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Gravity on a Large Scale—Does It Necessarily Look like It Does on a Small Scale?

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Abstract: The notion of a local inertial reference frame is thoroughly analyzed. Dynamics of a field of such frames is derived from the variational principle. It is shown that the resulting theory splits naturally into three sectors, one of which is purely gravitational. Field dynamics in this sector, equivalent to Einstein's vacuum equations, is obtained *unambiguously* and admits no *ad hoc* corrections. The cosmological constant is an essential element of this construction and cannot be removed. It has been shown that the second sector of this theory corresponds to electrodynamics, while the last sector could possibly describe dark matter.

Keywords: gravitational physic; gravity and inertia; affine variational principle; large scale structure of the universe

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1. Introduction

Classical, 19th century physics perfectly describes our everyday experience. However, extrapolating its validity by eight orders of magnitude, down to the atomic scale, is not correct, as we learned a hundred years ago, and a new paradigm (quantum physics) is needed. Meanwhile, on the other side of the scale, we try to extrapolate the validity of General Relativity from our planetary scale (where it works perfectly) by 20 orders of magnitude, up to the cosmological scale, with the hope that small ad hoc improvements (like non-linear corrections to the Hilbert Lagrangian density: $\mathcal{L}_{\text{Hilbert}} = \frac{1}{2k}\sqrt{|\det g|}R$) will suffice completely. However, such small tweaks are unlikely to solve the really big problems observed on a cosmological scale. These problems are usually referred to as the "existence of dark matter and dark energy". However, this nice terminology means nothing more than that "General relativity, extrapolated by 20 orders of magnitude, does not correctly describe physical reality".

At the dawn of Einstein's theory, a hundred years ago, the problem of Mercury's orbit could also have been easily solved with a slight tweak to the existing theory, for example, changing the behavior of the gravitational potential from $\frac{1}{r}$ to $\frac{1}{r^{1+\epsilon}}$. However, Einstein did not resort to such a correction but proposed a broader context in which the old Newtonian theory was guaranteed the status of the limit of the new theory in situations where the velocities of the interacting bodies are very small.

In this article, we present a new formulation of the General Theory of Relativity, based on a careful analysis of what an "inertial frame" is. The analysis of the concept of inertia was the starting point of both Newton's and Einstein's theories. The advantage of our formulation over the conventional, metric formulation, proposed by Hilbert, is its "rigidity": there is no room here for any *ad hoc* corrections to vacuum Einstein equations as far as pure gravity is concerned. Moreover, the cosmological constant is not an optional element of the theory but its essential and necessary feature. On the other hand, the theory of pure gravity is only one of the three intrinsic sectors of the whole theory. In Section 7, we show that the second sector describes properly the electromagnetic field, while the last sector could possibly describe the observed large-scale effects.

2. Local Inertiality

When Newton came to the conclusion that the equations of motion of a body under the force F are of second differential order (and not first, like in Aristotle's *Physica,* where the action of a force caused *velocity* and not *acceleration* of the body), he realized that even the concept of a force vector makes no sense—unless we measure it in a privileged coordinate system. This is why Newton's second law is preceded by his first law (underestimated in many textbooks), stating the existence of such privileged coordinate systems, which Newton called "inertial". Indeed, the first-order equation

$$\dot{y}^{\alpha} = F^{\alpha} \tag{1}$$

is invariant with respect to nonlinear transformations of coordinates because both of its sides behave like vectors (a "dot" is always used here to denote the derivative with respect to an independent time parameter "s", e.g., the biological proper time of an astronaut in a spacecraft traveling between stars). However, the second-order equation

$$\ddot{y}^{\alpha} = F^{\alpha}\,, \tag{2}$$

with the same force vector F^{α}, makes *a priori* no sense, because the force can acquire an arbitrary value $\widetilde{F}$, when rewritten in a new coordinate system (x^{μ}) (below, the Einstein summation convention will always be used) as follows:

$$\dot{y}^{\alpha} = \frac{\partial y^{\alpha}}{\partial x^{\mu}}\dot{x}^{\mu} \tag{3}$$

$$\ddot{y}^{\alpha} = \frac{\partial^2 y^{\alpha}}{\partial x^{\nu}\partial x^{\mu}}\dot{x}^{\nu}\dot{x}^{\mu} + \frac{\partial y^{\alpha}}{\partial x^{\mu}}\ddot{x}^{\mu} \tag{4}$$

$$F^{\lambda} := \ddot{y}^{\alpha}\frac{\partial x^{\lambda}}{\partial y^{\alpha}} = \frac{\partial x^{\lambda}}{\partial y^{\alpha}}\frac{\partial^2 y^{\alpha}}{\partial x^{\nu}\partial x^{\mu}}\dot{x}^{\nu}\dot{x}^{\mu} + \ddot{x}^{\lambda}\,. \tag{5}$$

Denoting by $\Gamma^{\lambda}_{\mu\nu}$ the table of derivatives between old and new coordinates:

$$\Gamma^{\lambda}_{\mu\nu} := \frac{\partial x^{\lambda}}{\partial y^{\alpha}}\frac{\partial^2 y^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} = \Gamma^{\lambda}_{\nu\mu}\,, \tag{6}$$

We obtain the value of the new force $\widetilde{F}$, acting in the new coordinate system:

$$\ddot{x}^{\lambda} = F^{\lambda} - \Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} =: \widetilde{F}^{\lambda}\,, \tag{7}$$

which differs from the original vector F^{λ}. If the coordinates (y^{α}) are inertial and the force F vanishes, that is, our spacecraft is free falling, $\ddot{y}^{\alpha} = 0$, then the coordinates (x^{μ}) are also inertial if and only if the second derivatives between them vanish, i.e., $\frac{\partial^2 y^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} = 0$. However, the "inertial reference frame" cannot be identified with just one coordinate system, because if coordinates (y^{α}) are "inertial", then any linear transformation of them gives us an equally good, inertial coordinate system, defining exactly the same reference frame. The answer of a pure mathematician would be that the "inertial frame of reference" is an affine (linear) structure of spacetime, and only linear coordinates should be used if we want to avoid paradoxes such as (7), i.e., if the force vector is to be uniquely determined. However, unlike a pure mathematician, whose task is not to solve equations but to prove the existence (or non-existence) of solutions, we physicists are obliged to solve equations, and for this purpose, we need coordinates and not only linear ones. Therefore, we propose the following description of the concept of an inertial frame. Let us declare two coordinate systems (y^{α}) and (x^{μ}) to be *equivalent*, if the following second derivatives vanish:

$$\left\{(y^{\alpha}) \;\sim\; \left(x^{\lambda}\right)\right\} \Longleftrightarrow \frac{\partial^2 y^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} = 0\,. \tag{8}$$

Equivalence relations provide a standard tool to construct new mathematical objects from the already known ones. The crucial example being the construction of rational numbers from the natural numbers. The rational number a "half" cannot be identified with just one fraction $\frac{1}{2}$, because other fractions, like $\frac{2}{4}$ or $\frac{3}{6}$, represent exactly the same "half". Hence a rational number has to be identified with the class of equivalent fractions. To work properly, the relation must be (1) symmetric (if $A \sim B$ then also $B \sim A$), (2) reflexive (every A is equivalent with itself: $A \sim A$) and (3) transitive ($A \sim B$ and $B \sim C$ imply $A \sim C$). Any set of mathematical objects equipped with such a relation splits naturally to disjoint classes of mutually equivalent objects. It is easy to convince oneself that the relation (8) is symmetric, reflexive and transitive, and hence, it is a genuine equivalence relation. Consequently, the set of all spacetime coordinate systems splits into disjoint classes of mutually equivalent systems, each of them being called "a reference frame". However, only one of them has been chosen by Nature as a privileged one and will be called "an inertial frame". Any coordinate system belonging to this privileged class will be called "an inertial coordinate system". This definition is obviously equivalent to the pure mathematician's approach on one side and to the Newton's physical intuition on the other.

