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Sedenion Algebra Model as an Extension of the Standard Model and Its Link to SU(5)

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Abstract: In the Standard Model, ad hoc hypotheses assume the existence of three generations of point-like leptons and quarks, which possess a point-like structure and follow the Dirac equation involving four anti-commutative matrices. In this work, we consider the sedenion hypercomplex algebra as an extension of the Standard Model and show its close link to SU(5), which is the underlying symmetry group for the grand unification theory (GUT). We first consider the direct-product quaternion model and the eight-element octonion algebra model. We show that neither the associative quaternion model nor the non-associative octonion model could generate three fermion generations. Instead, we show that the sedenion model, which contains three octonion sub-algebras, leads naturally to precisely three fermion generations. Moreover, we demonstrate the use of basis sedenion operators to construct twenty-four 5×5 generalized lambda matrices representing SU(5) generators, in analogy to the use of octonion basis operators to generate Gell-Mann's eight 3×3 lambda-matrix generators for SU(3). Thus, we provide a link between the sedenion algebra and Georgi and Glashow's SU(5) GUT model that unifies the electroweak and strong interactions for the Standard Model's elementary particles, which obey SU(3)⊗SU(2)⊗U(1) symmetry.

Keywords: Dirac equation; Standard Model; SU(5) GUT; sedenion; octonion; quaternion

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1. Introduction

The Dirac equation is the Standard Model's cornerstone for describing all fermionic elementary particles [1–3]. According to the ad hoc assumptions, the three generations of leptons and quarks are point-like objects without a physical size [4–6]. However, such a long-held conceptual picture of a point-like particle is inconsistent with observations. For example, it is unclear why there are precisely three generations of leptons and quarks [7] and why a point-like muon or tau can be much heavier than an electron [8,9]; and, for the six types of point-like quarks, why the charm, strange, top and bottom quarks are heavier than the up and down quarks [10,11]. In Dirac's theory of a relativistic electron, he coupled four 4×4 anti-commutative matrices $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$, equivalently, to the first-order derivatives in space and time to describe the wave [12,13]. Extending the operator techniques beyond Dirac's theory, one can include additional operators in higher dimensions to describe a particle's internal degrees of freedom. In this work, we analyze two types of higher-dimensional generalizations, using the direct-product operators constructed from Pauli's matrices versus hyper-complex operators beyond quaternions [14,15], which is equivalent to Dirac's 4×4 gamma matrices. Like the four-element quaternion algebra with three anti-commutative basis operators [16], the eight-element octonion algebra contains seven operators. Still, unlike the associative quaternion algebra, the octonion algebra is non-associative [17–20]. In this work, we examine the corresponding multiplication rules and compare their multiplication tables to clarify the similarities, differences and physical implications.

In Section 2.1, we briefly review Dirac's original model and its connection to the quaternion algebra. In Section 2.2, we consider a higher-dimensional model beyond Dirac's four anti-commutative gamma matrices. We analyze the group of 16 direct-product operators constructed from quaternions or, equivalently, Pauli's matrices and an identity operator, which are related to the 16 Dirac gamma. In Section 2.3, we discuss the non-associative 16-element sedenion algebra [21–26], and show it comprises three distinctive octonion algebras. We analyze its differences from the direct-product case and examine whether the corresponding mass–energy relation is consistent with Einstein's special relativity. In Section 2.5, we show how proper operator assignment of the operators from three octonion sub-algebras can naturally lead to the rise of three generations of leptons and quarks. Finally, in Section 2.6, we present a mapping of octonions to 8 SU(3) generators, and a mapping of sedenion operators to 24 generators for SU(5), which was proposed by Geogi and Glashow [27,28] for their GUT model for the grand unification of the Standard Model's elementary particles.

2. Theory

In this section, we present a theoretical analysis of an associative algebra model using Dirac's gamma matrices, which are related to direct-product operators constructed from quaternions or, equivalently, Pauli's matrices and an identity matrix, versus the non-associative algebra model based on octonion and sedenion operators. This work extends Dirac's original theory of the electron to three generations of leptons and quarks. It opens up the Standard Model beyond the ad hoc assumption of point-like elementary fermions. We show that sedenion algebra provides a pivotal link to the SU(5) symmetry of GUT. This hyper-complex operator model with higher degrees of freedom leads precisely to three generations of leptons/quarks with internal structures.

