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Article

Fixed Point Theorems: Exploring Applications in Fractional Differential Equations for Economic Growth

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Abstract: The aim of this research is to introduce two new notions, Θ-(Ξ,h)-contraction and rational (α,η)-ψ-interpolative contraction, in the setting of $\mathfrak{F}$-metric space and to establish corresponding fixed point theorems. To reinforce understanding and highlight the novelty of our findings, we provide a non-trivial example that not only supports the obtained results but also illuminates the established theory. Finally, we apply our main result to discuss the existence and uniqueness of solutions for a fractional differential equation describing an economic growth model.

Keywords: fixed point; $\mathfrak{F}$-metric space; Θ-(Ξ,h)-contraction; fractional differential equations; economic growth

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1. Introduction

Fixed point theory, a vibrant intersection of topology and analysis, offers a robust analytical tool with wide-ranging applications in pure and applied mathematics. By establishing key concepts and frameworks, it fuels ongoing research and development. At its heart lies the study of metric spaces, which define distances within sets. Countless impactive and elegant applications of this notion abound across various scientific fields ([1,2]). To broaden its reach, mathematicians have generalized metric spaces, leading to fruitful extensions of fixed point theory. Branciari [3] proposed the ingenious idea of a generalized metric, replacing the standard triangle inequality with a more general, four-term "rectangular inequality". This type of metric space, known as the rectangular metric space, is well-established in the literature. Bakhtin [4] introduced the concept of b-metric space in 1989, a notion further elucidated by Czerwik [5] in 1993. Unlike classical metrics, the b-metric lacks continuity within the topology it defines. Expanding upon these ideas, Jleli et al. [6] initiated the concept of $\mathfrak{F}$-metric space in 2018, presenting it as a generalization encompassing all previously mentioned metric spaces. These generalizations offer greater flexibility for modeling real-world phenomena and tackling problems that might not fit classical metric structures.

Stefan Banach [7] played a pioneering role in this theory, introducing the concept of contraction within the background of metric spaces and proving a fundamental fixed point theorem. His contributions have found significant applications in diverse domains, including optimization problems, differential equations, economics, and many other fields. Numerous research endeavors have been dedicated to enhancing and extending Banach's contraction principle through various avenues. Jleli et al. [8] introduced the notion of Θ-contractions as a significant generalization of traditional contractions. This concept goes beyond the simple distance criterion by incorporating an additional function, Θ, which captures the intricate interplay between distances for paired points in a metric space. This extension allows for a more nuanced and flexible analysis of contraction properties and proves to be a valuable tool for understanding the convergence behavior of iterative processes in broader settings. Samet et al. [9] further extended the reach of admissibility, a crucial concept in fixed point theory, by introducing the idea of α-admissibility. This innovation involves a "weighting" function, $\alpha(\mathfrak{a}, \mathfrak{b})$, that assigns varying importance to the contractiveness between different pairs of points. This allows for a more customized analysis that takes into account specific conditions and complexities within the metric space. Ansari et al. [10,11] further contributed to this domain by introducing C-class functions, paving the way for establishing fixed point results as generalizations of Banach's theorem in metric-like spaces and 0-f-orbitally complete partial metric spaces. These novel functions provide even greater adaptability and enable fixed point analysis within broader and more diverse mathematical structures. For a deeper exploration of these advancements and their implications, readers are encouraged to delve into references [12–20].

On the other side, fixed point theory has become an indispensable tool for the analysis and solution of fractional differential equations. Its ability to establish the existence and uniqueness of solutions, as well as its capacity to handle nonlinearities and provide constructive methods, has made it a cornerstone for the study of fractional differential equations across various scientific and engineering disciplines. Further, these equations offer a promising approach to economic growth modeling, providing a richer framework for capturing complex dynamics and memory effects. As research in this field progresses, equations are expected to assume growing significance for comprehending and predicting patterns of economic growth, informing policy decisions, and addressing economic challenges more effectively. In 2016, McTier [21] employed a fractional order approach for modeling the economic growth of both the United States and Italy, with a specific emphasis on their respective gross domestic products (GDPs). McTier incorporated key variables such as geographic expanse, cultivable terrain, population, attendance at educational institutions, total capital investment, exports of goods and services, overall government consumption spending, and currency and quasi-money to characterize the GDP. The findings indicated that fractional models exhibit superior performance compared to alternative approaches discussed in the existing literature. Subsequently, Tejado et al. [22] used fractional calculus to model the economic growth of Spain and Portugal. Later on, Ming et al. [23] utilized fractional calculus in models pertaining to the economic growth of China. Very recently, Johansyah et al. [24] gave a comprehensive review approach of applications of fractional differential equations in different economic growth models.

In this ongoing investigation, we propose the concepts of Θ-(Ξ,h)-contractions and rational (α,η)-ψ-interpolative contractions in the background of $\mathfrak{F}$-MS and obtain corresponding fixed point theorems. As an application, we apply the leading result to discuss the existence and uniqueness of solutions for a fractional differential equation.

2. Preliminaries

In this article, we employ the subsequent symbols: $\mathbb{R}$ represents the set of all real numbers, $\mathbb{R}^+$ denotes the set of all positive real numbers, and $\mathbb{N}$ indicates the set of all natural numbers. In the literature, numerous extensions of the renowned Banach contraction principle [7] can be found. This principle states that any self-mapping $\mathcal{B}$ defined on a complete metric space (X, d) and satisfying

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) \leq \tau d(\mathfrak{a}, \flat), \text{ where } \tau \in [0, 1)$$

for all $\mathfrak{a}, \flat \in X$ has a unique fixed point.

In the year 2014, Jleli et al. [8] introduced an innovative form of contraction and proved some novel fixed point theorems applicable to this type of contraction within the realm of generalized metric spaces.

Definition 1 ([8])**.** *Let Ω be the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ such that*

(Θ_1) *for all $t_1, t_2 \in \mathbb{R}^+$ such that $t_1 \leq t_2$, which implies $\Theta(t_1) \leq \Theta(t_2)$;*
(Θ_2) *for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \Theta(t_n) = 1$ if and only if $\lim_{n\to\infty}(t_n) = 0$;*
(Θ_3) *there exists $0 < q < 1$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\Theta(t)-1}{t^q} = l$.*

A mapping $\mathcal{B} : X \to X$ is termed a Θ-contraction if there exist a function Θ satisfying the conditions (Θ_1)–(Θ_3) and a constant $\tau \in (0,1)$ such that for all $\mathfrak{a}, \flat \in X$,

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) \neq 0 \implies \Theta(d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat)) \leq [\Theta(d(\mathfrak{a}, \flat))]^{\tau}.$$

Theorem 1 ([8])**.** *Suppose that $\mathcal{B} : X \to X$ is a Θ-contraction on complete metric space (X, d); then $\mathcal{B}$ has a unique fixed point.*

Bakhtin [4] introduced the concept of a b-metric by modifying the triangular inequality of the metric space in the following manner:
for all $\mathfrak{a}, \omega, \flat \in X$ and for some $s \geq 1$,

$$d(\mathfrak{a}, \flat) \leq s[d(\mathfrak{a}, \omega) + d(\omega, \flat)].$$

On the other hand, The concept of $\mathfrak{F}$-metric space ($\mathfrak{F}$-MS) originates from the work of Jleli et al. [6], as detailed in the following definition.

Let $\mathfrak{F}$ be the class of functions $\xi : (0, +\infty) \to \mathbb{R}$ satisfying
($\mathfrak{F}_1$) $0 < \mathfrak{a}_1 < \mathfrak{a}_2 \Rightarrow \xi(\mathfrak{a}_1) \leq \xi(\mathfrak{a}_2)$,
($\mathfrak{F}_2$) for all $\{\mathfrak{a}_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \mathfrak{a}_n = 0$ if and only if $\lim_{n\to\infty} \xi(\mathfrak{a}_n) = -\infty$.