Hence, Newton's first law can be formulated as follows: among all the spacetime reference frames (i.e., all classes of equivalent coordinate systems), there is a privileged one, which has been chosen by Nature. We call it the "inertial frame". The free-falling bodies move uniformly along a straight line (i.e., satisfy the equation $\ddot{y}^\alpha = 0$) only in the inertial frame. Newton's second law is valid only when calculated in this frame, whereas working in a non-inertial frame, we have to take into account the "fictitious forces" $\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ (centrifugal, Coriolis etc.), see formulaes (5)–(7).

In Newtonian physics, coordinates were supposed to be *global.* When Einstein was looking for the relativistically invariant description of gravity, besides for the extremal properties of *ether* (cf. [1]), the main difficulty (cf. [2]) was the notion of a reference frame[1]. After many failed attempts (e.g., "the frame of reference is determined by distant stars"), he understood that the correct description of any physical phenomenon must be *local* and tried to avoid *global* objects. Unfortunately, the mathematical tools which properly describe the notion of a *local* reference frame appeared in modern differential geometry much later (cf. [3]). We propose here the most natural description, namely the *local* version of Newton's first law. For this purpose, we define the *local* version "$\sim_{\mathbf{m}}$" of the equivalence relation (8) between *local* coordinate systems, at each spacetime point $\mathbf{m} \in M$ separately, where by M, we denote the spacetime. Two local coordinate systems (y^α) and (x^μ), defined in a neighborhood of $\mathbf{m} \in M$, are declared to be "equivalent at $\mathbf{m}$" if and only if their second derivatives vanish at this point:

$$\left\{ (y^\alpha) \;\sim_{\mathbf{m}}\; \left(x^\lambda\right) \right\} \Longleftrightarrow \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(\mathbf{m}) = 0\,. \tag{9}$$

Even if x^μ and y^α appear here in a non-symmetric way, it may be easily proven that, indeed, this is a genuine equivalence relation (symmetric, reflexive and transitive). The local inertial frame at $\mathbf{m} \in M$ can be identified with an entire class of equivalent local coordinate systems. In this way, Newton's first law (global) can be replaced by the Einstein first law (local): at each spacetime point $\mathbf{m}$, there is a privileged reference frame, chosen by Nature, which we call the "local inertial frame". Free-falling bodies satisfy, at this point, the simple equation of motion

$$\ddot{y}^\alpha(\mathbf{m}) = 0 \tag{10}$$

when described in any coordinate system (y^α) which is inertial at $\mathbf{m}$, i.e., it belongs to the privileged class $r = [(y^\alpha)]$. Using formulaes (5)–(7), we also know its equation of motion in *any coordinate system.*

Hence, knowing the inertial frame at each spacetime point $\mathbf{m} \in M$, i.e., knowing the field of inertial frames, we already know the motion of all gravitating test bodies, i.e., we know the active properties of the gravitational field (how the gravitational field acts on massive bodies), i.e., we know the gravitational field. The goal of this paper is to show that also its passive properties (how massive bodies act on gravitational field) can be easily (and uniquely) derived in this context. This means that also from this point of view, the gravitational field can be identified with the field of local inertial frames. Unlike in the Newton's physics, they are not given a priori but form a dynamical component of the physical reality. As will be seen in the sequel, the formulation of the theory of gravity based on this idea is perfectly equivalent to Einstein's theory, but it has the advantage over the conventional "metric formulation" that gravity appears as only one particular sector of a much more general theory in which—perhaps—there is also room for a natural description of the observed large-scale effects (dark matter, dark energy???), without resorting to ad hoc corrections to the Hilbert Lagrangian.

3. Field of Inertial Frames: The Coordinate Description

Denote by $\mathcal{R}(M)$ all the local reference frames, at all the spacetime points and by $\mathcal{R}_{\mathbf{m}}(M)$ all the reference frames at the point $\mathbf{m}$. Given any coordinate system (x^μ), all abstract geometric objects like vectors, covectors, tensors, etc., acquire a coordinate description (X^μ for vectors, A_μ for covectors, etc.). We are going to construct a similar coordinate description for reference frames (elements of $\mathcal{R}(M)$) in any coordinates (x^μ) we want to use. For any local reference system $[(y^\alpha)] = r \in \mathcal{R}_{\mathbf{m}}(M)$ at $\mathbf{m}$, consider the table of numbers (6)

$$\Gamma^\lambda_{\mu\nu} := \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(\mathbf{m})\,, \tag{11}$$

where the coordinate system (y^α) was chosen among those belonging to the equivalence class r. It is easy to observe that this collection of numbers uniquely characterizes the frame r. For this purpose, one can easily check that (1) the value of $\Gamma^\lambda_{\mu\nu}$ does not depend upon the choice of a representant (y^α) within the class r, and (2) for any choice of a table of numbers $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, there is a single class of coordinate systems $r \in \mathcal{R}_{\mathbf{m}}(M)$ corresponding to the right-hand side of (11), namely the one which contains the following representant:

$$y^\lambda := x^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu}x^\mu x^\nu\,. \tag{12}$$

(Without any loss of generality, we have assumed here that coordinates (x^μ) are *centered* at $\mathbf{m}$, i.e., that coordinates (m^μ) of the point $\mathbf{m}$ vanish. Otherwise, "x^μ" in the above formula should be replaced by "$x^\mu - m^\mu$"). This way, $\mathcal{R}(M)$ becomes a fiber bundle over M with coordinates $(x^\mu, \Gamma^\lambda_{\mu\nu})$, whereas the latter are global coordinates in every fiber $\mathcal{R}_{\mathbf{m}}(M)$. The gravitational field, i.e., the field of reference frames, acquires, therefore, the coordinate description in terms of 40 functions $\Gamma^\lambda_{\mu\nu}(x^\mu)$ and, hence, can be identified with the spacetime symmetric connection[2]. Due to Formula (12), the values $\Gamma^\lambda_{\mu\nu}(\mathbf{m})$ acquire a simple geometric interpretation: it is a quadratic correction which is necessary to "upgrade" our working coordinate system (x^μ) to an inertial system at $\mathbf{m}$.