2.1. Dirac Equation

According to Dirac's theory of the electron, when using the gamma matrices, one has

$$\begin{gathered}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\Psi=0,\mu=0,\ 1,\ 2,\ 3\\ p_k=-i\partial_k,p_0=i\partial_0\\ \gamma^k=i\sigma_2\otimes\sigma_k=\begin{pmatrix}0&\sigma_k\\-\sigma_k&0\end{pmatrix},\gamma^0=\sigma_3\otimes\sigma_0=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix}\end{gathered}\tag{1A}$$

where the natural unit $\hbar=c\ =1$ is used. Here, we define five 2×2 matrices, including Pauli's matrices $\sigma_1,\ \sigma_2\ ,\ \sigma_3$, and an identity matrix $\mathbf{I}_2$, as

$$\sigma_1\ =\ \begin{pmatrix}0&1\\1&0\end{pmatrix},\ \sigma_2\ =\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \sigma_3\ =\ \begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \sigma_0\ =\ \mathbf{I}_2\ =\ \begin{pmatrix}1&0\\0&1\end{pmatrix},\tag{1B}$$

Equivalently, using the matrices $\boldsymbol{\alpha}_k\ =\ \gamma^0\gamma^k$ and $\boldsymbol{\beta}\ =\ \gamma^0$, the Dirac equation can also be expressed as $E\Psi\ =(\ \boldsymbol{\alpha}\cdot\mathbf{P}+\boldsymbol{\beta}m)\Psi$. The identity matrix and three Pauli matrices form the basis operators for quaternion algebra. Based on Dirac's first-order differential equation in spacetime, one obtains Einstein's relativistic mass–energy relation $E^2=m^2+p^2$. These Dirac gamma matrices are used in the Dirac equation. However, the product of these matrices does not satisfy the closure property; for example, it does not belong to the same set. Therefore, these five operators do not form a group.

In the Standard Model, the same Dirac equation is used for all leptons and quarks; assuming these particles are point-like objects with an infinitely small volume, it does not offer physical explanations for why there are precisely three generations. To generalize the Dirac equation to higher dimensions, and to account for three fermion generations by incorporating internal structural dynamics, we consider in the following sections two modeling approaches, i.e., a direct-product matrix model of 16 associative 4×4 matrices versus the non-associative hype-complex algebra of 16 sedenion basis operators.

2.2. Associative Algebra of 16 Direct-Product Matrices $\{\boldsymbol{I}, \Gamma_1, \Gamma_2, \Gamma_3, \Theta_1, \boldsymbol{U}_1, \boldsymbol{U}_2, \boldsymbol{U}_3, \Theta_2, \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \Theta_3, \boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3\}$

To generalize Dirac's description of the electron using four Dirac gamma matrices $\gamma^\mu(\mu = 0,\ 1,\ 2,\ 3)$, we can define a set of direct-product operators using three Pauli matrices and an identity matrix as

$$\begin{aligned} &\sigma_{ij} \ \equiv \ \sigma_i \otimes \sigma_j \\ &\mathbf{U}_k = i\sigma_{1k}, \quad \mathbf{V}_k = i\sigma_{2k}\ \mathbf{W}_k = i\sigma_{3k}, \ \mathbf{\Gamma}_k = i\sigma_{0k} \quad \mathbf{\Theta}_k = i\sigma_{k0} \end{aligned} \tag{2}$$

One can show that the eight-element set of direct-product operators forms a group with the closure property for multiplication. Similarly, one can show that the other two sets $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_2,\ \mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3\}$ and $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_3,\ \mathbf{W}_1,\ \mathbf{W}_2,\ \mathbf{W}_3\}$ form a group for multiplication. These direct-product operators can be related to the sixteen direct-product matrices of quaternions. The multiplication table for the sixteen direct-product operators, which are related to Dirac's gamma matrices, is illustrated in Figure 1.

