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Fixed Point Results via α-Admissibility in Extended Fuzzy Rectangular b-Metric Spaces with Applications to Integral Equations

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Abstract: In this article, the concept of extended fuzzy rectangular b-metric space (EFR$_b$MS, for short) is initiated, and some fixed point results frequently used in the literature are generalized via α-admissibility in the setting of EFR$_b$MS. For the illustration of the work presented, some supporting examples and an application to the existence of solutions for a class of integral equations are also discussed.

Keywords: extended fuzzy rectangular b-metric spaces; α-η-β contractions; G-convergence; G-completeness; G-Cauchy sequences

MSC: 37C25; 03E72; 31A10

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1. Introduction

In 1965, Zadeh [1] introduced the concept of fuzzy set and fuzzy logic, providing a new context as an extension of the classical sets and logic. In ordinary set theory, an element either does or does not belong to a set under consideration, whereas, in the fuzzy logic, the bonding of an element to a set is expressed as a real number from the interval $[0,1]$. Since the establishment of this setting, a substantial number of pieces of literature have been developed in order to gain insight into the theory of fuzzy sets and their applications. Heilpern [2] introduced the concept of fuzzy mapping and provided some fixed point results for this type of mappings.

The concept of metric, a distance-measuring mapping, has been generalized in a number of ways in the last seven decades (see, for instance, [3,4]). If the distance between the elements can not be expressed by an exact real number, then the factor of inaccuracy is incorporated in the metric. A significant generalization of the class of metrics is molded by the idea of fuzzy metrics. In 1975, Kramosil and Michalek [5] presented the concept of fuzzy distance and fuzzy metrics using the concepts of fuzzy set and triangular norm (t-norm, for short). While in metric spaces, the distance between two objects is given by an exact real number, in fuzzy metric spaces, it is given by a "distribution function" that models the degree of possibility of the event in which two arbitrary objects are at a distance less than a certain real parameter.

In 1988, Grabiec [6] weakened the condition of completeness in fuzzy metric spaces by defining a notion of Cauchy sequence known as G-Cauchy sequence, and the corresponding completeness concept, known as G completeness. George and Veeramani [7] modified the definition of fuzzy metric space and proved that every fuzzy metric space induces a Hausdorff topology.

Gregori and Sapena [8] introduced fuzzy contractive mappings and proved Banach fixed point theorem for complete fuzzy metric spaces. Subsequently, several fixed point results for various types of contractive mappings in fuzzy metric spaces were established (see [9–18]). Nădăban [19] generalized the notion of b-metric space by introducing fuzzy b-metric spaces. Mehmood et al. [20,21] defined extended fuzzy b-metric spaces and fuzzy rectangular b-metric spaces and, thus, further extrapolated contractions to this general setting.

Samet et al. [22] established some fixed point results in metric spaces for α-ψ-contractions. Stimulated by this work, Gopal and Vetro [18] introduced the concept of α-ϕ-contractive mapping and established some theorems for G-complete fuzzy metric spaces in the sense of Grabiec [6]. Later, many researchers explored α-admissibility in fuzzy metric spaces (see [18,23,24]). We also mention the recent extension to spaces endowed with a graph given in [25].

Motivated by the works [20,24], we introduce the notion of extended fuzzy rectangular b-metric space (EFR$_b$MS) and establish some fixed point results via α-admissibility in the setting of EFR$_b$MS. Our work, being in a more general framework than the classes of "extended fuzzy b-metric spaces" and "rectangular fuzzy b-metric spaces", relaxes the triangle inequality of classical fuzzy metric spaces and generalizes the notion of distance. Consequently, more efficient techniques and algorithms can be devised for image filtering in such spaces. Some interesting applications of the relaxed triangle inequality and fuzzy metrics to the removal of image noise can be found in [26–31].

Our notions and results generalize some other concepts and fixed point results existing in the literature for fuzzy metric spaces. To demonstrate the validity of our results, some examples, along with an application for the existence of solutions to a class of integral equations, are provided.

2. Materials and Methods

In the following, some terms and definitions are provided, which will be needed in the sequel. Throughout this paper, $\mathbb{N}$ represents the set of positive integers, and all the sets under consideration are assumed to be non-empty.

Definition 1. ([32]) *Let $* : [0,1]^2 \to [0,1]$ be a (continuous) binary operation and $([0,1], \leq, *)$ be an ordered abelian topological monoid with unit 1, then $*$ is referred to as a continuous t-norm.*

Examples of some frequently used continuous t-norms are $a *_L b = \max\{a+b-1, 0\}$ (Lukasievicz t-norm), $a *_P b = ab$ (product t-norm) and $a *_m b = \min\{a, b\}$ (minimum t-norm).

Definition 2. ([5]) *Let * be a continuous t-norm and M a fuzzy set on $S \times S \times [0, \infty)$ which satisfies the following conditions, for all $p, q, u \in S$:*

$(KM1)$ $M(p,q,0) = 0$;
$(KM2)$ $M(p,q,\delta) = 1,\ \forall\, \delta > 0$ *iff* $p = q$;
$(KM3)$ $M(p,q,\delta) = M(q,p,\delta)$;
$(KM4)$ $M(p,q,\delta+t) \geq M(p,u,\delta) * M(u,q,t), \forall\, \delta, t > 0$;
$(KM5)$ $M(p,q,\cdot) : [0,\infty) \to [0,1]$ *is continuous;*
$(KM6)$ $\lim_{\delta\to\infty} M(p,q,\delta) = 1$.

Then the 3-tuple $(S, M, *)$ is termed as fuzzy metric space.

$M(p,q,\delta)$ indicates the degree of closeness between p and q with respect to $\delta \geq 0$.

Remark 1. *For $p \neq q$ and $\delta > 0$, it is always true that $0 < M(p,q,\delta) < 1$.*

Lemma 1. ([8]) *$M(p,q,\cdot)$ is non-decreasing for every fixed p, $q \in S$.*

Example 1. ([28]) *Consider the space (S, d), where d is a metric on S. A fuzzy set $M : S \times S \times [0, \infty) \to [0, 1]$ defined on (S, d) as follows is a fuzzy metric on S*

$$M(x, y, \delta) = \frac{k\delta^m}{k\delta^m + n * d(x, y)}, \text{ for every } x, y \in S \text{ and } \delta > 0.$$

Here k, m and n are positive real numbers, and $*$ is the product t-norm. This is a fuzzy metric induced by the metric d. In the above terms, a fuzzy metric is also defined if the minimum t-norm is used instead of the product t-norm.

For $k = m = n = 1$, it reduces to the *standard fuzzy metric*.

Definition 3. ([19]) *Let * be a continuous t-norm, $b \geq 1$ be a given real number and M be a fuzzy set on $S \times S \times [0, \infty)$ satisfying the following conditions, for all $p, q, u \in S$:*

(FbM_1) $M(p, q, 0) = 0$;
(FbM_2) $M(p, q, \delta) = 1, \forall\, \delta > 0$ *iff* $p = q$;
(FbM_3) $M(p, q, \delta) = M(q, p, \delta)$;
(FbM_4) $M(p, q, b(\delta + t)) \geq M(p, u, \delta) * M(u, q, t), \forall \delta, t > 0$;
(FbM_5) $M(p, q, \cdot) : (0, \infty) \to [0, 1]$ *is continuous and* $\lim_{\delta \to \infty} M(p, q, \delta) = 1$.