The identification of coefficients $\Gamma^\lambda_{\mu\nu}$ with the connection coefficients is a standard issue which was discussed in many texbooks. Indeed, to specify the spacetime connection, it is sufficient to specify at each spacetime point $\mathbf{m}$ a coordinate system (y^α) in which connection coefficients vanish (i.e., representing a local inertial frame). Once known, the connection coefficients in any other coordinate system (x^λ) are uniquely given by Formula (11). What is new in our approach is the physical interpretation of this object as the "field of local inertial frames" and its identification with the gravitational field. What is probably less known to non-specialists (but still well known to geometers) is the Formula (12) which enables the easy transition in an opposite direction: from the connection coefficients $\Gamma^\lambda_{\mu\nu}$ to the local inertial frame, represented by the coordinate system (y^α), in which the free-falling bodies satisfy the equation of motion (10).

It may happen—by chance—that our working system is inertial at the point $\mathbf{m}$. In such a case, we can take $y^\mu = x^\mu$, and Formula (11) gives immediately $\Gamma^\lambda_{\mu\nu}(\mathbf{m}) = 0$. We conclude that one can always "kill the connection coefficients $\Gamma^\lambda_{\mu\nu}$" at a single point, by an appropriate choice of a coordinate system, e.g., the one given by Formula (12).

The practical use of this approach is given by navigational problems. Here, the "loxodromy", i.e., the line which satisfies Equation (10) in geographic coordinates, is not the "true straight line" according to spherical geometry. This observation shows that geographic coordinates are not "inertial". (In fact, they are locally inertial at equator points!) To follow the "orthodrome", i.e., the great circle, which plays role of a straight line in spherical geometry, it is sufficient to find the correct "inertial frame" at each point of the sphere, which is a simple exercise in classical differential geometry.

4. Curvature Tensor

The connection is flat in a neighborhood of $\mathbf{m} \in M$ if there is a coordinate system (y^α), which is inertial not only at the point $\mathbf{m}$ but also in its neighborhood (like Newton's inertial coordinates). In other words, it is flat in a neighborhood if we are able to kill $\Gamma^\lambda_{\mu\nu}$ not only at a single point (which is always possible) but also in its neighbourhood. We already know that the zero-order corrections (i.e., "$(x^\mu - m^\nu)$" if the coordinates are not centered at $\mathbf{m}$ instead of (x^μ)) are irrelevant for this purpose because coordinates enter into (11) via their derivatives, exclusively. Hence, zero-order corrections of coordinates do not change the value of $\Gamma^\lambda_{\mu\nu}(\mathbf{m})$. Furthermore, first-order corrections (i.e., linear transformations of coordinates: $y^\alpha = A^\alpha{}_\mu x^\mu$) are irrelevant, because they cause the linear (tensorial) transformation of connection coefficients $\Gamma^\lambda_{\mu\nu}(\mathbf{m})$. Hence, they remain non-zero after such a transformation, if they were non-zero before. The second-order correction is *uniquely* given, if we want to kill $\Gamma^\lambda_{\mu\nu}(\mathbf{m})$. Hence, the only corrections which could carry out the task (to kill Γ's in a neighbourhood of $\mathbf{m}$) are 3rd and higher-order corrections:

$$y^\lambda := x^\lambda + \frac{1}{2}\Gamma^\lambda_{\mu\nu}(\mathbf{m})x^\mu x^\nu + \frac{1}{6}W^\lambda_{\mu\nu\kappa}x^\mu x^\nu x^\kappa + \frac{1}{24}U^\lambda_{\mu\nu\kappa\sigma}x^\mu x^\nu x^\kappa x^\sigma + \cdots . \tag{13}$$

(Again, without any loss of generality, we have assumed that (x^μ) vanish at $\mathbf{m}$). Furthermore, we can assume that the tables of coefficients W and U of a 3rd and 4th order polynomials are totally symmetric. They have no influence on the value of the new connection coefficients $\widetilde{\Gamma}^\lambda_{\mu\nu}(\mathbf{m})$, calculated in the corrected coordinate system (y^λ), because after being differentiated twice, they still vanish at $\mathbf{m} = (0, \ldots, 0)$. However, W changes the value of derivatives of Γ's at $\mathbf{m}$, which we denote by

$$\Gamma^\lambda_{\mu\nu\kappa} := \frac{\partial}{\partial x^\kappa}\Gamma^\lambda_{\mu\nu} \quad ; \quad \widetilde{\Gamma}^\lambda_{\mu\nu\kappa} := \frac{\partial}{\partial y^\kappa}\widetilde{\Gamma}^\lambda_{\mu\nu} . \tag{14}$$

It may be easily calculated that, after correction (13), we obtain

$$\widetilde{\Gamma}^\lambda_{\mu\nu}(\mathbf{m}) = 0 \quad ; \quad \widetilde{\Gamma}^\lambda_{\mu\nu\kappa}(\mathbf{m}) = \Gamma^\lambda_{\mu\nu\kappa}(\mathbf{m}) + W^\lambda_{\mu\nu\kappa} . \tag{15}$$

(On the other hand, the 4th order corrections U, and possible higher-order corrections, are irrelevant here, because after being differentiated 3 times, they vanish at $\mathbf{m}$). One could conclude erroneously that by choosing appropriately the 3rd order correction W, we are able to kill derivatives of Γ at $\mathbf{m}$. However, because of the total symmetry of W, we are able to kill only their totally symmetric part $\widetilde{\Gamma}^\lambda_{(\mu\nu\kappa)}$. We conclude, that the remaining part, namely

$$K^\lambda_{\mu\nu\kappa} := \Gamma^\lambda_{\mu\nu\kappa} - \Gamma^\lambda_{(\mu\nu\kappa)} , \tag{16}$$

calculated in an inertial frame, cannot be killed and remains as an indelible "obstruction" to the possibility of killing Γ in the vicinity of $\mathbf{m}$, i.e., as a measure of non-flatness. We call this remaining part a *curvature tensor.* Hence, the curvature tensor is simply the table of partial derivatives of connection coefficients, calculated in an inertial frame, with its totally symmetric (non-tensorial!) part subtracted.

It is a matter of simple calculations that formula (16), valid in an inertial coordinate system, implies the following formula:

$$K^{\lambda}_{\mu\nu\kappa} = \Gamma^{\lambda}_{\mu\nu\kappa} - \Gamma^{\lambda}_{(\mu\nu\kappa)} + \left(\Gamma^{\lambda}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\sigma(\kappa}\Gamma^{\sigma}_{\mu\nu)}\right) \tag{17}$$

$$= \Gamma^{\lambda}_{\mu\nu\kappa} + \Gamma^{\lambda}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\nu} - \left(\Gamma^{\lambda}_{(\mu\nu\kappa)} + \Gamma^{\lambda}_{\sigma(\kappa}\Gamma^{\sigma}_{\mu\nu)}\right), \tag{18}$$

which is valid in arbitrary coordinates. Due to the above definition, the curvature tensor fulfills ex definitione the following identities:

$$K^{\lambda}_{\mu\nu\kappa} = K^{\lambda}_{\nu\mu\kappa} \quad ; \quad K^{\lambda}_{(\mu\nu\kappa)} = 0\,. \tag{19}$$

