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The Dynamical and Kinetic Equations of Four-Five-Six-Wave Resonance for Ocean Surface Gravity Waves in Water with a Finite Depth

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Abstract: Based on the Hamilton canonical equations for ocean surface waves with four-five-six-wave resonance conditions, the determinate dynamical equation of four-five-six-wave resonances for ocean surface gravity waves in water with a finite depth is established, thus leading to the elimination of the nonresonant second-, third-, fourth-, and fifth-order nonlinear terms though a suitable canonical transformation. The four kernels of the equation and the 18 coefficients of the transformation are expressed in explicit form in terms of the expansion coefficients of the gravity wave Hamiltonian in integral-power series in normal variables. The possibilities of the existence of integrals of motion for the wave momentum and the wave action are discussed, particularly the special integrals for the latter. For ocean surface capillary–gravity waves on a fluid with a finite depth, the sixth-order expansion coefficients of the Hamiltonian in integral-power series in normal variables are concretely provided, thus naturally including the classical fifth-order kinetic energy expansion coefficients given by Krasitskii.

Keywords: dynamical equation; kinetic equation; four-five-six-wave resonance; surface gravity waves; finite water depth

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1. Introduction

The development of ocean surface wave theories has focused on the mechanism of wave–wave resonance [1–5]. Two integro-differential equations [6], based on this mechanism, are central to wave turbulence theory. One of them is the dynamical equation (also known as the "Zakharov equation") [7,8], which is based on the Hamiltonian function of wave energy and the linear dispersion relation, and the other one is the kinetic equation (also known as the "Hasselmann equation") [9,10], which is based on the wave numbers (or wave action) [7,9–17]. The former is deterministic, and the latter is stochastic. The broad-banded wave turbulence theory is contrasted with the classical narrow-banded high-order Stokes waves with a nonlinear dispersion relation [18] and its extended theories; however, it is also implicated in [19,20]. At present, the wave turbulence theories of ocean surface waves are limited to the three-wave resonance of surface capillary waves in deep water [21] and the four-wave [9] and four-five-wave resonance [7] of surface gravity waves in a finite water depth. Therefore, the next step is to establish a wave turbulence theory of four-five-six-wave resonance for surface gravity waves in a finite water depth.

The existing spatial scale of the ocean can satisfy the six-wave resonance interactions. If a wave length, a wave steepness [19], and a horizontal scale of the ocean are taken as typical values, respectively, 100 m, 0.2, and 2×10^6 m, then the number of resonance waves n is $\frac{100}{\varepsilon^n} = 2\times10^6$, $n = \frac{\lg 0.5-4}{\lg\varepsilon} \approx 6.15$. It can be expected that the steeper the resonance waves are, the higher the order of the resonance waves are, and the higher the number of the resonance waves are until the resonance waves are broken [22]. If the effect of the surface capillary waves is considered separately [21,23], the resonant number of ocean surface capillary waves can reach infinity in a wide ocean area, which makes it possible to construct a high-order ocean surface capillary–gravity wave system with six-wave resonance. Therefore, a six-wave resonance system for ocean surface gravity waves is necessary. The fifth-order Korteweg–de Vries equation of solitary wave phenomena originating from water waves may admit an infinite number of locally propagating waves under some singular conditions [24].

It has been shown that the higher-order effect is not only possible but also observed [22]. A quite common phenomenon seen by people on the sea or river surface under the action of fresh wind comprises the so-called 'horseshoe' or 'crescent-shaped' patterns, which appear, owing to the class II instability for the five-wave and higher-order interactions [25–27]. Low-order theories sometimes distort the real physical picture and produce apparent chaos, which disappears at a higher order. Therefore, the high-order approximations of the water–wave problem are necessary [28] for reliable studies of the transition from order to chaos. In the one-dimensional case, the amplitudes of four-wave interactions in the effective Hamiltonian are exactly equal to zero, and the five-wave interaction is the first nonvanishing term [29,30], i.e., an effective five-wave Hamiltonian [31]. It is also found that the six-wave amplitude of the compact 1D Zakharov equation is not zero; that is, the equation is not integrable [32]. The high-order spectral (HOS) of numerical simulation [19] can be effective, and the theory of high-order wave resonance should be more so.

If we look at other waves besides water waves, one-dimensional six-wave resonance has already appeared in light wave turbulence [33] and quantum wave turbulence [34]. To this end, the classical dynamical equation and its kinetic equation [7] of the classical four-five-wave resonance of the surface gravity waves in a finite water depth are extended to the four-five-six-wave resonance. At the same time, the classical fifth-order expansion of the Hamiltonian in integral-power series in a normal variable for ocean surface capillary–gravity waves in a finite water depth is extended to the sixth order, thus laying a necessary foundation for the dynamical equation and its kinetic equation for surface capillary–gravity waves in a finite water depth in the future.

The nonlinear Schrödinger equations play important roles in constructing the mathematical model of ocean surface gravity waves, and they can be derived from the dynamical equations [8,19,35]. Based on the results, we can establish a fifth-order nonlinear Schrödinger equation of ocean surface gravity waves. Afterward, the corresponding abundant solutions can be obtained through various methods [36–40].

In this paper, we follow Krasitskii's method to ensure that the coefficients of the two equations obtained meet the desired symmetries. The paper contains six sections. In Section 2, the sixth-order integral-power series of Hamiltonian and canonical transformation are introduced. In Section 3, the expansion of the Hamiltonian with accuracy up to the sixth-order terms for ocean surface capillary–gravity waves with a finite water depth is presented. In Section 4, the dynamical equations of ocean surface gravity wave that contain four-five-six-wave interactions and the forms of the coefficients concerning the canonical transformation are obtained. In Section 5, taking the method of quasi-Gaussian approximation, the kinetic equations of four-five-six-wave resonance for ocean surface gravity waves are established. Finally, the potential applications are discussed in Section 6.

2. Background and the Expansion Form of the Hamiltonian

The Hamiltonian description of ocean surface waves was introduced by Zakharov [8] as follows:

$$\frac{\partial\zeta(\boldsymbol{x},t)}{\partial t} = \frac{\delta H}{\delta\psi(\boldsymbol{x},t)}, \ \frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = -\frac{\delta H}{\delta\zeta(\boldsymbol{x},t)}, \tag{1}$$

where $\boldsymbol{x} = (x, y)$ is the horizontal coordinate, $\zeta(\boldsymbol{x}, t)$ is the vertical displacement, $\phi(\boldsymbol{x}, z, t)$ is the velocity potential, $\psi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \zeta, t)$ is the velocity potential evaluated on the surface, z is the vertical coordinate directed upwards with its origin on the undisturbed surface $z = 0$, $\frac{\delta H}{\delta\psi}$ is the functional derivative of H with respect to ψ, and H is the Hamiltonian (the total energy) expressed in $\zeta(\boldsymbol{x},t)$ and $\psi(\boldsymbol{x},t)$. Here, the Hamiltonian $(H = K + \Pi)$ is the sum of the kinetic (K) and potential (Π) energies divided by the fluid density. These are given by

$$K = \frac{1}{2}\int\int_{-h}^{\zeta}\left[(\nabla\phi)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]dzd\boldsymbol{x}, \tag{2a}$$

$$\Pi = \frac{1}{2}g\int\zeta^2 d\boldsymbol{x} + \gamma\int\left[\left(1+(\nabla\zeta)^2\right)^{\frac{1}{2}} - 1\right]d\boldsymbol{x}, \tag{2b}$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ is the horizontal gradient operator, h is the constant depth of water, g is the acceleration due to gravity, and γ is the ratio of the surface tension coefficient to the fluid density. The velocity potential $\phi(\boldsymbol{x},z,t)$ must satisfy the following equations:

$$\nabla^2\phi + \frac{\partial^2\phi}{\partial z^2} = 0, \quad -\infty < x, y < +\infty, -h < z < \zeta(\boldsymbol{x},t), \tag{3a}$$

$$\frac{\partial\phi}{\partial z} = 0, \quad z = -h. \tag{3b}$$

The Fourier transforms of $\zeta(\boldsymbol{x},t)$ and $\psi(\boldsymbol{x},t)$ are introduced by

$$\zeta(\boldsymbol{x},t) = \frac{1}{2\pi}\int\zeta(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}, \zeta(\boldsymbol{k},t) = \zeta^*(-\boldsymbol{k},t), \tag{4a}$$

$$\psi(\boldsymbol{x},t) = \frac{1}{2\pi}\int\psi(\boldsymbol{k},t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{k}, \psi(\boldsymbol{k},t) = \psi^*(-\boldsymbol{k},t), \tag{4b}$$

where $\zeta(\boldsymbol{k},t)$ and $\psi(\boldsymbol{k},t)$ are, respectively, the Fourier transforms of $\zeta(\boldsymbol{x},t)$ and $\psi(\boldsymbol{x},t)$, $\boldsymbol{k} = (k_x, k_y)$ is the horizontal wave vector, and the asterisk denotes the complex conjugate, $\mathrm{i} = \sqrt{-1}$. In the following, the time variable t is omitted for simplicity. Please note that the Fourier transformation is the canonical one; then, the canonical Equations (1) can be transformed into the canonical ones with the pair of canonically conjugate variables $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ as follows:

$$\frac{\partial\zeta(\boldsymbol{k})}{\partial t} = \frac{\delta H}{\delta\psi^*(\boldsymbol{k})}, \quad \frac{\partial\psi(\boldsymbol{k})}{\partial t} = -\frac{\delta H}{\delta\zeta^*(\boldsymbol{k})}. \tag{5}$$

To normalize Equation (5), the canonical transformation is introduced by

$$\zeta(\boldsymbol{k}) = \mathcal{U}(\boldsymbol{k})[a(\boldsymbol{k}) + a^*(-\boldsymbol{k})], \ \psi(\boldsymbol{k}) = -i\mathcal{N}(\boldsymbol{k})[a(\boldsymbol{k}) - a^*(-\boldsymbol{k})], \tag{6}$$

where $\mathcal{U}(\boldsymbol{k}) = \left[\frac{q(\boldsymbol{k})}{2\omega(\boldsymbol{k})}\right]^{\frac{1}{2}}$, $\mathcal{N}(\boldsymbol{k}) = \left[\frac{\omega(\boldsymbol{k})}{2q(\boldsymbol{k})}\right]^{\frac{1}{2}}$, and $\omega(\boldsymbol{k})$ is the linear dispersion relation defined by

$$\omega^2(\boldsymbol{k}) = \tau(\boldsymbol{k})q(\boldsymbol{k}), \ \tau(\boldsymbol{k}) = g + \gamma|\boldsymbol{k}|^2, \ q(\boldsymbol{k}) = |\boldsymbol{k}|\tanh(|\boldsymbol{k}|h). \tag{7}$$

Equations (5) can be reduced to a single equation

$$\mathrm{i}\frac{\partial a(\boldsymbol{k})}{\partial t} = \frac{\delta H}{\delta a^*(\boldsymbol{k})} \tag{8}$$

using (6). Now, the Hamiltonian H is a functional of $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. In this work, attention is paid to waves of small but finite amplitudes (weakly nonlinear waves). Assuming small wave slopes, we can formally expand the Hamiltonian $H = H(a, a^*)$ into integral power series in powers of a and a^*(see in Section 3). Here, we consider this expansion with accuracy up to and including the sixth-order terms as follows:

$$H = H_2 + H_3 + H_4 + H_5 + H_6, \tag{9}$$

where $H_j (j = 3, 4, 5, 6)$ stands for the Hamiltonian of j-wave interactions, and

$$H_2 = \int \omega_0 a_0^* a_0 dk_0, \tag{10a}$$

$$\begin{aligned} H_3 = &\int U^{(1)}_{0,1,2}(a_0^* a_1 a_2 + c.c.)\delta_{0-1-2} dk_{012} \\ &+ \frac{1}{3}\int U^{(3)}_{0,1,2}(a_0^* a_1^* a_2^* + c.c.)\delta_{0+1+2} dk_{012}, \end{aligned} \tag{10b}$$

$$\begin{aligned} H_4 = &\int V^{(1)}_{0,1,2,3}(a_0^* a_1 a_2 a_3 + c.c.)\delta_{0-1-2-3} dk_{0123} \\ &+ \frac{1}{2}\int V^{(2)}_{0,1,2,3} a_0^* a_1^* a_2 a_3 \delta_{0+1-2-3} dk_{0123} \\ &+ \frac{1}{4}\int V^{(4)}_{0,1,2,3}(a_0^* a_1^* a_2^* a_3^* + c.c.)\delta_{0+1+2+3} dk_{0123}, \end{aligned} \tag{10c}$$

$$\begin{aligned} H_5 = &\int W^{(1)}_{0,1,2,3,4}(a_0^* a_1 a_2 a_3 a_4 + c.c.)\delta_{0-1-2-3-4} dk_{01234} \\ &+ \frac{1}{2}\int W^{(2)}_{0,1,2,3,4}(a_0^* a_1^* a_2 a_3 a_4 + c.c.)\delta_{0+1-2-3-4} dk_{01234} \\ &+ \frac{1}{5}\int W^{(5)}_{0,1,2,3,4}(a_0^* a_1^* a_2^* a_3^* a_4^* + c.c.)\delta_{0+1+2+3+4} dk_{01234}, \end{aligned} \tag{10d}$$

$$\begin{aligned} H_6 = &\int X^{(1)}_{0,1,2,3,4,5}(a_0^* a_1 a_2 a_3 a_4 a_5 + c.c.)\delta_{0-1-2-3-4-5} dk_{012345} \\ &+ \frac{1}{2}\int X^{(2)}_{0,1,2,3,4,5}(a_0^* a_1^* a_2 a_3 a_4 a_5 + c.c.)\delta_{0+1-2-3-4-5} dk_{012345} \\ &+ \frac{1}{3}\int X^{(3)}_{0,1,2,3,4,5} a_0^* a_1^* a_2^* a_3 a_4 a_5 \delta_{0+1+2-3-4-5} dk_{012345} \\ &+ \frac{1}{6}\int X^{(6)}_{0,1,2,3,4,5}(a_0^* a_1^* a_2^* a_3^* a_4^* a_5^* + c.c.)\delta_{0+1+2+3+4+5} dk_{012345}, \end{aligned} \tag{10e}$$

where $U^{(n)}, V^{(n)}$, and $W^{(n)}$ were given by Krasitskii [7], and $X^{(n)}$ will be given in Section 3. In the compact notation, the arguments $\boldsymbol{k}_j$ in a, $U^{(n)}$,$V^{(n)}$, $W^{(n)}$, $X^{(n)}$, and δ-functions are replaced by the subscript j, with the subscript zero assigned to $\boldsymbol{k}$. For example, $a_j = a(\boldsymbol{k}_j, t)$, $\omega_j = \omega(\boldsymbol{k}_j), \delta_{0-1-2} = \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2), U^{(n)}_{0,1,2} = U^{(n)}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2), dk_0 = d\boldsymbol{k}, dk_{012} = d\boldsymbol{k}_0 d\boldsymbol{k}_1 d\boldsymbol{k}_2$, etc., and the integral signs denote corresponding multiple integrals with the limits from $-\infty$ to $+\infty$.

It is important to assume that the coefficients $U^{(n)}, V^{(n)}, W^{(n)}$, and $X^{(n)}$ satisfy the natural symmetry conditions, which ensure that the Hamiltonian coefficients do not change as the integral variable subscripts change. For example, the coefficient $U^{(1)}_{0,1,2}$ is symmetric under the transposition of the arguments 1 and 2, $U^{(3)}_{0,1,2}$ under all transpositions of 0, 1, and 2, $V^{(1)}_{0,1,2,3}$ under the transpositions of 1, 2, and 3, $V^{(2)}_{0,1,2,3}$ under the transpositions of the arguments inside the groups $(0, 1)$ and $(3, 4)$, $V^{(4)}_{0,1,2,3}$ under the transpositions of all its indices, $W^{(1)}_{0,1,2,3,4}$ under all transpositions of 1, 2, 3, and 4, $W^{(2)}_{0,1,2,3,4}$ under the transpositions of the arguments inside the groups $(0, 1)$ and $(2, 3, 4)$, $X^{(1)}_{0,1,2,3,4,5}$ under all transpositions of 1, 2, 3, 4, and 5, $X^{(2)}_{0,1,2,3,4,5}$ under the transpositions of the arguments inside the groups $(0, 1)$ and $(2, 3, 4, 5)$, and so on. The coefficients should ensure the Hamiltonian is a real quantity. In view of this, there are additional symmetry conditions

$$V^{(2)}_{0,1,2,3} = V^{(2)}_{2,3,0,1}, \quad X^{(3)}_{0,1,2,3,4,5} = X^{(3)}_{3,4,5,0,1,2}. \tag{11}$$

It should be mentioned here that the coefficients calculated directly are asymmetric, so it is necessary to pay attention to skills in the calculation process (the definite integral has nothing to do with the form of the integral variable) to construct the symmetric coefficients (see Section 3).

