with binomial and/or multinomial coefficients, which shows that the "coefficient extraction method" is indeed powerful. Compared with the existing literature on related research topics, this gives a comprehensive and systematic treatment of the harmonic series with convergence rate "1/2". Except for a few known formulae explicitly specified, most of the displayed identities are new, including eight remarkable ones conjectured by sun [16,17]. The authors believe that these series may find potential applications in mathematics (particularly, number theory [21] and combinatorial analysis [22,23]), physics (standing waves in strings [24]), and computer sciences (analysis of algorithms [25]).

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