Definition 2 ([6])**.** *Let $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$. Suppose that there exists $(\xi, \hbar) \in \mathfrak{F} \times [0, +\infty)$ such that*

(D_1) $(\mathfrak{a}, \flat) \in X \times X$, $d(\mathfrak{a}, \flat) = 0$ if and only if $\mathfrak{a} = \flat$;
(D_2) $d(\mathfrak{a}, \flat) = d(\flat, \mathfrak{a})$ for all $\mathfrak{a}, \flat \in X$;
(D_3) for all $(\mathfrak{a}, \flat) \in X \times X$ and $(\mathfrak{a}_i)_{i=1}^{N} \subset X$, with $(\mathfrak{a}_1, \mathfrak{a}_N) = (\mathfrak{a}, \flat)$ for all $N \geq 2$, we have

$$d(\mathfrak{a}, \flat) > 0 \Rightarrow \xi(d(\mathfrak{a}, \flat)) \leq \xi(\sum_{i=1}^{N-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1})) + \hbar.$$

Subsequently, (X, d) is termed an $\mathfrak{F}$-MS.

Example 1 ([6])**.** *Consider $X = \mathbb{N}$ and let the mapping $d : X \times X \to [0, +\infty)$ be defined by*

$$d(\mathfrak{a}, \flat) = \begin{cases} (\mathfrak{a} - \flat)^2 & \text{if } (\mathfrak{a}, \flat) \in [0,3] \times [0,3] \\ |\mathfrak{a} - \flat| & \text{if } (\mathfrak{a}, \flat) \notin [0,3] \times [0,3] \end{cases}$$

for all $(\mathfrak{a}, \flat) \in X \times X$. Then, the pair (X, d) forms an $\mathfrak{F}$-MS, where $\xi : (0, +\infty) \to \mathbb{R}$ is defined as $\xi(t) = \ln(t)$ for $t > 0$, and $\hbar = \ln(3)$.

Definition 3 ([6])**.** *Let (X, d) be an $\mathfrak{F}$-MS.*
(i) A sequence $\{\mathfrak{a}_n\} \subseteq X$ is called an $\mathfrak{F}$-convergent if

$$\lim_{n\to\infty} d(\mathfrak{a}_n, \mathfrak{a}) = 0.$$

(ii) A sequence $\{\mathfrak{a}_n\}$ is an $\mathfrak{F}$-Cauchy if

$$\lim_{n,m\to\infty} d(\mathfrak{a}_n, \mathfrak{a}_m) = 0.$$

Definition 4 ([6])**.** *Let (X, d) be an $\mathfrak{F}$-MS and $\mathcal{B} : X \to X$. Then $\mathcal{B}$ is professed to be continuous at a specific point $\mathfrak{a} \in X$ if for each sequence $\{\mathfrak{a}_n\}$ in X converging to $\mathfrak{a}$, the sequence $\mathcal{B}\mathfrak{a}_n$ converges to $\mathcal{B}\mathfrak{a}$. Additionally, the mapping $\mathcal{B}$ is deemed continuous on X if it exhibits continuity at every point $\mathfrak{a} \in X$.*

In their work, Samet et al. [9] gave the concept of α-admissibility in a manner unique to their investigation.

Definition 5. *A mapping $\mathcal{B} : X \to X$ is known as an α-admissible mapping if*

$$\alpha(\mathfrak{a}, \flat) \geq 1 \quad \textit{implies} \quad \alpha(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) \geq 1.$$

Recently, Ansari et al. [11] used the the following pair (Ξ, h) of functions in contractive inequalities and established some results.

Definition 6 ([11])**.** *Let $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and Ξ:$\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$; then we say that the pair of the functions Ξ and h are C-class functions if the ensuing conditions are fulfilled:*
(i) $x \geq 1$ implies $h(1, y) \leq h(x, y)$;
(ii) $0 \leq \ell \leq 1$ implies $\Xi(\ell, \wp) \leq \Xi(1, \wp)$;
(iii) $h(1, y) \leq \Xi(\ell, \wp)$ implies $y \leq \ell\wp$
for all $y, \ell, \wp \in \mathbb{R}^+$.

Example 2. *Define $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and Ξ:$\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x, y) = y$ and $\Xi(\ell, \wp) = \ell\wp$; then the pair Ξ and h are C-class functions.*

3. Results and Discussions

3.1. Fixed Point Results for Θ-(Ξ,h)-Contractions

To facilitate our investigation in this subsection, we introduce a formal definition for Θ-(Ξ,h)-contractions, setting the stage for their impactful utilization within the context of $\mathfrak{F}$-MS (X, d).

Definition 7. *A mapping $\mathcal{B} : (X, d) \to (X, d)$ is designated as a Θ-(Ξ,h)-contraction if there exist a constant $\tau \in (0, 1)$, the functions $\Theta \in \Omega$, $\alpha : X \times X \to \mathbb{R}^+$, and the C-class functions Ξ and h satisfying the condition:*

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) > 0 \Longrightarrow h(\alpha(\mathfrak{a}, \flat)), \Theta(d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat))) \leq \Xi\big(1, [\Theta(d(\mathfrak{a}, \flat))]^\tau\big) \tag{1}$$

for all $\mathfrak{a}, \flat \in X$.

Theorem 2. *Let (X, d) be a complete $\mathfrak{F}$-MS; $\mathcal{B} : X \to X$ is a Θ-(Ξ, h)-contraction. Suppose the following conditions are met:*
(i) $\mathcal{B}$ is an α-admissible mapping;
(ii) There exists a point $\mathfrak{a}_0 \in X$ such that $\alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq 1$;
(iii) Either $\mathcal{B} : X \to X$ is continuous, or, if $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq 1$, then $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq 1$ for all $n \in \mathbb{N}$.
Then $\mathcal{B}$ has a fixed point. Furthermore, if $\alpha(\mathfrak{a}, \mu) \geq 1$ for all $\mathfrak{a}, \mu \in Fix(\mathcal{B})$, then the fixed point is unique.

Proof. Let $\mathfrak{a}_0 \in X$ be an arbitrary point such that

$$\alpha(\mathfrak{a}_0, \mathfrak{a}_1) = \alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq 1. \tag{2}$$

Now, we define a sequence $\{\mathfrak{a}_n\}$ in this way:

$$\mathfrak{a}_1 = \mathcal{B}\mathfrak{a}_0, \cdots, \mathfrak{a}_{n+1} = \mathcal{B}\mathfrak{a}_n = \mathcal{B}^{n+1}\mathfrak{a}_0 \tag{3}$$

for all $n \geq 0$. By condition (i) and (2), we have

$$\alpha(\mathfrak{a}_1, \mathfrak{a}_2) = \alpha(\mathcal{B}\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_1) \geq 1.$$