By virtue of (8) and (9), the following evolution equation is obtained:

$$\mathrm{i}\frac{\partial a(\boldsymbol{k})}{\partial t}=\frac{\delta H_2}{\delta a^*(\boldsymbol{k})}+\frac{\delta H_3}{\delta a^*(\boldsymbol{k})}+\frac{\delta H_4}{\delta a^*(\boldsymbol{k})}+\frac{\delta H_5}{\delta a^*(\boldsymbol{k})}+\frac{\delta H_6}{\delta a^*(\boldsymbol{k})}, \tag{12}$$

where

$$\frac{\delta H_2}{\delta a^*(\boldsymbol{k})}=\omega_0 a_0, \tag{13a}$$

$$\begin{aligned}\frac{\delta H_3}{\delta a^*(\boldsymbol{k})}=&\int U^{(1)}_{0,1,2}a_1a_2\delta_{0-1-2}dk_{12}+2\int U^{(1)}_{2,1,0}a_1^*a_2\delta_{0+1-2}dk_{12}\\&+\int U^{(3)}_{0,1,2}a_1^*a_2^*\delta_{0+1+2}dk_{12},\end{aligned} \tag{13b}$$

$$\begin{aligned}\frac{\delta H_4}{\delta a^*(\boldsymbol{k})}=&\int V^{(1)}_{0,1,2,3}a_1a_2a_3\delta_{0-1-2-3}dk_{123}+\int V^{(2)}_{0,1,2,3}a_1^*a_2a_3\delta_{0+1-2-3}dk_{123}\\&+3\int V^{(1)}_{3,2,1,0}a_1^*a_2^*a_3\delta_{0+1+2-3}dk_{123}+\int V^{(4)}_{0,1,2,3}a_1^*a_2^*a_3^*\delta_{0+1+2+3}dk_{123},\end{aligned} \tag{13c}$$

$$\begin{aligned}\frac{\delta H_5}{\delta a^*(\boldsymbol{k})}=&\int W^{(1)}_{0,1,2,3,4}a_1a_2a_3a_4\delta_{0-1-2-3-4}dk_{1234}\\&+\int W^{(2)}_{0,1,2,3,4}a_1^*a_2a_3a_4\delta_{0+1-2-3-4}dk_{1234}\\&+\frac{3}{2}\int W^{(2)}_{4,3,2,1,0}a_1^*a_2^*a_3a_4\delta_{0+1+2-3-4}dk_{1234}\\&+4\int W^{(1)}_{4,3,2,1,0}a_1^*a_2^*a_3^*a_4\delta_{0+1+2+3-4}dk_{1234}\\&+\int W^{(5)}_{0,1,2,3,4}a_1^*a_2^*a_3^*a_4^*\delta_{0+1+2+3+4}dk_{1234},\end{aligned} \tag{13d}$$

$$\begin{aligned}\frac{\delta H_6}{\delta a^*(\boldsymbol{k})}=&\int X^{(1)}_{0,1,2,3,4,5}a_1a_2a_3a_4a_5\delta_{0-1-2-3-4-5}dk_{12345}\\&+\int X^{(2)}_{0,1,2,3,4,5}a_1^*a_2a_3a_4a_5\delta_{0+1-2-3-4-5}dk_{12345}\\&+\int X^{(3)}_{0,1,2,3,4,5}a_1^*a_2^*a_3a_4a_5\delta_{0+1+2-3-4-5}dk_{12345}\\&+2\int X^{(2)}_{5,4,3,2,1,0}a_1^*a_2^*a_3^*a_4a_5\delta_{0+1+2+3-4-5}dk_{12345}\\&+5\int X^{(1)}_{5,4,3,2,1,0}a_1^*a_2^*a_3^*a_4^*a_5\delta_{0+1+2+3+4-5}dk_{12345}\\&+\int X^{(6)}_{0,1,2,3,4,5}a_1^*a_2^*a_3^*a_4^*a_5^*\delta_{0+1+2+3+4+5}dk_{12345}.\end{aligned} \tag{13e}$$

To simplify the expression of the Hamiltonian, the canonical transformation $a(\boldsymbol{k})=a[b(\boldsymbol{k}),b^*(\boldsymbol{k})]$ was introduced [7]. Thus, we can obtain a new Hamiltonian $\tilde{H}$ that satisfies

$$\mathrm{i}\frac{\partial b(\boldsymbol{k})}{\partial t}=\frac{\delta\tilde{H}}{\delta b^*(\boldsymbol{k})}, \tag{14}$$

where the canonical transformation satisfies Poisson's brackets,

$$\int\left[\frac{\delta a(\boldsymbol{k})}{\delta b(\boldsymbol{q})}\frac{\delta a(\boldsymbol{k}')}{\delta b^*(\boldsymbol{q})}-\frac{\delta a(\boldsymbol{k})}{\delta b^*(\boldsymbol{q})}\frac{\delta a(\boldsymbol{k}')}{\delta b(\boldsymbol{q})}\right]d\boldsymbol{q}=0, \tag{15a}$$

$$\int\left[\frac{\delta a(\boldsymbol{k})}{\delta b(\boldsymbol{q})}\frac{\delta a^*(\boldsymbol{k}')}{\delta b^*(\boldsymbol{q})}-\frac{\delta a(\boldsymbol{k})}{\delta b^*(\boldsymbol{q})}\frac{\delta a^*(\boldsymbol{k}')}{\delta b(\boldsymbol{q})}\right]d\boldsymbol{q}=\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{15b}$$

Here, $\delta(\boldsymbol{k}-\boldsymbol{k}')$ on the right side of (15b) stands for the Dirac function.

According to previously established canonical transformation principles, the form of canonical transformation is postulated by

$$\begin{aligned}
a_0 =& b_0 + \int A^{(1)}_{0,1,2} b_1 b_2 \delta_{0-1-2} dk_{12} + \int A^{(2)}_{0,1,2} b_1^* b_2 \delta_{0+1-2} dk_{12} \\
&+ \int A^{(3)}_{0,1,2} b_1^* b_2^* \delta_{0+1+2} dk_{12} + \int B^{(1)}_{0,1,2,3} b_1 b_2 b_3 \delta_{0-1-2-3} dk_{123} \\
&+ \int B^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3} dk_{123} + \int B^{(3)}_{0,1,2,3} b_1^* b_2^* b_3 \delta_{0+1+2-3} dk_{123} \\
&+ \int B^{(4)}_{0,1,2,3} b_1^* b_2^* b_3^* \delta_{0+1+2+3} dk_{123} \\
&+ \int C^{(1)}_{0,1,2,3,4} b_1 b_2 b_3 b_4 \delta_{0-1-2-3-4} dk_{1234} \\
&+ \int C^{(2)}_{0,1,2,3,4} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4} dk_{1234} \\
&+ \int C^{(3)}_{0,1,2,3,4} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4} dk_{1234} \\
&+ \int C^{(4)}_{0,1,2,3,4} b_1^* b_2^* b_3^* b_4 \delta_{0+1+2+3-4} dk_{1234} \\
&+ \int C^{(5)}_{0,1,2,3,4} b_1^* b_2^* b_3^* b_4^* \delta_{0+1+2+3+4} dk_{1234} \\
&+ \int D^{(1)}_{0,1,2,3,4,5} b_1 b_2 b_3 b_4 b_5 \delta_{0-1-2-3-4-5} dk_{12345} \\
&+ \int D^{(2)}_{0,1,2,3,4,5} b_1^* b_2 b_3 b_4 b_5 \delta_{0+1-2-3-4-5} dk_{12345} \\
&+ \int D^{(3)}_{0,1,2,3,4,5} b_1^* b_2^* b_3 b_4 b_5 \delta_{0+1+2-3-4-5} dk_{12345} \\
&+ \int D^{(4)}_{0,1,2,3,4,5} b_1^* b_2^* b_3^* b_4 b_5 \delta_{0+1+2+3-4-5} dk_{12345} \\
&+ \int D^{(5)}_{0,1,2,3,4,5} b_1^* b_2^* b_3^* b_4^* b_5 \delta_{0+1+2+3+4-5} dk_{12345} \\
&+ \int D^{(6)}_{0,1,2,3,4,5} b_1^* b_2^* b_3^* b_4^* b_5^* \delta_{0+1+2+3+4+5} dk_{12345}.
\end{aligned} \tag{16}$$

Supposing that $A^{(n)}, B^{(n)}, C^{(n)}$, and $D^{(n)}$ satisfy the natural symmetry conditions, their detailed expressions will be given in Section 4. Then, the new Hamiltonian $\tilde{H}$ can be obtained by substituting (16) into (7). In the process, the coefficients $U^{(n)}, V^{(n)}, W^{(n)}$, and $X^{(n)}$ are replaced, respectively, by $\tilde{U}^{(n)}, \tilde{V}^{(n)}$, $\tilde{W}^{(n)}$, and $\tilde{X}^{(n)}$ and satisfy symmetry conditions like the previous ones.

On the basis of the transformation, as mentioned above, and the resonance conditions, and choosing appropriate $A^{(n)}, B^{(n)}, C^{(n)}$, and $D^{(n)}$, one can deduce the dynamical equation of ocean surface waves concerning three-four-five-six-wave resonance by eliminating the nonresonant terms at the right-hand side of Equation (12). The dynamical equation of ocean surface gravity waves that contain four-five-wave resonance interactions was obtained by Krasitskii [7]. In Section 4, the corresponding dynamical equation, which contains four-five-six-wave resonance interactions, will be obtained.

3. The Six-Order Expansion Coefficients of the Hamiltonian

In this section, the precise form of the expansion coefficients $U^{(n)}, V^{(n)}, W^{(n)}$, and $X^{(n)}$ are obtained by complex calculations. The first three were given by Krasitskii [7]. Here, the expansion of H with accuracy up to the sixth-order terms for ocean surface capillary–gravity waves with a finite water depth is presented. The general solution of the Laplace equation satisfying the bottom boundary condition can be presented by the following Fourier integral:

$$\phi(\boldsymbol{x}, z) = \frac{1}{2\pi} \int \phi(\boldsymbol{k}) \frac{\cosh[|\boldsymbol{k}|(z+h)]}{\sinh(|\boldsymbol{k}|h)} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} dk_0, \;\; \phi(\boldsymbol{k}) = \phi^*(-\boldsymbol{k}). \tag{17}$$

3.1. Expending the Hamiltonian H in Powers of $\zeta(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ with Accuracy Up to the Sixth Order

Using the expression of the kinetic K(2a) and (17), it can be shown that

$$\begin{aligned}&\int_{-h}^{\zeta}\left[(\nabla\phi)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]dz\\&\quad=-\frac{1}{(2\pi)^2}\int\int[(\boldsymbol{k}\cdot\boldsymbol{k}_1)I^+-|\boldsymbol{k}||\boldsymbol{k}_1|I^-]\phi(\boldsymbol{k})\phi(\boldsymbol{k}_1)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}_1)\cdot\boldsymbol{x}}dk_0dk_1,\end{aligned}\tag{18}$$

where

$$I^s=\frac{n\sinh[m(h+\zeta)]+sm\sinh[n(h+\zeta)]}{mn[\cosh(mh)-\cosh(nh)]},s=\pm,\tag{19}$$

with $m=|\boldsymbol{k}|+|\boldsymbol{k}_1|, n=|\boldsymbol{k}|-|\boldsymbol{k}_1|$.

Assuming $|\boldsymbol{k}\zeta|$ is small (weak nonlinearity), and we can replace the hyperbolic sines in I^s by their Taylor's expansion up to the fourth order $[(m\zeta^4)$ and $(n\zeta^4)]$, which is sufficient for presenting the kinetic energy with accuracy up to the sixth-order terms. Thus, using (4), the expansion of the kinetic energy is presented by

$$\begin{aligned}K=&\frac{1}{2}\int K^{(2)}\phi_0\phi_0^*dk_0-\frac{1}{2(2\pi)}\int K^{(3)}\phi_0\phi_1\zeta_2\delta_{0+1+2}dk_{012}\\&-\frac{1}{2(2\pi)^2}\int K^{(4)}\phi_0\phi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\&-\frac{1}{2(2\pi)^3}\int K^{(5)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}\\&-\frac{1}{2(2\pi)^4}\int K^{(6)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345},\end{aligned}\tag{20}$$

where

$$\begin{aligned}K^{(2)}&=|\boldsymbol{k}|\coth(|\boldsymbol{k}|h),\ K^{(3)}=\coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-q_0q_1],\\K^{(4)}&=\frac{1}{2}\coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)\{[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}_1|^2]q_0+[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}|^2]q_1\},\\K^{(5)}&=\frac{1}{6}\coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)[(|\boldsymbol{k}|^2+|\boldsymbol{k}_1|^2)(\boldsymbol{k}\cdot\boldsymbol{k}_1)\\&\quad-2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2-|\boldsymbol{k}-\boldsymbol{k}_1|^2q_0q_1],\\K^{(6)}&=\frac{1}{24}\coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)\{[|\boldsymbol{k}|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1-3|\boldsymbol{k}_1|^2)\\&\quad+|\boldsymbol{k}_1|^2(3\boldsymbol{k}\cdot\boldsymbol{k}_1-|\boldsymbol{k}_1|^2)]q_0+[|\boldsymbol{k}_1|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1-3|\boldsymbol{k}|^2)\\&\quad+|\boldsymbol{k}|^2(3\boldsymbol{k}\cdot\boldsymbol{k}_1-|\boldsymbol{k}|^2)]q_1\},\end{aligned}$$

where $q_0=q(\boldsymbol{k})$ is given by (7).

The expansion of potential energy Π is trivial, and the result is

$$\begin{aligned}\Pi=&\frac{1}{2}\int\tau_0\zeta_0^*\zeta_0dk_0+\int\Pi^{(4)}_{0,1,2,3}\zeta_0\zeta_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\&+\int\Pi^{(6)}_{0,1,2,3,4,5}\zeta_0\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+5+6}dk_{012345},\end{aligned}\tag{21}$$

where

$$\begin{aligned}\Pi^{(4)}_{0,1,2,3}&=-\frac{\gamma}{24(2\pi)^2}[(\boldsymbol{k}\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)+(\boldsymbol{k}\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)\\&\quad+(\boldsymbol{k}\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)]\\&\equiv-\frac{\gamma}{24(2\pi)^2}\mathfrak{k}_{0,1,2,3},\end{aligned}\tag{22}$$

$$\Pi^{(6)}_{0,1,2,3,4,5} = \frac{\gamma}{240(2\pi)^4}[(\boldsymbol{k}\cdot\boldsymbol{k}_1)\mathfrak{k}_{2,3,4,5} + (\boldsymbol{k}\cdot\boldsymbol{k}_2)\mathfrak{k}_{1,3,4,5} + (\boldsymbol{k}\cdot\boldsymbol{k}_3)\mathfrak{k}_{1,2,4,5} + (\boldsymbol{k}\cdot\boldsymbol{k}_4)\mathfrak{k}_{1,2,3,5} + (\boldsymbol{k}\cdot\boldsymbol{k}_5)\mathfrak{k}_{1,2,3,4}]. \tag{23}$$

The compact notation $\tau_0 = \tau(\boldsymbol{k})$ is given by (7). We now have obtained the expansion of H in terms of $\zeta(\boldsymbol{k})$ and $\phi(\boldsymbol{k})$ with accuracy up to the sixth-order terms.

3.2. Expressing $\phi(\boldsymbol{k})$ through $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ with Accuracy Up to the Fifth Order

From (17), we can obtain

$$\psi(\boldsymbol{x}) = \frac{1}{2\pi}\int \phi(\boldsymbol{k},t)\frac{\cosh[|\boldsymbol{k}|(\zeta+h)]}{\sinh(|\boldsymbol{k}|h)}e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}dk_0. \tag{24}$$

The hyperbolic cosine in (24) can be replaced by Taylor's expansion up to the fourth order $[(|\boldsymbol{k}|\zeta)^4]$. Using (4), Equation (24) yields

$$\begin{aligned}\psi_0 =& \coth(|\boldsymbol{k}|h)\phi_0 + \frac{1}{2\pi}\int |\boldsymbol{k}|\phi_1\zeta_2\delta_{0-1-2}dk_{12}\\ &+ \frac{1}{(2\pi)^2}\int \frac{1}{2}|\boldsymbol{k}_1|^2\coth(|\boldsymbol{k}_1|h)\phi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\\ &+ \frac{1}{(2\pi)^3}\int \frac{1}{6}|\boldsymbol{k}_1|^3\phi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\\ &+ \frac{1}{(2\pi)^4}\int \frac{1}{24}|\boldsymbol{k}_1|^4\coth(|\boldsymbol{k}_1|h)\phi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345}.\end{aligned} \tag{25}$$

This equation can be solved by the iterative method. The solution of (25), after proper symmetrization, is

$$\begin{aligned}\phi_0 =& \tanh(|\boldsymbol{k}|h)\bigg[\psi_0 - \frac{1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\\ &- \frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\\ &- \frac{1}{(2\pi)^3}\int \Phi^{(4)}_{0,1,2,3,4}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\\ &- \frac{1}{(2\pi)^4}\int \Phi^{(5)}_{0,1,2,3,4,5}\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345}\bigg],\end{aligned} \tag{26}$$

where

$$\Phi^{(3)}_{0,1,2,3} = \frac{1}{2}(|\boldsymbol{k}_1|^2 - q_1q_{0-2} - q_1q_{0-3}), \tag{27}$$

$$\begin{aligned}\Phi^{(4)}_{0,1,2,3,4} =& \frac{1}{6}\bigg[|\boldsymbol{k}_1|^2q_1 - q_1\left(\sum_{r=2}^{4}|\boldsymbol{k}_1+\boldsymbol{k}_r|^2\right) - 2q_{0-2}\Phi^{(3)}_{0-2,1,3,4}\\ &- 2q_{0-3}\Phi^{(3)}_{0-3,1,2,4} - 2q_{0-4}\Phi^{(3)}_{0-4,1,2,3}\bigg]\end{aligned} \tag{28}$$

$$\begin{aligned}=& \frac{1}{6}\bigg[q_1(|\boldsymbol{k}_1|^2 - \sum_{r=2}^{4}|\boldsymbol{k}_1+\boldsymbol{k}_r|^2) - |\boldsymbol{k}_1|^2(q_{0-2}+q_{0-3}+q_{0-4})\\ &+ \sum_{r=2}^{4}q_1q_{0-r}(\sum_{s=2,s\neq r}^{4}q_{1+s})\bigg],\end{aligned} \tag{29}$$

$$\Phi^{(5)}_{0,1,2,3,4,5} = \frac{1}{24}\bigg\{|\boldsymbol{k}_1|^4 - \sum_{r=2}^{5}|\boldsymbol{k}_1+\boldsymbol{k}_r|^2q_1q_{1+r}$$

$$
\begin{aligned}
&- 2 \sum_{2\le r<s\le 5} |\boldsymbol{k}_{1+r+s}|^2 \Phi^{(3)}_{1+r+s,1,r,s} - 6q_{0-2}\Phi^{(4)}_{0-2,1,3,4,5} \\
&- 6q_{0-3}\Phi^{(4)}_{0-3,1,2,4,5} - 6q_{0-4}\Phi^{(4)}_{0-4,1,2,3,5} - 6q_{0-5}\Phi^{(4)}_{0-5,1,2,3,4} \Big\} \\
=&\frac{1}{24}\Big\{ |\boldsymbol{k}_1|^4 - \sum_{r=2}^{5} |\boldsymbol{k}_1+\boldsymbol{k}_r|^2 q_1 q_{1+r} - |\boldsymbol{k}_1|^2 (\sum_{2\le r<s\le 5} |\boldsymbol{k}_{1+r+s}|^2) \\
&+ \sum_{r=2}^{5}\left(q_1 q_{1+r} \sum_{s=2,s\neq r}^{5} |\boldsymbol{k}_{1+r+s}|^2 \right) \\
&+ \sum_{r=2}^{5}\left(|\boldsymbol{k}_1+\boldsymbol{k}_r|^2 q_1 \sum_{s=2,s\neq r}^{5} q_{0-s} \right) \\
&- q_1|\boldsymbol{k}_1|^2 \left(\sum_{r=2}^{5} q_{0-r} \right) + |\boldsymbol{k}_1|^2 \sum_{r=2}^{5} \left[q_{0-r} \sum_{s=2,s\neq r}^{5} (q_{0-r-s}) \right] \\
&- \sum_{r=2}^{5} q_1 q_{0-r} \left[\sum_{s=2,s\neq r}^{5} q_{0-r-s} \left(\sum_{t=2,t\neq r,t\neq s}^{5} q_{1+t} \right) \right] \Big\}.
\end{aligned} \tag{30}
$$

Now, the relation of $\phi(\boldsymbol{k})$ to $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ is obtained.