Then the 3-tuple $(S, M, *)$ is termed as fuzzy b-metric space.

Definition 4. ([21]) *Let * be a continuous t-norm, $\vartheta : S \times S \to [1, \infty)$ be a given function and M be a fuzzy set on $S \times S \times [0, \infty)$ satisfying the following conditions, for all $p, q, u \in S$:*

(FbM_1) $M(p, q, 0) = 0$;
(FbM_2) $M(p, q, \delta) = 1, \forall\, \delta > 0$ *iff* $p = q$;
(FbM_3) $M(p, q, \delta) = M(q, p, \delta)$;
(FbM_4) $M(p, q, \vartheta(p, q)(\delta + t)) \geq M(p, u, \delta) * M(u, q, t), \forall \delta, t > 0$;
(FbM_5) $M(p, q, \cdot) : (0, \infty) \to [0, 1]$ *is continuous and* $\lim_{\delta \to \infty} M(p, q, \delta) = 1$.

Then the 3-tuple $(S, M, *)$ is called an extended fuzzy b-metric space.

Chugh and Kumar [33] defined the notion of fuzzy rectangular metric space as follows:

Definition 5. *Let * be a continuous t-norm and M be a fuzzy set on $S \times S \times [0, \infty)$ satisfying the following conditions, for all $p, q, u, v \in S$ and $\delta, \mu, \kappa > 0$:*

(FRM_1) $M(p, q, 0) = 0$;
(FRM_2) $M(p, q, \delta) = 1, \forall\, \delta > 0$ *iff* $p = q$;
(FRM_3) $M(p, q, \delta) = M(q, p, \delta)$;
(FRM_4) $M(p, q, \delta + \mu + \kappa) \geq M(p, u, \delta) * M(u, v, \mu) * M(v, q, \kappa)$, *for all* $u, v \in S \backslash \{p, q\}$;
(FRM_5) $M(p, q, \cdot) : (0, \infty) \to [0, 1]$ *is continuous and*

$$\lim_{\delta \to \infty} M(p, q, \delta) = 1.$$

Then the ordered triple $(S, M, *)$ is called a fuzzy rectangular metric space.

Mehmood et al. [20] proposed the concept of fuzzy rectangular b-metric space as follows:

Definition 6. *Let * be a continuous t-norm, $b \geq 1$ be a given real number and M be a fuzzy set on $S \times S \times [0, \infty)$ satisfying the following conditions, for all $p, q, r, s \in S$ and $\delta, \mu, \kappa > 0$:*

(FR_bM_1) $M(p, q, 0) = 0$;
(FR_bM_2) $M(p, q, \delta) = 1, \forall\, \delta > 0$ *iff* $p = q$;
(FR_bM_3) $M(p, q, \delta) = M(q, p, \delta)$;
(FR_bM_4) $M(p, q, b(\delta + \mu + \kappa)) \geq M(p, r, \delta) * M(r, s, \mu) * M(s, q, \kappa), \ \forall\, r, s \in S \backslash \{p, q\}$.

(FR_bM_5) $M(p,q,\cdot):(0,\infty)\to[0,1]$ *is continuous and*

$$\lim_{\delta\to\infty} M(p,q,\delta)=1.$$

Then, the ordered triple $(S,M,*)$ is called a fuzzy rectangular b-metric space.

Example 2. ([20]) *Let (S,d) be a rectangular b-metric space and $M:S^2\times[0,\infty)\to[0,1]$ be defined as*

$$M(a,b,\delta)=\begin{cases}\frac{\delta}{\delta+d(a,b)} & \text{if } \delta>0,\\ 0 & \text{if } \delta=0.\end{cases}$$

Then $(S,M,*)$ is a fuzzy rectangular b-metric space with $*_m$ as t-norm.

Remark 2. *Condition (FR_bM_2) of Definition 6 can be, equivalently, stated as*

$$M(p,p,\delta)=1,\ \forall\, p\in S \text{ and } \delta>0, \text{ and } M(p,q,\delta)<1,\ \forall\ p\neq q \text{ and } \delta>0.$$

3. Results

By pursuing the idea of fuzzy rectangular b-metric space presented by Mehmood et al. [20], we introduce the notion of extended fuzzy rectangular b-metric space and generalize some fixed point results via α-η-β contractions.

Definition 7. *Let * be a continuous t-norm, $\vartheta:S\times S\to[1,\infty)$ be a given function and M be a fuzzy set on $S\times S\times[0,\infty)$ satisfying the following conditions, for all $p,q,u,v\in S$ and $\delta,\mu,\kappa>0$:*

(FR_bM_1) $M(p,q,0)=0$;
(FR_bM_2) $M(p,q,\delta)=1,\ \forall\,\delta>0$ *iff* $p=q$;
(FR_bM_3) $M(p,q,\delta)=M(q,p,\delta)$;
(FR_bM_4) $M(p,q,\vartheta(p,q)(\delta+\mu+\kappa))\geq M(p,u,\delta)*M(u,v,\mu)*M(v,q,\kappa)$, *for all* $u,v\in S\backslash\{p,q\}$;
(FR_bM_5) $M(p,q,\cdot):(0,\infty)\to[0,1]$ *is continuous and* $\lim_{\delta\to\infty}M(p,q,\delta)=1$.

Then, the 3-tuple $(S,M,*)$ is called an extended fuzzy rectangular b-metric space.

For $\vartheta(p,q)=b\geq1$, Definition 7 reduces to Definition 6. In the following, we exemplify Definition 7.

Example 3. *Let $S=\{1,2,3,4\}$ and define a b-metric $d:S\times S\to[0,\infty)$ as $d(m,n)=(m-n)^2$. Let $\vartheta:S\times S\to[1,\infty)$ be defined as $\vartheta(m,n)=1+m+n$ and*

$$M(m,n,\delta)=\begin{cases}\frac{\delta}{\delta+d(m,n)} & \text{if } \delta>0,\\ 0 & \text{if } \delta=0.\end{cases}$$

Then, $(S,M,*_m)$ is an extended fuzzy rectangular b-metric space, where $*_m$ is the minimum t-norm.
As $d(\theta,\theta)=0$ for all $\theta\in S$, $d(\theta,\theta-1)=d(\theta-1,\theta)=1$ for all $\theta\in S\backslash\{1\}$,
$d(\theta,\theta-2)=d(\theta-2,\theta)=4$ for all $\theta\in S\backslash\{1,2\}$ and $d(1,4)=d(4,1)=9$.
Also
$\vartheta(1,1)=3,\ \ \vartheta(2,2)=5,\ \ \vartheta(3,3)=7$ *and* $\vartheta(4,4)=9$,
$\vartheta(1,2)=4=\vartheta(2,1),\ \ \vartheta(1,3)=5=\vartheta(3,1)$,
$\vartheta(1,4)=6=\vartheta(4,1),\ \ \vartheta(2,3)=6=\vartheta(3,2)$,
$\vartheta(2,4)=7=\vartheta(4,2),\ \ \vartheta(3,4)=8=\vartheta(4,3)$.