By continuing this process, we get

$$\alpha(\mathfrak{a}_{n-1}, \mathfrak{a}_n) = \alpha(\mathcal{B}\mathfrak{a}_{n-2}, \mathcal{B}\mathfrak{a}_{n-1}) \geq 1 \tag{4}$$

for all $n \in \mathbb{N} \cup \{0\}$. If $\mathfrak{a}_{n_0} = \mathfrak{a}_{n_0+1}$ holds true for a certain $n_0 \in \mathbb{N} \cup \{0\}$, it becomes clear that $\mathfrak{a}_{n_0}$ qualifies as a fixed point of $\mathcal{B}$. Consequently, let us consider the scenario where $\mathfrak{a}_n \neq \mathfrak{a}_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. In this case, we suppose that:

$$d(\mathcal{B}\mathfrak{a}_{n-1}, \mathcal{B}\mathfrak{a}_n) = d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) > 0$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, it follows from (1) that we have

$$\begin{aligned} & h(1, \Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1}))) \\ = \ & h(1, \Theta(d(\mathcal{B}\mathfrak{a}_{n-1}, \mathcal{B}\mathfrak{a}_n))) \\ \leq \ & h(\alpha(\mathfrak{a}_{n-1}, \mathfrak{a}_n)), \Theta(d(\mathcal{B}\mathfrak{a}_{n-1}, \mathcal{B}\mathfrak{a}_n))) \\ \leq \ & \Xi(1, [\Theta(d(\mathfrak{a}_{n-1}, \mathfrak{a}_n))]^\tau), \end{aligned}$$

which entails that

$$\Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1})) \leq [\Theta(d(\mathfrak{a}_{n-1}, \mathfrak{a}_n))]^\tau. \tag{5}$$

Repeatedly applying inequality (5) yields

$$\Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1})) \leq [\Theta(d(\mathfrak{a}_{n-1}, \mathfrak{a}_n))]^\tau \leq ... \leq [\Theta(d(\mathfrak{a}_0, \mathfrak{a}_1))]^{\tau^n}. \tag{6}$$

Letting $n \to \infty$ in (6), we get

$$\lim_{n\to\infty} \Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1})) = 1.$$

By (Θ_2), we have

$$\lim_{n\to\infty} d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) = 0. \tag{7}$$

From (Θ_3), there exist $q \in (0,1)$ and $l \in (0,\infty)$ such that

$$\lim_{n\to\infty} \frac{\Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1})) - 1}{[d(\mathfrak{a}_n, \mathfrak{a}_{n+1})]^q} = l. \tag{8}$$

Consider $\lambda \in (0, l)$. According to the definition of a limit, there exists a natural number n_0 such that

$$[d(\mathfrak{a}_n, \mathfrak{a}_{n+1})]^q \leq \lambda^{-1}[\Theta(d(\mathfrak{a}_n, \mathfrak{a}_{n+1})) - 1] \tag{9}$$

for all $n > n_0$. Employing (6) along with the previously mentioned inequality, we infer

$$n[d(\mathfrak{a}_n, \mathfrak{a}_{n+1})]^q \leq \lambda^{-1} n \Big([\Theta(d(\mathfrak{a}_0, \mathfrak{a}_1))]^{\tau^n} - 1 \Big).$$

This implies that

$$\lim_{n\to\infty} n[d(\mathfrak{a}_n, \mathfrak{a}_{n+1})]^q = \lim_{n\to\infty} n[d(\mathfrak{a}_0, \mathfrak{a}_1)]^h = 0.$$

Then, there exists $n_1 \in \mathbb{N}$ such that

$$d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \frac{1}{n^{\frac{1}{q}}} \tag{10}$$

for $n > n_1$. This yields

$$\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{q}}} \tag{11}$$

for $m > n$. Since $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}$ is convergent,

$$0 < \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{q}}} < \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}} < \delta$$

for $n > n_1$. Consider a fixed $\epsilon > 0$ and let $(\xi, \hbar) \in \mathfrak{F} \times [0, +\infty)$ such that the condition (D_3) is satisfied. According to $(\mathfrak{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \text{ implies } \xi(t) < \xi(t) - \hbar. \tag{12}$$

Hence, by (11), (12), and $(\mathfrak{F}_1)$, we have

$$\xi\left(\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1})\right) \leq \xi\left(\sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{q}}}\right) \leq \xi\left(\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}\right) < \xi(\epsilon) - \hbar \tag{13}$$

given that $m > n \geq n_1$. Now, in accordance with condition (D_3) and (13) for $d(\mathfrak{a}_n, \mathfrak{a}_m) > 0$, $m > n \geq n_1$, we obtain

$$\begin{aligned} \xi(d(\mathfrak{a}_n, \mathfrak{a}_m)) &\leq \xi\left(\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1})\right) + \hbar \\ &\leq \xi\left(\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{q}}}\right) + \hbar \\ &< \xi(\epsilon), \end{aligned}$$

which, from $(\mathfrak{F}_1)$, gives that

$$d(\mathfrak{a}_n, \mathfrak{a}_m) < \epsilon,$$

for all $m > n \geq n_1$; hence, $\{\mathfrak{a}_n\}$ forms a Cauchy sequence in (X, d). Given that (X, d) is complete, there exists $\mathfrak{a}^* \in X$ such that $\lim_{n\to\infty} \mathfrak{a}_n \to \mathfrak{a}^*$. Now, we demonstrate that $\mathfrak{a}^* = \mathcal{B}\mathfrak{a}^*$. Assuming $\mathcal{B} : X \to X$ is continuous, then $\mathcal{B}\mathfrak{a}_n \to \mathcal{B}\mathfrak{a}^*$ as $n \to \infty$. Therefore,

$$\mathcal{B}\mathfrak{a}^* = \lim_{n\to\infty} \mathcal{B}\mathfrak{a}_n = \lim_{n\to\infty} \mathfrak{a}_{n+1} = \mathfrak{a}^*.$$

Now, if $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then according to supposition (iii), it follows that $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq 1$ for all $n \in \mathbb{N}$. Assuming the contrary, if $\mathfrak{a}^*$ is not the fixed point of $\mathcal{B}$, then $d(\mathcal{B}\mathfrak{a}^*, \mathfrak{a}^*) \neq 0$. Referring to (1), we obtain

$$\begin{aligned} & h(1, \Theta(d(\mathfrak{a}_{n+1}, \mathcal{B}\mathfrak{a}^*))) \\ = \; & h(1, \Theta(d(\mathcal{B}\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*))) \\ \leq \; & h(\alpha(\mathfrak{a}_n, \mathfrak{a}^*)), \Theta(d(\mathcal{B}\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*))) \\ \leq \; & \Xi(1, [\Theta(d(\mathfrak{a}_n, \mathfrak{a}^*))]^{\tau}), \end{aligned}$$

which infers that

$$\Theta(d(\mathfrak{a}_{n+1}, \mathcal{B}\mathfrak{a}^*)) \leq [\Theta(d(\mathfrak{a}_n, \mathfrak{a}^*))]^{\tau}.$$

Allowing $n \to \infty$ in the aforementioned inequality and leveraging the continuity of Θ and d, we get

$$\Theta(d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}^*)) = 1,$$

which implies by (Θ_2) that we have $d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}^*) = 0$, which is a contradiction. Thus, $\mathfrak{a}^* = \mathcal{B}\mathfrak{a}^*$ and $\mathfrak{a}^*$ is a fixed point of $\mathcal{B}$. Now let $\mathfrak{a}^/$ be another fixed point of $\mathcal{B}$ such that $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^* \neq \mathfrak{a}^/ = \mathcal{B}\mathfrak{a}^/$. Then by the assumption, we obtain $\alpha\left(\mathfrak{a}^*, \mathfrak{a}^/\right) \geq 1$. Now, by (1), we have

$$\begin{aligned}
& h\Big(1,\Theta\Big(d(\mathfrak{a}^*,\mathfrak{a}^/)\Big)\Big) \\
= \; & h\Big(1,\Theta\Big(d(\mathcal{B}\mathfrak{a}^*,\mathcal{B}\mathfrak{a}^/)\Big)\Big) \\
\leq \; & h\Big(\alpha\Big(\mathfrak{a}^*,\mathfrak{a}^/\Big),\Theta\Big(d(\mathcal{B}\mathfrak{a}^*,\mathcal{B}\mathfrak{a}^/)\Big)\Big) \\
\leq \; & \Xi\Big(1,\Big[\Theta\Big(d(\mathfrak{a}^*,\mathfrak{a}^/)\Big)\Big]^{\tau}\Big),
\end{aligned}$$

which implies that

$$\Theta\Big(d(\mathfrak{a}^*,\mathfrak{a}^/)\Big) \leq \Big[\Theta\Big(d(\mathfrak{a}^*,\mathfrak{a}^/)\Big)\Big]^{\tau} < \Theta\Big(d(\mathfrak{a}^*,\mathfrak{a}^/)\Big),$$

which is a contradiction because $\tau < 1$. Thus, $\mathfrak{a}^* = \mathfrak{a}^/$, and the fixed point is unique. □

The following outcome represents the primary finding of Ahmad et al. [16] and is a direct implication of our main Theorem 2.