3.3. Presenting H through the Canonically Conjugate Variables $\zeta(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ with Accuracy Up to the Sixth Order

Substituting (26) into (20) and retaining therein the terms up to the sixth order leads to

$$
\begin{aligned}
K = &\frac{1}{2}\int q_0\psi_0\psi_0^* dk_0 + \int E^{(3)}_{0,1,2}\psi_0\psi_1\zeta_2\delta_{0+1+2} dk_{012} \\
&+ \int E^{(4)}_{0,1,2,3}\psi_0\psi_1\zeta_2\zeta_3\delta_{0+1+2+3} dk_{0123} \\
&+ \int E^{(5)}_{0,1,2,3,4}\psi_0\psi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4} dk_{01234} \\
&+ \int E^{(6)}_{0,1,2,3,4,5}\psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5} dk_{012345},
\end{aligned} \tag{31}
$$

where

$$
E^{(3)}_{0,1,2} = -\frac{1}{2(2\pi)}[(\boldsymbol{k}\cdot\boldsymbol{k}_1) + q_0 q_1], \tag{32}
$$

$$
\begin{aligned}
E^{(4)}_{0,1,2,3} = &-\frac{1}{8(2\pi)^2}\Big[2q_0|\boldsymbol{k}_1|^2 + 2q_1|\boldsymbol{k}|^2 \\
&- q_0 q_1 (q_{0+2} + q_{0+3} + q_{1+2} + q_{1+3}) \Big],
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
E^{(5)}_{0,1,2,3,4} = &-\frac{1}{12(2\pi)^3}\Big[2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2 - q_0|\boldsymbol{k}_1|^2 (\sum_{r=2}^{4} q_{0+r}) - q_1|\boldsymbol{k}|^2 (\sum_{r=2}^{4} q_{1+r}) \\
&+ \frac{1}{2} q_0 q_1 \left(|\boldsymbol{k}|^2 - \sum_{r=2}^{4} |\boldsymbol{k}_{0+r}|^2 + |\boldsymbol{k}_1|^2 - \sum_{r=2}^{4} |\boldsymbol{k}_{1+r}|^2 \right) \\
&+ q_0 q_1 \sum_{r=2}^{4} q_{0+r} \left(\sum_{t=2,t\neq r}^{4} q_{1+t} \right) \Big],
\end{aligned} \tag{34}
$$

$$
E^{(6)}_{0,1,2,3,4,5} = -\frac{1}{48(2\pi)^4}\Big\{ \Big[E^{(6)}_1 + E^{(6)}_2 + E^{(6)}_3 \Big] + (0 \Leftrightarrow 1) \Big\}, \tag{35}
$$

with

$$E_1^{(6)} = q_0|\boldsymbol{k}_1|^4 - q_0|\boldsymbol{k}_1|^2\left(\sum_{2\leq r<s\leq 5}^{5}|\boldsymbol{k}_{0+r+s}|^2\right) - |\boldsymbol{k}|^2|\boldsymbol{k}_1|^2\left[\sum_{2\leq r<s\leq 5}^{5} q_{0+r+s}\right], \tag{36a}$$

$$E_2^{(6)} = \sum_{r=2}^{5} q_0 q_1 q_{0+r}\left(3|\boldsymbol{k}_1|^2 - \sum_{t=2,t\neq r}^{5}|\boldsymbol{k}_{1+t}|^2\right) - 2\sum_{r=2}^{5} q_0 q_{0+r}\left(\sum_{t=2,t\neq r}^{5} q_{0+r+t}\right)|\boldsymbol{k}_1|^2, \tag{36b}$$

$$E_3^{(6)} = -\sum_{r=2}^{5} q_0 q_1 q_{0+r}\left[\sum_{s=2,s\neq r}^{5} q_{0+r+s}\left(\sum_{t=2,t\neq r,t\neq s}^{5} q_{1+t}\right)\right]. \tag{36c}$$

Here, $(0 \Leftrightarrow 1)$, standing for the first term in the curly braces of Equation (35), should be repeated, and the indices 0 and 1 interchanged, such that the right-hand side of Equation (35) should be symmetrized with respect to arguments $(0,1)$. The detailed process of deriving $E^{(4)}_{0,1,2,3}$, $E^{(5)}_{0,1,2,3,4}$, and $E^{(6)}_{0,1,2,3,4,5}$ is given in Appendix A.

Please note that the coefficients $E^{(4)}_{0,1,2,3}$ and $E^{(5)}_{0,1,2,3,4}$ are identical to the classical ones given by Krasitskii [7].

Using (21) and (31), we can obtain

$$\begin{aligned}\frac{\partial\zeta_0}{\partial t} = {} & q_0\psi_0 + 2\int E^{(3)}_{-0,1,2}\psi_1\zeta_2\delta_{0-1-2}dk_{12} \\ & + 2\int E^{(4)}_{-0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123} \\ & + 2\int E^{(5)}_{-0,1,2,3,4}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234} \\ & + 2\int E^{(6)}_{-0,1,2,3,4,5}\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345},\end{aligned} \tag{37}$$

$$\begin{aligned}\frac{\partial\psi_0}{\partial t} = {} & -\tau_0\zeta_0 - \int E^{(3)}_{1,2,-0}\psi_1\psi_2\delta_{0-1-2}dk_{12} \\ & - 2\int E^{(4)}_{1,2,3,-0}\psi_1\psi_2\zeta_3\delta_{0-1-2-3}dk_{123} \\ & - 4\int \Pi^{(4)}_{1,2,3,-0}\zeta_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123} \\ & - 3\int E^{(5)}_{1,2,3,4,-0}\psi_1\psi_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234} \\ & - 4\int E^{(6)}_{1,2,3,4,5,-0}\psi_1\psi_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345} \\ & - 6\int \Pi^{(6)}_{1,2,3,4,5,-0}\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345}.\end{aligned} \tag{38}$$

Analogous to Krasitskii [7], the system of Equations (37) and (38) is more convenient for the numerical study of combined four-five-six-wave interactions than the dynamical Equation (47).

3.4. *The Forms of Coefficients $U^{(n)}$, $V^{(n)}$, $W^{(n)}$, and $X^{(n)}$*

Finally, combining Equations $(5), (6), (21)$ and (31), the forms of the coefficients $[U^{(n)}$, $V^{(n)}, W^{(n)}$,, and $X^{(n)}]$ can be determined. The forms of $U^{(n)}$, $V^{(n)}$, and $W^{(n)}$ were given by Krasitskii [7]. After proper symmetrization, we obtain the following expressions concerning $X^{(n)}$:

$$\begin{aligned}
X^{(1)}_{0,1,2,3,4,5} = & \frac{1}{5}(-X_{-0,1,2,3,4,5} - X_{-0,2,1,3,4,5} - X_{-0,3,1,2,4,5} \\
& - X_{-0,4,1,2,3,5} - X_{-0,5,1,2,3,4} + X_{1,2,-0,3,4,5} + X_{1,3,-0,2,4,5} \\
& + X_{1,4,-0,2,3,5} + X_{1,5,-0,2,3,4} + X_{2,3,-0,1,4,5} + X_{2,4,-0,1,3,5} \\
& + X_{2,5,-0,1,3,4} + X_{3,4,-0,1,2,5} + X_{3,5,-0,1,2,4} + X_{4,5,-0,1,2,3}) \\
& - 6\Gamma_{0,1,2,3,4,5},
\end{aligned} \tag{39}$$

$$\begin{aligned}
X^{(2)}_{0,1,2,3,4,5} = & \frac{2}{3}(X_{-0,-1,2,3,4,5} - X_{-0,2,-1,3,4,5} - X_{-0,3,-1,2,4,5} \\
& - X_{-0,4,-1,2,3,5} - X_{-0,5,-1,2,3,4} - X_{-1,2,-0,3,4,5} - X_{-1,3,-0,2,4,5} \\
& - X_{-1,4,-0,2,3,5} - X_{-1,5,-0,2,3,4} + X_{2,3,-0,-1,4,5} + X_{2,4,-0,-1,3,5} \\
& + X_{2,5,-0,-1,3,4} + X_{3,4,-0,-1,2,5} + X_{3,5,-0,-1,2,4} + X_{4,5,-0,-1,2,3}) \\
& + 20\Gamma_{0,1,2,3,4,5},
\end{aligned} \tag{40}$$

$$\begin{aligned}
X^{(3)}_{0,1,2,3,4,5} = & X_{-0,-1,-2,3,4,5} + X_{-0,-2,-1,3,4,5} - X_{-0,3,-1,-2,4,5} \\
& - X_{-0,4,-1,-2,3,5} - X_{-0,5,-1,-2,3,4} - X_{-1,-2,-0,3,4,5} - X_{-1,3,-0,-2,4,5} \\
& - X_{-1,4,-0,-2,3,5} - X_{-1,5,-0,-2,3,4} + X_{-2,3,-0,-1,4,5} + X_{-2,4,-0,-1,3,5} \\
& + X_{-2,5,-0,-1,3,4} + X_{3,4,-0,-1,-2,5} + X_{3,5,-0,-1,-2,4} + X_{4,5,-0,-1,-2,3} \\
& - 30\Gamma_{0,1,2,3,4,5},
\end{aligned} \tag{41}$$

$$\begin{aligned}
X^{(6)}_{0,1,2,3,4,5} = & \frac{1}{5}(X_{0,1,2,3,4,5} + X_{0,2,1,3,4,5} + X_{0,3,1,2,4,5} \\
& + X_{0,4,1,2,3,5} + X_{0,5,1,2,3,4} + X_{1,2,0,3,4,5} + X_{1,3,0,2,4,5} + X_{1,4,0,2,3,5} \\
& + X_{1,5,0,2,3,4} + X_{2,3,0,1,4,5} + X_{2,4,0,1,3,5} + X_{2,5,0,1,3,4} + X_{3,4,0,1,2,5} \\
& + X_{3,5,0,1,2,4} + X_{4,5,0,1,2,3}) + 6\Gamma_{0,1,2,3,4,5},
\end{aligned} \tag{42}$$

where

$$X_{0,1,2,3,4,5} = -3\mathcal{N}_0\mathcal{N}_1\mathcal{U}_2\mathcal{U}_3\mathcal{U}_4\mathcal{U}_5 E^{(6)}_{0,1,2,3,4,5},$$

$$\Gamma_{0,1,2,3,4,5} = \mathcal{U}_0\mathcal{U}_1\mathcal{U}_2\mathcal{U}_3\mathcal{U}_4\mathcal{U}_5 \Pi^{(6)}_{0,1,2,3,4,5}.$$

The coefficients $X^{(n)}$ satisfy all the necessary symmetry conditions.

4. The Dynamical Equation

In this section, we pay attention to ocean surface gravity waves. The dynamical equations that contain four-five-six-wave interactions and the forms of the coefficients concerning the canonical transformation $[A^{(n)}, B^{(n)}, C^{(n)}, D^{(n)}]$ are presented. The methods of deducing the dynamical equations are as follows: (i) starting directly from the canonical transformation conditions (15) [41]; (ii) substituting the transformation (16) into (9) [7]; (iii) using the Hilbert transformation to transform the canonical variables [30]. In the following, method (ii) is adopted to calculate the coefficients of the canonical transformation and the dynamical equation. The forms of $A^{(n)}$, $B^{(n)}$, and $C^{(n)}$ were obtained by Krasitskii [7]. Considering the symmetrization of the coefficients, Janssen [42] gave a new calculation method concerning $B^{(2)}$. The six-wave resonance conditions of ocean surface gravity waves are as follows [1]:

$$\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 + \boldsymbol{k}_5, \tag{43a}$$

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4) + \omega(\boldsymbol{k}_5), \tag{43b}$$

$$\boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4 + \boldsymbol{k}_5, \tag{44a}$$

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4) + \omega(\boldsymbol{k}_5), \tag{44b}$$

$$\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{k}_4 + \boldsymbol{k}_5, \tag{45a}$$

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) + \omega(\boldsymbol{k}_3) = \omega(\boldsymbol{k}_4) + \omega(\boldsymbol{k}_5). \tag{45b}$$

The following process is to determine the coefficients $A^{(n)}, B^{(n)}, C^{(n)}$, and $D^{(n)}$, so that the new Hamiltonian is simplified when the coefficients are expressed in terms of b. This process implies removing nonresonant terms from the Hamiltonian. Thus, after a successful determination of the function in (16), the new Hamiltonian $\tilde{H}$, which we wish to obtain, has the form

$$\begin{aligned}\tilde{H} = &\int \omega_0 b_0^* b_0 dk_0 + \frac{1}{2}\int \tilde{V}^{(2)}_{0,1,2,3} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3} dk_{0123} \\ &+ \frac{1}{2}\int \tilde{W}^{(2)}_{0,1,2,3,4}(b_0^* b_1^* b_2 b_3 b_4 + c.c.)\delta_{0+1-2-3-4} dk_{01234} \\ &+ \frac{1}{2}\int \tilde{X}^{(2)}_{0,1,2,3,4,5}(b_0^* b_1^* b_2 b_3 b_4 b_5 + c.c.)\delta_{0+1-2-3-4-5} dk_{012345} \\ &+ \frac{1}{3}\int \tilde{X}^{(3)}_{0,1,2,3,4,5} b_0^* b_1^* b_2^* b_3 b_4 b_5 \delta_{0+1+2-3-4-5} dk_{012345}.\end{aligned} \tag{46}$$

Given this form of Hamiltonian and Equation (14), the dynamical equation of the four-five-six-wave resonance for ocean surface gravity waves in water of a finite depth takes the following form:

$$\begin{aligned}\mathrm{i}\frac{\partial b_0}{\partial t} = &\frac{\delta \tilde{H}}{\delta b_0^*} = \omega_0 b_0 + \int \tilde{V}^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3} dk_{123} \\ &+ \int \tilde{W}^{(2)}_{0,1,2,3,4} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4} dk_{1234} \\ &+ \frac{3}{2}\int \tilde{W}^{(2)}_{4,3,2,1,0} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4} dk_{1234} \\ &+ \int \tilde{X}^{(2)}_{0,1,2,3,4,5} b_1^* b_2 b_3 b_4 b_5 \delta_{0+1-2-3-4-5} dk_{12345} \\ &+ \int \tilde{X}^{(3)}_{0,1,2,3,4,5} b_1^* b_2^* b_3 b_4 b_5 \delta_{0+1+2-3-4-5} dk_{12345} \\ &+ 2\int \tilde{X}^{(2)}_{5,4,3,2,1,0} b_1^* b_2^* b_3^* b_4 b_5 \delta_{0+1+2+3-4-5} dk_{12345}.\end{aligned} \tag{47}$$

The following describes how to obtain the kernel functions $\tilde{X}^{(n)}$ and the coefficients $D^{(n)}$. We should substitute transformation (16) into (12) and substitute the derivatives arising from $\frac{\partial b_j}{\partial t}$ from Equation (47). After combining the like terms and proper symmetrization, we can obtain 18 algebraic equations concerning the coefficients, in which six equations of $D^{(n)}$ can be written as follows:

$$Y^{(1)}_{0,1,2,3,4,5} + X^{(1)}_{0,1,2,3,4,5} + \Delta_{0-1-2-3-4-5} D^{(1)}_{0,1,2,3,4,5} = 0, \tag{48}$$

$$\tilde{X}^{(2)}_{0,1,2,3,4,5} = Y^{(2)}_{0,1,2,3,4,5} + X^{(2)}_{0,1,2,3,4,5} + \Delta_{0+1-2-3-4-5} D^{(2)}_{0,1,2,3,4,5}, \tag{49}$$

$$\tilde{X}^{(3)}_{0,1,2,3,4,5} = Y^{(3)}_{0,1,2,3,4,5} + X^{(3)}_{0,1,2,3,4,5} + \Delta_{0+1+2-3-4-5} D^{(3)}_{0,1,2,3,4,5}, \tag{50}$$

$$2\tilde{X}^{(2)}_{5,4,3,2,1,0} = Y^{(4)}_{0,1,2,3,4,5} + 2X^{(2)}_{5,4,3,2,1,0} + \Delta_{0+1+2+3-4-5} D^{(4)}_{0,1,2,3,4,5}, \tag{51}$$

$$Y^{(5)}_{0,1,2,3,4,5} + 5X^{(1)}_{5,4,3,2,1,0} + \Delta_{0+1+2+3+4-5} D^{(5)}_{0,1,2,3,4,5} = 0, \tag{52}$$

$$Y^{(6)}_{0,1,2,3,4,5} + X^{(6)}_{0,1,2,3,4,5} + \Delta_{0+1+2+3+4+5} D^{(6)}_{0,1,2,3,4,5} = 0, \tag{53}$$

where $\Delta_{n\pm m\pm p\pm q\pm r\pm s} = \boldsymbol{k}_n \pm \boldsymbol{k}_m \pm \boldsymbol{k}_p \pm \boldsymbol{k}_q \pm \boldsymbol{k}_r \pm \boldsymbol{k}_s$. The detailed expression of function $Y^{(n)}_{0,1,2,3,4,5}(n = 1,2,3,4,5,6)$ is given in Appendix B. The coefficients $D^{(n)}$ can be divided into two classes: nonresonant ones ($D^{(1)}$,$D^{(5)}$, $D^{(6)}$) and resonant ones ($D^{(2)}$, $D^{(3)}$, $D^{(4)}$). The nonresonant coefficients can be obtained directly from Equations (48), (52) and (53):

$$D^{(1)}_{0,1,2,3,4,5} = -\frac{Y^{(1)}_{0,1,2,3,4,5} + X^{(1)}_{0,1,2,3,4,5}}{\Delta_{0-1-2-3-4-5}}, \tag{54}$$

$$D^{(5)}_{0,1,2,3,4,5} = -\frac{Y^{(5)}_{0,1,2,3,4,5} + 5X^{(1)}_{5,4,3,2,1,0}}{\Delta_{0+1+2+3+4-5}}, \tag{55}$$

$$D^{(6)}_{0,1,2,3,4,5} = -\frac{Y^{(6)}_{0,1,2,3,4,5} + X^{(6)}_{0,1,2,3,4,5}}{\Delta_{0+1+2+3+4+5}}. \tag{56}$$

We now turn to obtain the resonant coefficients ($D^{(2)}$, $D^{(3)}$, $D^{(4)}$) and the kernels of the six-wave dynamical equation($\tilde{X}^{(2)}_{0,1,2,3,4,5}$, $\tilde{X}^{(3)}_{0,1,2,3,4,5}$).