Conditions $(FR_bM_1), (FR_bM_2), (FR_bM_3)$, and (FR_bM_5) trivially hold. In the following, we prove that property (FR_bM_4) is valid.
Indeed, for all $\delta, \mu, \kappa > 0$, we have

$$\begin{aligned} M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) &= \frac{\vartheta(1,2)(\delta+\mu+\kappa)}{\vartheta(1,2)(\delta+\mu+\kappa)+d(1,2)} \\ &= \frac{4(\delta+\mu+\kappa)}{4(\delta+\mu+\kappa)+1} = 1 - \frac{1}{4(\delta+\mu+\kappa)+1}, \end{aligned}$$

$$\begin{aligned} M(1,3,\delta) &= \frac{\delta}{\delta+4} = 1 - \frac{4}{\delta+4}, \\ M(3,4,\mu) &= \frac{\mu}{\mu+1} = 1 - \frac{1}{\mu+1}, \\ M(4,2,\kappa) &= \frac{\kappa}{\kappa+4} = 1 - \frac{4}{\kappa+4}. \end{aligned}$$

Now,

$$\begin{aligned} M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) &= 1 - \frac{1}{4(\delta+\mu+\kappa)+1} \\ &= 1 - \frac{4}{16\delta+16\mu+16\kappa+4} \\ &> 1 - \frac{4}{16\delta+4} > 1 - \frac{4}{\delta+4} \\ &= M(1,3,\delta). \end{aligned}$$

Furthermore,

$$\begin{aligned} M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) &> M(3,4,\mu), \\ M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) &> M(4,2,\kappa). \end{aligned}$$

Therefore,

$$M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) \geq \min\{M(1,3,\delta), M(3,4,\mu), M(4,2,\kappa)\},$$

that is,

$$M(1,2,\vartheta(1,2)(\delta+\mu+\kappa)) \geq M(1,3,\delta) *_m M(3,4,\mu) *_m M(4,2,\kappa).$$

Similarly, it can be shown that

$$\begin{aligned} M(1,3,\vartheta(1,3)(\delta+\mu+\kappa)) &\geq M(1,2,\delta) *_m M(2,4,\mu) *_m M(4,3,\kappa), \\ M(1,4,\vartheta(1,4)(\delta+\mu+\kappa)) &\geq M(1,2,\delta) *_m M(2,3,\mu) *_m M(3,4,\kappa), \\ M(2,3,\vartheta(2,3)(\delta+\mu+\kappa)) &\geq M(2,1,\delta) *_m M(1,4,\mu) *_m M(4,3,\kappa), \\ M(2,4,\vartheta(2,4)(\delta+\mu+\kappa)) &\geq M(2,1,\delta) *_m M(1,3,\mu) *_m M(3,4,\kappa), \\ M(3,4,\vartheta(3,4)(\delta+\mu+\kappa)) &\geq M(3,1,\delta) *_m M(1,2,\mu) *_m M(2,4,\kappa). \end{aligned}$$

Therefore,

$$\begin{aligned} &M(m,n,\vartheta(m,n)(\delta+\mu+\kappa)) \geq M(m,p,\delta) *_m M(p,q,\mu) *_m M(q,n,\kappa), \\ &\textit{for every } p,q \in S\backslash\{m,n\} \textit{ and } \delta,\mu,\kappa > 0. \end{aligned}$$

Hence, $(S, M, *_m)$ is an extended fuzzy rectangular b-metric space.

Definition 8. *In an extended fuzzy rectangular b-metric space $(S, M, *)$, we state that a sequence $\{\nu_n\}_{n\in\mathbb{N}}$:*

1. *converges to $\nu \in S$ if $\lim\limits_{n\to\infty} M(\nu_n, \nu, \delta) = 1$, for every $\delta > 0$;*
2. *is M-Cauchy if, for every $\epsilon \in (0,1)$ and $\delta > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $M(\nu_n, \nu_m, \delta) > 1 - \epsilon$, for every $m, n \geq n_\epsilon$;*
3. *is G-Cauchy if $\lim\limits_{n\to\infty} M(\nu_{n+k}, \nu_n, \delta) = 1$, for all $\delta > 0$ and each $k \in \mathbb{N}$.*

Definition 9. *An EFR_bMS is G complete (resp., M complete) if every G-Cauchy (resp., M-Cauchy) sequence converges in it.*

Definition 10. *Let $(S, M, *)$ be an EFR_bMS and $\alpha : S^2 \times (0,\infty) \to [0,\infty)$ be a mapping. Then $\mathcal{L} : S \to S$ is called α-admissible if, for all $a, b \in S$ and $\delta > 0$,*

$$\alpha(a, b, \delta) \geq 1 \Rightarrow \alpha(\mathcal{L}a, \mathcal{L}b, \delta) \geq 1. \tag{1}$$

Definition 11. *Let $(S, M, *)$ be an EFR_bMS and $\alpha, \eta : S^2 \times (0,\infty) \to [0,\infty)$ be two functions. Then, $\mathcal{L} : S \to S$ is said to be α-η-admissible if, for all $a, b \in S$ and $\delta > 0$,*

$$\alpha(a, b, \delta) \geq \eta(a, b, \delta) \Rightarrow \alpha(\mathcal{L}a, \mathcal{L}b, \delta) \geq \eta(\mathcal{L}a, \mathcal{L}b, \delta). \tag{2}$$

For $\eta(a, b, \delta) = 1$, this definition reduces to Definition 10. $\mathcal{L}$ is called η-subadmissible if $\alpha(a, b, \delta) = 1$.

Definition 12. *Let $(S, M, *)$ be an EFR_bMS. Let $\alpha, \eta : S^2 \times (0,\infty) \to [0,\infty)$ be two functions. Then, $\mathcal{L} : S \to S$ is called an α-η-β contraction if there is some function $\beta : [0,1] \to [1,\infty)$ such that, for any sequence $\{r_n\} \subset [0,1]$, it is satisfied the condition $\beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ when $n \to \infty$, and, moreover, for all $a, b \in S$ and $\delta > 0$*

$$\begin{aligned} &\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta) \geq \eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta) \\ &\Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \beta\big(M(a, b, \delta)\big)N(a, b, \delta), \end{aligned} \tag{3}$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$.

Theorem 1. *Let $(S, M, *)$ be a G-complete EFR_bMS, and $\mathcal{L} : S \to S$ be an α-η-β contraction such that:*

(i) $\mathcal{L}$ is α-η-admissible;
(ii) There is some $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq \eta(a_0, \mathcal{L}a_0, \delta)$ for all $\delta > 0$;
(iii) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq \eta(a_n, a_{n+1}, \delta)$ for all $n \in \mathbb{N}$, $\delta > 0$ and $lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq \eta(a, \mathcal{L}a, \delta)$ for all $\delta > 0$.

Suppose also that M is such that

(FR_bM_6) $M(p, q, \cdot) : (0,\infty) \to [0,1]$ is continuous uniformly for $(p, q) \in [0,1] \times [0,1]$ and $\lim_{\delta\to\infty} M(p, q, \delta) = 1$ uniformly for $(p, q) \in [0,1] \times [0,1]$.

Then $\mathcal{L}$ has a unique fixed point.