On the other hand, the commonly known Riemann tensor

$$R^{\lambda}_{\mu\nu\kappa} := \Gamma^{\lambda}_{\mu\kappa\nu} - \Gamma^{\lambda}_{\mu\nu\kappa} + \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\lambda}_{\sigma\kappa}\Gamma^{\sigma}_{\mu\nu}\,, \tag{20}$$

fulfills identities

$$R^{\lambda}_{\mu\nu\kappa} = -R^{\lambda}_{\mu\kappa\nu} \quad ; \quad R^{\lambda}_{[\mu\nu\kappa]} = 0\,, \tag{21}$$

where the bracket "[ ]" denotes the complete *antisymmetrization* of the tensor. In fact, both the curvature and the Riemann tensors carry the same information because of the following, obvious, identities:

$$R^{\lambda}_{\mu\nu\kappa} = -2K^{\lambda}_{\mu[\nu\kappa]} = K^{\lambda}_{\mu\kappa\nu} - K^{\lambda}_{\mu\nu\kappa} \quad ; \quad K^{\lambda}_{\mu\nu\kappa} = -\frac{2}{3}R^{\lambda}_{(\mu\nu)\kappa} = -\frac{1}{3}\left(R^{\lambda}_{\mu\nu\kappa} + R^{\lambda}_{\nu\mu\kappa}\right). \tag{22}$$

As will be seen in the sequel, the use of the tensor K, instead of R, although totally equivalent, simplifies considerably the variational description of the field dynamics.

5. Field Dynamics—A Variational Approach

Leaving aside all preconceptions (and prejudices), we expect that the dynamics of the gravitational field Γ follows from a variational principle with the Lagrangian density $\mathcal{L}$ depending upon the field and its derivatives as follows:

$$\mathcal{L} = \mathcal{L}\left(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu\kappa}\right). \tag{23}$$

(The above quantity could, in principle, depend also upon additional matter fields and their derivatives—cf. [4–9]—but we limit ourselves here to the vacuum case, where matter is absent). To simplify considerably the further analysis of the field dynamics, it is useful to introduce the following auxiliary quantity $\mathcal{P}^{\mu\nu\kappa}_{\lambda}$, called by physically oriented authors (cf. [10] or [11]) *the momentum canonically conjugate* with the field configuration $\Gamma^{\lambda}_{\mu\nu}$:

$$\mathcal{P}^{\mu\nu\kappa}_{\lambda} := \frac{\partial\mathcal{L}}{\partial\Gamma^{\lambda}_{\mu\nu\kappa}}\,. \tag{24}$$

The use of this quantity enables us to write variational equations resulting from $\mathcal{L}$ (Euler–Lagrange equations or, simply, field equations of the theory) in the following, compact way:

$$\delta\mathcal{L}\left(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu\kappa}\right) = \left(\partial_{\kappa}\,\mathcal{P}^{\mu\nu\kappa}_{\lambda}\right)\delta\,\Gamma^{\lambda}_{\mu\nu} + \mathcal{P}^{\mu\nu\kappa}_{\lambda}\,\delta\,\Gamma^{\lambda}_{\mu\nu\kappa}\,, \tag{25}$$

where, according to a longstanding tradition going back to Euler and Lagrange, we consider any one-parameter family of field configurations $\Gamma^{\lambda}_{\mu\nu}(x^{\kappa}, \epsilon)$ and denote by $\delta := \frac{\partial}{\partial\epsilon}$ the derivative with respect to the parameter ϵ. Indeed, formula (25) contains both the definition of the momentum (24) (second term), and also the Euler–Lagrange equations of the theory (first term):

$$\left\{\partial_\kappa \frac{\partial \mathcal{L}}{\partial \Gamma^\lambda_{\mu\nu\kappa}} =\right\} \partial_\kappa \mathcal{P}_\lambda^{\mu\nu\kappa} = \frac{\partial \mathcal{L}}{\partial \Gamma^\lambda_{\mu\nu}} . \tag{26}$$

However, if we want field equations to be coordinate-invariant, the Lagrangian $\mathcal{L}$ cannot depend upon those variables in an arbitrary way but via the only invariant tensor which can be manufactured of them, i.e., the curvature tensor (or, equivalently, the Riemann tensor), exclusively as follows:

$$\mathcal{L} = \mathcal{L}\left(K^\lambda_{\mu\nu\kappa}\right) . \tag{27}$$

Choosing here K (instead of R) and using (17), we obtain

$$\frac{\partial \mathcal{L}}{\partial K^\lambda_{\mu\nu\kappa}} = \frac{\partial \mathcal{L}}{\partial \Gamma^\lambda_{\mu\nu\kappa}} = \mathcal{P}_\lambda^{\mu\nu\kappa} , \tag{28}$$

which highly simplifies the derivation of field Equations (cf. [4], where the affine variational principle was derived for the first time with the use of the Riemann tensor. Derivation was much longer and much more complicated there because the relation between the canonical momenta $\mathcal{P}_\lambda^{\mu\nu\kappa}$ and the derivatives of $\mathcal{L}$ with respect to $R^\lambda_{\mu\nu\kappa}$ is much more complicated than just (28)). This formula implies that the canonical momenta $\mathcal{P}$ inherit the symmetries from the symmetries (19) of the curvature tensor K as follows:

$$\mathcal{P}_\lambda^{\mu\nu\kappa} = \mathcal{P}_\lambda^{\nu\mu\kappa} \quad ; \quad \mathcal{P}_\lambda^{(\mu\nu\kappa)} = 0 . \tag{29}$$

It is a matter of simple calculations that we can rewrite (25) in an equivalent form as follows:

$$\delta\mathcal{L}\left(K^\lambda_{\mu\nu\kappa}\right) = \left(\nabla_\kappa \mathcal{P}_\lambda^{\mu\nu\kappa}\right) \delta\,\Gamma^\lambda_{\mu\nu} + \mathcal{P}_\lambda^{\mu\nu\kappa}\,\delta\,K^\lambda_{\mu\nu\kappa} . \tag{30}$$