Corollary 1. *Let (X,d) be a complete $\mathfrak{F}$-MS and $\mathcal{B}: X \to X$. Suppose that there exists a constant $\tau \in (0,1)$ and the functions $\Theta \in \Omega$, $\alpha : X \times X \to \mathbb{R}^+$ such that $(\mathfrak{a},\flat) \geq 1$ and*

$$d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat) > 0 \Longrightarrow \Theta(d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)) \leq [\Theta(d(\mathfrak{a},\flat))]^{\tau}$$

for all $\mathfrak{a},\flat \in X$. Suppose that the subsequent conditions are met:
(i) $\mathcal{B}$ is an α-admissible mapping;
(ii) There exists a point $\mathfrak{a}_0 \in X$ such that $\alpha(\mathfrak{a}_0,\mathcal{B}\mathfrak{a}_0) \geq 1$;
(iii) Either $\mathcal{B}: X \to X$ is continuous, or, if $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n,\mathfrak{a}_{n+1}) \geq 1$, then $\alpha(\mathfrak{a}_n,\mathfrak{a}^*) \geq 1$ for all $n \in \mathbb{N}$.
Then $\mathcal{B}$ has a fixed point. Furthermore, if $\alpha(\mathfrak{a},\preceq)) \geq 1$ for all $\mathfrak{a},\mu \in Fix(\mathcal{B})$, then the fixed point is unique.

Proof. Define $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x,y) = y$ and $\Xi(\ell,\wp) = \ell\wp$ in Theorem 2. □

Corollary 2. *Consider a complete $\mathfrak{F}$-MS (X,d), where $\mathcal{B}: X \to X$ is continuous. Assume that there exists a constant $\tau \in (0,1)$ and a function $\Theta \in \Omega$ such that*

$$d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat) > 0 \Longrightarrow \Theta(d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)) \leq [\Theta(d(\mathfrak{a},\flat))]^{\tau}$$

for all $\mathfrak{a},\flat \in X$. Then $\mathcal{B}$ has a unique fixed point.

Proof. Define $\alpha : X \times X \to \mathbb{R}^+$, $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\Xi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $\alpha(\mathfrak{a},\flat) = 1$, $h(x,y) = y$ and $\Xi(\ell,\wp) = \ell\wp$ in Theorem 2. □

Example 3. *Let us define the sequence $\{\mathfrak{a}_n\}$ in the following manner.*
$\mathfrak{a}_1 = \ln(1)$
$\mathfrak{a}_2 = \ln(3)$
.
.
.
$\mathfrak{a}_n = \ln(1+3+5+\ldots+(2n-1)) = 2\ln(n)$ *for all $n \in \mathbb{N}$. Consider the set $X = \{\mathfrak{a}_n : n \in \mathbb{N}\}$ along with the $\mathfrak{F}$ metric as specified by*

$$d(\mathfrak{a},\flat) = \begin{cases} e^{|\mathfrak{a}-\flat|}, \; if \;\; \mathfrak{a} \neq \flat \\ 0, \; if \;\; \mathfrak{a} = \flat \end{cases}$$

with $\xi(t) = \frac{-1}{t}$ and $\hbar = 1$.

Consequently, (X,d) forms a complete $\mathfrak{F}$-MS. Let us define the mapping $\mathcal{B}: X \to X$ as follows

$$\mathcal{B}(\mathfrak{a}_n) = \begin{cases} \mathfrak{a}_1, & \text{if } n = 1, \\ \mathfrak{a}_{n-1}, & \text{if } n > 1 \end{cases}$$

and $\alpha : X \times X \to [1,+\infty)$ by

$$\alpha(\mathfrak{a}_n, \mathfrak{a}_m) = \begin{cases} 1, \text{if } \mathfrak{a}_n \neq \mathfrak{a}_m \\ 0, \text{ if } \mathfrak{a}_n = \mathfrak{a}_m \end{cases}.$$

Define $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and Ξ:$\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by $h(x,y) = y$ and $\Xi(\ell, \wp) = \ell\wp$. Clearly,

$$\lim_{n \longrightarrow \infty} \frac{d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_1))}{d(\mathfrak{a}_n, \mathfrak{a}_1)} = 1.$$

Then $\mathcal{B}$ is not a contraction in the sense of [8]. Consider the mapping $\Theta : \mathbb{R} \to [1,+\infty)$ given by $\Theta(t) = e^t$, $t > 0$. Demonstrating that Θ belongs to the set Ω is straightforward. Now, we prove $\mathcal{B}$ is an Θ-(Ξ,h)-contraction: that is, $d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m)) \neq 0$ implies

$$\tau + \ln d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m))) + d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m))) \leq \ln d(\mathfrak{a}_n, \mathfrak{a}_m) + d(\mathfrak{a}_n, \mathfrak{a}_m)$$

for $\tau > 0$. The condition stated above is identical to

$$d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m)) \neq 0 \Longrightarrow e^{\tau + \ln d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m))) + d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m)))} \leq e^{\ln d(\mathfrak{a}_n, \mathfrak{a}_m) + d(\mathfrak{a}_n, \mathfrak{a}_m)}.$$

So we have to check that

$$d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m)) \neq 0 \Longrightarrow \frac{d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m))}{d(\mathfrak{a}_n, \mathfrak{a}_m)} e^{d(\mathcal{B}(\mathfrak{a}_n), \mathcal{B}(\mathfrak{a}_m)) - d(\mathfrak{a}_n, \mathfrak{a}_m)} \leq e^{-\tau}.$$

For all $m \in \mathbb{N}$, $m \geq 2$, we get

$$\begin{aligned} d(\mathcal{B}(\mathfrak{a}_m), \mathcal{B}(\mathfrak{a}_1)) &\neq 0 \Longrightarrow \frac{d(\mathcal{B}(\mathfrak{a}_m), \mathcal{B}(\mathfrak{a}_1))}{d(\mathfrak{a}_m, \mathfrak{a}_1)} e^{d(\mathcal{B}(\mathfrak{a}_m), \mathcal{B}(\mathfrak{a}_1)) - d(\mathfrak{a}_m, \mathfrak{a}_1)} \leq e^{-\tau} \\ & \quad \frac{d(\mathfrak{a}_{m-1}, \mathfrak{a}_1)}{d(\mathfrak{a}_m, \mathfrak{a}_1)} e^{d(\mathfrak{a}_{m-1}, \mathfrak{a}_1) - d(\mathfrak{a}_m, \mathfrak{a}_1)} \\ &= \frac{e^{\mathfrak{a}_{m-1} - \mathfrak{a}_1}}{e^{\mathfrak{a}_m - \mathfrak{a}_1}} e^{e^{\mathfrak{a}_{m-1} - \mathfrak{a}_1} - e^{\mathfrak{a}_m - \mathfrak{a}_1}} \\ &= \frac{e^{\ln(m-1)^2}}{e^{\ln(m)^2}} e^{e^{\ln(m-1)^2} - e^{\ln(m)^2}} \\ &= \frac{(m-1)^2}{m^2} e^{-2m+1} < e^{-1}. \end{aligned}$$

Therefore, the inequality (1) holds true with $\tau = 1 > 0$. Consequently, $\mathcal{B}$ established as an F-(Ξ,h)-contraction. Hence, Theorem 4 deduces that $\mathfrak{a} = ln(1)$ stands as the unique fixed point of $\mathcal{B}$.