First, by the symmetry properties, Equation (49) leads to

$$\Delta_{0+1-2-3-4-5}[D^{(2)}_{0,1,2,3,4,5} - D^{(2)}_{1,0,2,3,4,5}] + Y^{(2)}_{0,1,2,3,4,5} - Y^{(2)}_{1,0,2,3,4,5} = 0. \tag{57}$$

The general solution of Equation (57) is symmetric under transpositions within the groups (2,3,4,5) and can be expressed by $\Lambda_{0,1,2,3,4,5} + \lambda_{0,1,2,3,4,5}$, where the particular solution $\Lambda_{0,1,2,3,4,5}$ is symmetric under the transpositions within the groups (2,3,4,5), and the function $\lambda_{0,1,2,3,4,5}$ is symmetric under the transpositions within the groups (0,1) and (2,3,4,5). It is worth pointing out that the choice of function $\lambda_{0,1,2,3,4,5}$ is free. In the following, we choose $\lambda_{0,1,2,3,4,5} = 0$, and the coefficient $D^{(2)}$ is a suitable particular solution of Equation (57), which is symmetric under the transpositions within the groups (2,3,4,5) and non-singular when $\Delta_{0+1-2-3-4-5} \to 0$. We construct a particular solution:

$$D^{(2)}_{0,1,2,3,4,5} = -\frac{1}{2\Delta_{0+1-2-3-4-5}}[Y^{(2)}_{0,1,2,3,4,5} - Y^{(2)}_{1,0,2,3,4,5}]. \tag{58}$$

The solution (58) is formally singular for resonant condition $\Delta_{0+1-2-3-4-5} = 0$; however, it is possible to show that each term of function $Y^{(2)}$ can factor out $\Delta_{0+1-2-3-4-5}$, which means that the solution (58) is non-singular when $\Delta_{0+1-2-3-4-5} \to 0$. This yields

$$Y^{(2)}_{0,1,2,3,4,5} - Y^{(2)}_{1,0,2,3,4,5} = -2\Delta_{0+1-2-3-4-5}D^{(2)}_{0,1,2,3,4,5}. \tag{59}$$

Please note that the antisymmetry of $D^{(2)}_{0,1,2,3,4,5}$ under $0, 1$ means

$$D^{(2)}_{0,1,2,3,4,5} = -D^{(2)}_{1,0,2,3,4,5}. \tag{60}$$

From (49) and (60), we obtain

$$\tilde{X}^{(2)}_{0,1,2,3,4,5} = \frac{1}{2}[Y^{(2)}_{0,1,2,3,4,5} + Y^{(2)}_{1,0,2,3,4,5}] + X^{(2)}_{0,1,2,3,4,5}. \tag{61}$$

This representation of the kernel($\tilde{X}^{(2)}_{0,1,2,3,4,5}$) already possesses all the necessary symmetry properties in explicit form.

Second, using (50), we turn to the determination of the canonical transformation coefficients $D^{(3)}$ as follows:

$$\Delta_{0+1+2-3-4-5}[D^{(3)}_{0,1,2,3,4,5} + D^{(3)}_{5,4,3,2,1,0}] + Y^{(3)}_{0,1,2,3,4,5} - Y^{(3)}_{5,4,3,2,1,0} = 0. \tag{62}$$

Analogous to the above, we construct a particular solution as follows:

$$D^{(3)}_{0,1,2,3,4,5} = -\frac{1}{6\Delta_{0+1+2-3-4-5}}[5Y^{(3)}_{0,1,2,3,4,5} - Y^{(3)}_{1,0,2,3,4,5} - Y^{(3)}_{2,0,1,3,4,5}$$

$$- Y^{(3)}_{3,4,5,0,1,2} - Y^{(3)}_{4,3,5,0,1,2} - Y^{(3)}_{5,3,4,0,1,2}]. \tag{63}$$

The solution (63) is formally singular for resonant condition $\Delta_{0+1+2-3-4-5} = 0$; however, it is possible to show that each term of the function $Y^{(3)}$ can factor out $\Delta_{0+1+2-3-4-5}$, which implies that the solution (63) is non-singular when $\Delta_{0+1+2-3-4-5} \to 0$. Substituting (63) into (50) gives

$$\tilde{X}^{(3)}_{0,1,2,3,4,5} = \frac{1}{6}[Y^{(3)}_{0,1,2,3,4,5} + Y^{(3)}_{1,0,2,3,4,5} + Y^{(3)}_{2,0,1,3,4,5} + Y^{(3)}_{3,4,5,0,1,2} + Y^{(3)}_{4,3,5,0,1,2} + Y^{(3)}_{5,3,4,0,1,2}] + X^{(3)}_{0,1,2,3,4,5}. \tag{64}$$

Finally, it remains to determine the coefficient $D^{(4)}$. Combining Equations (49) and (51) leads to

$$D^{(4)}_{5,4,3,2,1,0} = \frac{1}{\Delta_{0+1-2-3-4-5}}[Y^{(4)}_{5,4,3,2,1,0} - 2Y^{(2)}_{0,1,2,3,4,5}] - 2D^{(2)}_{0,1,2,3,4,5}. \tag{65}$$

Adding (65) to the equation arising from interchanging the indices 0 and 1 of each term in (65) yields

$$D^{(4)}_{5,4,3,2,1,0} = \frac{1}{2\Delta_{0+1-2-3-4-5}}\{[Y^{(4)}_{5,4,3,2,1,0} - 2Y^{(2)}_{0,1,2,3,4,5}] + [Y^{(4)}_{5,4,3,2,0,1} - 2Y^{(2)}_{1,0,2,3,4,5}]\}. \tag{66}$$

Please note that each term of the function $[Y^{(4)}_{5,4,3,2,1,0} - 2Y^{(2)}_{0,1,2,3,4,5}]$ can factor out $\Delta_{0+1+2+3-4-5}$; then, the right side of (66) is non-singular. Equation (65) yields

$$D^{(4)}_{0,1,2,3,4,5} = \frac{-1}{2\Delta_{0+1-2-3-4-5}}\{[Y^{(4)}_{0,1,2,3,4,5} - 2Y^{(2)}_{5,4,3,2,1,0}] + [Y^{(4)}_{0,1,2,3,5,4} - 2Y^{(2)}_{4,5,3,2,1,0}]\}. \tag{67}$$

To summarize, all the coefficients of the canonical transformation and the kernels of the dynamical equation are determined.

Obviously, Equation (47) conserves the energy. Now, we consider the integrals of motion $I = \int r_0 b_0 b_0^* dk_0$. For $r_0 = \boldsymbol{k}$, the quantity I is the wave momentum, and for $r_0 = 1$, it is the wave action. From (47), we obtain

$$\begin{aligned} 2\mathrm{i}\frac{\partial I}{\partial t} = &\int (r_0 + r_1 - r_2 - r_3)\tilde{V}^{(2)}_{0,1,2,3} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3} dk_{0123} \\ &+ \int (r_0 + r_1 - r_2 - r_3 - r_4)\tilde{W}^{(2)}_{0,1,2,3,4}(b_0^* b_1^* b_2 b_3 b_4 - c.c.) \\ &\times \delta_{0+1-2-3-4} dk_{01234} \\ &+ \int (r_0 + r_1 - r_2 - r_3 - r_4 - r_5)\tilde{X}^{(2)}_{0,1,2,3,4,5}(b_0^* b_1^* b_2 b_3 b_4 b_5 - c.c.) \\ &\times \delta_{0+1-2-3-4-5} dk_{012345} \\ &+ \frac{2}{3}\int (r_0 + r_1 + r_2 - r_3 - r_4 - r_5)\tilde{X}^{(3)}_{0,1,2,3,4,5} b_0^* b_1^* b_2^* b_3 b_4 b_5 \\ &\times \delta_{0+1+2-3-4-5} dk_{012345}. \end{aligned} \tag{68}$$

From (43)–(45) and (68), we can find that the dynamical equation for four-five-six-wave resonance (47) only conserves the momentum; however, the one for four-wave resonance conserves both the momentum and the action. It is worth noting that the wave action is not conservative when we only consider six-wave resonance.

5. The Kinetic Equation

This section focuses on deducing the kinetic equation for ocean surface gravity waves in water of a finite depth. First, some definitions are introduced. We define the observable wave number spectrum $F(\boldsymbol{k})$ of a horizontally uniform random wave field by

$$F(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\frac{\omega(\boldsymbol{k})}{g}N(\boldsymbol{k}),\ < a(\boldsymbol{k})a^*(\boldsymbol{k}') >= N(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{69}$$

where the angle brackets imply an ensemble average. By analogy with $F(\boldsymbol{k})$ and $N(\boldsymbol{k})$, $f(\boldsymbol{k})$ and $n(\boldsymbol{k})$ are defined by

$$f(\boldsymbol{k}) = \frac{1}{(2\pi)^2}\frac{\omega(\boldsymbol{k})}{g}n(\boldsymbol{k}),\ < b(\boldsymbol{k})b^*(\boldsymbol{k}') >= n(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{70}$$

where $f(\boldsymbol{k})$ and $n(\boldsymbol{k})$ are the weak-interaction spectra.

The calculating of $\frac{\partial n(\boldsymbol{k})}{\partial t}$ is usually performed using the following steps: (1) multiplying (47) by $b^*(\boldsymbol{k})$ and multiplying the complex conjugate equation of (47) by $b(\boldsymbol{k}')$; (2) subtracting the latter from the former; (3) averaging and setting $\boldsymbol{k}' = \boldsymbol{k}$. We can obtain

$$\begin{aligned}\frac{\partial n(\boldsymbol{k})}{\partial t} &= 2\int \tilde{V}^{(2)}_{0,1,2,3}\,\mathrm{Im}[M^{(4)}_{0,1,2,3}]dk_{123}\\ &+2\int \tilde{W}^{(2)}_{0,1,2,3,4}\,\mathrm{Im}[M^{(5)}_{0,1,2,3,4}]dk_{1234}\\ &-3\int \tilde{W}^{(2)}_{4,3,2,1,0}\,\mathrm{Im}[M^{(5)}_{4,3,2,1,0}]dk_{1234}\\ &+2\int \tilde{S}^{(2)}_{0,1,2,3,4,5}\,\mathrm{Im}[M^{(6-1)}_{0,1,2,3,4,5}]dk_{12345}\\ &-4\int \tilde{X}^{(2)}_{5,4,3,2,1,0}\,\mathrm{Im}[M^{(6-1)}_{5,4,3,2,1,0}]dk_{12345}\\ &+2\int \tilde{X}^{(3)}_{0,1,2,3,4,5}\,\mathrm{Im}[M^{(6-2)}_{0,1,2,3,4,5}]dk_{12345},\end{aligned} \tag{71}$$

where $M^{(4)}_{0,1,2,3} =< b_0^*b_1^*b_2b_3 >$, $M^{(5)}_{0,1,2,3,4} =< b_0^*b_1^*b_2b_3b_4 >$, $M^{(6-1)}_{0,1,2,3,4,5} =< b_0^*b_1^*b_2b_3b_4b_5 >$, and $M^{(6-2)}_{0,1,2,3,4,5} =< b_0^*b_1^*b_2^*b_3b_4b_5 >$ are the correlators. In order to close Equation (71), we will apply the quasi-Gaussian approximation. Replacing the linear term $\omega_0 b_0$ in (47) with $(\omega_0 - i\varepsilon)b_0(\varepsilon > 0)$, we can construct the evolution equations of $M^{(4)}, M^{(5)}, M^{(6-1)}$, and $M^{(6-2)}$ as follows:

$$\begin{aligned}\frac{\partial}{\partial t}M^{(4)}_{0,1,2,3} &= \mathrm{i}(\omega_0+\omega_1-\omega_2-\omega_3+4i\varepsilon)M^{(4)}_{0,1,2,3}\\ &+\{\tilde{V}^{(2)}M^{(6)}\}+\{\tilde{W}^{(2)}M^{(7)}\}+\{\tilde{S}^{(2)}M^{(8-1)}\}\\ &+\{\tilde{S}^{(3)}M^{(8-2)}\},\end{aligned} \tag{72}$$

$$\begin{aligned}\frac{\partial}{\partial t}M^{(5)}_{0,1,2,3,4} &= \mathrm{i}(\omega_0+\omega_1-\omega_2-\omega_3-\omega_4+5i\varepsilon)M^{(5)}_{0,1,2,3,4}\\ &+\{\tilde{V}^{(2)}M^{(7)}\}+\{\tilde{W}^{(2)}M^{(8)}\}+\{\tilde{X}^{(2)}M^{(9-1)}\}\\ &+\{\tilde{X}^{(3)}M^{(9-2)}\},\end{aligned} \tag{73}$$

$$\begin{aligned}\frac{\partial}{\partial t}M^{(6-1)}_{0,1,2,3,4,5} &= \mathrm{i}(\omega_0+\omega_1-\omega_2-\omega_3-\omega_4-\omega_5+6i\varepsilon)M^{(6-1)}_{0,1,2,3,4,5}\\ &+\{\tilde{V}^{(2)}M^{(8)}\}+\{\tilde{W}^{(2)}M^{(9)}\}+\{\tilde{X}^{(2)}M^{(10-1)}\}\\ &+\{\tilde{X}^{(3)}M^{(10-2)}\},\end{aligned} \tag{74}$$

$$\begin{aligned}\frac{\partial}{\partial t}M^{(6-2)}_{0,1,2,3,4,5} &= \mathrm{i}(\omega_0+\omega_1+\omega_2-\omega_3-\omega_4-\omega_5+6i\varepsilon)M^{(6-2)}_{0,1,2,3,4,5}\\ &+\{\tilde{V}^{(2)}M^{(8)}\}+\{\tilde{W}^{(2)}M^{(9)}\}+\{\tilde{X}^{(2)}M^{(10-1)}\}\end{aligned} \tag{75}$$

$$+\{\tilde{X}^{(3)}M^{(10-2)}\},$$

where $\{\tilde{V}^{(2)}M^{(6)}\}$ is the sum of the integrals containing the products of $\tilde{V}^{(2)}M^{(6)}$. The other expressions in brackets have a similar meaning. Next, the quasi-Gaussian approximation is introduced: (1) $M^{(7)}, M^{(9)}$ is assumed to be zero; (2) the 6-8-10-order correlators are expressed through the two-order correlators (Gaussian random process) as follows:

$$\begin{aligned}< b_0^* b_1^* b_2^* b_3 b_4 b_5 > &= n_0 n_1 n_2 [\delta_{0,1,2,3,4,5} + \delta_{0,2,1,3,4,5} + \delta_{1,0,2,3,4,5} \\ &\quad + \delta_{1,2,0,3,4,5} + \delta_{2,1,0,3,4,5} + \delta_{2,0,1,3,4,5}],\end{aligned} \tag{76}$$

$$\begin{aligned}< b_0^* b_1^* b_2^* b_3^* b_4 b_5 b_6 b_7 > &= n_0 n_1 n_2 n_3 [\delta_{0,1,2,3,4,5,6,7} + \delta_{3,1,2,0,4,5,6,7} \\ &\quad + \delta_{0,3,2,1,4,5,6,7} + \delta_{0,1,3,2,4,5,6,7}],\end{aligned} \tag{77}$$

$$\begin{aligned}< b_0^* b_1^* b_2^* b_3^* b_4^* b_5 b_6 b_7 b_8 b_9 > &= n_0 n_1 n_2 n_3 n_4 [\delta_{0,1,2,3,4,5,6,7,8,9} \\ &\quad + \delta_{4,1,2,3,0,5,6,7,8,9} + \delta_{0,4,2,3,1,5,6,7,8,9} \\ &\quad + \delta_{0,1,4,3,2,5,6,7,8,9} + \delta_{0,1,2,4,3,5,6,7,8,9}],\end{aligned} \tag{78}$$

where

$$\delta_{0-1} = \delta(\boldsymbol{k}_0 - \boldsymbol{k}_1), \tag{79a}$$

$$\delta_{0,1,2,3,4,5} = \delta_{0-3}\delta_{1-4}\delta_{2-5}, \tag{79b}$$

$$\begin{aligned}\delta_{0,1,2,3,4,5,6,7} &= (\delta_{0,1,2,4,5,6} + \delta_{0,2,1,4,5,6} + \delta_{1,0,2,4,5,6} \\ &\quad + \delta_{1,2,0,4,5,6} + \delta_{2,1,0,4,5,6} + \delta_{2,0,1,4,5,6})\delta_{3-7},\end{aligned} \tag{79c}$$

$$\begin{aligned}\delta_{0,1,2,3,4,5,6,7,8,9} &= (\delta_{0,1,2,3,5,6,7,8} + \delta_{3,1,2,0,5,6,7,8} \\ &\quad + \delta_{0,3,2,1,5,6,7,8} + \delta_{0,1,3,2,5,6,7,8})\delta_{4-9}.\end{aligned} \tag{79d}$$

The $2n$-order correlator contains $n!$ terms. In the cases of $n = 2, 3$ and $n = 4$, we can refer the reader to the respective work of Zakharov et al. [11] and Krasitskii [43]. With these assumptions, Equations (71)–(75) are closed.

Using formula $\mathrm{Im}(x + \mathrm{i}\varepsilon)^{-1} \to -\pi\delta(x)(\varepsilon \to 0)$ and the correlators assumed to be changing slowly (we neglect the time derivative), substituting Equations (72)–(75) into (71), we can obtain the precise forms of $M^{(4)}$, $M^{(5)}$, $M^{(6-1)}$, and $M^{(6-2)}$. Thus, the Hamiltonian kinetic equation of four-five-six-wave resonance is

$$\begin{aligned}\frac{\partial n(\boldsymbol{k})}{\partial t} &= 4\pi \int [\tilde{V}^{(2)}_{0,1,2,3}]^2 n_0 n_1 n_2 n_3 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}) \\ &\quad \times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\delta_{0+1-2-3} dk_{123} \\ &\quad + 12\pi \int [\tilde{W}^{(2)}_{0,1,2,3,4}]^2 n_0 n_1 n_2 n_3 n_4 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}) \\ &\quad \times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4)\delta_{0+1-2-3-4} dk_{1234} \\ &\quad + 18\pi \int [\tilde{W}^{(2)}_{4,3,2,1,0}]^2 n_0 n_1 n_2 n_3 n_4 (\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}) \\ &\quad \times \delta(\omega_0 + \omega_1 + \omega_2 - \omega_3 - \omega_4)\delta_{0+1+2-3-4} dk_{1234} \\ &\quad + 48\pi \int [\tilde{X}^{(2)}_{0,1,2,3,4,5}]^2 n_0 n_1 n_2 n_3 n_4 n_5 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \\ &\quad \times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4 - \omega_5)\delta_{0+1-2-3-4-5} dk_{12345} \\ &\quad + 48\pi \int [\tilde{X}^{(3)}_{0,1,2,3,4,5}]^2 n_0 n_1 n_2 n_3 n_4 n_5 (\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \\ &\quad \times \delta(\omega_0 + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5)\delta_{0+1+2-3-4-5} dk_{12345} \\ &\quad + 96\pi \int [\tilde{X}^{(2)}_{5,4,3,2,1,0}]^2 n_0 n_1 n_2 n_3 n_4 n_5 (\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \\ &\quad \times \delta(\omega_0 + \omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5)\delta_{0+1+2+3-4-5} dk_{12345}.\end{aligned} \tag{80}$$

Equation (80) contains the kinetic equation of four-five-wave resonance given by Krasitskii [7]. In the approximation applied, each kinetic integral depends only on itself.