Proof. Define an iterative scheme $\{a_n\}$ by $a_{n+1} = \mathcal{L}a_n$, for $n \geq 0$, where $a_0 \in S$ is such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq \eta(a_0, \mathcal{L}a_0, \delta)$. Suppose that $a_{n+1} \neq a_n$ for every $n \geq 0$; if not, then a_n is a fixed point of $\mathcal{L}$. From $\alpha(a_0, \mathcal{L}a_0, \delta) \geq \eta(a_0, \mathcal{L}a_0, \delta)$ along with the α-η-admissibility of $\mathcal{L}$, we have

$$\alpha(a_1, a_2, \delta) = \alpha(\mathcal{L}a_0, \mathcal{L}a_1, \delta) \geq \eta(\mathcal{L}a_0, \mathcal{L}a_1, \delta) = \eta(a_1, a_2, \delta),$$

therefore,

$$\alpha(a_0, \mathcal{L}a_0, \delta)\alpha(a_1, \mathcal{L}a_1, \delta) \geq \eta(a_0, \mathcal{L}a_0, \delta)\eta(a_1, \mathcal{L}a_1, \delta).$$

Proceeding in the same way, we have

$$\alpha(a_{n-1}, \mathcal{L}a_{n-1}, \delta)\alpha(a_n, \mathcal{L}a_n, \delta) \geq \eta(a_{n-1}, \mathcal{L}a_{n-1}, \delta)\eta(a_n, \mathcal{L}a_n, \delta),$$

for all $n \in \mathbb{N}$ and $\delta > 0$. Using (3), we have

$$\begin{aligned} M(a_n, a_{n+1}, \delta) &= M(\mathcal{L}a_{n-1}, \mathcal{L}a_n, \delta) \\ &\geq \beta\big(M(a_{n-1}, a_n, \delta)\big)N(a_{n-1}, a_n, \delta). \end{aligned}$$

Here,

$$\begin{aligned} N(a_{n-1}, a_n, \delta) &= \min\big\{M(a_{n-1}, a_n, \delta), \max\{M(a_{n-1}, \mathcal{L}a_{n-1}, \delta), M(a_n, \mathcal{L}a_n, \delta)\}\big\} \\ &= \min\big\{M(a_{n-1}, a_n, \delta), \max\{M(a_{n-1}, a_n, \delta), M(a_n, a_{n+1}, \delta)\}\big\}. \end{aligned}$$

In each of the two possible cases above, it is true that

$$N(a_{n-1}, a_n, \delta) = M(a_{n-1}, a_n, \delta) \text{ for all } n \in \mathbb{N} \text{ and } \delta > 0.$$

Therefore,

$$\begin{aligned} M(a_n, a_{n+1}, \delta) &\geq \beta\big(M(a_{n-1}, a_n, \delta)\big)M(a_{n-1}, a_n, \delta) \\ &\geq M(a_{n-1}, a_n, \delta). \end{aligned} \tag{4}$$

It means that $\{M(a_n, a_{n+1}, \delta)\}$ is an increasing sequence in (0, 1]. Furthermore, from (4), it follows that

$$\begin{aligned} &\frac{M(a_n, a_{n+1}, \delta)}{M(a_{n-1}, a_n, \delta)} \geq \beta\big(M(a_{n-1}, a_n, \delta)\big) \geq 1 \\ &\Rightarrow \lim_{n\to\infty} \beta\big(M(a_{n-1}, a_n, \delta)\big) = 1 \\ &\Rightarrow \lim_{n\to\infty} M(a_{n-1}, a_n, \delta) = 1. \end{aligned} \tag{5}$$

Now, we are going to prove that $\{a_n\}$ is a Cauchy sequence. Suppose it is not. Then, there will be some $r \in (0,1)$ and $\delta_0 > 0$ such that, for all $p \geq 1$, there exist $m(p), n(p) \in \mathbb{N}$ where $m(p) > n(p) \geq p$ with $m(p)$ being the least integer, which exceeds $n(p)$ and satisfying

$$M\left(a_{m(p)}, a_{n(p)}, \frac{\delta_0}{\vartheta(a_{m(p)}, a_{n(p)})}\right) \leq 1 - r.$$

Note that, from (5), $m(p) \geq n(p) + 2$ for p large. Therefore,

$$M\left(a_{m(p)-2}, a_{n(p)}, \frac{\delta_0}{\vartheta(a_{m(p)}, a_{n(p)})}\right) > 1 - r, \tag{6}$$

which is deduced if $m(p) - 2$ exceeds $n(p)$, or trivially obtained if both indexes are equal.

Using (FR_bM_4), we have

$$\begin{aligned}
1-r &\geq M\left(a_{m(p)},a_{n(p)},\frac{\delta_0}{\vartheta(a_{m(p)},a_{n(p)})}\right)\\
&\geq M\left(a_{m(p)},a_{m(p)-1},\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}\right)\\
&* M\left(a_{m(p)-1},a_{m(p)-2},\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}\right)\\
&* M\left(a_{m(p)-2},a_{n(p)},\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}\right).
\end{aligned} \tag{7}$$

Now, if $\overline{\delta}:=\limsup_{p\to\infty}\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}<+\infty$, then

$$\begin{aligned}
&M\left(a_{m(p)},a_{m(p)-1},\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}\right)\\
&= M\left(a_{m(p)},a_{m(p)-1},\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}\right)\\
&\quad -M\left(a_{m(p)},a_{m(p)-1},\overline{\delta}\right)+M\left(a_{m(p)},a_{m(p)-1},\overline{\delta}\right),
\end{aligned}$$

and similarly for the second term. Using (5) and (6) and (FR_bM_6), for p large enough, we have, from (7),

$$1-r>1-r,$$

which is a contradiction. If $\limsup_{p\to\infty}\frac{\delta_0}{3\vartheta(a_{m(p)},a_{n(p)})}=+\infty$, then, by (FR_bM_6), we obtain the same conclusion. Hence $\{a_n\}$ is a Cauchy sequence. The G-completeness of $(S,M,*)$ ensures the existence of some $\tilde{a}\in S$ such that

$$\lim_{n\to\infty}a_n\to\tilde{a}\Rightarrow\lim_{n\to\infty}M(a_n,\tilde{a},\delta)=1 \text{ for all } \delta>0.$$

From (iii), it follows that

$$\begin{aligned}
&\alpha(\tilde{a},\mathcal{L}\tilde{a},\delta)\geq\eta(\tilde{a},\mathcal{L}\tilde{a},\delta)\ \ \forall\delta>0\\
&\Rightarrow\alpha(a_n,\mathcal{L}a_n,\delta)\alpha(\tilde{a},\mathcal{L}\tilde{a},\delta)\geq\eta(a_n,\mathcal{L}a_n,\delta)\eta(\tilde{a},\mathcal{L}\tilde{a},\delta) \text{ for all } n\in\mathbb{N}\cup\{0\} \text{ and } \delta>0.
\end{aligned}$$

Using the hypothesis (i) of the theorem we have, for all $\delta>0$,

$$M(\mathcal{L}\tilde{a},\mathcal{L}a_n,\delta)=M(\mathcal{L}\tilde{a},a_{n+1},\delta)\geq\beta\big(M(\tilde{a},a_n,\delta)\big)N(\tilde{a},a_n,\delta). \tag{8}$$