3.2. Fixed Point Results for Interpolative Contractions

The concept of α-ψ-contractive mappings was established in 2012 by Samet et al. [9]. These mappings utilize a family Ψ of non-decreasing functions $\psi : [0,+\infty) \to [0,+\infty)$. A crucial property of these functions is that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th iterate of ψ.

The subsequent lemma is widely recognized.

Lemma 1. *If $\psi \in \Psi$, then the following hold:*
(i) $(\psi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in (0,+\infty)$;

(ii) $\psi(t) < t$ *for all* $t > 0$;
(iii) $\psi(t) = 0$ *iff* $t = 0$.

Theorem 3 ([9])**.** *Let* (X, d) *be a complete metric space and* $\mathcal{B} : X \to X$ *be* α*-admissible mapping. Assume that*

$$\alpha(\mathfrak{a},\mathfrak{b})d(\mathcal{B}\mathfrak{a}, \mathcal{B}\mathfrak{b}) \leq \psi(d(\mathfrak{a},\mathfrak{b}))$$

for all $\mathfrak{a},\mathfrak{b} \in X$, *where* $\psi \in \Psi$. *Also, suppose that*

(i) *There exists* $\mathfrak{a}_0 \in X$ *such that* $\alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq 1$;
(ii) *Either* $\mathcal{B}$ *is continuous, or, for any sequence* $\{\mathfrak{a}_n\}$ *in* X *with* $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $\mathfrak{a}_n \to \mathfrak{a}^*$ *as* $n \to +\infty$, *we have* $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq 1$ *for all* $n \in \mathbb{N}$.

Then $\mathcal{B}$ *has a fixed point.*

Here we define the notion of rational (α,η)-ψ-interpolative contraction in the context of $\mathfrak{F}$-MS (X, d).

Definition 8. *A mapping* $\mathcal{B} : X \to X$ *is described as a rational* (α,η)*-*ψ*-interpolative contraction if there exist the functions* $\alpha, \eta : X \times X \longrightarrow [0, +\infty)$, $\psi \in \Psi$ *and a constant* $\lambda \in [0, 1)$ *such that*

$$\alpha(\mathfrak{a},\mathfrak{b}) \geq \eta(\mathfrak{a},\mathfrak{b}), \tag{14}$$

which implies

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\mathfrak{b}) \leq \psi\left[d(\mathfrak{a}, \mathfrak{b})^{\lambda}.\left(\frac{d(\mathfrak{a}, \mathcal{B}\mathfrak{a})d(\mathfrak{a}, \mathcal{B}\mathfrak{b}) + d(\mathfrak{b}, \mathcal{B}\mathfrak{b})d(\mathfrak{b}, \mathcal{B}\mathfrak{a})}{\max\{d(\mathfrak{a}, \mathcal{B}\mathfrak{b}), d(\mathfrak{b}, \mathcal{B}\mathfrak{a})\}}\right)^{1-\lambda}\right], \tag{15}$$

for all $\mathfrak{a}, \mathfrak{b} \in X \backslash Fix(\mathcal{B})$.

Theorem 4. *Let* $\mathcal{B} : X \to X$ *be a rational* (α,η)*-*ψ*-interpolative contraction. Let us consider the following conditions to be true:*
(i) $\mathcal{B}$ *is an* α*-admissible mapping with respect to* η;
(ii) There exists a point $\mathfrak{a}_0 \in X$ *such that* $\alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq \eta(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0)$;
(iii) $\mathcal{B} : X \to X$ *is continuous.*
Then there exists $\mathfrak{a}^* \in X$ *such that* $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Let $\mathfrak{a}_0 \in X$ be an arbitrary point such that

$$\alpha(\mathfrak{a}_0, \mathfrak{a}_1) = \alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq \eta(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) = \eta(\mathfrak{a}_0, \mathfrak{a}_1). \tag{16}$$

Now we define a sequence $\{\mathfrak{a}_n\}$ in this way:

$$\mathfrak{a}_1 = \mathcal{B}\mathfrak{a}_0, \cdots, \mathfrak{a}_{n+1} = \mathcal{B}\mathfrak{a}_n = \mathcal{B}^{n+1}\mathfrak{a}_0, \tag{17}$$

for all $n \geq 0$. By hypothesis (i) and inequality (16), we have

$$\alpha(\mathfrak{a}_1, \mathfrak{a}_2) = \alpha(\mathcal{B}\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_1) \geq \eta(\mathcal{B}\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_1) = \eta(\mathfrak{a}_1, \mathfrak{a}_2).$$

Following this procedure, we ultimately achieve

$$\alpha(\mathfrak{a}_{n-1}, \mathfrak{a}_n) = \alpha(\mathcal{B}\mathfrak{a}_{n-2}, \mathcal{B}\mathfrak{a}_{n-1}) \geq \eta(\mathcal{B}\mathfrak{a}_{n-2}, \mathcal{B}\mathfrak{a}_{n-1}) = \eta(\mathfrak{a}_{n-1}, \mathfrak{a}_n).$$

for all $n \in \mathbb{N} \cup \{0\}$. Now, if $\mathfrak{a}_{n_0} = \mathfrak{a}_{n_0+1}$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then clearly $\mathfrak{a}_{n_0}$ is a fixed point of $\mathcal{B}$. Thus, we assume that $\mathfrak{a}_n \neq \mathfrak{a}_{n+1}$, holds true for every $n \in \mathbb{N} \cup \{0\}$. Therefore, we assume that

$$d(\mathcal{B}\mathfrak{a}_{n-1}, \mathcal{B}\mathfrak{a}_n) = d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) > 0,$$

for all $n \in \mathbb{N} \cup \{0\}$. From (14) and (15), we get

$$\begin{aligned} d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) &= d(\mathcal{B}\mathfrak{a}_{n-1}, \mathcal{B}\mathfrak{a}_n) \leq \psi\left[\begin{array}{c} d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)^{\lambda} \\ \cdot\left(\frac{d(\mathfrak{a}_{n-1},\mathcal{B}\mathfrak{a}_{n-1})d(\mathfrak{a}_{n-1},\mathcal{B}\mathfrak{a}_n)+d(\mathfrak{a}_n,\mathcal{B}\mathfrak{a}_n)d(\mathfrak{a}_n,\mathcal{B}\mathfrak{a}_{n-1})}{\max\{d(\mathfrak{a}_{n-1},\mathcal{B}\mathfrak{a}_n),d(\mathfrak{a}_n,\mathcal{B}\mathfrak{a}_{n-1})\}}\right)^{1-\lambda} \end{array}\right] \\ &= \psi\left[\begin{array}{c} d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)^{\lambda} \\ \cdot\left(\frac{d(\mathfrak{a}_{n-1},\mathfrak{a}_n)d(\mathfrak{a}_{n-1},\mathfrak{a}_{n+1})+d(\mathfrak{a}_n,\mathfrak{a}_{n+1})d(\mathfrak{a}_n,\mathfrak{a}_n)}{\max\{d(\mathfrak{a}_{n-1},\mathfrak{a}_{n+1}),d(\mathfrak{a}_n,\mathfrak{a}_n)\}}\right)^{1-\lambda} \end{array}\right] \\ &= \psi\left[d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)^{\lambda}.(d(\mathfrak{a}_{n-1}, \mathfrak{a}_n))^{1-\lambda}\right] \\ &= \psi[d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)] \end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \psi[d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)] \tag{18}$$