Finally, we consider the possibility of the existence of conservation laws of the form $J = \int r_0 n_0 dk_0$. For $r_0 = \boldsymbol{k}$, J is the mean momentum of the random wave field, for $r_0 = \omega_0$, it is the mean potential energy, and for $r_0 = 1$, it is the mean wave action. From Equation (80), we obtain

$$\begin{aligned}
\frac{\partial J}{\partial t} &= \pi \int [\tilde{V}^{(2)}_{0,1,2,3}]^2 (r_0 + r_1 - r_2 - r_3) n_0 n_1 n_2 n_3 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}) \\
&\times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta_{0+1-2-3} dk_{123} \\
&+ 6\pi \int [\tilde{W}^{(2)}_{0,1,2,3,4}]^2 (r_0 + r_1 - r_2 - r_3 - r_4) n_0 n_1 n_2 n_3 n_4 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} \\
&- \frac{1}{n_3} - \frac{1}{n_4}) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4) \delta_{0+1-2-3-4} dk_{1234} \\
&+ 48\pi \int [\tilde{X}^{(2)}_{0,1,2,3,4,5}]^2 (r_0 + r_1 - r_2 - r_3 - r_4 - r_5) n_0 n_1 n_2 n_3 n_4 n_5 \\
&\times (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4 - \omega_5) \\
&\times \delta_{0+1-2-3-4-5} dk_{12345} \\
&+ 8\pi \int [\tilde{X}^{(3)}_{0,1,2,3,4,5}]^2 (r_0 + r_1 + r_2 - r_3 - r_4 - r_5) n_0 n_1 n_2 n_3 n_4 n_5 \\
&\times (\frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \delta(\omega_0 + \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5) \\
&\times \delta_{0+1+2-3-4-5} dk_{12345}.
\end{aligned} \tag{81}$$

It is seen from this equation that (1) the Hamilton kinetic equation of four-five-six-wave resonance conserves the momentum and the energy, and (2) only the Hamilton kinetic equation of four-wave resonance conserves the wave action. The action evolution is governed by Equation (81):

$$\begin{aligned}
\frac{\partial}{\partial t} \int n_0 d\boldsymbol{k}_0 &= -6\pi \int [\tilde{W}^{(2)}_{0,1,2,3,4}]^2 n_0 n_1 n_2 n_3 n_4 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4}) \\
&\times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4) \delta_{0+1-2-3-4} dk_{1234} \\
&- 48\pi \int [\tilde{X}^{(2)}_{0,1,2,3,4,5}]^2 n_0 n_1 n_2 n_3 n_4 n_5 (\frac{1}{n_0} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} - \frac{1}{n_4} - \frac{1}{n_5}) \\
&\times \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3 - \omega_4 - \omega_5) \delta_{0+1-2-3-4-5} dk_{12345}.
\end{aligned} \tag{82}$$

6. Discussion and Conclusions

The wave turbulence theory of ocean surface gravity waves has been advanced from the classical four-five-wave resonance [7] to the four-five-six-wave ones in this work, i.e., the dynamical Equation (47) and its kinetic Equation (80), which are hard won and have another basic foothold at a deep level. This introduces a series of new changes and extensions to the basic solutions of the Kolmogorov–Zakharov spectra [11], instability [25–27], nonlinear Schrödinger equation [8,44], etc., and also provides a theoretical platform for an extension of the practical and extensive level for wave–current interactions [45,46], slowly varying depths [47,48], surface capillary–gravity waves [21,23], atmosphere–ocean coupling [49,50], etc.

If we look back at the wave turbulence theory in this work, we find that it has a serious theoretical defect, just like the classical wave turbulence theory [7,9]: it adopts the linear dispersion relation instead of the matching nonlinear dispersion relation [20,51]. This can only be addressed in the future.

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Appendix A. Derivation of the Three Expressions for the Kinetic Energy in (31)

Let

$$\begin{aligned} K^{(3)} &= \coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)K^{(3)}_{0,1}, \\ K^{(4)} &= \coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)K^{(4)}_{0,1}, \\ K^{(5)} &= \coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)K^{(5)}_{0,1}, \\ K^{(6)} &= \coth(|\boldsymbol{k}|h)\coth(|\boldsymbol{k}_1|h)K^{(6)}_{0,1}. \end{aligned} \tag{A1}$$

In Section 3.3, we know that the terms $E^{(4)}_{0,1,2,3}$, $E^{(5)}_{0,1,2,3,4}$ and $E^{(6)}_{0,1,2,3,4,5}$ are determined by substituting (26) into (20) and retaining therein the terms up to the sixth order. Here, the four-waves resonance coefficient $E^{(4)}_{0,1,2,3}$ is determined by the fourth-order terms of the first three terms of the right-hand side of Equation (20) after substitution; the five-waves resonance coefficient $E^{(5)}_{0,1,2,3,4}$ is determined by the fifth-order terms of first four terms of the right-hand side of Equation (20) after substitution; the six-waves resonance coefficient $E^{(5)}_{0,1,2,3,4}$ is determined by the sixth-order terms of the right-hand side of Equation (20) after substitution.

Appendix A.1. The Calculation of $E^{(4)}_{0,1,2,3}$

The $E^{(4)}_{0,1,2,3}$ is composed of three components as follows:

(1) The first component is the fourth-order terms coefficient of substituting the (26) into $\frac{1}{2}\int K^{(2)}\phi_0\phi_0^* dk_0$. The process of calculation is shown below.

$$\begin{aligned} &\frac{1}{2}\int K^{(2)}\tanh^2(|\boldsymbol{k}|h)\psi_0^*\Big(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\Big)dk_0 \\ &+\frac{1}{2}\int K^{(2)}\tanh^2(|\boldsymbol{k}|h)\psi_0\Big(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1^*\zeta_2^*\zeta_3^*\delta_{3+2+1-0}dk_{123}\Big)dk_0 \\ &+\frac{1}{2}\int K^{(2)}\tanh^2(|\boldsymbol{k}|h)\frac{1}{(2\pi)^2}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\int q_1\psi_1^*\zeta_2^*\delta_{0-1-2}dk_{12}dk_0 \\ &=-\frac{1}{2(2\pi)^2}\int\Big[q_0(\Phi^{(3)}_{-0,1,2,3}+\Phi^{(3)}_{0,-1,-2,-3})-q_{0+3}q_0q_1\Big] \\ &\times\psi_0\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}. \end{aligned} \tag{A2}$$

The points of calculation are as follows:

$$\begin{aligned} \frac{1}{2}\int K^{(2)}\phi_0\phi_0^* dk_0 = &\frac{1}{2}\int q_0\psi_0^*\Big(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\Big)dk_0 \\ &+\frac{1}{2}\int q_0\psi_0\Big(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1^*\zeta_2^*\zeta_3^*\delta_{3+2+1-0}dk_{123}\Big)dk_0 \\ &+\frac{1}{2}\int q_0\frac{1}{(2\pi)^2}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12} \\ &\times\int q_1\psi_1^*\zeta_2^*\delta_{2+1-0}dk_{12}dk_0. \end{aligned} \tag{A3}$$

The first two terms on the right-hand side of (A3) are

$$
\begin{aligned}
&-\frac{1}{2(2\pi)^2}\int q_0\left[\Phi^{(3)}_{0,1,2,3}\psi_{-0}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}+\Phi^{(3)}_{0,1,2,3}\psi_0\psi_{-1}\zeta_{-2}\zeta_{-3}\delta_{3+2+1-0}\right]dk_{0123}\\
&=-\frac{1}{2(2\pi)^2}\int q_0(\Phi^{(3)}_{-0,1,2,3}+\Phi^{(3)}_{0,-1,-2,-3})\psi_0\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}.
\end{aligned} \tag{A4}
$$

The last term on the right-hand side of (A3) is

$$
\begin{aligned}
&\frac{1}{2(2\pi)^2}\int q_0\int q_n\psi_n\zeta_m\delta_{0-n-m}dk_{nm}\int q_1\psi_1^*\zeta_2^*\delta_{2+1-0}dk_{12}dk_0\\
&\qquad=\frac{1}{2(2\pi)^2}\int q_{n+m}q_n\psi_n\zeta_m q_1\psi_1^*\zeta_2^*\delta_{-n-m+1+2}dk_{nm12}(n\to 0, m\to 3)\\
&\qquad=\frac{1}{2(2\pi)^2}\int q_{0+3}q_0q_1\psi_0\psi_{-1}\zeta_{-2}\zeta_3\delta_{-n-m+1+2}dk_{0123}\\
&\qquad=\frac{1}{2(2\pi)^2}\int q_{0+3}q_0q_1\psi_0\psi_1\zeta_1\zeta_3\delta_{0+1+2+3}dk_{0123}.
\end{aligned} \tag{A5}
$$

Similar calculations will be used several times below, but we will not go into details.

(2) The second component is the fourth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)}\int K^{(3)}\phi_0\phi_1\zeta_2\delta_{0+1+2}dk_{012}$.

$$
\begin{aligned}
&-\frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\psi_0(-\frac{1}{2\pi}\int q_0\psi_0\zeta_2\delta_{1-0-2}dk_{02})\zeta_2\delta_{0+1+2}dk_{012}\\
&-\frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\psi_1(-\frac{1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12})\zeta_2\delta_{0+1+2}dk_{012}\\
&=\frac{1}{2(2\pi)^2}\int\left(K^{(3)}_{0,1+3}q_1+K^{(3)}_{0+3,1}q_0\right)\psi_0\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}.
\end{aligned} \tag{A6}
$$

(3) The third component is the fourth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^2}\int K^{(4)}\phi_0\phi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}$.

$$
-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\psi_0\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}. \tag{A7}
$$

From the above, it follows that

$$
\begin{aligned}
E^{(4)}_{0,1,2,3}=\frac{1}{2(2\pi)^2}\Big\{&-q_0(\Phi^{(3)}_{-0,1,2,3}+\Phi^{(3)}_{0,-1,-2,-3})+q_{0+3}q_0q_1\\
&+[(\boldsymbol{k}\cdot\boldsymbol{k}_{1+3})-q_0q_{1+3}]q_1+[(\boldsymbol{k}_{0+3}\cdot\boldsymbol{k}_1)-q_{0+3}q_1]q_0\\
&-\frac{1}{2}[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}_1|^2]q_0+[(\boldsymbol{k}\cdot\boldsymbol{k}_1)-|\boldsymbol{k}|^2]q_1\Big\}.
\end{aligned} \tag{A8}
$$

By proper symmetrization, (A8) yields

$$
\begin{aligned}
E^{(4)}_{0,1,2,3}=&-\frac{1}{8(2\pi)^2}\Big[-2q_0\boldsymbol{k}_1\cdot(\boldsymbol{k}+\boldsymbol{k}_2+\boldsymbol{k}_3)-2q_1\boldsymbol{k}_0\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\\
&-q_0q_1(q_{0+2}+q_{0+3}+q_{1+2}+q_{1+3})\Big]\\
=&-\frac{1}{8(2\pi)^2}\Big[2q_0|\boldsymbol{k}_1|^2+2q_1|\boldsymbol{k}|^2-q_0q_1(q_{0+2}+q_{0+3}+q_{1+2}+q_{1+3})\Big].
\end{aligned} \tag{A9}
$$

Appendix A.2. The Calculation of $E^{(5)}_{0,1,2,3,4}$

The $E^{(5)}_{0,1,2,3,4}$ is composed of four components as follows:

(1) The first component is the fifth-order terms coefficient of substituting the (26) into $\frac{1}{2}\int K^{(2)}\phi_0\phi_0^* dk_0$.

$$
\begin{aligned}
&\frac{1}{2}\int q_0\left(\frac{-1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\right)\\
&\times\left(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right)^* dk_0\\
&+\frac{1}{2}\int q_0\psi_0\left(-\frac{1}{(2\pi)^3}\int \Phi^{(4)}_{0,1,2,3,4}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\right)^* dk_0\\
&+\frac{1}{2}\int q_0\left(\frac{-1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\right)^*\\
&\times\left(-\frac{1}{(2\pi)^2}\int \Phi^{(3)}_{0,1,2,3}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right)dk_0\\
&+\frac{1}{2}\int q_0\psi_0^*\left(-\frac{1}{(2\pi)^3}\int \Phi^{(4)}_{0,1,2,3,4}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\right)dk_0\\
&=\frac{1}{2(2\pi)^3}\int\Big[q_0q_{0+4}\left(\Phi^{(3)}_{0+4,-1,-2,-3}+\Phi^{(3)}_{-0-4,1,2,3}\right)\\
&-q_0\left(\Phi^{(4)}_{0,-1,-2,-3,-4}+\Phi^{(4)}_{-0,1,2,3,4}\right)\Big]\psi_0\psi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}.
\end{aligned}
\tag{A10}
$$

(2) The second component is the fifth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)}\int K^{(3)}\phi_0\phi_1\zeta_2\delta_{0+1+2}dk_{012}$.

$$
\begin{aligned}
&-\frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\psi_0\left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{1,n,m,r}\psi_n\zeta_m\zeta_r\delta_{1-n-m-r}dk_{nmr}\right)\\
&\times\zeta_2\delta_{0+1+2}dk_{012}\\
&-\frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{0-r-s}dk_{rs}\right)\\
&\times\left(-\frac{1}{2\pi}\int q_n\psi_n\zeta_m\delta_{1-n-m}dk_{nm}\right)\zeta_2\delta_{0+1+2}dk_{012}\\
&-\frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{0,n,m,r}\psi_n\zeta_m\zeta_r\delta_{0-n-m-r}dk_{nmr}\right)\psi_1\zeta_2dk_{012}\\
=\;&\frac{1}{2(2\pi)^3}\int\Big[K^{(3)}_{0,-0-2}\Phi^{(3)}_{-0-2,1,3,4}-q_0q_1K^{(3)}_{0+3,1+4}+K^{(3)}_{-1-2,1}\Phi^{(3)}_{-1-2,0,3,4}\Big]\\
&\times\psi_0\psi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}.
\end{aligned}
\tag{A11}
$$

(3) The third component is the fifth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^2}\int K^{(4)}\phi_0\phi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}$.

$$
\begin{aligned}
&-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\psi_0\left(-\frac{1}{2\pi}\int q_n\psi_n\zeta_m\delta_{1-n-m}dk_{nm}\right)\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{0-r-s}dk_{rs}\right)\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&=\frac{1}{2(2\pi)^3}\int\Big[q_1K^{(4)}_{0,1+4}+q_0K^{(4)}_{0+4,1}\Big]\psi_0\psi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}.
\end{aligned}
\tag{A12}
$$

(4) The fourth component is the fifth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^3}\int K^{(5)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}$.

$$-\frac{1}{2(2\pi)^3}\int K^{(5)}_{0,1}\psi_0\psi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}. \tag{A13}$$

From the above, it follows that

$$\begin{aligned} E^{(5)}_{0,1,2,3,4} =&\frac{1}{2(2\pi)^3}\Bigg\{\bigg[q_0q_{0+4}\Big(\Phi^{(3)}_{0+4,-1,-2,-3}+\Phi^{(3)}_{-0-4,1,2,3}\Big) \\ &-q_0\Big(\Phi^{(4)}_{0,-1,-2,-3,-4}+\Phi^{(4)}_{-0,1,2,3,4}\Big)\bigg]+\bigg[K^{(3)}_{0,-0-2}\Phi^{(3)}_{-0-2,1,3,4} \\ &+K^{(3)}_{-1-2,1}\Phi^{(3)}_{-1-2,0,3,4}-q_0q_1K^{(3)}_{0+3,1+4}\bigg] \\ &+\bigg[q_1K^{(4)}_{0,1+4}+q_0K^{(4)}_{0+4,1}\bigg]-K^{(5)}_{0,1}\Bigg\}. \end{aligned} \tag{A14}$$

By proper symmetrization, the terms on the right-hand side in (A14) can be expressed as the following four equations, respectively.