Therefore, using (FR_bM_4) and (8), we have

$$\begin{aligned}
M(\mathcal{L}\tilde{a},\tilde{a},\delta) &\geq M\left(\mathcal{L}\tilde{a},a_{n+1},\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)*M\left(a_{n+1},a_n,\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)*M\left(a_n,\tilde{a},\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)\\
&\geq \beta\left(M\left(a_n,\tilde{a},\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)\right)N\left(\tilde{a},a_n,\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)*M\left(a_{n+1},a_n,\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right)\\
&* M\left(a_n,\tilde{a},\frac{\delta}{3\vartheta(\mathcal{L}\tilde{a},\tilde{a})}\right),
\end{aligned} \tag{9}$$

where

$$N\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right)$$
$$= \min\left\{M\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right), \max\left\{M\left(\tilde{a}, \mathcal{L}\tilde{a}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right), M\left(a_n, a_{n+1}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right)\right\}\right\}.$$

Besides, (5) implies

$$\lim_{n\to\infty} N\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) = 1. \tag{10}$$

Using the hypotheses of the theorem along with (5), (10), and (9) implies that

$$\lim_{n\to\infty} M(\mathcal{L}\tilde{a}, \tilde{a}, \delta) = 1 \text{ for every } \delta > 0.$$

Hence, $\mathcal{L}\tilde{a} = \tilde{a}$ and $\tilde{a}$ is a fixed point of $\mathcal{L}$.

To show the uniqueness, suppose that $b \neq \tilde{a}$ is another fixed point of $\mathcal{L}$. Then

$$M(\tilde{a}, b, \delta) < 1 \Rightarrow \beta\big(M(\tilde{a}, b, \delta)\big) > 1. \tag{11}$$

Therefore, from (3), it follows that

$$M(\tilde{a}, b, \delta) = M(\mathcal{L}\tilde{a}, \mathcal{L}b, \delta) \geq \beta\big(M(\tilde{a}, b, \delta)\big)N(\tilde{a}, b, \delta), \tag{12}$$

where

$$\begin{aligned} N(\tilde{a}, b, \delta) &= \min\left\{M(\tilde{a}, b, \delta), \max\{M(\tilde{a}, \mathcal{L}\tilde{a}, \delta), M(b, \mathcal{L}b, \delta)\}\right\} \\ &= \min\left\{M(\tilde{a}, b, \delta), \max\{M(\tilde{a}, \tilde{a}, \delta), M(b, b, \delta)\}\right\} \\ &= M(\tilde{a}, b, \delta). \end{aligned}$$

Finally, (11) and (12) imply that

$$M(\tilde{a}, b, \delta) \geq \beta\big(M(\tilde{a}, b, \delta)\big)M(\tilde{a}, b, \delta) > M(\tilde{a}, b, \delta),$$

which is a contradiction. Hence, $\tilde{a} = b$. □

Letting $\eta(a, b, \delta) = 1$ in the above theorem, we have the following result.

Corollary 1. *Let $(S, M, *)$ be a G-complete EFR$_b$MS, $\mathcal{L} : S \to S$ be an α-admissible mapping and $\alpha : S^2 \times (0, \infty) \to [0, \infty)$, $\beta : [0, 1] \to [1, \infty)$ be two functions such that, for any $\{r_n\} \subset [0, 1]$, $\beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ and, for all $a, b \in S$ and $\delta > 0$, the following is satisfied:*

$$\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta) \geq 1 \Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \beta\big(M(a, b, \delta)\big)N(a, b, \delta),$$

where $N(a, b, \delta) = \min\left\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\right\}$. Assume that (FR_bM_6) and the following conditions hold:

(i) There is some $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then, $\alpha(a, \mathcal{L}a, \delta) \geq 1$, for all $\delta > 0$.

Then, $\mathcal{L}$ has a unique fixed point.

Corollary 2. *Let $\mathcal{L} : S \to S$ be an α-admissible mapping, where $(S, M, *)$ is a G-complete EFR$_b$MS and $\alpha : S^2 \times (0, \infty) \to [0, \infty)$, $\beta : [0, 1] \to [1, \infty)$ are two functions such that, for any $\{r_n\} \subset [0, 1], \beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ and, for all $a, b \in S$ and $\delta > 0$, the following is satisfied:*

$$\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta)M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \beta\big(M(a, b, \delta)\big)N(a, b, \delta),$$
$$\textit{where } N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}.$$

Furthermore, suppose that (FR_bM_6) and the following conditions hold:

(i) There is some $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq 1$, for all $\delta > 0$.

Then, $\mathcal{L}$ has a unique fixed point.

For $\alpha(a, b, \delta) = 1$, the following corollary is deduced from the above theorem.

Corollary 3. *Let $\mathcal{L} : S \to S$ be an η-subadmissible mapping, where $(S, M, *)$ is a G-complete EFR$_b$MS. Let $\beta : [0, 1] \to [1, \infty)$ be a function such that, for any $\{r_n\} \subset [0, 1], \beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ and, for all $a, b \in S$ and $\delta > 0$, the following statement is true:*

$$\eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta) \leq 1 \Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \beta\big(M(a, b, \delta)\big)N(a, b, \delta),$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$. Moreover, assume that (FR_bM_6) and the following conditions hold:

(i) There is some $a_0 \in S$ such that $\eta(a_0, \mathcal{L}a_0, \delta) \leq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\eta(a_n, a_{n+1}, \delta) \leq 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = a$, then $\eta(a, \mathcal{L}a, \delta) \leq 1$, for all $\delta > 0$.

Then, there is a unique $x \in S$ such that $\mathcal{L}x = x$.

Corollary 4. *Let $\mathcal{L} : S \to S$ be an η-subadmissible mapping, $(S, M, *)$ be a G-complete EFR$_b$MS and the function $\beta : [0, 1] \to [1, \infty)$ be such that, for any sequence $\{r_n\} \subset [0, 1], \beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ and, for all $a, b \in S$, $\delta > 0$, the following condition is true:*

$$M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta)\beta\big(M(a, b, \delta)\big)N(a, b, \delta),$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$. Furthermore, assume that (FR_bM_6) and the following conditions hold:

(i) There is some $a_0 \in S$ such that $\eta(a_0, \mathcal{L}a_0, \delta) \leq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\eta(a_n, a_{n+1}, \delta) \leq 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = a$, then $\eta(a, \mathcal{L}a, \delta) \leq 1$, for all $\delta > 0$.

Then $\mathcal{L}$ has a unique fixed point.

Remark 3. *Due to the definition of the sequence $\{a_n\}$ in the proof of Theorem 1, the condition (iii) in Theorem 1 can be replaced by the following condition:*

$(iii)^*$ For $\{a_n\} \subset S$, if $\alpha(a_n, \mathcal{L}a_n, \delta) \geq \eta(a_n, \mathcal{L}a_n, \delta)$ for all $n \in \mathbb{N}$, $\delta > 0$ and $\lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq \eta(a, \mathcal{L}a, \delta)$ for all $\delta > 0$.

Similarly, in Corollaries 1 and 2, condition (ii) can be replaced by the following condition:

$(ii)^*$ For $\{a_n\} \subset S$, if $\alpha(a_n, \mathcal{L}a_n, \delta) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq 1$, for all $\delta > 0$.