for all $n \in \mathbb{N}$. Following this approach, we obtain

$$d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \leq \psi[d(\mathfrak{a}_{n-1}, \mathfrak{a}_n)] \leq \psi[\psi(d(\mathfrak{a}_{n-2}, \mathfrak{a}_{n-1}))] \leq \ldots \leq \psi^n[d(\mathfrak{a}_0, \mathfrak{a}_1)]$$

for all $n \in \mathbb{N}$, which yields

$$\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i(d(\mathfrak{a}_0, \mathfrak{a}_1)) \tag{19}$$

for $m > n$. We consider a fixed positive value ϵ. Additionally, let $n(\epsilon)$ be a natural number satisfying the condition that $\sum_{n \geq n(\delta)} \psi^i(d(\mathfrak{a}_0, \mathfrak{a}_1)) < \epsilon$. Furthermore, given an arbitrary element $(\xi, \hbar) \in \mathcal{F} \times [0, +\infty)$, if condition (D_3) holds, then due to property $(\mathfrak{F}_2)$, there exists a positive value δ such that

$$0 < t < \delta \text{ implies } \xi(t) < \xi(\delta) - \hbar. \tag{20}$$

Hence, by (19), (20), and $(\mathfrak{F}_1)$, we have

$$\xi\left(\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1})\right) \leq \xi\left(\sum_{i=n}^{m-1} \psi^i(d(\mathfrak{a}_0, \mathfrak{a}_1))\right) \leq \xi\left(\sum_{n \geq n(\delta)} \psi^i(d(\mathfrak{a}_0, \mathfrak{a}_1))\right) < \xi(\epsilon) - \hbar \tag{21}$$

for $m > n \geq N$. Applying (D_3) and (21), we get $d(\mathfrak{a}_n, \mathfrak{a}_m) > 0$, $m > n \geq N$, which implies

$$\xi(d(\mathfrak{a}_n, \mathfrak{a}_m)) \leq \xi\left(\sum_{i=n}^{m-1} d(\mathfrak{a}_i, \mathfrak{a}_{i+1})\right) + \hbar < \xi(\epsilon).$$

This, combined with property $(\mathcal{F}_1)$, implies that $d(\mathfrak{a}_n, \mathfrak{a}_m) < \epsilon$, $m > n \geq N$. This establishes the $\mathcal{F}$-Cauchy property for the sequence $\{\mathfrak{a}_n\}$. As (X, d) is complete, a limit element $\mathfrak{a}^*$ exists in X such that the sequence $\{x_n\}$ converges to x^*, which can be written as

$$\lim_{n \to \infty} d(\mathfrak{a}_n, \mathfrak{a}^*) = 0. \tag{22}$$

Now, we show that $\mathfrak{a}^* = \mathcal{B}\mathfrak{a}^*$. Since $\mathcal{B} : X \to X$ is continuous, we have $\mathcal{B}\mathfrak{a}_n \to \mathcal{B}\mathfrak{a}^*$ as $n \to \infty$. Thus,

$$\mathcal{B}\mathfrak{a}^* = \lim_{n \to \infty} \mathcal{B}\mathfrak{a}_n = \lim_{n \to \infty} \mathfrak{a}_{n+1} = \mathfrak{a}^*.$$

□

Theorem 5. *Let (X, d) be a complete $\mathfrak{F}$-MS and $\mathcal{B} : X \to X$ ba a rational (α,η)-ψ-interpolative contraction. Let us establish the following premises.*

(i) $\mathcal{B}$ is an α-admissible mapping with respect to η;

(ii) There exists a point $\mathfrak{a}_0 \in X$ such that $\alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq \eta(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0)$;

(iii) If $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq \eta(\mathfrak{a}_n, \mathfrak{a}_{n+1})$ for all $n \in \mathbb{N}$, then $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq \eta(\mathfrak{a}_n, \mathfrak{a}^*)$ for all $n \in \mathbb{N}$.

Then there exists $\mathfrak{a}^* \in X$ such that $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Following a similar approach to the proof of Theorem 4, we get that $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq \eta(\mathfrak{a}_n, \mathfrak{a}_{n+1})$ for all $n \in \mathbb{N}$. Then, by hypothesis (iii), we have $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq \eta(\mathfrak{a}_n, \mathfrak{a}^*)$ for all $n \in \mathbb{N}$. To establish a contradiction, let us assume that $\mathfrak{a}^*$ is not a fixed point of $\mathcal{B}$. Then $d(\mathcal{B}\mathfrak{a}^*, \mathfrak{a}^*) \neq 0$. Now by ($D_3$) and (14), we thave

$$\begin{aligned}
\xi(d(\mathcal{B}\mathfrak{a}^*, \mathfrak{a}^*)) &\leq \xi(d(\mathcal{B}\mathfrak{a}^*, \mathcal{B}\mathfrak{a}_n) + d(\mathcal{B}\mathfrak{a}_n, \mathfrak{a}^*)) + \hbar \\
&\leq \xi\left(\begin{array}{c} \psi\left[d(\mathfrak{a}^*, \mathfrak{a}_n)^{\lambda} . \left(\frac{d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}^*) d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}_n) + d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}_n) d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)}{\max\{d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}_n), d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)\}} \right)^{1-\lambda} \right] \\ + d(\mathcal{B}\mathfrak{a}_n, \mathfrak{a}^*) \end{array} \right) + \hbar \\
&= \xi\left(\begin{array}{c} \psi\left[d(\mathfrak{a}^*, \mathfrak{a}_n)^{\lambda} . \left(\frac{d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}^*) d(\mathfrak{a}^*, \mathfrak{a}_{n+1}) + d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)}{\max\{d(\mathfrak{a}^*, \mathfrak{a}_{n+1}), d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)\}} \right)^{1-\lambda} \right] \\ + d(\mathfrak{a}_{n+1}, \mathfrak{a}^*) \end{array} \right) + \hbar.
\end{aligned} \tag{23}$$

Taking $n \to \infty$ in the preceding inequality and considering the fact that $\lim_{n\to\infty} d(\mathfrak{a}_n, \mathfrak{a}^*) = 0$ together with $\lim_{n\to\infty} d(\mathfrak{a}_{n+1}, \mathfrak{a}^*) = 0$ and

$$\lim_{n\to\infty} \xi\left(\begin{array}{c} \psi\left[d(\mathfrak{a}^*, \mathfrak{a}_n)^{\lambda} . \left(\frac{d(\mathfrak{a}^*, \mathcal{B}\mathfrak{a}^*) d(\mathfrak{a}^*, \mathfrak{a}_{n+1}) + d(\mathfrak{a}_n, \mathfrak{a}_{n+1}) d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)}{\max\{d(\mathfrak{a}^*, \mathfrak{a}_{n+1}), d(\mathfrak{a}_n, \mathcal{B}\mathfrak{a}^*)\}} \right)^{1-\lambda} \right] \\ + d(\mathfrak{a}_{n+1}, \mathfrak{a}^*) \end{array} \right) + \hbar = -\infty.$$

Thus, by (23), we have $\xi(d(\mathcal{B}\mathfrak{a}^*, \mathfrak{a}^*)) = -\infty$. Hence, by ($\mathfrak{F}_2$), we have $d(\mathcal{B}\mathfrak{a}^*, \mathfrak{a}^*) = 0$, which is a contradiction. Thus, $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^*$. □

Corollary 3. *Let (X, d) be a complete $\mathfrak{F}$-MS and $\mathcal{B} : X \to X$. Assume that there exist the functions $\alpha : X \times X \longrightarrow [0, +\infty)$, $\psi \in \Psi$ and the constant $\lambda \in [0, 1)$ such that*

$$\alpha(\mathfrak{a}, \flat) \geq 1,$$

which implies

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) \leq \psi\left[d(\mathfrak{a}, \flat)^{\lambda} . \left(\frac{d(\mathfrak{a}, \mathcal{B}\mathfrak{a}) d(\mathfrak{a}, \mathcal{B}\flat) + d(\flat, \mathcal{B}\flat) d(\flat, \mathcal{B}\mathfrak{a})}{\max\{d(\mathfrak{a}, \mathcal{B}\flat), d(\flat, \mathcal{B}\mathfrak{a})\}} \right)^{1-\lambda} \right],$$

for all $\mathfrak{a}, \flat \in X \backslash Fix(\mathcal{B})$.