(1)

$$\begin{aligned} &\bigg[q_0q_{0+4}\Big(\Phi^{(3)}_{0+4,-1,-2,-3}+\Phi^{(3)}_{-0-4,1,2,3}\Big)-q_0\Big(\Phi^{(4)}_{0,-1,-2,-3,-4}+\Phi^{(4)}_{-0,1,2,3,4}\Big)\bigg] \\ &=\frac{1}{6}\bigg[q_0|\boldsymbol{k}_1|^2(\sum_{r=2}^{4}q_{0+r})+q_1|\boldsymbol{k}|^2(\sum_{r=2}^{4}q_{1+r})-2q_0q_1\sum_{r=2}^{4}q_{0+r}\Bigg(\sum_{t=2,t\neq r}^{4}q_{1+t}\Bigg) \\ &-q_0q_1\Bigg(|\boldsymbol{k}_1|^2-\sum_{r=2}^{4}|\boldsymbol{k}_{1+r}|^2\Bigg)+q_0|\boldsymbol{k}_1|^2(\sum_{r=2}^{4}q_{0+r})-q_0q_1\Bigg(|\boldsymbol{k}|^2-\sum_{r=2}^{4}|\boldsymbol{k}_{0+r}|^2\Bigg) \\ &+q_1|\boldsymbol{k}|^2(\sum_{r=2}^{4}q_{1+r})-2q_0q_1\sum_{r=2}^{4}q_{0+r}\Bigg(\sum_{t=2,t\neq r}^{4}q_{1+t}\Bigg)\bigg]. \end{aligned} \tag{A15}$$

(2)

$$\begin{aligned} &\bigg[K^{(3)}_{0,-0-2}\Phi^{(3)}_{-0-2,1,3,4}-q_0q_1K^{(3)}_{0+3,1+4}+K^{(3)}_{-1-2,1}\Phi^{(3)}_{-1-2,0,3,4}\bigg] \\ &=\frac{1}{6}\bigg[(|\boldsymbol{k}_1|^2+|\boldsymbol{k}|^2)(\boldsymbol{k}\cdot\boldsymbol{k}_1)-4|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2+q_1\sum_{r=2}^{4}(\boldsymbol{k}\cdot\boldsymbol{k}_{0+r})\Bigg(\sum_{t=2,t\neq r}^{4}q_{1+t}\Bigg) \\ &+q_0\sum_{r=2}^{4}(\boldsymbol{k}_1\cdot\boldsymbol{k}_{1+r})\Bigg(\sum_{t=2,t\neq r}^{4}q_{0+t}\Bigg)+3q_0q_1\sum_{r=2}^{4}q_{0+r}\Bigg(\sum_{t=2,t\neq r}^{4}q_{1+t}\Bigg) \\ &-q_0q_1\sum_{r=2}^{4}\boldsymbol{k}_{0+r}\cdot\Bigg(\sum_{t=2,t\neq r}^{4}\boldsymbol{k}_{1+t}\Bigg)-q_0|\boldsymbol{k}_1|^2(\sum_{r=2}^{4}q_{0+r})-q_1|\boldsymbol{k}|^2(\sum_{r=2}^{4}q_{1+r})\bigg]. \end{aligned} \tag{A16}$$

(3)

$$\begin{aligned} &\bigg[q_1K^{(4)}_{0,1+4}+q_0K^{(4)}_{0+4,1}\bigg] \\ &=\frac{1}{6}\bigg[q_0q_1(\boldsymbol{k}\cdot\sum_{r=2}^{4}\boldsymbol{k}_{1+r}-\sum_{r=2}^{4}|\boldsymbol{k}_{1+r}|^2)+\sum_{r=2}^{4}q_1q_{1+r}(\boldsymbol{k}\cdot\boldsymbol{k}_{1+r}) \\ &-q_1|\boldsymbol{k}|^2(\sum_{r=2}^{4}q_{1+r})\bigg]+\frac{1}{6}\bigg[q_0q_1(\boldsymbol{k}_1\cdot\sum_{r=2}^{4}\boldsymbol{k}_{0+r}-\sum_{r=2}^{4}|\boldsymbol{k}_{0+r}|^2) \end{aligned} \tag{A17}$$

$$+\sum_{r=2}^{4} q_0 q_{0+r}(\boldsymbol{k}_1\cdot\boldsymbol{k}_{0+r}) - q_0|\boldsymbol{k}_1|^2(\sum_{r=2}^{4} q_{0+r})\Big].$$

(4)

$$-K_{0,1}^{(5)} = -\frac{1}{6}[(|\boldsymbol{k}|^2+|\boldsymbol{k}_1|^2)(\boldsymbol{k}\cdot\boldsymbol{k}_1) - 2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2 - |\boldsymbol{k}-\boldsymbol{k}_1|^2 q_0 q_1]. \tag{A18}$$

From (A.15)–(A.18), it follows that

$$\begin{aligned} E_{0,1,2,3,4}^{(5)} = &-\frac{1}{12(2\pi)^3}\Bigg[2|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2 - q_0|\boldsymbol{k}_1|^2(\sum_{r=2}^{4} q_{0+r}) - q_1|\boldsymbol{k}|^2(\sum_{r=2}^{4} q_{1+r}) \\ &+\frac{1}{2}q_0q_1\left(|\boldsymbol{k}|^2 - \sum_{r=2}^{4}|\boldsymbol{k}_{0+r}|^2 + |\boldsymbol{k}_1|^2 - \sum_{r=2}^{4}|\boldsymbol{k}_{1+r}|^2\right) \\ &+ q_0q_1\sum_{r=2}^{4} q_{0+r}\left(\sum_{t=2,t\neq r}^{4} q_{1+t}\right)\Bigg]. \end{aligned} \tag{A19}$$

Appendix A.3. The Calculation of $E_{0,1,2,3,4,5}^{(6)}$

The $E_{0,1,2,3,4,5}^{(6)}$ is composed of five components as follows

(1) The first component is the sixth-order terms coefficient of substituting the (26) into $\frac{1}{2}\int K^{(2)}\phi_0\phi_0^* dk_0$.

$$\begin{aligned} &\frac{1}{2}\int q_0\left(\frac{-1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\right) \\ &\times\left(-\frac{1}{(2\pi)^3}\int \Phi_{0,1,2,3,4}^{(4)}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\right)^* dk_0 \\ &+\frac{1}{2}\int q_0\psi_0\left(-\frac{1}{(2\pi)^4}\int \Phi_{0,1,2,3,4,5}^{(5)}\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345}\right)^* dk_0 \\ &+\frac{1}{2}\int q_0\left(-\frac{1}{(2\pi)^2}\int \Phi_{0,1,2,3}^{(3)}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right)^* \\ &\times\left(-\frac{1}{(2\pi)^2}\int \Phi_{0,1,2,3}^{(3)}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right)dk_0 \\ &+\frac{1}{2}\int q_0\left(\frac{-1}{2\pi}\int q_1\psi_1\zeta_2\delta_{0-1-2}dk_{12}\right)^* \\ &\times\left(-\frac{1}{(2\pi)^3}\int \Phi_{0,1,2,3,4}^{(4)}\psi_1\zeta_2\zeta_3\zeta_4\delta_{0-1-2-3-4}dk_{1234}\right)dk_0 \\ &+\frac{1}{2}\int q_0\psi_0^*\left(-\frac{1}{(2\pi)^4}\int \Phi_{0,1,2,3,4,5}^{(5)}\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0-1-2-3-4-5}dk_{12345}\right)dk_0 \\ &+\frac{1}{2}\int q_0\left(-\frac{1}{(2\pi)^2}\int \Phi_{0,1,2,3}^{(3)}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right) \\ &\times\left(-\frac{1}{(2\pi)^2}\int \Phi_{0,1,2,3}^{(3)}\psi_1\zeta_2\zeta_3\delta_{0-1-2-3}dk_{123}\right)^* dk_0 \\ &=\frac{1}{2(2\pi)^4}\int\Big[q_0q_{0+5}\left(\Phi_{0+5,-1,-2,-3,-4}^{(4)}+\Phi_{-0-5,1,2,3,4}^{(4)}\right) \\ &-q_0\left(\Phi_{0,-1,-2,-3,-4,-5}^{(5)}+\Phi_{-0,1,2,3,4,5}^{(5)}\right)+2q_{0+4+5}\Phi_{-0-4-5,-0,-4,-5}^{(3)}\Phi_{-0-4-5,1,2,3}^{(3)}\Big] \\ &\times\psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}. \end{aligned} \tag{A20}$$

(2) The second component is the sixth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)}\int K^{(3)}\phi_0\phi_1\zeta_2\delta_{0+1+2}dk_{012}$.

$$\begin{aligned}
&- \frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\psi_0\left(\frac{-1}{(2\pi)^3}\int \Phi^{(4)}_{1,n,m,r,t}\psi_n\zeta_m\zeta_r\zeta_t\delta_{1-n-m-r-t}dk_{nmrt}\right)\\
&\times \zeta_2\delta_{0+1+2}dk_{012}\\
&- \frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{0-r-s}dk_{rs}\right)\\
&\times \left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{1,n,m,r}\psi_n\zeta_m\zeta_r\delta_{1-n-m-r}dk_{nmr}\right)\zeta_2\delta_{0+1+2}dk_{012}\\
&- \frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{1-r-s}dk_{rs}\right)\\
&\times \left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{0,n,m,r}\psi_n\zeta_m\zeta_r\delta_{0-n-m-r}dk_{nmr}\right)\zeta_2\delta_{0+1+2}dk_{012}\\
&- \frac{1}{2(2\pi)}\int K^{(3)}_{0,1}\left(\frac{-1}{(2\pi)^3}\int \Phi^{(4)}_{0,n,m,r,t}\psi_n\zeta_m\zeta_r\zeta_t\delta_{0-n-m-r-t}dk_{nmrt}\right)\psi_1\zeta_2dk_{012}\\
&= \frac{1}{2(2\pi)^4}\int\Big[K^{(3)}_{0,-0-2}\Phi^{(4)}_{-0-2,1,3,4,5}+K^{(3)}_{-1-2,1}\Phi^{(4)}_{-1-2,0,3,4,5}\\
&- q_0\left(K^{(3)}_{0+3,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5}+K^{(3)}_{1+4+5,0+3}\Phi^{(3)}_{1+4+5,1,4,5}\right)\Big]\\
&\times \psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}.
\end{aligned} \tag{A21}$$

(3) The third component is the sixth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^2}\int K^{(4)}\phi_0\phi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}$.

$$\begin{aligned}
&-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\psi_0\left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{1,n,m,r}\psi_n\zeta_m\zeta_r\delta_{1-n-m-r}dk_{nmr}\right)\\
&\times \zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{0-r-s}dk_{rs}\right)\\
&\times\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{1-r-s}dk_{rs}\right)\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&-\frac{1}{2(2\pi)^2}\int K^{(4)}_{0,1}\left(\frac{-1}{(2\pi)^2}\int \Phi^{(3)}_{0,n,m,r}\psi_n\zeta_m\zeta_r\delta_{0-n-m-r}dk_{nmr}\right)\\
&\times \psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&=\frac{1}{2(2\pi)^4}\int\left[K^{(4)}_{0,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5}+K^{(4)}_{0+4+5,1}\Phi^{(3)}_{0+4+5,0,4,5}-q_0q_1K^{(4)}_{0+4,1+5}\right]\\
&\times \psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}.
\end{aligned} \tag{A22}$$

(4) The fourth component is the sixth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^3}\int K^{(5)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}$.

$$\begin{aligned}
&-\frac{1}{2(2\pi)^3}\int K^{(5)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\delta_{0+1+2+3+4}dk_{01234}\\
&=-\frac{1}{2(2\pi)^3}\int K^{(5)}_{0,1}\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{0-r-s}dk_{rs}\right)\psi_1\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&-\frac{1}{2(2\pi)^3}\int K^{(5)}_{0,1}\psi_0\left(-\frac{1}{2\pi}\int q_r\psi_r\zeta_s\delta_{1-r-s}dk_{rs}\right)\zeta_2\zeta_3\delta_{0+1+2+3}dk_{0123}\\
&=\frac{1}{2(2\pi)^4}\int\left[q_0K^{(5)}_{0+5,1}+q_1K^{(5)}_{0,1+5}\right]\psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}.
\end{aligned} \tag{A23}$$

(5) The fifth component is the sixth-order terms coefficient of substituting the (26) into $-\frac{1}{2(2\pi)^4}\int K^{(6)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}$.

$$\begin{aligned}&-\frac{1}{2(2\pi)^4}\int K^{(6)}\phi_0\phi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}\\&\quad=-\frac{1}{2(2\pi)^4}\int K^{(6)}_{0,1}\psi_0\psi_1\zeta_2\zeta_3\zeta_4\zeta_5\delta_{0+1+2+3+4+5}dk_{012345}.\end{aligned}\tag{A24}$$

From the above, it follows that

$$\begin{aligned}E^{(6)}_{0,1,2,3,4,5}=\frac{1}{2(2\pi)^4}\Big\{&q_0q_{0+5}\Big(\Phi^{(4)}_{0+5,-1,-2,-3,-4}+\Phi^{(4)}_{-0-5,1,2,3,4}\Big)\\&-q_0\Big(\Phi^{(5)}_{0,-1,-2,-3,-4,-5}+\Phi^{(5)}_{-0,1,2,3,4,5}\Big)\\&+2q_{0+4+5}\Phi^{(3)}_{-0-4-5,-0,-4,-5}\Phi^{(3)}_{-0-4-5,1,2,3}\\&+K^{(3)}_{0,-0-2}\Phi^{(4)}_{-0-2,1,3,4,5}+K^{(3)}_{-1-2,1}\Phi^{(4)}_{-1-2,0,3,4,5}\\&-q_0\Big(K^{(4)}_{0+3,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5}+K^{(4)}_{1+4+5,0+3}\Phi^{(3)}_{1+4+5,1,4,5}\Big)\\&+K^{(4)}_{0,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5}+K^{(4)}_{0+4+5,1}\Phi^{(3)}_{0+4+5,0,4,5}\\&-q_0q_1K^{(4)}_{0+4,1+5}+q_0K^{(5)}_{0+5,1}+q_1K^{(5)}_{0,1+5}-K^{(6)}_{0,1}\Big\}.\end{aligned}\tag{A25}$$

The main purpose of the following is to simplify the right-hand side of the above equation. After proper symmetric transformation, the right term of the (A25) can be expressed into the following nine simplified equations:

(1)

$$\begin{aligned}&q_0q_{0+5}\Big(\Phi^{(4)}_{0+5,-1,-2,-3,-4}+\Phi^{(4)}_{-0-5,1,2,3,4}\Big)\\&=\Bigg\{\frac{1}{24}q_0q_{0+5}\Bigg[q_1|\boldsymbol{k}_1|^2-q_1\sum_{r=2}^{4}|\boldsymbol{k}_{1+r}|^2-|\boldsymbol{k}_1|^2\Bigg(\sum_{r=2}^{4}q_{0+5+r}\Bigg)\\&+\sum_{r=2}^{4}q_1q_{0+5+r}\Bigg(\sum_{t=2,t\neq r}^{4}q_{1+t}\Bigg)\Bigg]\Bigg\}+(5\Leftrightarrow 2)+(5\Leftrightarrow 3)+(5\Leftrightarrow 4)\\&+(0\Leftrightarrow 1)+(0\Leftrightarrow 1,5\Leftrightarrow 2)+(0\Leftrightarrow 1,5\Leftrightarrow 3)+(0\Leftrightarrow 1,5\Leftrightarrow 4).\end{aligned}\tag{A26}$$

In this context, where $(m\Leftrightarrow n)$ standing for the terms in curly braces on the right-hand side of (A29) should be repeated, and the indices m and n interchanged. The similarities in the following text will not be specified.

(2)

$$\begin{aligned}&-q_0\Big(\Phi^{(5)}_{0,-1,-2,-3,-4,-5}+\Phi^{(5)}_{-0,1,2,3,4,5}\Big)\\&=\frac{-q_0}{24}\Bigg\{|\boldsymbol{k}_1|^4-\sum_{r=2}^{5}|\boldsymbol{k}_1+\boldsymbol{k}_r|^2q_1q_{1+r}-|\boldsymbol{k}_1|^2\Bigg(\sum_{2\le r<s\le 5}|\boldsymbol{k}_{1+r+s}|^2\Bigg)\\&+\sum_{r=2}^{5}\Bigg(q_1q_{1+r}\sum_{s=2,s\neq r}^{5}|\boldsymbol{k}_{1+r+s}|^2\Bigg)+\sum_{r=2}^{5}\Bigg(|\boldsymbol{k}_1+\boldsymbol{k}_r|^2q_1\sum_{s=2,s\neq r}^{5}q_{0+s}\Bigg)\\&-q_1|\boldsymbol{k}_1|^2\Bigg(\sum_{r=2}^{5}q_{0+r}\Bigg)+|\boldsymbol{k}_1|^2\sum_{r=2}^{5}\Bigg[q_{0+r}\sum_{s=2,s\neq r}^{5}(q_{0+r+s})\Bigg]\end{aligned}\tag{A27}$$

$$-\sum_{r=2}^{5} q_1 q_{0+r}\left[\sum_{s=2,s\neq r}^{5} q_{0+r+s}\left(\sum_{t=2,t\neq r,t\neq s}^{5} q_{1+t}\right)\right]\Bigg\} + (0 \Leftrightarrow 1).$$

(3)

$$\begin{aligned}
&2q_{0+4+5}\Phi^{(3)}_{-0-4-5,-0,-4,-5}\Phi^{(3)}_{-0-4-5,1,2,3}\\
&=\frac{1}{24}q_{0+4+5}\Big(|\boldsymbol{k}|^2 - q_0q_{0+4} - q_0q_{0+5}\Big)(|\boldsymbol{k}_1|^2 - q_1q_{1+3} - q_1q_{1+2})\\
&+((4,5)\Leftrightarrow(2,3)) + ((4,5)\Leftrightarrow(2,4)) + ((4,5)\Leftrightarrow(2,5)) + ((4,5)\Leftrightarrow(3,4))\\
&+((4,5)\Leftrightarrow(3,5)) + (0\Leftrightarrow 1) + (0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,3))\\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,4)) + (0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,5))\\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,4)) + (0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,5)).
\end{aligned} \tag{A28}$$

(4)

$$\begin{aligned}
&K^{(3)}_{0,-0-2}\Phi^{(4)}_{-0-2,1,3,4,5} + K^{(3)}_{-1-2,1}\Phi^{(4)}_{-1-2,0,3,4,5}\\
&=\frac{1}{24}\Bigg\{\Big(-|\boldsymbol{k}|^2 - \boldsymbol{k}\cdot\boldsymbol{k}_2 - q_0q_{0+2}\Big)\Bigg[q_1|\boldsymbol{k}_1|^2 - q_1\sum_{r=3}^{5}|\boldsymbol{k}_{1+r}|^2\\
&-|\boldsymbol{k}_1|^2\left(\sum_{r=3}^{5} q_{0+2+r}\right) + \sum_{r=3}^{5} q_1q_{0+2+r}\left(\sum_{t=3,t\neq r}^{5} q_{1+t}\right)\Bigg]\Bigg\} + (2\Leftrightarrow 3)\\
&+(2\Leftrightarrow 4) + (2\Leftrightarrow 5) + (0\Leftrightarrow 1) + (0\Leftrightarrow 1,2\Leftrightarrow 3)\\
&+(0\Leftrightarrow 1,2\Leftrightarrow 4) + (0\Leftrightarrow 1,2\Leftrightarrow 5).
\end{aligned} \tag{A29}$$