In Corollaries 3 and 4, the condition (ii) can be replaced by the following condition:

$(ii)^*$ For $\{a_n\} \subset S$, if $\eta(a_n, \mathcal{L}a_n, \delta) \leq 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} a_n = a$, then $\eta(a, \mathcal{L}a, \delta) \leq 1$, for all $\delta > 0$.

The following example supports Theorem 1.

Example 4. *Let $S = [0, \infty)$ and $M(a, b, \delta) = e^{\frac{-(a-b)^2}{\delta}}$ for all $a, b \in S$, $\delta > 0$, with $r * s = rs$ for $r, s \in [0, 1]$. It can be easily verified that $(S, M, *)$ is a G-complete EFR$_b$MS. Define $\mathcal{L} : S \to S$ by*

$$\mathcal{L}(a) = \begin{cases} \frac{a}{2} & \text{if } a \in [0,1], \\ \ln(a + e - 1) & \text{if } a \in (1, \infty). \end{cases}$$

Consider the mappings $\alpha, \eta : S \times S \times (0, \infty) \to [0, \infty)$ given, respectively, by

$$\alpha(a, b, \delta) = \begin{cases} 3 & \text{if } a, b \in [0,1], \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\eta(a, b, \delta) = \begin{cases} 2 & \text{if } a, b \in [0,1], \\ 1 & \text{otherwise}. \end{cases}$$

For $a, b \in S$, if

$$\alpha(a, b, \delta) \geq \eta(a, b, \delta),$$

then $a, b \in [0, 1]$, and, for every $u \in [0, 1]$, we have $\mathcal{L}u < 1$, therefore

$$\alpha(\mathcal{L}a, \mathcal{L}b, \delta) \geq \eta(\mathcal{L}a, \mathcal{L}b, \delta).$$

That is, $\mathcal{L}$ is α-η-admissible. Furthermore, $\alpha(a, \mathcal{L}a, \delta) \geq \eta(a, \mathcal{L}a, \delta)$ for $a \in [0, 1]$. If $\{a_n\} \subset S$ satisfies that $\alpha(a_n, \mathcal{L}a_n, \delta) \geq \eta(a_n, \mathcal{L}a_n, \delta)$ for all $n \in \mathbb{N}$, $\delta > 0$, and $\lim_{n\to\infty} a_n \to a$, then $\{a_n\} \subset [0, 1]$, hence $a \in [0, 1]$. This implies that $\alpha(a, \mathcal{L}a, \delta) \geq \eta(a, \mathcal{L}a, \delta)$, for all $\delta > 0$.

On the other hand, obviously

$$\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta) \geq \eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta) \quad \Rightarrow a, b \in [0, 1].$$

Choose $\beta : [0, 1] \to [1, \infty)$ as a function such that $\beta(x) \leq x^{-\frac{3}{4}}$, for $x > 0$, and that, for any sequence $\{r_n\} \subset [0, 1]$, it is satisfied the condition $\beta(r_n) \to 1 \Leftrightarrow r_n \to 1$ when $n \to \infty$. Hence, $\beta(e^{\frac{-(a-b)^2}{\delta}}) \leq e^{\frac{3(a-b)^2}{4\delta}}$, and, therefore,

$$\begin{aligned} M(\mathcal{L}a, \mathcal{L}b, \delta) &= e^{\frac{-(\mathcal{L}a - \mathcal{L}b)^2}{\delta}} \\ &= e^{\frac{-\frac{1}{4}(a-b)^2}{\delta}} \\ &\geq \beta(e^{\frac{-(a-b)^2}{\delta}})e^{\frac{-(a-b)^2}{\delta}} \\ &= \beta\big(M(a, b, \delta)\big)M(a, b, \delta) \\ &\geq \beta\big(M(a, b, \delta)\big)N(a, b, \delta). \end{aligned}$$

Note that (FR_bM_6) holds since M is continuous on $[0, 1] \times [0, 1] \times (0, \infty)$ and $1 \geq M(a, b, \delta) \geq e^{-\frac{1}{\delta}}$, $\delta > 0$.

To summarize, all conditions of Theorem 1 are fulfilled. Clearly, $0 \in S$ is the only fixed point of $\mathcal{L}$.

4. α-η-ψ Contractions in Extended Fuzzy Rectangular b-Metric Spaces

Let Ψ be the collection of all continuous and non-decreasing mappings $\psi : [0, 1] \to [0, 1]$ such that $\psi(t) > t$ for all $t \in (0, 1)$.

Definition 13. *Let $\mathcal{L} : S \to S$, $(S, M, *)$ be an EFR$_b$MS and $\alpha, \eta : S^2 \times (0, \infty) \to [0, \infty)$ be functions such that, for all $a, b \in S$ and $\delta > 0$,*

$$\begin{aligned}\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta) &\geq \eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta) \\ &\Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \psi\big(N(a, b, \delta)\big),\end{aligned} \tag{13}$$

with $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$ and $\psi \in \Psi$. Then $\mathcal{L}$ is called an α-η-ψ contraction.

In the following, we establish a fixed point theorem for α-η-ψ contractions in EFR$_b$MS.

Theorem 2. *Let $(S, M, *)$ be a G-complete EFR$_b$MS and $\mathcal{L} : S \to S$ be an α-η-ψ contraction such that:*

(a) $\mathcal{L}$ is α-η-admissible;
(b) There exists $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq \eta(a_0, \mathcal{L}a_0, \delta)$ for all $\delta > 0$;
(c) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq \eta(a_n, a_{n+1}, \delta)$ for all $n \in \mathbb{N}$, $\delta > 0$ and $lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq \eta(a, \mathcal{L}a, \delta)$ for all $\delta > 0$.

Then, there is a unique $\tilde{a} \in S$ such that $\mathcal{L}\tilde{a} = \tilde{a}$.

Proof. Arguing as in the proof of Theorem 1, we have

$$\alpha(a_{n-1}, \mathcal{L}a_{n-1}, \delta)\alpha(a_n, \mathcal{L}a_n, \delta) \geq \eta(a_{n-1}, \mathcal{L}a_{n-1}, \delta)\eta(a_n, \mathcal{L}a_n, \delta) \text{ for all } n \in \mathbb{N} \text{ and } \delta > 0.$$

Using (13), we have

$$\begin{aligned} M(a_n, a_{n+1}, \delta) &= M(\mathcal{L}a_{n-1}, \mathcal{L}a_n, \delta) \\ &\geq \psi\big(N(a_{n-1}, a_n, \delta)\big),\end{aligned} \tag{14}$$

where

$$\begin{aligned} N(a_{n-1}, a_n, \delta) &= \min\big\{M(a_{n-1}, a_n, \delta), \max\{M(a_{n-1}, \mathcal{L}a_{n-1}, \delta), M(a_n, \mathcal{L}a_n, \delta)\}\big\} \\ &= \min\big\{M(a_{n-1}, a_n, \delta), \max\{M(a_{n-1}, a_n, \delta), M(a_n, a_{n+1}, \delta)\}\big\}.\end{aligned}$$

In each of the two above possible cases, it is true that

$$N(a_{n-1}, a_n, \delta) = M(a_{n-1}, a_n, \delta), \text{ for all } n \in \mathbb{N} \text{ and } \delta > 0.$$

Therefore, from (14), we have

$$M(a_n, a_{n+1}, \delta) \geq \psi\big(M(a_{n-1}, a_n, \delta)\big) > M(a_{n-1}, a_n, \delta).$$

This means that $\{M(a_n, a_{n+1}, \delta)\}$ is an increasing sequence in (0,1].