(i) $\mathcal{B}$ is an α-admissible mapping;

(ii) There exists a point $\mathfrak{a}_0 \in X$ such that $\alpha(\mathfrak{a}_0, \mathcal{B}\mathfrak{a}_0) \geq 1$;

(iii) $\mathcal{B} : X \to X$ is continuous, or, if $\{\mathfrak{a}_n\}$ is a sequence in X such that $\mathfrak{a}_n \to \mathfrak{a}^*$ and $\alpha(\mathfrak{a}_n, \mathfrak{a}_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\mathfrak{a}_n, \mathfrak{a}^*) \geq 1$ for all $n \in \mathbb{N}$.

Then there exists $\mathfrak{a}^* \in X$ such that $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Define $\eta : X \times X \longrightarrow [0, +\infty)$ by $\eta(\mathfrak{a}, \flat) = 1$ in Theorems 4 and 5. □

Corollary 4. *Let (X, d) be a complete $\mathfrak{F}$-MS, and let $\mathcal{B} : X \to X$ be a continuous function. Assume that there exist the function $\psi \in \Psi$ and the constant $\lambda \in [0, 1)$ such that*

$$d(\mathcal{B}\mathfrak{a}, \mathcal{B}\flat) \leq \psi\left[d(\mathfrak{a}, \flat)^{\lambda} . \left(\frac{d(\mathfrak{a}, \mathcal{B}\mathfrak{a}) d(\mathfrak{a}, \mathcal{B}\flat) + d(\flat, \mathcal{B}\flat) d(\flat, \mathcal{B}\mathfrak{a})}{\max\{d(\mathfrak{a}, \mathcal{B}\flat), d(\flat, \mathcal{B}\mathfrak{a})\}} \right)^{1-\lambda} \right]$$

for all $\mathfrak{a}, \flat \in X \backslash Fix(\mathcal{B})$. Then there exists $\mathfrak{a}^* \in X$ such that $\mathcal{B}\mathfrak{a}^* = \mathfrak{a}^*$.

Proof. Take $\alpha, \eta : X \times X \longrightarrow [0, +\infty)$ by $\alpha(\mathfrak{a},\flat) = \eta(\mathfrak{a},\flat) = 1$ in Theorems 4 and 5. $\square$

4. Applications

Fractional differential equations have emerged as powerful tools, with applications spanning various fields of science and engineering. Caputo fractional differential equations present a promising avenue for building more accurate and insightful economic growth models. By capturing memory effects and offering greater flexibility, Caputo fractional differential equations pave the way for a deeper understanding of economic dynamics and informed policy decisions (see [21–24]).

In the context of economic growth, the fractional differential equation

$$^{C}D^{\eta}(\mathfrak{a}(t)) = g(t, \mathfrak{a}(t)), \ (0 < t < 1, \ 1 < \eta \leq 2) \tag{24}$$

subject to the integral boundary conditions

$$\mathfrak{a}(0) = 0\,, \ I\mathfrak{a}(1) = \mathfrak{a}^{/}(0), \tag{25}$$

where $^{C}D^{\eta}g(t)$ represents the Caputo fractional derivative with respect to order η as defined by

$$^{C}D^{\eta}g(t) = \frac{1}{\Gamma(j-\eta)} \int_0^t (t-s)^{j-\eta-1} g^{j}(s)ds,$$

$(j - 1 < \eta < j, \ j = [\eta] + 1)$ and $g : [0, 1] \times \mathbb{R} \to [0, +\infty)$ is a continuous function and $I^{\eta}g$ denotes the Riemann–Liouville fractional integral of order η of a continuous function $f : \mathbb{R}^{+} \to \mathbb{R}$ given by

$$I^{\eta}f(t) = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} f(s)ds,$$

can be applied to model and analyze various aspects of economic dynamics. The variable $\mathfrak{a}(t)$ could represent the GDP or another economic indicator that characterizes the economic health of a country or region. The right-hand side consists of a nonlinear function $g(t, \mathfrak{a}(t))$ that encompasses various factors contributing to economic growth. This may include investment levels, innovation, education, government spending, and other elements influencing the overall economic output. The fractional order η reflects the memory and non-local effects in the economic system. It captures the impact of historical economic conditions on the current state, acknowledging that the rate of change of economic indicators might depend on past values.The condition $\mathfrak{a}(0) = 0$ could represent a starting point of economic activity or output at the beginning of the observation period, and $I\mathfrak{a}(1) = \mathfrak{a}^{/}(0)$ could signify a connection between the accumulated value of the economic variable over a certain period (from 0 to 1) and the rate of change of the economic variable at the beginning of the observation period.

Consider X=$\{\mathfrak{a} : \mathfrak{a} \in C([0, 1], \mathbb{R})\}$ with supremum norm $\|\mathfrak{a}\|_{\infty} = \sup_{t\in[0,1]} |\mathfrak{a}(t)|$. Then the Banach space $(X,\|\cdot\|_{\infty})$ provided with the $\mathfrak{F}$-metric d as elaborated by

$$d(\mathfrak{a}, \flat) = \|\mathfrak{a} - \flat\|_{\infty} = \sup_{t\in[0,1]} |\mathfrak{a}(t) - \flat(t)|$$

for $\mathfrak{a}, \flat \in X$ is metric space as well as an $\mathfrak{F}$-MS.

Theorem 6. *Consider the nonlinear fractional differential Equation (24). Let $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function. Under the premise that the requirements enumerated herein are fulfilled:*

(i) The function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(ii) There exists some $\sigma \in [1, \infty)$ such that the function $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies the following inequality

$$|g(t, \mathfrak{a}) - g(t, \flat)| \leq \frac{\Gamma(\eta + 1)}{4} e^{-\sigma} |\mathfrak{a} - \flat|$$

for all $\mathfrak{a}, \flat \in C([0, 1]$ and for all $t \in [0, 1]$;

(iii) There exists $\mathfrak{a}_0 \in C([0,1],\mathbb{R})$ such that $\zeta(\mathfrak{a}_0(t), \mathcal{B}\mathfrak{a}_0(t)) > 0$ for all $t \in [0,1]$, where a mapping $\mathcal{B} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is defined by

$$\begin{aligned}\mathcal{B}\mathfrak{a}(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{a}(s))ds \\ &\quad + \frac{2t}{\Gamma(\eta)}\int_0^1 \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{a}(m))dm\right)ds\end{aligned}$$

for $t \in [0,1]$;

(iv) For each $t \in [0,1]$ and $\mathfrak{a}, \flat \in C([0,1],\mathbb{R})$, $\zeta(\mathfrak{a}(t), \flat(t)) > 0$ implies that $\zeta(\mathcal{B}\mathfrak{a}(t), \mathcal{B}\flat(t)) > 0$;

(v) For $\{\mathfrak{a}_n\} \subseteq C([0,1],\mathbb{R})$ such that $\mathfrak{a}_n \to \mathfrak{a}$ in $C([0,1],\mathbb{R})$ and $\zeta(\mathfrak{a}_n, \mathfrak{a}_{n+1}) > 0$ for all $n \in \mathbb{N}$, then $\zeta(\mathfrak{a}_n, \mathfrak{a}) > 0$ for all $n \in \mathbb{N}$;
then, (24) has at least one solution.