I	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3	Θ_2	V_1	V_2	V_3	Θ_3	W_1	W_2	W_3
Γ_1	$-I$	Γ_3	$-\Gamma_2$	U_1	$-\Theta_1$	$-U_3$	U_2	V_1	$-\Theta_2$	$-V_3$	V_2	$-W_1$	Θ_3	W_3	$-W_2$
Γ_2	$-\Gamma_3$	$-I$	Γ_1	U_2	U_3	$-\Theta_1$	$-U_1$	V_2	V_3	$-\Theta_2$	$-V_1$	$-W_2$	$-W_3$	Θ_3	W_1
Γ_3	Γ_2	$-\Gamma_1$	$-I$	U_3	$-U_2$	U_1	$-\Theta_1$	V_3	$-V_2$	V_1	$-\Theta_2$	$-W_3$	W_2	$-W_1$	Θ_3
Θ_1	$-U_1$	$-U_2$	$-U_3$	$-I$	Γ_1	Γ_2	Γ_3	Θ_3	W_1	W_2	W_3	$-\Theta_2$	$-V_1$	$-V_2$	$-V_3$
U_1	Θ_1	$-U_3$	U_2	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2	W_1	$-\Theta_3$	W_3	$-W_2$	V_1	$-\Theta_2$	V_3	$-V_2$
U_2	U_3	Θ_1	$-U_1$	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$	W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	$-\Theta_2$	V_1
U_3	$-U_2$	U_1	Θ_1	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$	W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	$-\Theta_2$
Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_3$	$-W_1$	$-W_2$	$-W_3$	$-I$	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3
V_1	Θ_2	$-V_3$	V_2	$-W_1$	Θ_3	W_3	$-W_2$	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2	$-U_1$	Θ_1	U_3	$-U_2$
V_2	V_3	Θ_2	$-V_1$	$-W_2$	$-W_3$	Θ_3	W_1	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$	$-U_2$	$-U_3$	Θ_1	U_1
V_3	$-V_2$	V_1	Θ_2	$-W_3$	W_2	$-W_1$	Θ_3	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$	$-U_3$	U_2	$-U_1$	Θ_1
Θ_3	W_1	W_2	W_3	Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_1$	U_1	U_2	U_3	$-I$	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$
W_1	$-\Theta_3$	W_3	$-W_2$	V_1	Θ_2	V_3	$-V_2$	$-U_1$	$-\Theta_1$	U_3	$-U_2$	Γ_1	$-I$	Γ_3	$-\Gamma_2$
W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	Θ_2	V_1	$-U_2$	$-U_3$	$-\Theta_1$	U_1	Γ_2	$-\Gamma_3$	$-I$	Γ_1
W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	Θ_2	$-U_3$	U_2	$-U_1$	$-\Theta_1$	Γ_3	Γ_2	$-\Gamma_1$	$-I$

Figure 1. The color-coded multiplication table for 16 direct-product operators. Among the 16 operators, the 4th, 8th and 12th operators are commutative with other operators, although most are anti-commutative. These properties differ from those of sedenion algebra. This 16-element group contains three 8-element sub-groups with closure properties for multiplication, namely, $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_1,\ \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$, $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_2,\ \mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3\}$ and $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_3,\ \mathbf{W}_1,\ \mathbf{W}_2,\ \mathbf{W}_3\}$. Each domain contains quaternion triplets following cyclic multiplication rules.

Here, we summarize their multiplication rules for these sixteen direct-product operators:

$$\begin{aligned} &\mathbf{U}_i\mathbf{U}_j = -\varepsilon_{ijk}\mathbf{\Gamma}_k - \delta_{ij}\mathbf{I}, \ \ \mathbf{V}_i\mathbf{V}_j = -\varepsilon_{ijk}\mathbf{\Gamma}_k - \delta_{ij}\mathbf{I}, \ \ \mathbf{W}_i\mathbf{W}_j = -\varepsilon_{ijk}\mathbf{\Gamma}_k - \delta_{ij}\mathbf{I} \\ &\mathbf{\Gamma}_i\mathbf{\Gamma}_j = -\varepsilon_{ijk}\mathbf{\Gamma}_k - \delta_{ij}\ \mathbf{I}, \quad \mathbf{\Theta}_i\ \mathbf{\Theta}_j = -\varepsilon_{ijk}\mathbf{\Theta}_k - \delta_{ij}\ \mathbf{I} \\ &\mathbf{U}_i\ \mathbf{V}_j = -i\varepsilon_{ijk}\mathbf{W}_k - \delta_{ij}\ \mathbf{\Theta}_3, \\ &\mathbf{V}_j\ \mathbf{U}_i = -i\varepsilon_{ijk}\mathbf{W}_k + \delta_{ij}\ \mathbf{\Theta}_3 \\ &\mathbf{V}_i\mathbf{W}_j = -\ i\varepsilon_{ijk}\mathbf{U}_k + \delta_{ij}\mathbf{\Theta}_1, \\ &\mathbf{W}_j\ \mathbf{V}_i = -i\varepsilon_{ijk}\mathbf{U}_k + \ \delta_{ij}\mathbf{\Theta}_1\ \ , \\ &\mathbf{W}_i\mathbf{U}_j = -i\varepsilon_{ijk}\mathbf{V}_k + \delta_{ij}\mathbf{\Theta}_2 \end{aligned} \tag{3A}$$