Let $\lim_{n\to\infty} M(a_n, a_{n+1}, \delta) = \ell(\delta)$. We prove that $\ell(\delta) = 1$ for all $\delta > 0$. Suppose there is some $\delta_0 > 0$ for which $\ell(\delta_0) < 1$. Letting $n \to \infty$ and using the definition of ψ, (14) gives the following contradiction

$$\ell(\delta_0) \geq \psi\big(\ell(\delta_0)\big) > \ell(\delta_0).$$

Hence,

$$\lim_{n\to\infty} M(a_n, a_{n+1}, \delta) = 1, \text{ for all } \delta > 0. \tag{15}$$

Using a similar argument as the proof of Theorem 1, it can be shown that the sequence $\{a_n\}$ is of Cauchy type.

As $(S, M, *)$ is complete, there will be some $\tilde{a} \in S$ such that

$$\lim_{n\to\infty} a_n \to \tilde{a} \Rightarrow \lim_{n\to\infty} M(a_n, \tilde{a}, \delta) = 1 \text{ for all } \delta > 0.$$

From (c), we have

$$\alpha(\tilde{a}, \mathcal{L}\tilde{a}, \delta) \geq \eta(\tilde{a}, \mathcal{L}\tilde{a}, \delta)$$
$$\Rightarrow \alpha(a_n, \mathcal{L}a_n, \delta)\alpha(\tilde{a}, \mathcal{L}\tilde{a}, \delta) \geq \eta(a_n, \mathcal{L}a_n, \delta)\eta(\tilde{a}, \mathcal{L}\tilde{a}, \delta) \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } \delta > 0.$$

Using the α-η-ψ contractivity hypothesis of the theorem, we have, for all $\delta > 0$,

$$M(\mathcal{L}\tilde{a}, \mathcal{L}a_n, \delta) \geq \psi\big(N(\tilde{a}, a_n, \delta)\big). \tag{16}$$

Therefore, using (FR_bM_4) and (16), and fixing $\vartheta(\mathcal{L}\tilde{a}, \tilde{a}) > 0$, we have

$$\begin{aligned} M(\mathcal{L}\tilde{a}, \tilde{a}, \delta) &\geq M\left(\mathcal{L}\tilde{a}, a_{n+1}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) * M\left(a_{n+1}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) * M\left(a_n, \tilde{a}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) \\ &\geq \psi\left(N\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right)\right) * M\left(a_{n+1}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) * M\left(a_n, \tilde{a}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right), \end{aligned} \tag{17}$$

where

$$\begin{aligned} &N\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) \\ &= \min\left\{M\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right), \max\left\{M\left(\tilde{a}, \mathcal{L}\tilde{a}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right), M\left(a_n, a_{n+1}, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right)\right\}\right\}. \end{aligned}$$

At this point, (15) implies

$$\lim_{n\to\infty} N\left(\tilde{a}, a_n, \frac{\delta}{3\vartheta(\mathcal{L}\tilde{a}, \tilde{a})}\right) = 1. \tag{18}$$

Using the hypotheses of the theorem along with (15), (18) and (17) implies that

$$\lim_{n\to\infty} M(\mathcal{L}\tilde{a}, \tilde{a}, \delta) = 1, \text{ for all } \delta > 0.$$

Hence, $\mathcal{L}\tilde{a} = \tilde{a}$ and $\tilde{a}$ is a fixed point of $\mathcal{L}$.
To show the uniqueness, suppose that $b \neq \tilde{a}$ is another fixed point of $\mathcal{L}$. Then

$$\mathcal{L}\tilde{a} = \tilde{a} \neq b = \mathcal{L}b, \text{ therefore } M(\tilde{a}, b, \delta) < 1.$$

Hence,

$$M(\tilde{a}, b, \delta) \quad = \quad M(\mathcal{L}\tilde{a}, \mathcal{L}b, \delta) \geq \psi\big(M(\tilde{a}, b, \delta)\big) > M(\tilde{a}, b, \delta),$$

which is not possible. Hence, $\tilde{a} = b$. □

Placing $\eta(a, b, \delta) = 1$ in Theorem 2, we obtain the following corollaries:

Corollary 5. *Let $(S, M, *)$ be a G-complete EFR$_b$MS, and $\mathcal{L} : S \to S$ be an α-admissible mapping such that*

$$\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta) \geq 1 \Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \psi\big(N(a, b, \delta)\big),$$

where $N(a, b, \delta) = \min\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\}$ for all $a, b \in S$ and $\delta > 0$. Assume that the following conditions hold:

(a) There is some $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq 1$ for all $\delta > 0$;
(b) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq 1$ for all $n \in \mathbb{N}$, $\delta > 0$ and $lim_{n\to\infty} a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq 1$ for all $\delta > 0$.

Then, $\mathcal{L}$ has a unique fixed point.

Corollary 6. *Let $(S, M, *)$ be a G-complete EFR$_b$MS, and $\mathcal{L} : S \to S$ be an α-admissible mapping such that*

$$\alpha(a, \mathcal{L}a, \delta)\alpha(b, \mathcal{L}b, \delta)M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \psi\big(N(a, b, \delta)\big),$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$ for all $a, b \in S$ and $\delta > 0$. Assume that the following conditions hold:

(a) There is some $a_0 \in S$ such that $\alpha(a_0, \mathcal{L}a_0, \delta) \geq 1$ for all $\delta > 0$;
(b) For $\{a_n\} \subset S$, if $\alpha(a_n, a_{n+1}, \delta) \geq 1$ for all $n \in \mathbb{N}$, $\delta > 0$ and $lim_{n\to\infty}a_n = a$, then $\alpha(a, \mathcal{L}a, \delta) \geq 1$ for all $\delta > 0$.

Then, there is a unique $\tilde{a} \in S$ such that $\mathcal{L}\tilde{a} = \tilde{a}$.

Letting $\alpha(a, b, \delta) = 1$ in Theorem 2, we obtain the following corollaries.

Corollary 7. *Let $(S, M, *)$ be a G-complete EFR$_b$MS, and $\mathcal{L} : S \to S$ be an η-subadmissible mapping such that*

$$\eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta) \leq 1 \Rightarrow M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \psi\big(N(a, b, \delta)\big),$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$ for all $a, b \in S$ and $\delta > 0$. Suppose that:

(i) There is some $a_0 \in S$ such that $\eta(a_0, \mathcal{L}a_0, \delta) \leq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\eta(a_n, a_{n+1}, \delta) \leq 1$ for all $n \in \mathbb{N}$, $\delta > 0$ and $\lim_{n\to\infty} a_n = a$, then $\eta(a, \mathcal{L}a, \delta) \leq 1$ for all $\delta > 0$.

Then, there is a unique $\tilde{a} \in S$ for which $\mathcal{L}\tilde{a} = \tilde{a}$.