Indeed, upgrading the partial derivative ∂_κ in Formula (25) to the covariant (with respect to the connection Γ) derivative ∇_κ, we have to add extra terms, exactly the same ones that we must subtract in order to upgrade the partial derivative $\Gamma^\lambda_{\mu\nu\kappa}$ of the connection in Formula (25) to the curvature tensor $K^\lambda_{\mu\nu\kappa}$. In the case of the vacuum, density $\mathcal{L}$ does not contain any matter field, i.e., Γ does not appear outside of K (otherwise, it could appear in the covariant derivatives of matter fields). Hence, Equation (30) implies the universal field equations:

$$\nabla_\kappa \mathcal{P}_\lambda^{\mu\nu\kappa} = 0 . \tag{31}$$

To fully analyze the field dynamics, we must know the relation between curvature $K^\lambda_{\mu\nu\kappa}$ and the momentum $\mathcal{P}_\lambda^{\mu\nu\kappa}$. For this purpose, we observe (cf. [12]) that the curvature tensor splits naturally into three irreducible sectors:

$$K^\lambda_{\mu\nu\kappa} = -\frac{1}{9}\left(\delta^\lambda_\mu K_{\nu\kappa} + \delta^\lambda_\nu K_{\mu\kappa} - 2\delta^\lambda_\kappa K_{\mu\nu}\right) - \frac{1}{5}\left(\delta^\lambda_\mu F_{\nu\kappa} + \delta^\lambda_\nu F_{\nu\kappa}\right) + U^\lambda_{\mu\nu\kappa} , \tag{32}$$

where $K_{\mu\nu}$ and $F_{\mu\nu}$ are, respectively, the symmetric and the antisymmetric part of the Ricci tensor:

$$R_{\mu\nu} := R^\lambda_{\mu\lambda\nu} = K_{\mu\nu} + F_{\mu\nu} \quad ; \quad K_{\mu\nu} = R_{(\mu\nu)} = \frac{3}{2}K^\lambda_{\mu\nu\lambda} \quad ; \quad F_{\mu\nu} = R_{[\mu\nu]} = -K^\lambda_{\lambda[\mu\nu]} , \tag{33}$$

whereas U is the remaining, traceless part of the curvature (which reduces to the Weyl tensor if Γ is the Levi–Civita metric connection). Similarly, also the momentum tensor can be decomposed (cf. [12]) as a sum of three irreducible terms:

$$P_\lambda^{\mu\nu\kappa} = \left(\delta^\kappa_\lambda \pi^{\mu\nu} - \delta^{(\mu}_\lambda \pi^{\nu)\kappa}\right) - \frac{1}{2}\left(\delta^\mu_\lambda \mathcal{F}^{\nu\kappa} + \delta^\nu_\lambda \mathcal{F}^{\mu\kappa}\right) + p_\lambda^{\mu\nu\kappa} , \tag{34}$$

where

$$\pi^{\mu\nu} = \frac{1}{3} P^{\mu\nu\lambda}_{\lambda} \quad ; \quad \mathcal{F}^{\mu\nu} = -\frac{2}{5} P^{\lambda[\mu\nu]}_{\lambda} \tag{35}$$

and $p^{\mu\nu\kappa}_{\lambda}$ is the the remaining, traceless part of $P^{\mu\nu\kappa}_{\lambda}$. (We stress that (32) and (34) are purely algebraic, trivial identities).

Plugging these decompositions into the generating formula (30), we obtain

$$\begin{aligned} P^{\mu\nu\kappa}_{\lambda} \delta K^{\lambda}_{\mu\nu\kappa} &= -\frac{1}{9} P^{\mu\nu\kappa}_{\lambda} \delta\left(\delta^{\lambda}_{\mu} K_{\nu\kappa} + \delta^{\lambda}_{\nu} K_{\mu\kappa} - 2\delta^{\lambda}_{\kappa} K_{\mu\nu}\right) - \frac{1}{5} P^{\mu\nu\kappa}_{\lambda} \delta\left(\delta^{\lambda}_{\mu} F_{\nu\kappa} + \delta^{\lambda}_{\nu} F_{\mu\kappa}\right) + P^{\mu\nu\kappa}_{\lambda} \delta U^{\lambda}_{\mu\nu\kappa} \\ &= \pi^{\mu\nu} \delta K_{\mu\nu} + \mathcal{F}^{\mu\nu} \delta F_{\mu\nu} + p^{\mu\nu\kappa}_{\lambda} \delta U^{\lambda}_{\mu\nu\kappa} \,, \end{aligned} \tag{36}$$

and, hence, the whole field dynamics follows from the following, new version of the generating Formula (30)

$$\delta \mathcal{L}\left(K_{\mu\nu}, F_{\mu\nu}, U^{\lambda}_{\mu\nu\kappa}\right) = \left(\nabla_{\kappa} \mathcal{P}^{\mu\nu\kappa}_{\lambda}\right) \delta \Gamma^{\lambda}_{\mu\nu} + \pi^{\mu\nu} \delta K_{\mu\nu} + \mathcal{F}^{\mu\nu} \delta F_{\mu\nu} + p^{\mu\nu\kappa}_{\lambda} \delta U^{\lambda}_{\mu\nu\kappa} \,. \tag{37}$$

We conclude that the complete theory is composed of the three physical sectors, each of them described by the particular sensitivity of the Lagrangian density $\mathcal{L}$ to one of the three irreducible parts of the curvature: K, F and U, respectively.

6. Gravity as We Know It Today

The first, natural conjecture about the structure of the "World-Lagrangian-density" $\mathcal{L}$ would be its total insensitivity to other parts of the curvature that the symmetric part $K_{\mu\nu}$ of Ricci , i.e.,

$$\mathcal{L} = \mathcal{L}(K_{\mu\nu}) \Longleftrightarrow \left\{ \mathcal{F}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} = 0 \ \text{ and } \ p^{\mu\nu\kappa}_{\lambda} = \frac{\partial \mathcal{L}}{\partial U^{\lambda}_{\mu\nu\kappa}} = 0 \right\}. \tag{38}$$

If this is true, we have consequently, due to (34),

$$P^{\mu\nu\kappa}_{\lambda} = \left(\delta^{\kappa}_{\lambda} \pi^{\mu\nu} - \delta^{(\mu}_{\lambda} \pi^{\nu)\kappa}\right) \quad \text{where} \quad \pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial K_{\mu\nu}} \,, \tag{39}$$

(the last equation being a consequence of (37)) and the universal (in vacuum) field Equation (31) reduces to

$$\nabla_{\kappa} \pi^{\mu\nu} = 0 \,. \tag{40}$$

To solve this equation, let us observe that π is not a tensor but the tensor density (because $\mathcal{L}$ is a scalar density). Denote by $g^{\mu\nu}$, the corresponding contravariant tensor

$$g^{\mu\nu} := \frac{1}{2k} \frac{\pi^{\mu\nu}}{|\det \pi|} \Longleftrightarrow \pi^{\mu\nu} = \frac{1}{2k} \sqrt{|\det g_{\alpha\beta}} \, g^{\mu\nu} \,, \tag{41}$$

where $g_{\alpha\beta}$ is its inverse (covariant) tensor. Already in 1938 V.A. Fock observed in his famous book [13] that both the Lagrangian and the Hamiltonian descriptions of General Relativity Theory simplify considerably if we represent metric structure of spacetime by its contravariant density $\pi^{\mu\nu}$, instead of the conventional representation by the covariant tensor $g_{\mu\nu}$. (The constant "k" has been introduced here for dimensional reasons. It encodes in a proper way the gravitational constant. In the geometric system of units, see "red pages" in the monograph [14], we have $k = 8\pi$). Now, Equation (40) is equivalent to

$$\nabla_{\kappa} g_{\mu\nu} = 0 \,, \tag{42}$$

and the only solution is the metric connection given by the Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_{\mu} g_{\nu\kappa} + \partial_{\nu} g_{\mu\kappa} - \partial_{\kappa} g_{\mu\nu}\right) , \tag{43}$$

which implies the vanishing of the antisymmetric part $F_{\mu\nu} = 0$ of the Ricci tensor as follows:

$$K_{\mu\nu} = R_{\mu\nu} \,. \tag{44}$$

On the other hand, there is not much freedom in "inventing" the value of $\mathcal{L}$ in this, purely gravitational, case because the only scalar density which can be manufactured from the covariant tensor $K_{\mu\nu}$ is $\sqrt{|\det K|}$. Hence, we have *unambiguously*

$$\mathcal{L} = C \cdot \sqrt{|\det K|} \,, \tag{45}$$

where the constant C is necessary because of dimensional reasons ($\mathcal{L}$ must have the dimension of the action density, but the actual dimension of C depends upon the system of units and, moreover, upon the convention concerning the dimensionality of coordinates (x^μ)). Hence, the "momentum" $\pi^{\mu\nu}$ given by (39) equals

$$\left\{ \frac{1}{2k} \sqrt{|\det g_{\alpha\beta}|}\, g^{\mu\nu} = \right\} \pi^{\mu\nu} = C \cdot \frac{\partial \sqrt{|\det K|}}{\partial K_{\mu\nu}} = \frac{C}{2} \sqrt{|\det K|} \left(K^{-1} \right)^{\mu\nu} , \tag{46}$$

and consequently,

$$\begin{aligned} \sqrt{|\det g_{\alpha\beta}|} g^{\mu\nu} &= Ck \sqrt{|\det K|} \left(K^{-1} \right)^{\mu\nu} && (47) \\ \sqrt{|\det g_{\alpha\beta}|} &= (Ck)^2 \sqrt{|\det K|} && (48) \\ Ck\, g^{\mu\nu} &= \left(K^{-1} \right)^{\mu\nu} && (49) \\ \frac{1}{Ck}\, g_{\mu\nu} &= K_{\mu\nu} = R_{\mu\nu} \,. && (50) \end{aligned}$$

Comparing it with the vacuum Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \quad \Longleftrightarrow \quad R_{\mu\nu} = \Lambda g_{\mu\nu} = K_{\mu\nu} \,, \tag{51}$$

we see that—to be able to interpret this theory as a conventional theory of gravity—the tensor $g_{\mu\nu}$ must be interpreted as the spacetime metric tensor, whereas

$$\Lambda = \frac{1}{kC} \tag{52}$$

is the cosmological constant. Consequently, $C = \frac{1}{k\Lambda}$. We see that, unlike in the conventional metric formulation of General Relativity, the dynamic equations of the theory presented here follow *unambiguously* from its geometric structure: the Lagrangian density (45) is *unique* here, unlike in the conventional metric formulation, where there is an infinite number of Lagrangian densities which can be manufactured from the metric tensor and its first and second derivatives. Moreover, the cosmological constant is *necessary* here. By imposing anthropological constraints on the Ricci signature $R_{\mu\nu}$, we can rule out unwanted, non-anthropological (i.e., having a non-wanted signature) solutions to the theory; although, they should probably be treated more seriously than simply "non-existent".

7. Electromagnetic Sector of the Theory

It is hard to believe that the constraints (38) (i.e., $\mathcal{F}^{\mu\nu} = 0$ and $p_\lambda^{\mu\nu\kappa} = 0$) are fundamental, but rather, they describe solutions which we meet in everyday life, where the value of these quantities is very small with respect to the remaining components (39) of the momentum. In other words, the solutions of the theory that we encounter on the anthropomorphic scale are characterized by the following structure of the momentum $\mathcal{P}$:

$$P_{\lambda}^{\mu\nu\kappa} = \frac{1}{2k}\sqrt{|\det g|}\left(\delta^{\kappa}_{\lambda}g^{\mu\nu} - \delta^{(\mu}_{\lambda}g^{\nu)\kappa}\right). \tag{53}$$

Removing the first of these constraints, we admit Lagrangian densities which can depend upon the whole Ricci tensor.

In 1918, Hermann Weyl already conjectured (see [15]) that the electrodynamic field $f_{\mu\nu}$ could somehow be related to non-metricity of the connection. As a measure of the non-metricity, we can use the antisymmetric part $F_{\mu\nu}$ of the Ricci because of the following identity:

$$F_{\mu\nu} = \partial_{\mu}\Gamma^{\lambda}_{\nu\lambda} - \partial_{\nu}\Gamma^{\lambda}_{\mu\lambda}\,, \tag{54}$$

which means that the trace $\Gamma^{\lambda}_{\nu\lambda}$ of the connection coefficients plays a role similar to electrodynamic f-potential A_{ν}. This observation implies a priori the first pair of Maxwell equations:

$$\partial_{[\lambda}F_{\mu\nu]} \equiv 0\,. \tag{55}$$

Moreover, for a metric connection (43), we have

$$\Gamma^{\lambda}_{\lambda\nu} = \frac{1}{2}g^{\lambda\kappa}(\partial_{\lambda}g_{\nu\kappa} + \partial_{\nu}g_{\lambda\kappa} - \partial_{\kappa}g_{\lambda\nu}) = \frac{1}{2}g^{\lambda\kappa}\,\partial_{\nu}g_{\lambda\kappa} = \partial_{\nu}\left(\ln\sqrt{\det g}\right), \tag{56}$$

and consequently, $F_{\mu\nu} = 0$ because of (54).

However, F can not be directly identified with the electromagnetic field f because of dimensional reasons. Weyl was unable to find uniquely this identification and to capture the second pair of the Maxwell equations as an intrinsic property of their framework. Below, we prove that this is almost uniquely possible within our framework.

We have, therefore, at our disposal only three scalar densities which can be manufactured out of $R_{\mu\nu}$, namely (1) $\sqrt{|\det R|}$, (2) $\sqrt{|\det K|}$ and (3) $\sqrt{|\det F|}$. Limiting ourselves to the weak-field region, where gravitational field satisfying vacuum Equation (50) prevails, we are going to treat $F_{\mu\nu}$ as its 1st order perturbation. For this purpose, we can skip the quantity number (3) as being a higher-order function of F. Hence, we have at our disposal the following family of conjectures:

$$\mathcal{L} = A\cdot\sqrt{|\det R|} + B\cdot\sqrt{|\det K|} = A\cdot\sqrt{|\det(K+F)|} + B\cdot\sqrt{|\det K|} \tag{57}$$

Treating F as a 1st order correction to K, we see that the zero-order approximation equals

$$\mathcal{L} = (A+B)\cdot\sqrt{|\det K|}\,. \tag{58}$$

As proved in the previous Section, the zero-order approximation of the theory is, therefore, the vacuum Einstein theory with a cosmological constant equal to

$$\Lambda = \frac{1}{k(A+B)}\,. \tag{59}$$

Assuming that the gravitational part (50) of the solution prevails, i.e., that we have

$$\det(K_{\mu\nu} + F_{\mu\nu}) \simeq \det(\Lambda g_{\mu\nu} + F_{\mu\nu}) = \Lambda^4\cdot\det(g_{\mu\lambda})\cdot\det\left(\delta^{\lambda}_{\nu} + \frac{1}{\Lambda}g^{\lambda\mu}F_{\mu\nu}\right), \tag{60}$$

we can use the following algebraic identity (cf. [7]) involving (1) a symmetric—**A**—and (2) an antisymmetric—**B**—matrix in four dimensions, namely

$$\det(\mathbb{I} + \mathbf{A}\cdot\mathbf{B}) = 1 - \frac{1}{2}Tr(\mathbf{A}\cdot\mathbf{B})^2 + \frac{1}{8}\left(Tr(\mathbf{A}\cdot\mathbf{B})^2\right)^2 - \frac{1}{4}Tr(\mathbf{A}\cdot\mathbf{B})^4\,. \tag{61}$$