(5)

$$\begin{aligned}
&-q_0\Big(K^{(4)}_{0+3,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5} + K^{(4)}_{1+4+5,0+3}\Phi^{(3)}_{1+4+5,1,4,5}\Big)\\
&=\frac{-q_0}{24}\Big[(\boldsymbol{k}_{0+3}\cdot\boldsymbol{k}_{1+4+5} - q_{0+3}q_{1+4+5})\Big(|\boldsymbol{k}_1|^2 - q_1q_{1+4} - q_1q_{1+5}\Big)\Big]\\
&+((4,5)\Leftrightarrow(2,4)) + ((4,5)\Leftrightarrow(2,5))\\
&+(3\Leftrightarrow 2) + (3\Leftrightarrow 2,(4,5)\Leftrightarrow(3,4)) + (3\Leftrightarrow 2,(4,5)\Leftrightarrow(3,5))\\
&+(3\Leftrightarrow 4,(4,5)\Leftrightarrow(3,5)) + (3\Leftrightarrow 4,(4,5)\Leftrightarrow(2,3))\\
&+(3\Leftrightarrow 4,(4,5)\Leftrightarrow(2,5)) + (3\Leftrightarrow 5,(4,5)\Leftrightarrow(3,4))\\
&+(3\Leftrightarrow 5,(4,5)\Leftrightarrow(2,3)) + (3\Leftrightarrow 5,(4,5)\Leftrightarrow(2,4))\\
&+(0\Leftrightarrow 1) + (0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,4))\\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,5)) + (0\Leftrightarrow 1,3\Leftrightarrow 2)\\
&+(0\Leftrightarrow 1,3\Leftrightarrow 2,(4,5)\Leftrightarrow(3,4)) + (0\Leftrightarrow 1,3\Leftrightarrow 2,(4,5)\Leftrightarrow(3,5))\\
&+(0\Leftrightarrow 1,3\Leftrightarrow 4,(4,5)\Leftrightarrow(3,5)) + (0\Leftrightarrow 1,3\Leftrightarrow 4,(4,5)\Leftrightarrow(2,3))\\
&+(0\Leftrightarrow 1,3\Leftrightarrow 4,(4,5)\Leftrightarrow(2,5)) + (0\Leftrightarrow 1,3\Leftrightarrow 5,(4,5)\Leftrightarrow(3,4))\\
&+(0\Leftrightarrow 1,3\Leftrightarrow 5,(4,5)\Leftrightarrow(2,3)) + (0\Leftrightarrow 1,3\Leftrightarrow 5,(4,5)\Leftrightarrow(2,4)).
\end{aligned} \tag{A30}$$

(6)

$$\begin{aligned}
&K^{(4)}_{0,1+4+5}\Phi^{(3)}_{1+4+5,1,4,5} + K^{(4)}_{0+4+5,1}\Phi^{(3)}_{0+4+5,0,4,5}\\
&=\frac{1}{24}\Big\{\Big[\Big(\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5} - |\boldsymbol{k}_{1+4+5}|^2\Big)q_0 + \Big(\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5} - |\boldsymbol{k}|^2\Big)q_{1+4+5}\Big]\\
&\times\Big(|\boldsymbol{k}_1|^2 - q_1q_{1+4} - q_1q_{1+5}\Big)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+((4,5)\Leftrightarrow(2,3))+((4,5)\Leftrightarrow(2,4))+((4,5)\Leftrightarrow(2,5)) \\
&+((4,5)\Leftrightarrow(3,4))+((4,5)\Leftrightarrow(3,5))+(0\Leftrightarrow 1) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,3))+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,4)) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,5))+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,4)) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,5)).
\end{aligned} \tag{A31}
$$

(7)

$$
\begin{aligned}
&-q_0q_1K^{(4)}_{0+4,1+5} \\
=&\frac{-1}{24}q_0q_1\Big\{\Big[(\boldsymbol{k}_{0+4}\cdot\boldsymbol{k}_{1+5})-|\boldsymbol{k}_{1+5}|^2\Big]q_{0+4}+\Big[(\boldsymbol{k}_{0+4}\cdot\boldsymbol{k}_{1+5})-|\boldsymbol{k}_{0+4}|^2\Big]q_{1+5}\Big\} \\
&+((4,5)\Leftrightarrow(2,3))+((4,5)\Leftrightarrow(2,4))+((4,5)\Leftrightarrow(2,5)) \\
&+((4,5)\Leftrightarrow(3,4))+((4,5)\Leftrightarrow(3,5))+(0\Leftrightarrow 1) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,3))+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,4)) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(2,5))+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,4)) \\
&+(0\Leftrightarrow 1,(4,5)\Leftrightarrow(3,5)).
\end{aligned} \tag{A32}
$$

(8)

$$
\begin{aligned}
&q_0K^{(5)}_{0+5,1}+q_1K^{(5)}_{0,1+5} \\
=&\frac{1}{24}q_0\Big[(|\boldsymbol{k}_{0+5}|^2+|\boldsymbol{k}_1|^2)(\boldsymbol{k}_{0+5}\cdot\boldsymbol{k}_1)-2|\boldsymbol{k}_{0+5}|^2|\boldsymbol{k}_1|^2-|\boldsymbol{k}_{0+5}-\boldsymbol{k}_1|^2q_{0+5}q_1\Big] \\
&+(5\Leftrightarrow 2)+(5\Leftrightarrow 3)+(5\Leftrightarrow 4)+(0\Leftrightarrow 1)+(0\Leftrightarrow 1,5\Leftrightarrow 2) \\
&+(0\Leftrightarrow 1,5\Leftrightarrow 3)+(0\Leftrightarrow 1,5\Leftrightarrow 4).
\end{aligned} \tag{A33}
$$

(9)

$$
\begin{aligned}
-K^{(6)}_{0,1}=&-\frac{1}{24}\{[|\boldsymbol{k}|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1-3|\boldsymbol{k}_1|^2)+|\boldsymbol{k}_1|^2(3\boldsymbol{k}\cdot\boldsymbol{k}_1-|\boldsymbol{k}_1|^2)]q_0 \\
&+[|\boldsymbol{k}_1|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1-3|\boldsymbol{k}|^2)+|\boldsymbol{k}|^2(3\boldsymbol{k}\cdot\boldsymbol{k}_1-|\boldsymbol{k}|^2)]q_1\}.
\end{aligned} \tag{A34}
$$

Substituting (A26)–(A34) into (A25), and combined with the resonance conditions, after the combination of similar terms we can have

$$
\begin{aligned}
&48(2\pi)^4E^{(6)}_{0,1,2,3,4,5} \\
=&-q_0|\boldsymbol{k}_1|^4+q_0|\boldsymbol{k}_1|^2\left(\sum_{2\le r<s\le 5}^{5}|\boldsymbol{k}_{0+r+s}|^2\right) \\
&+|\boldsymbol{k}|^2|\boldsymbol{k}_1|^2\left[\sum_{2\le r<s\le 5}^{5}q_{0+r+s}\right] \\
&-\sum_{r=2}^{5}q_0q_1q_{0+r}\left(3|\boldsymbol{k}_1|^2-\sum_{t=2,t\ne r}^{5}|\boldsymbol{k}_{1+t}|^2\right) \\
&+2\sum_{r=2}^{5}q_0q_{0+r}\left(\sum_{t=2,t\ne r}^{5}q_{0+r+t}\right)|\boldsymbol{k}_1|^2 \\
&+\sum_{r=2}^{5}q_0q_1q_{0+r}\left[\sum_{s=2,s\ne r}^{5}q_{0+r+s}\left(\sum_{t=2,t\ne r,t\ne s}^{5}q_{1+t}\right)\right]+(0\Leftrightarrow 1).
\end{aligned} \tag{A35}
$$

Merging of the similar terms is the key to simplification (3.91). There are eight types of merging as follows:

(1) The terms similar to $|\boldsymbol{k}_1|^2\left(\sum_{r=3}^{5} q_{0+2+r}\right)(\boldsymbol{k}\cdot\boldsymbol{k}_2)$ of (A29) are transformed to $q_{0+2+r}|\boldsymbol{k}_1|^2(\boldsymbol{k}\cdot\boldsymbol{k}_2+\boldsymbol{k}\cdot\boldsymbol{k}_r)$, and add to $q_{1+4+5}\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5}|\boldsymbol{k}_1|^2 = q_{0+2+3}\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5}|\boldsymbol{k}_1|^2$ of (A31).

(2) The terms similar to $-q_1|\boldsymbol{k}_1|^2(\boldsymbol{k}\cdot\boldsymbol{k}_2)$ of (A29) add to the terms $-q_0|\boldsymbol{k}|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1)$ of (A.34).

(3) The terms similar to $q_1\left(\sum_{r=3}^{5}|\boldsymbol{k}_{1+r}|^2\right)(\boldsymbol{k}\cdot\boldsymbol{k}_2)$ of (A29) add to the terms $q_0|\boldsymbol{k}_{0+5}|^2(\boldsymbol{k}_{0+5}\cdot\boldsymbol{k}_1)$ of (A33).

(4) The terms similar to $-q_0(\boldsymbol{k}_{0+3}\cdot\boldsymbol{k}_{1+4+5})|\boldsymbol{k}_1|^2$ of (A30) add to the terms $q_0\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5}|\boldsymbol{k}_1|^2$ of (A31).

(5) The terms similar to $q_0|\boldsymbol{k}_1|^2(\boldsymbol{k}_{0+5}\cdot\boldsymbol{k}_1)$ of (A33) add to the terms $-3q_0|\boldsymbol{k}_1|^2(\boldsymbol{k}\cdot\boldsymbol{k}_1)$ in (A34).

(6) The terms similar to $(\boldsymbol{k}_{0+3}\cdot\boldsymbol{k}_{1+4+5})q_0q_1(q_{1+4}+q_{1+5})$ of (A30) add to the terms $-(\boldsymbol{k}_{0+4}\cdot\boldsymbol{k}_{1+5})q_0q_1(q_{0+4}+q_{1+5})$ of (A32) .

(7) The terms similar to $-(\boldsymbol{k}\cdot\boldsymbol{k}_2)\left[\sum_{r=3}^{5} q_1q_{0+2+r}\left(\sum_{t=3,t\neq r}^{5} q_{1+t}\right)\right]$ of (A29) add to the terms $-(\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5})q_{1+4+5}q_1(q_{1+4}+q_{1+5})$ of (A.31) .

(8) The terms similar to $-(\boldsymbol{k}\cdot\boldsymbol{k}_{1+4+5})q_0q_1(q_{1+4}+q_{1+5})$ of (A31) add to the terms $\sum_{r=2}^{5} -q_0q_1q_{0+r}|\boldsymbol{k}_1|^2, -\sum_{r=2}^{5} q_0q_1q_{0+r}|\boldsymbol{k}_{0+r}-\boldsymbol{k}_1|^2$, and $\sum_{r=2}^{5} q_0q_1q_{0+r}|\boldsymbol{k}_{0+r}|^2$.

Appendix B. Derivation of the Expression Forms of the Function $Y^{(n)}_{0,1,2,3,4,5}(n=1,2,3,4,5,6)$ in (48)–(53)

For simplicity, introducing a function $\mathcal{F}$ implies that the $\mathcal{F}(U_{0,1,\cdots,m})_{(n,\cdots,m)}$ stands for a function which is symmetric under transpositions within the group $(n,\cdots,m)$, and it contain $(m-n+1)!$ terms. For example,

$$\begin{aligned}\mathcal{F}(C_{(1,2,3,4)})_{(1,2,3,4)} = \frac{1}{24}\big[&(C_{(1,2,3,4)}+C_{(1,2,4,3)}+C_{(1,3,4,2)}+C_{(1,3,2,4)}\\&+C_{(1,4,2,3)}+C_{(1,4,3,2)}+C_{(2,1,3,4)}+C_{(2,1,4,3)}\\&+C_{(2,3,1,4)}+C_{(2,3,4,1)}+C_{(2,4,1,3)}+C_{(2,4,3,1)}\\&+C_{(3,1,2,4)}+C_{(3,1,4,2)}+C_{(3,2,1,4)}+C_{(3,2,4,1)}\\&+C_{(3,4,1,2)}+C_{(3,4,2,1)}+C_{(4,1,2,3)}+C_{(4,1,3,2)}\\&+C_{(4,2,1,3)}+C_{(4,2,3,1)}+C_{(4,3,1,2)}+C_{(4,3,2,1)}\big].\end{aligned} \tag{A36}$$

In the following, we present the forms of functions $Y^{(n)}_{0,1,2,3,4,5}(n=1,2,3,4,5,6)$.

$$\begin{aligned}Y^{(1)}_{0,1,2,3,4,5} = 2\mathcal{F}(&U^{(1)}_{0,1,0-1}C^{(1)}_{0-1,2,3,4,5}+U^{(1)}_{1,0,1-0}C^{(4)}_{1-0,2,3,4,5}\\&+U^{(1)}_{0,1+2,3+4+5}A^{(1)}_{1+2,1,2}B^{(1)}_{3+4+5,3,4,5}\\&+U^{(1)}_{3+4+5,0,-1-2}A^{(3)}_{-1-2,1,2}B^{(1)}_{3+4+5,3,4,5}\\&+U^{(1)}_{1+2,0,-3-4-5}A^{(1)}_{1+2,1,2}B^{(4)}_{-3-4-5,3,4,5})_{(1,2,3,4,5)}\\&+\mathcal{F}(U^{(3)}_{0,-1-2,-3-4-5}A^{(3)}_{-1-2,1,2}B^{(4)}_{-3-4-5,3,4,5})_{(1,2,3,4,5)}\\&+3\mathcal{F}(V^{(1)}_{0,1,2,3+4+5}B^{(1)}_{3+4+5,3,4,5})_{(1,2,3,4,5)}\\&+3\mathcal{F}(V^{(1)}_{0,1,2+3,4+5}A^{(1)}_{2+3,2,3}A^{(1)}_{4+5,4,5})_{(1,2,3,4,5)}\\&+\mathcal{F}(V^{(2)}_{0,1,2,-3-4-5}B^{(4)}_{-3-4-5,3,4,5})_{(1,2,3,4,5)}\\&+2\mathcal{F}(V^{(1)}_{0,1,-2-3,4+5}A^{(3)}_{-2-3,2,3}A^{(1)}_{4+5,4,5})_{(1,2,3,4,5)}\\&+3\mathcal{F}(V^{(3)}_{1,0,-2-3,-4-5}A^{(3)}_{-2-3,2,3}A^{(3)}_{-4-5,4,5})_{(1,2,3,4,5)}\\&+4\mathcal{F}(W^{(1)}_{0,1,2,3,4+5}A^{(1)}_{4+5,4,5})_{(1,2,3,4,5)}\end{aligned} \tag{A37}$$

$$+\mathcal{F}(W^{(2)}_{0,1,2,3,-4-5}A^{(3)}_{-4-5,4,5})_{(1,2,3,4,5)},$$

$$\begin{aligned}
Y^{(2)}_{0,1,2,3,4,5} = {} & 2\mathcal{F}[U^{(1)}_{0,2,0-2}C^{(2)}_{0-2,1,3,4,5} + U^{(1)}_{0,2+3,-1+4+5}A^{(1)}_{2+3,2,3}B^{(2)}_{-1+4+5,1,4,5} \\
& + U^{(1)}_{0,-1+2,3+4+5}A^{(2)}_{-1+2,1,2}B^{(1)}_{3+4+5,3,4,5} + U^{(1)}_{0+1,1,0}C^{(1)}_{0+1,2,3,4,5} \\
& + U^{(1)}_{3+4+5,1-2,0}A^{(2)}_{1-2,1,2}B^{(1)}_{3+4+5,3,4,5} \\
& + U^{(1)}_{4+5-1,-2-3,0}A^{(3)}_{-2-3,2,3}B^{(2)}_{4+5-1,1,4,5} \\
& + U^{(1)}_{2-1,-3-4-5,0}A^{(2)}_{2-1,1,2}B^{(4)}_{-3-4-5,3,4,5} \\
& + U^{(1)}_{2+3,1-4-5,0}A^{(1)}_{2+3,2,3}B^{(3)}_{1-4-5,1,4,5} \\
& + U^{(1)}_{2,2-0,0}C^{(4)}_{2-0,1,3,4,5} + U^{(3)}_{0,1,-0-1}C^{(5)}_{-0-1,2,3,4,5} \\
& + U^{(3)}_{0,1-2,-3-4-5}A^{(2)}_{1-2,2,1}B^{(4)}_{-3-4-5,3,4,5} \\
& + U^{(3)}_{0,-2-3,1-4-5}A^{(3)}_{-2-3,2,3}B^{(3)}_{1-4-5,1,4,5}]_{(2,3,4,5)} \\
& + 3\mathcal{F}[V^{(1)}_{0,2,3,4+5-1}B^{(2)}_{4+5-1,1,4,5} + V^{(1)}_{0,5,3+4,2-1}A^{(1)}_{3+4,3,4}A^{(2)}_{2-1,2,1}]_{(2,3,4,5)} \\
& + \mathcal{F}[V^{(2)}_{0,2,3,4+5-1}B^{(3)}_{4+5-1,4,5,1} \\
& + 2V^{(2)}_{0,1,2,3+4+5}B^{(1)}_{3+4+5,3,4,5}]_{(2,3,4,5)} \\
& + 6\mathcal{F}[V^{(1)}_{3,1-2,-4-5,0}A^{(3)}_{-4-5,4,5}A^{(2)}_{1-2,2,1} \\
& + V^{(1)}_{4+5,-2-3,1,0}A^{(1)}_{4+5,4,5}A^{(3)}_{-2-3,2,3} \\
& + V^{(1)}_{3,-2-4-5,1,0}B^{(4)}_{-2-4-5,2,4,5}]_{(2,3,4,5)} \\
& + 3\mathcal{F}[V^{(4)}_{0,1,-2-3,-4-5}A^{(3)}_{-2-3,2,3}A^{(3)}_{-4-5,4,5}]_{(2,3,4,5)} \\
& + \mathcal{F}[4W^{(1)}_{0,2,3,4,5-1}A^{(2)}_{5-1,5,1} + W^{(2)}_{0,1-5,2,3,4}A^{(2)}_{1-5,5,1} \\
& + 3W^{(2)}_{0,1,2,3,4+5}A^{(1)}_{4+5,4,5} + 3W^{(2)}_{4,3,1,-2-5,0}A^{(3)}_{-2-5,2,5}]_{(2,3,4,5)} \\
& - 2\mathcal{F}[A^{(1)}_{0,2,0-2}\tilde{W}^{(2)}_{0-2,1,3,4,5}]_{(2,3,4,5)} \\
& - 3\mathcal{F}[B^{(1)}_{0,0-2-3,2,3}\tilde{V}^{(2)}_{0-2-3,1,4,5}]_{(2,3,4,5)},
\end{aligned} \tag{A38}$$

$$Y^{(3)}_{0,1,2,3,4,5} = \mathcal{F}[Y3^{(3)}_{0,1,2,3,4,5}]_{(1,2)}, \tag{A39}$$