Corollary 8. *Let $\mathcal{L} : S \to S$ be an η-subadmissible mapping, $(S, M, *)$ be a G-complete EFR$_b$MS and suppose that*

$$M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \eta(a, \mathcal{L}a, \delta)\eta(b, \mathcal{L}b, \delta)\psi\big(N(a, b, \delta)\big) \text{ for all } a, b \in S \text{ and } \delta > 0,$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$. Suppose also that:

(i) There is some $a_0 \in S$ for which $\eta(a_0, \mathcal{L}a_0, \delta) \leq 1$ for all $\delta > 0$;
(ii) For $\{a_n\} \subset S$, if $\eta(a_n, a_{n+1}, \delta) \leq 1$ for all $n \in \mathbb{N}$, $\delta > 0$ and $\lim_{n\to\infty} a_n = a$, then $\eta(a, \mathcal{L}a, \delta) \leq 1$ for all $\delta > 0$.

Then, $\mathcal{L}$ has a unique fixed point.

Remark 4. *Similar comments given in Remark 3 apply to Theorem 2 and Corollaries 5–8.*

Taking $\alpha(a, b, \delta) = 1$ in Corollary 6 and $\eta(a, b, \delta) = 1$ in Corollary 8, we obtain the following corollary, which is the result by Mihet [34] for EFR$_b$MS.

Corollary 9. *Let $(S, M, *)$ be a G-complete EFR$_b$MS and $\mathcal{L} : S \to S$. Assume that*

$$M(\mathcal{L}a, \mathcal{L}b, \delta) \geq \psi\big(N(a, b, \delta)\big) \text{ for all } a, b \in S \text{ and } \delta > 0,$$

where $N(a, b, \delta) = \min\big\{M(a, b, \delta), \max\{M(a, \mathcal{L}a, \delta), M(b, \mathcal{L}b, \delta)\}\big\}$. Then $\mathcal{L}$ has a unique fixed point.

Example 5. *Theorem 2 is also applicable to Example 4 with any continuous and nondecreasing mapping* $\psi : [0,1] \to [0,1]$ *such that* $t^{\frac{1}{4}} \geq \psi(t) > t$ *for all* $t \in (0,1)$. *Indeed,*

$$\begin{aligned} M(\mathcal{L}a, \mathcal{L}b, \delta) &= e^{\frac{-(\mathcal{L}a-\mathcal{L}b)^2}{\delta}} \\ &= e^{\frac{-\frac{1}{4}(a-b)^2}{\delta}} \\ &\geq \psi\left(e^{\frac{-(a-b)^2}{\delta}}\right) \\ &= \psi(M(a,b,\delta)) \\ &\geq \psi(N(a,b,\delta)). \end{aligned}$$

5. Application to Integral Equations

Integral equations find applications in a variety of scientific fields, such as biology, chemistry, physics, or engineering. Furthermore, fuzzy integral equations constitute one of the important branches of fuzzy analysis theory and play a major role in numerical analysis. One of the approaches followed for the study of integral equations is the application of fixed point theory directly to the mapping defined by the right-hand side of the equation, or by the development of homotopy methods, which are largely considered in fixed point theory. In particular, for its connection with the study of fuzzy integral problems, we highlight a very recent paper [35], in which the author proposes a homotopy analysis method to find an approximate solution of the two-dimensional non-linear fuzzy Volterra integral equation. We also refer the reader to [18,20,23,36,37] for other related works.

We apply our theory of fixed point to ensure the existence of solutions to the following type of integral equations:

$$u(t) = f(t) + \int_0^t H(t,s,u(s))ds, \; t \in [0,b], \tag{19}$$

where $b > 0$. The Banach space $C([0,b],\mathbb{R})$ of all real continuous functions defined on $[0,b]$, with norm $\|u\| := \sup_{s\in[0,b]} |u(s)|$ for every $u \in C([0,b],\mathbb{R})$, can be considered as a fuzzy Banach space [38] (for more details concerning the relation between Banach spaces and fuzzy Banach spaces, see [39]). Consider the fuzzy metric on $C([0,b],\mathbb{R})$ given by

$$M(u,v,\delta) = e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\delta}},$$

for all $u, v \in C([0,b],\mathbb{R})$ and $\delta > 0$, furnished with the t-norm $*_p$ defined as $x *_p y = xy$ for all $x, y \in [0,1]$. Then $C([0,b],\mathbb{R}, M, *_p)$ is a G-complete EFR$_b$MS.

In the following, we discuss the existence of solutions for the integral equations of the form (19).

Theorem 3. *Let* $P : C([0,b],\mathbb{R}) \to C([0,b],\mathbb{R})$ *be the integral operator given by*

$$[P(u)](t) = f(t) + \int_0^t H(t,s,u(s))ds, \; u \in C([0,b],\mathbb{R}), \; t \in [0,b],$$

where $f \in C([0,b],\mathbb{R})$ *and* $H \in C([0,b] \times [0,b] \times \mathbb{R}, \mathbb{R})$ *satisfies the following condition:*

(i) *There exists a continuous and non-decreasing mapping* $\psi : [0,1] \to [0,1]$ *with* $\psi(t) > t$ *for all* $t \in (0,1)$*, such that, for all* $u, v \in C([0,b],\mathbb{R})$*, and every* $\delta > 0$*,*

$$\sup_{s\in[0,b]} \left(\int_0^s |H(s,r,u(r)) - H(s,r,v(r))| dr \right)^2 \leq -\ln\left(\psi\left(e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\delta}} \right) \right)^{\delta}.$$

Then, the integral Equation (19) has a solution $u^* \in C([0,b],\mathbb{R})$*.*

Proof. For all $u, v \in C([0,b],\mathbb{R})$, and $\delta > 0$, we have

$$\begin{aligned} M(P(u),P(v),\delta) &= e^{-\frac{\sup_{s\in[0,b]} |[P(u)](s)-[P(v)](s)|^2}{\delta}} \\ &\geq e^{-\frac{\sup_{s\in[0,b]} \left(\int_0^s |H(s,r,u(r))-H(s,r,v(r))|dr\right)^2}{\delta}} \\ &\geq \psi\left(e^{-\frac{\sup_{s\in[0,b]} |u(s)-v(s)|^2}{\delta}} \right) \\ &= \psi(M(u,v,\delta)) \\ &\geq \psi(N(u,v,\delta)). \end{aligned}$$

Hence, using Theorem 2, P has a fixed point $u^* \in C([0,b],\mathbb{R})$, which is a solution to the integral Equation (19). □

6. Discussion

We proposed the notion of extended fuzzy rectangular b-metric space and proved some results concerning the existence and uniqueness of fixed points via α-η-β and α-η-ψ contractions. Our framework, being more general than the classes of "extended fuzzy b-metric spaces" and "rectangular fuzzy b-metric spaces", relaxes the triangle inequality of classical fuzzy metric spaces. Consequently, our notions and results generalize some other concepts and fixed point results existing in the literature for fuzzy metric spaces. On the other hand, the relaxed triangle inequality can give some interesting applications to the removal of image noise, as shown in [26,27,29–31], so that the new concepts may lead to further investigation and applications. We also presented some examples and illustrated the implication of the new results in the study of the existence of solutions for a class of integral equations.

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