Here, "$\mathbb{I}$" denotes the unit matrix, and "Tr" stands for the "trace" operation. In linear approximation, i.e., when $\mathbf{B} = \frac{1}{\Lambda}F$ is much smaller than $\mathbf{A} = g$, we can reject the last two terms which are of order four in F and leave only the quadratic term:

$$Tr(\mathbf{A} \cdot \mathbf{B})^2 = \frac{1}{\Lambda^2} g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu} F_{\nu\mu} = -\frac{1}{\Lambda^2} g^{\mu\alpha} F_{\mu\nu} F_{\alpha\beta} g^{\nu\beta} \,. \tag{62}$$

Hence,

$$\sqrt{|\det(K_{\mu\nu} + F_{\mu\nu})|} \simeq \Lambda^2 \cdot \sqrt{|\det g_{\mu\lambda}|} \cdot \left(1 + \frac{1}{\Lambda^2} g^{\mu\alpha} F_{\mu\nu} F_{\alpha\beta} g^{\nu\beta}\right)^{\frac{1}{2}} \tag{63}$$

$$\simeq \sqrt{|\det g_{\mu\lambda}|} \cdot \left(\Lambda^2 + \frac{1}{2} g^{\mu\alpha} F_{\mu\nu} F_{\alpha\beta} g^{\nu\beta}\right) . \tag{64}$$

We conclude that given a purely gravitational zero-order approximation (51) of the complete solution, the first-order approximation follows from the following Lagrangian density for the field F:

$$\begin{aligned} \mathcal{L} &= A \cdot \sqrt{|\det(K + F)|} + B \cdot \sqrt{|\det K|} \\ &= \sqrt{|\det g_{\mu\lambda}|} \cdot \left((A + B)\Lambda^2 + \frac{A}{2} g^{\mu\alpha} F_{\mu\nu} F_{\alpha\beta} g^{\nu\beta}\right) \\ &= \frac{\Lambda}{k} \cdot \sqrt{|\det g_{\mu\lambda}|} - \frac{1}{4}\sqrt{|\det g_{\mu\lambda}|}\left(-2A g^{\mu\alpha} F_{\mu\nu} F_{\alpha\beta} g^{\nu\beta}\right) . \end{aligned} \tag{65}$$

Comparing it with the Lagrangian of electromagnetic field $f_{\mu\nu}$ on a given metric background,

$$\mathcal{L}_{\text{el-mag}} = -\frac{1}{4}\sqrt{|\det g_{\mu\lambda}|}\left(g^{\mu\alpha} f_{\mu\nu} f_{\alpha\beta} g^{\nu\beta}\right), \tag{66}$$

we see that in the weak field region our theory agrees with the Maxwell electrodynamics provided that (1) the constant $A = \frac{1}{k\Lambda} - B$ (see (59)) is negative ($-A = |A|$) and (2) the electromagnetic field is defined as

$$f_{\mu\nu} := \sqrt{2|A|} \cdot F_{\mu\nu} = \sqrt{2\left|\frac{1}{k\Lambda} - B\right|} \cdot F_{\mu\nu} = \sqrt{2\left|\frac{\Lambda}{k} - B\Lambda^2\right|} \cdot \frac{1}{\Lambda} F_{\mu\nu} \,. \tag{67}$$

For both negative and positive values of the cosmological constant Λ (which is assumed to be known from astronomical observations), it is possible to choose the constant B in such a way that (1) A is negative and (2) the value of $\frac{1}{\Lambda}F$ is much smaller than g, or equivalently,

$$\frac{1}{\Lambda} \cdot F_{\mu\nu} = \left(\sqrt{2\left|\frac{\Lambda}{k} - B\Lambda^2\right|}\right)^{-1} \cdot f_{\mu\nu} \ll g_{\mu\nu} \,, \tag{68}$$

for electromagnetic fields which we encounter in our anthropomorphic scale.

Note that for slightly stronger fields, when the approximation method (60) cannot be used, the full Lagrangian $\sqrt{\det(K + F)}$ leads to Born–Infeld electrodynamics (cf. [16]), for which Maxwell electrodynamics provides the weak field approximation.

In this approach, electromagnetism is just one sector of a much larger "field of inertial frames" $\Gamma^{\lambda}_{\mu\nu}$, the other sector of which is gravity. The fact that it decouples entirely from gravity in our anthropomorphic scale is similar to the same phenomenon which occurs in hydrodynamics: For a barotropic fluid, the pressure p and its density ρ (or its proper volume $v := \frac{1}{\rho}$) are related by the constitutive equation

$$\mathrm{d}U(v) = -p \cdot \mathrm{d}v \,, \tag{69}$$

where U represents the internal energy (per one mole of the liquid). For most engineering applications, a linear approximation of the constitutive equation

$$\Delta U = -(p_0 + \Delta p)\Delta v \,, \tag{70}$$

resulting from expanding the function U around some equilibrium state (v_0, p_0) up to second-order terms, is sufficient:

$$U(v_0 + \Delta v) \simeq U(v_0) + U'(v_0) \cdot \Delta v + \frac{1}{2}U''(\Delta v)^2 \quad \Longrightarrow \quad \left\{ \begin{array}{c} -p_0 = U'(v_0) \\ -\Delta p = \frac{1}{2}U'' \cdot \Delta v \end{array} \right. . \tag{71}$$

The zero-order term $U(v_0)$ in this expansion is irrelevant (in physics we never measure the absolute value of energy, only work, i.e., the energy difference between two physical states). The first-order term only confirms that the general physical law (69) (or, equivalently (70)) also holds for the equilibrium state (v_0, p_o). Finally, the second-order term gives us the relation between Δv and Δp in a linear approximation as follows:

$$-\Delta p = H \Delta v \,,$$

where the quantity $H := \frac{1}{2}U''(v_0)$ represents the "rigidity", i.e., the fluid sensitivity to the changes of its proper volume (i.e., of its density). When this sensitivity becomes enormous, then the model of an incompressible fluid is more suitable, because Δv is too small to be really observed and, consequently, $v = v_0$ remains (almost) unchanged. In this model, the pressure p decouples from v on the kinematical level and becomes an independent physical quantity. In our theory, the constant $\sqrt{2|A|}$ (very big, of the order of Λ^{-1}, see Formula (67)) plays the role of the "rigidity", i.e., the "sensitivity of the electromagnetism to non-metricity of the connection", the latter being too small to be measured at the anthropomorphic scale. This is why electromagnetic field decouples from gravity in our everyday life.