where

$$\begin{aligned}
Y3^{(3)}_{0,1,2,3,4,5} = {} & 2\mathcal{F}[U^{(1)}_{0,3,0-3}C^{(3)}_{0-3,1,2,4,5} + U^{(1)}_{0,3+4,5-1-2}A^{(1)}_{3+4,3,4}B^{(3)}_{5-1-2,1,2,5} \\
& + U^{(1)}_{0,-1-2,3+4+5}A^{(3)}_{-1-2,1,2}B^{(1)}_{3+4+5,3,4,5} \\
& + U^{(1)}_{0,3-1,4+5-2}A^{(2)}_{3-1,1,3}B^{(2)}_{4+5-2,2,4,5} \\
& + U^{(2)}_{0+1,1,0}C^{(2)}_{0+1,2,3,4,5} + U^{(1)}_{3+4+5,1+2,0}A^{(1)}_{1+2,1,2}B^{(1)}_{3+4+5,3,4,5} \\
& + U^{(1)}_{4+5-2,1-3,0}A^{(2)}_{1-3,1,3}B^{(2)}_{4+5-2,2,4,5} \\
& + U^{(1)}_{4+5,1+2-3,0}A^{(1)}_{4+5,4,5}B^{(2)}_{1+2-3,3,2,1} \\
& + U^{(1)}_{3-1,2-4-5,0}A^{(2)}_{3-1,1,3}B^{(2)}_{2-4-5,5,4,2} \\
& + U^{(1)}_{-1-2,-3-4-5,0}A^{(3)}_{-1-2,1,2}B^{(4)}_{-3-4-5,3,4,5} \\
& + U^{(1)}_{3-1-2,-4-5,0}A^{(3)}_{-4-5,4,5}B^{(3)}_{3-1-2,1,2,3} + U^{(3)}_{3,0+3,0}C^{(3)}_{0+3,1,2,4,5} \\
& + U^{(3)}_{0,1,-0-1}C^{(4)}_{-0-1,3,4,5,2} + U^{(3)}_{0,1+2,-3-4-5}A^{(1)}_{1+2,1,2}B^{(4)}_{-3-4-5,3,4,5}
\end{aligned}$$

$$
\begin{aligned}
&+ U^{(3)}_{0,1-3,2-4-5} A^{(2)}_{1-3,3,1} B^{(3)}_{2-4-5,4,5,2} \\
&+ U^{(3)}_{0,4-5,1+2-3} A^{(3)}_{-4-5,4,5} B^{(2)}_{1+2-3,3,2,1}]_{(3,4,5)} \\
&+ 3\mathcal{F}[V^{(1)}_{0,4,5,3-1-2} B^{(3)}_{3-1-2,1,2,3} + V^{(1)}_{0,3,4+5,-1-2} A^{(1)}_{4+5,4,5} A^{(3)}_{-1-2,1,2} \\
&+ V^{(1)}_{0,3,4-1,5-2} A^{(2)}_{4-1,1,4} A^{(2)}_{5-2,2,5}]_{(3,4,5)} \\
&+ 2\mathcal{F}[V^{(2)}_{0,1,3,4+5-2} B^{(2)}_{4+5-2,2,4,5} + \frac{1}{2} V^{(2)}_{0,1+2-3,4,5} B^{(2)}_{1+2-3,3,1,2} \\
&+ V^{(2)}_{0,1,4+5,3-2} A^{(1)}_{4+5,4,5} A^{(2)}_{3-2,2,3} + V^{(2)}_{0,1+2,3+4,5} A^{(1)}_{1+2,1,2} A^{(1)}_{3+4,3,4} \\
&+ V^{(2)}_{0,1-3,4-2,5} A^{(2)}_{1-3,3,1} A^{(2)}_{4-2,2,4} \\
&+ V^{(2)}_{0,-4-5,-1-2,3} A^{(3)}_{-4-5,4,5} A^{(3)}_{-1-2,1,2}]_{(3,4,5)} \\
&+ 3\mathcal{F}[V^{(1)}_{3+4+5,2,1,0} B^{(1)}_{3+4+5,3,4,5} + V^{(1)}_{3,2,1-4-5,0} B^{(3)}_{1-4-5,4,5,1} \\
&+ V^{(1)}_{3,2-4-5,1,0} B^{(3)}_{2-4-5,4,5,2} \\
&+ V^{(1)}_{4+5,2-3,1,0} A^{(1)}_{4+5,4,5} A^{(2)}_{2-3,3,2} + V^{(1)}_{3-2,-4-5,1,0} A^{(3)}_{-4-5,4,5} A^{(2)}_{3-2,2,3} \\
&+ V^{(1)}_{3,-4-5,1+2,0} A^{(1)}_{-4-5,4,5} A^{(3)}_{1+2,1,2} \\
&+ V^{(1)}_{3,4-1,5-2,0} A^{(2)}_{4-1,1,4} A^{(2)}_{5-2,2,5}]_{(3,4,5)} \\
&+ 3\mathcal{F}[V^{(4)}_{0,1,2,-3-4-5} B^{(4)}_{-3-4-5,3,4,5} \\
&+ 2V^{(4)}_{0,1,2-3,-4-5} A^{(2)}_{2-3,3,2} A^{(3)}_{-4-5,4,5}]_{(3,4,5)} \\
&+ \mathcal{F}[4W^{(1)}_{0,3,4,5,-1-2} A^{(3)}_{-1-2,1,2} + 3W^{(2)}_{0,1,3,4,5-2} A^{(2)}_{5-2,2,5} \\
&+ W_{0,1+2,3,4,5} A^{(1)}_{1+2,1,2} + 3W^{(2)}_{4+5,3,2,1,0} A^{(1)}_{4+5,4,5} \\
&+ 3W^{(2)}_{4,3,2,1-5,0} A^{(2)}_{1-5,5,1} + 3W^{(1)}_{4,-3-5,2,1,0} A^{(3)}_{-3-5,3,5}]_{(3,4,5)} \\
&- \mathcal{F}[3A^{(1)}_{0,3,0-3} \tilde{W}^{(2)}_{5,4,2,1,0-3} - \frac{3}{2} A^{(2)}_{0,3-0,3} \tilde{W}^{(2)}_{1,2,4,5,3-0} \\
&+ A^{(2)}_{0,1,0+1} \tilde{W}^{(2)}_{0+1,2,3,4,5} - B^{(2)}_{0,1+2-3,4,5} \tilde{V}^{(2)}_{1+2-3,3,1,2} \\
&+ 2B^{(2)}_{0,1,4+5-2,3} \tilde{V}^{(2)}_{4+5-2,2,4,5}]_{(3,4,5)},
\end{aligned}
\tag{A40}
$$

$$
Y^{(4)}_{0,1,2,3,4,5} = \mathcal{F}[Y4^{(4)}_{0,1,2,3,4,5}]_{(1,2,3)}, \tag{A41}
$$

where

$$
\begin{aligned}
Y4^{(4)}_{0,1,2,3,4,5} = {} & 2\mathcal{F}[U_{0,1,0-5} C^{(4)}_{0-5,1,2,3,4} + U^{(1)}_{0,4+5,-1-2-3} A^{(1)}_{4+5,4,5} B^{(4)}_{-1-2-3,1,2,3} \\
&+ U^{(1)}_{0,4-1,5-2-3} A^{(2)}_{4-1,1,4} B^{(3)}_{5-2-3,2,3,5} \\
&+ U^{(1)}_{0,-1-2,4+5-3} A^{(3)}_{-1-2,1,2} B^{(2)}_{4+5-3,3,4,5} \\
&+ U^{(1)}_{0+1,1,0} C^{(3)}_{0+1,2,3,4,5} + U^{(1)}_{4+5-3,1+2,0} A^{(1)}_{1+2,1,2} B^{(2)}_{4+5-3,3,4,5} \\
&+ U^{(1)}_{5-2-3,1-4,0} A^{(2)}_{1-4,4,1} B^{(3)}_{5-2-3,2,3,5} \\
&+ U^{(1)}_{-1-2-3,-4-5,0} A^{(3)}_{-4-5,4,5} B^{(4)}_{-1-2-3,1,2,3} \\
&+ U^{(1)}_{4+5,1+2+3,0} A^{(1)}_{4+5,4,5} B^{(1)}_{1+2+3,1,2,3} \\
&+ U^{(1)}_{4-1,2+3-5,0} A^{(2)}_{4-1,1,4} B^{(2)}_{2+3-5,2,3,5} \\
&+ U^{(1)}_{-1-2,3-4-5,0} A^{(3)}_{-1-2,1,2} B^{(3)}_{3-4-5,3,4,5} + U^{(1)}_{5,5-0,0} C^{(2)}_{5-0,1,2,3,4} \\
&+ U^{(3)}_{0,1,-0-1} C^{(3)}_{-0-1,4,5,2,3} + U^{(3)}_{0,1+2,3-4-5} A^{(1)}_{1+2,1,2} B^{(3)}_{3-4-5,3,4,5}
\end{aligned}
$$

$$
\begin{aligned}
&+ U^{(3)}_{0,4-1,2+3-5} A^{(2)}_{4-1,1,4} B^{(2)}_{2+3-5,5,2,3} \\
&+ U^{(3)}_{0,1+2+3,-4-5} A^{(3)}_{-4-5,4,5} B^{(1)}_{1+2+3,1,2,3}]_{(4,5)} \\
&+ 3\mathcal{F}[V^{(1)}_{0,4,5,-1-2-3} B^{(4)}_{-1-2-3,1,2,3} \\
&+ 2V^{(1)}_{0,5,4-1,-2-3} A^{(2)}_{4-1,1,4} A^{(3)}_{-2-3,2,3}]_{(4,5)} \\
&+ \mathcal{F}[2V^{(2)}_{0,1,4+5,-2-3} A^{(1)}_{4+5,4,5} A^{(3)}_{-2-3,2,3} + V^{(2)}_{0,1,4-2,5-3} A^{(2)}_{4-2,2,4} A^{(2)}_{5-3,3,5} \\
&+ 2V^{(2)}_{0,1,4,5-2-3} B^{(3)}_{5-2-3,2,3,5} + 2V^{(2)}_{0,1+2,4-3,5} A^{(1)}_{1+2,1,2} A^{(2)}_{4-3,3,4} \\
&+ 2V^{(2)}_{0,4-3,-1-2,5} A^{(3)}_{-1-2,1,2} A^{(2)}_{4-3,3,4} + V^{(2)}_{0,1+2+3,4,5} B^{(1)}_{1+2+3,1,2,3}]_{(4,5)} \\
&+ 3\mathcal{F}[V^{(1)}_{4+5-3,2,1,0} B^{(2)}_{4+5-3,3,4,5} + 2V^{(1)}_{4+5,2+3,1,0} A^{(1)}_{2+3,2,3} A^{(1)}_{4+5,4,5} \\
&+ 2V^{(1)}_{5-3,2-4,1,0} A^{(2)}_{2-4,2,4} A^{(2)}_{5-3,3,5} \\
&+ 2V^{(1)}_{-2-3,-4-3,1,0} A^{(3)}_{-2-3,2,3} A^{(3)}_{-4-5,4,5} \\
&+ 2V^{(1)}_{5,3-4,1+2,0} A^{(1)}_{1+2,1,2} A^{(2)}_{3-4,4,3} + 2V^{(1)}_{5,2+3-4,1,0} B^{(2)}_{2+3-4,2,3,4}]_{(4,5)} \\
&+ 3\mathcal{F}[V^{(4)}_{0,1,2,3-4-5} B^{(3)}_{3-4-5,4,5,3} + 2V^{(4)}_{0,1,2+3,-4-5} A^{(1)}_{2+3,2,3} A^{(3)}_{-4-5,4,5} \\
&+ V^{(4)}_{0,1,2-4,3-5} A^{(2)}_{2-4,4,2} A^{(2)}_{3-5,5,3}]_{(4,5)} \\
&+ \mathcal{F}[3W^{(1)}_{0,1,-2-3,4,5} A^{(3)}_{-2-3,2,3} + 3W^{(2)}_{5,4-3,2,1,0} A^{(2)}_{4-3,3,4} \\
&+ 12W^{(1)}_{4,3,2,1-5,0} A^{(2)}_{1-5,5,1} + 4W^{(1)}_{4+5,3,2,1,0} A^{(1)}_{4+5,4,5} \\
&+ 4W^{(5)}_{0,1,2,3,-4-5} A^{(3)}_{-4-5,4,5}]_{(4,5)} \\
&+ \mathcal{F}[A^{2}_{0,4-0,4} \tilde{W}^{(2)}_{4-0,5,1,2,3} - \frac{3}{2} A^{(2)}_{0,1,0+1} \tilde{W}^{(2)}_{5,4,3,2,0+1} \\
&+ 3A^{(3)}_{0,2,-0-2} \tilde{W}_{1,3,4,5,-0-2} \\
&+ 2B^{(3)}_{0,1+3-4,2,5} \tilde{V}^{(2)}_{1+3-4,4,1,3} - B^{(3)}_{0,1,2,4+5-3} \tilde{V}^{(2)}_{4+5-3,3,4,5}]_{(4,5)},
\end{aligned} \tag{A42}
$$

$$
\begin{aligned}
Y^{(5)}_{0,1,2,3,4,5} = {} & 2\mathcal{F}[U^{(1)}_{0,5,0-5} C^{(5)}_{0-5,1,2,3,4} + U^{(1)}_{0,5-1,-2-3-4} A^{(2)}_{5-1,1,5} B^{(4)}_{-2-3-4,2,3,4} \\
&+ U^{(1)}_{0,-1-2,5-3-4} A^{(3)}_{-1-2,1,2} B^{(3)}_{5-3-4,3,4,5} + U^{(1)}_{0+1,1,0} C^{(4)}_{0+1,2,3,4,5} \\
&+ U^{(1)}_{5-3-4,1+2,0} A^{(1)}_{1+2,1,2} B^{(3)}_{5-3-4,3,4,5} \\
&+ U^{(1)}_{-2-3-4,1-5,0} A^{(2)}_{1-5,5,1} B^{(4)}_{-2-3-4,2,3,4} \\
&+ U^{(1)}_{5-4,1+2+3,0} A^{(2)}_{5-4,4,5} B^{(1)}_{1+2+3,1,2,3} \\
&+ U^{(1)}_{-1-2,3+4-5,0} A^{(3)}_{-1-2,1,2} B^{(2)}_{3+4-5,5,4,3} \\
&+ U^{(1)}_{5,5-0,0} C^{(1)}_{5-0,1,2,3,4} + U^{(3)}_{0,1+2,3+4-5} A^{(1)}_{1+2,1,2} B^{(2)}_{3+4-5,3,4,5} \\
&+ U^{(3)}_{0,1-5,2+3+4} A^{(2)}_{1-5,5,1} B^{(1)}_{2+3+4,2,3,4}]_{(1,2,3,4)} \\
&+ \mathcal{F}[3V^{(1)}_{5,-1-2,-3-4} A^{(3)}_{-1-2,1,2} A^{(3)}_{-3-4,3,4} + 2V^{(2)}_{0,1,5,-2-3-4} B^{(4)}_{-2-3-4,2,3,4} \\
&+ 2V^{(2)}_{0,1,5-2,-3-4} A^{(2)}_{5-2,2,5} A^{(3)}_{-3-4,3,4} + 2V^{(2)}_{0,1+2,-3-4,5} A^{(1)}_{1+2,1,2} A^{(3)}_{-3-4,3,4} \\
&+ 3V^{(1)}_{5-3-4,2,1,0} B^{(3)}_{5-3-4,3,4,5} + 6V^{(1)}_{5,2+3+4,2,0} B^{(1)}_{2+3+4,2,3,4} \\
&+ 6V^{(1)}_{5-4,2+3,1,0} A^{(1)}_{2+3,2,3} A^{(2)}_{5-4,4,5} + 6V^{(1)}_{-2-3,4-5,1,0} A^{(2)}_{4-5,5,4} A^{(3)}_{-2-3,2,3} \\
&+ 3V^{(1)}_{5,3+4,1+2,0} A^{(1)}_{1+2,1,2} A^{(1)}_{3+4,3,4} + 3V^{(4)}_{0,1,2,3+4-5} B^{(2)}_{3+4-5,5,3,4} \\
&+ 6V^{(4)}_{0,1,2+3,4-5} A^{(1)}_{2+3,2,3} A^{(2)}_{4-5,5,4}]_{(1,2,3,4)}
\end{aligned} \tag{A43}
$$

$$
\begin{aligned}
&+\mathcal{F}[3W^{(1)}_{5,-3-4,2,1,0}A^{(3)}_{-3-4,3,4}+4W^{(1)}_{5-4,3,2,1,0}A^{(2)}_{5-4,4,5}\\
&+12W^{(1)}_{5,3+4,2,1,0}A^{(1)}_{3+4,3,4}+4W^{(5)}_{0,1,2,3,4-5}A^{(2)}_{4-5,5,4}]_{(1,2,3,4)}\\
&+\mathcal{F}[2A^{(3)}_{0,2,-0-2}\tilde{W}^{(2)}_{-0-2,5,1,3,4}+4B^{(4)}_{0,2,3,1+4-5}\tilde{V}^{(2)}_{1+4-5,5,1,4}]_{(1,2,3,4)},
\end{aligned}
$$

$$
\begin{aligned}
Y^{(6)}_{0,1,2,3,4,5}=\;&2\mathcal{F}[U^{(1)}_{0,-1-2,-3-4-5}A^{(3)}_{-1-2,1,2}B^{(4)}_{-3-4-5,3,4,5}+U^{(1)}_{0+1,1,0}C^{(5)}_{0+1,2,3,4,5}\\
&+U^{(1)}_{-3-4-5,1+2,0}A^{(1)}_{1+2,1,2}B^{(4)}_{-3-4-5,3,4,5}\\
&+U^{(1)}_{-4-5,1+2+3,0}A^{(3)}_{-4-5,4,5}B^{(1)}_{1+2+3,1,2,3}\\
&+U^{(3)}_{0,1,-0-1}C^{(1)}_{-0-1,2,3,4,5}+U^{(3)}_{0,1+2,3+4+5}A^{(1)}_{1+2,1,2}B^{(1)}_{3+4+5,3,4,5}]_{(1,2,3,4,5)}\\
&+\mathcal{F}[V^{(2)}_{0,1,-2-3,-4-5}A^{(3)}_{-2-3,2,3}A^{(3)}_{-4-5,4,5}+3V^{(1)}_{-3-4-5,2,1,0}B^{(4)}_{-3-4-5,3,4,5}\\
&+6V^{(1)}_{-4-5,2+3,1,0}A^{(1)}_{2+3,2,3}A^{(3)}_{-4-5,4,5}+3V^{(4)}_{0,1,2,3+4+5}B^{(1)}_{3+4+5,3,4,5}\\
&+3V^{(4)}_{0,1,2+3,4+5}A^{(1)}_{2+3,2,3}A^{(1)}_{4+5,4,5}]_{(1,2,3,4,5)}\\
&+4\mathcal{F}[W^{(1)}_{-4-5,3,2,1,0}A^{(3)}_{-4-5,4,5}+W^{(5)}_{0,1,2,3,4+5}A^{(1)}_{4+5,4,5}]_{(1,2,3,4,5)}.
\end{aligned}
\tag{A44}
$$

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