Proof. A straightforward observation reveals that an element $\mathfrak{a} \in X$ satisfies Equation (24) if and only if it also satisfies the integral equation

$$\begin{aligned}\mathfrak{a}(t) &= \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{a}(s))ds \\ &\quad + \frac{2t}{\Gamma(\eta)}\int_0^1 \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{a}(m))dm\right)ds\end{aligned}$$

for $t \in [0,1]$. Now, let $\mathfrak{a}, \flat \in X$ such that $\zeta(\mathfrak{a}(t), \flat(t)) > 0$, for all $t \in [0,1]$. By (iii), we have

$$\begin{aligned}|\mathcal{B}\mathfrak{a}(t) - \mathcal{B}\flat(t)| &= \left| \begin{array}{l} \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\mathfrak{a}(s))ds - \frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1} g(s,\flat(s))ds \\ \qquad + \frac{2t}{\Gamma(\eta)}\int_0^1 \left(\int_0^s (s-m)^{\eta-1} g(m,\mathfrak{a}(m))dm\right)ds \\ \qquad - \frac{2t}{\Gamma(\eta)}\int_0^1 \left(\int_0^s (s-m)^{\eta-1} g(m,\flat(m))dm\right)ds \end{array} \right| \\ &\leq \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1} |g(s,\mathfrak{a}(s)) - g(s,\flat(s))|ds \\ &\quad + \frac{2t}{\Gamma(\eta)}\int_0^1 \left(\int_0^s (s-m)^{\eta-1} |g(m,\mathfrak{a}(m)) - g(m,\flat(m))|dm\right)ds,\end{aligned}$$

which implies that

$$\begin{aligned}|\mathcal{B}\mathfrak{a}(t) - \mathcal{B}\flat(t)| &\leq \frac{1}{\Gamma(\eta)}\int_0^t |t-s|^{\eta-1} \frac{\Gamma(\eta+1)}{4} e^{-\sigma} |\mathfrak{a}(s) - \flat(s)|ds \\ &\quad + \frac{2}{\Gamma(\eta)}\int_0^1 \left(\int_0^s |s-m|^{\eta-1} \frac{\Gamma(\eta+1)}{4} e^{-\sigma} |\mathfrak{a}(m) - \flat(m)|dm\right)ds \\ &= e^{-\sigma}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\int_0^t |t-s|^{\eta-1} |\mathfrak{a}(s) - \flat(s)|ds \\ &\quad + 2e^{-\sigma}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)}\int_0^1 \left(\int_0^s |s-m|^{\eta-1} |\mathfrak{a}(m) - \flat(m)|dm\right)ds \\ &\leq e^{-\sigma}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)} d(\mathfrak{a},\flat)\int_0^t |t-s|^{\eta-1} ds \\ &\quad + 2e^{-\sigma}\frac{\Gamma(\eta+1)}{4\Gamma(\eta)} d(\mathfrak{a},\flat)\int_0^1 \left(\int_0^s |s-m|^{\eta-1} dm\right)ds\end{aligned}$$

$$\begin{aligned}
&\leq e^{-\sigma}\frac{\Gamma(\eta)\Gamma(\eta+1)}{4\Gamma(\eta)\Gamma(\eta+1)}d(\mathfrak{a},\flat) \\
&\quad +2e^{-\sigma}B(\eta+1,1)\frac{\Gamma(\eta)\Gamma(\eta+1)}{4\Gamma(\eta)\Gamma(\eta+1)}d(\mathfrak{a},\flat) \\
&\leq \frac{e^{-\sigma}}{4}d(\mathfrak{a},\flat)+\frac{e^{-\sigma}}{2}d(\mathfrak{a},\flat)
\end{aligned}$$

considering the Beta function (denoted by B), the aforementioned inequality yields

$$d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)\leq e^{-\sigma}d(\mathfrak{a},\flat).$$

Applying the square root function to both sides and then exponentiating the resulting equation, we obtain

$$e^{\sqrt{d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)}}\leq e^{\sqrt{e^{-\sigma}d(\mathfrak{a},\flat)}}.$$

That is,

$$e^{\sqrt{d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)}}\leq\left(e^{\sqrt{d(\mathfrak{a},\flat)}}\right)^{\tau}$$

where $\tau=\sqrt{e^{-\sigma}}<1$. We now introduce a function, denoted by $\Theta:(0,+\infty)\to\mathbb{R}$, which is defined as $\Theta(t)=e^{\sqrt{t}}$ for each $t>0$. Then $\Theta\in\Omega$ and

$$\Theta(d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat))\leq[\Theta d(\mathfrak{a},\flat)]^{\tau} \tag{26}$$

for all $\mathfrak{a},\flat\in X$ and $d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)>0$. Now define $h:\mathbb{R}^{+}\times\mathbb{R}\to\mathbb{R}$ and Ξ:$\mathbb{R}^{+}\times\mathbb{R}\to\mathbb{R}$ by $h(x,y)=y$ and $\Xi(\ell,\wp)=\ell\wp$. Then the functions Ξ and h are C-class functions. Also, define $\alpha:X\times X\to\mathbb{R}^{+}$ by

$$\alpha(\mathfrak{a},\flat)=\begin{cases}1\text{ if }\zeta(\mathfrak{a}(t),\flat(t))>0,t\in[0,1],\\ 0,\text{ otherwise.}\end{cases}$$

Then from (26) and the above concepts, we have

$$h(\alpha(\mathfrak{a},\flat)),\Theta(d(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)))\leq\Xi\big(1,[\Theta d(\mathfrak{a},\flat)]^{\tau}\big).$$

Now, by using condition (iv), we have

$$\alpha(\mathfrak{a},\flat)\geq 1\Longrightarrow\zeta(\mathfrak{a}(t),\flat(t))>0,$$

which implies

$$\zeta(\mathcal{B}\mathfrak{a}(t),\mathcal{B}\flat(t))>0\text{ implies }\alpha(\mathcal{B}\mathfrak{a},\mathcal{B}\flat)\geq 1$$

for all $\mathfrak{a},\flat\in X$. Hence, $\mathcal{B}$ is an α-admissible. Also, from (iii), there exists $\mathfrak{a}_0\in X$ such that $\alpha(\mathfrak{a}_0,\mathcal{B}\mathfrak{a}_0)\geq 1$. Finally, we obtain simply that assertion (v) of Theorem 4 is satisfied. Hence, as an application of Theorem 4, we conclude the existence of $\mathfrak{a}^{*}\in X$ such that $\mathfrak{a}^{*}=\mathcal{B}\mathfrak{a}^{*}$. Thus, $\mathfrak{a}^{*}$ is a solution of (24). □

5. Conclusions

In this research article, we have defined two new concepts, Θ-(Ξ,h)-contraction and rational (α,η)-ψ-interpolative contraction, in the context of $\mathfrak{F}$-MS and established corresponding fixed point results. To solidify understanding and showcase the novelty of our findings, we have furnished an illustrative example that not only corroborates the obtained results but also sheds light on the established theory. Finally, we applied our leading theorem to discuss the existence and uniqueness of solutions for a fractional differential equation describing an economic growth model. This application not only highlights the practical relevance of our results but also opens avenues for further exploration in diverse scientific domains.

Fixed points of multi-valued mappings and fuzzy mappings satisfying Θ-(Ξ,h)-contraction and rational (α,η)-ψ-interpolative contraction in the framework of $\mathfrak{F}$-MS can be found as future work. This quest will yield the solutions for fractional differential inclusion problems, connecting theoretical breakthroughs to practical applications.

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