The idea to unify gravity and electromagnetism appears very often in Einstein's writings. The most mature version of their unification schemes was presented in [17] (see also the review article [18]). He proposes their unification at the level of the "non-symmetric metric tensor", the symmetric part of which would describe gravity, while the anti-symmetric part would somehow be identified with the Faraday tensor $f_{\mu\nu}$. The unification proposed by us in the present Section does not occur at the level of the metric (which is geometrically prohibited), but at the level of the Ricci curvature, whose symmetric (gravity) and antisymmetric (electromagnetism) parts have the same footings and are equally admissible. From this point of view, our construction is similar to Hermann Weyl's idea [15], with the difference that we do not assume a priori any particular form of connection (namely, its compatibility with the conformal structure of spacetime), but we consequently derive its dynamics from the variational principle. Other proposals have recently appeared (cf. e.g., [19,20]) which we consider less convincing.

8. Conclusions

To describe the properties of spacetime in the medium scale, the above two sectors, which properly describe gravity and electromagnetism, are sufficient. The dynamics of the field in the purely gravitational sector follows *unambiguously* from the geometric structure of the field. Furthermore, the electromagnetic sector obtains its dynamics in the weak field region uniquely from the geometrical structure of the Γ field, while the strong field follows the non-linear Born–Infeld dynamics. However, of course, the main problem is whether the last sector of the above theory can be useful for describing physical reality. Unlike in the conventional metric formulation of gravity, there is not much choice here to construct $\mathcal{L}$ that depends in a non-trivial way on the entire curvature tensor, not just the Ricci tensor. Indeed, the metric tensor is not available when constructing the Lagrangian density $\mathcal{L}$ but must be reconstructed from Equation (39) once $\mathcal{L}$ is already known. Therefore, unlike the conventional metric formulation, where we may contract the curvature and the metric in many non-equivalent ways, here there are very few ways to construct $\mathcal{L}$. We suppose that this sector would describe the large-scale phenomena, commonly attributed to the mysterious "dark matter" or "dark energy".

In any case, it is hard to believe that Nature would neglect the last sector of the theory, whose two sectors describe gravity and electromagnetism in such a simple way, leaving us no freedom to choose the Lagrangian density of the field, (i.e., the field dynamics).

A natural assumption is that U in (32) is very small compared to K in our tiny corner of the Universe in which we live. Moreover, it is likely that this field is practically constant on our anthropomorphic scale (in both timelike and spatial directions), and this is why we do not observe it directly. However, due to its "ubiquity", its impact on the global shape of spacetime can be significant. In any case, the slowly changing field U can be probably treated as constant in the description of both gravity and electromagnetism in the middle scale. However, in the large (cosmological) scale, this field follows its slow dynamics, causing changes in the values of fundamental physical constants (such as the cosmological constant Λ), which was already predicted by P. A. M. Dirac long time ago. The analysis of the possible dynamical effects of the complete theory, based on conjecture (27), is in progress and will be presented soon.

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Notes

[1] Furthermore, he was already aware of the fact that there is no natural splitting of "spacetime" into "space" and "time" and, whence, equations of motion (2) must always be four-dimensional, like, e.g., in electrodynamics, where we have: $F^{\lambda} = q \cdot f^{\lambda\kappa} u_{\kappa}$.

[2] In this context "generalizations" towards non-symmetric connections is a nonsense, because such a connection is not an irreducible object. It splits canonically into two disjoint terms: (1) a symmetric connection and (2) a tensor (torsion). The tensor fields appear in this framework as matter fields, but a field of inertial frames defines uniquely a *symmetric* connection, which we identify with gravitational field.

References

1. Einstein, A. *Über den Äther*; Schweizerische Naturforschende Gessellschaft: 1924; Volume 105.
2. Einstein, A. Autobiographical Notes. In *Albert Einstein: Philosopher-Scientist*; Schild, P.A., Ed.; Northwestern University Press: Evanstone, IL, USA, 1949.
3. Einstein, A. Nichteuklidische Geometrie und Physik. In *Die Neue Rundschau, XXXVI. Jahrgang der freien Bühne, Band 1*; 1925.
4. Kijowski, J. On a new variational principle in general relativity and the energy of gravitational field. *Gen. Relat. Grav.* **1978**, *9*, 857–881. [CrossRef]
5. Kijowski, J.; Tulczyjew, W.M. *A Symplectic Framework for Field Theories*; Springer Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 1979; Volume 107.
6. Kijowski, J.; Werpachowski, R. Universality of affine formulation in General Relativity. *Rep. Math. Phys.* **2007**, *59*, 1–31. [CrossRef]
7. Bąk, B. Variational formulations of General Relativity. *Acta Phys. Pol. B Proc. Suppl.* **2022**, *15*, 1-A1. [CrossRef]
8. Kijowski, J. A simple Derivation of Canonical Structure and quasi-local Hamiltonians in General Relativity. *Gen. Relat. Grav.* **1997**, *29*, 307–343. [CrossRef]
9. Chruściel, P.T.; Jezierski, J.; Kijowski, J. *Hamiltonian Field Theory in the Radiating Regime*; Springer Lecture Notes in Physics; Springer: Berlin/Heidelberg, Germany, 2001; Volume 70, 174p.
10. Bogoliubov, N.N.; Shirkov, D.V. *Introduction to the Theory of Quantized Fields*; John Wiley & Sons: Hoboken, NJ, USA, 1959; Original, Russian Version Appeared in 1976.
11. Białynicki-Birula, I.; Białynicka-Birula, Z. *Quantum Electrodynamics*; Pergamon: Oxford, UK, 1975.
12. Kijowski, J. Universality of the Einstein theory of gravitation. *Int. J. Geom. Methods Mod. Phys.* **2016**, *13*, 1640008. [CrossRef]

13. Fock, V.A. *The Theory of Space, Time and Gravitation*; Pergamon: Oxford, UK, 1964; Original, Russian Version Appeared in 1938.
14. Misner, C.W.; Thorne, K.S.; Wheeler, J.A.; *Gravitation*; W.H. Freeman and Company: San Francisco, CA, USA, 1973.
15. Weyl, H. Gravitation und Elektrizität. In *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*; 1918; pp. 465–480.
16. Born, M.; Infeld, L. Foundations of the New Field Theory. *Proc. R. Soc. Math. Phys. Eng. Sci.* **1934**, *144*, 425–451. [CrossRef]
17. Einstein, A. *Einheitliche Feldtheorie von Gravitation und Elektizität*; Verlag der Koeniglich-Preussichen Akademie der Wissenschaften: Berlin, Germany, 1925; pp. 414–419.
18. Goenner, H.F.M. On the History of Unified Field Theories. *Living Rev. Relativ.* **2004**, *7*, 2. [CrossRef] [PubMed]
19. Poplawski, N.J. On the Nonsymmetric Purely Affine Gravity. *Mod. Phys. A* **2007**, *22*, 2701–2720. [CrossRef]
20. Poplawski, N.J. Gravitation, Elektromagnetism and the Cosmological Constant in Purely Affine Gravity. *Found. Phys.* **2009**, *39*, 307–330. [CrossRef]