and

$$\begin{aligned}
&\mathbf{U}_i\mathbf{\Gamma}_j = -\sigma_{1i}\sigma_{0j} = -i\varepsilon_{ijk}\sigma_{1k} - \delta_{ij}\sigma_{10} = -\ \varepsilon_{ijk}\mathbf{U}_k + i\delta_{ij}\Theta_1 \\
&\mathbf{\Gamma}_j\mathbf{U}_i = -\sigma_{0j}\sigma_{1i} = i\varepsilon_{ijk}\sigma_{1k} - \delta_{ij}\sigma_{10} = \varepsilon_{ijk}\mathbf{U}_k + i\delta_{ij}\Theta_1 \\
&\mathbf{V}_i\mathbf{\Gamma}_j = -\varepsilon_{ijk}\mathbf{V}_k + i\delta_{ij}\mathbf{\Theta}_2 \\
&\mathbf{\Gamma}_j\mathbf{V}_i = \varepsilon_{ijk}\mathbf{V}_k + i\delta_{ij}\mathbf{\Theta}_2 \\
&\mathbf{W}_i\mathbf{\Gamma}_j = -\varepsilon_{ijk}\mathbf{W}_k + i\delta_{ij}\Theta_3 \\
&\mathbf{\Gamma}_j\mathbf{W}_i = \varepsilon_{ijk}\mathbf{W}_k + i\delta_{ij}\Theta_3
\end{aligned} \tag{3B}$$

and

$$\begin{aligned}
&\mathbf{\Gamma}_i\mathbf{\Gamma}_j = -\ \varepsilon_{ijk}\mathbf{\Gamma}_k - \delta_{ij}\ \mathbf{I},\quad \mathbf{\Theta}_i\ \mathbf{\Theta}_j = -\varepsilon_{ijk}\mathbf{\Theta}_k - \delta_{ij}\ \mathbf{I} \\
&\mathbf{\Theta}_1\mathbf{\Gamma}_k\ = \mathbf{\Gamma}_k\ \mathbf{\Theta}_1 = i\ \mathbf{U}_k,\quad \mathbf{\Theta}_2\ \mathbf{\Gamma}_k = \ \mathbf{\Gamma}_k\mathbf{\Theta}_2 = i\ \mathbf{V}_k,\quad \mathbf{\Theta}_3\mathbf{\Gamma}_k\ = \mathbf{\Gamma}_k\mathbf{\Theta}_3\ =\ i\ \mathbf{W}_k \\
&\mathbf{U}_k\mathbf{\Theta}_1 = \mathbf{\Theta}_1\mathbf{U}_k = i\mathbf{\Gamma}_k,\quad \mathbf{V}_k\mathbf{\Theta}_2 = \mathbf{\Theta}_2\mathbf{V}_k = i\mathbf{\Gamma}_k,\ \mathbf{W}_k\mathbf{\Theta}_3 = \mathbf{\Theta}_3\mathbf{W}_k = i\mathbf{\Gamma}_k \\
&\mathbf{U}_k\ \Theta_2 = -\Theta_2\mathbf{U}_k = -\mathbf{W}_k,\quad \mathbf{U}_k\ \Theta_3 = -\Theta_3\mathbf{U}_k = \mathbf{V}_k \\
&\mathbf{V}_k\ \Theta_1 = -\Theta_1\mathbf{V}_k = \mathbf{W}_k,\quad \mathbf{V}_k\ \Theta_3 = -\Theta_3\mathbf{V}_k = -\mathbf{U}_k \\
&\mathbf{W}_k\ \Theta_1 = -\Theta_1\mathbf{W}_k - \mathbf{V}_k,\quad \mathbf{W}_k\ \Theta_2 = -\Theta_2\mathbf{W}_k = \mathbf{U}_k\ .
\end{aligned} \tag{3C}$$

This 16-element group contains three 8-element subgroups, namely, $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_1,\ \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$, $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_2,\ \mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3\}$ and $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_3,\ \mathbf{W}_1,\ \mathbf{W}_2,\ \mathbf{W}_3\}$. These three subgroups with eight elements of 4×4 matrices satisfy the closure and associative properties. They differ from the non-associative octonion algebra, which will be discussed in detail later in Section 2.3.

Here, we examine the mass–energy relation based on the operators according to the first sub-group $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_1,\ \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$. We propose to extend Dirac's equation involving four gamma matrices to higher dimensions involving eight direct-product operators as

$$m_0\ =\ iE\mathbf{\Theta}_1\ + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k\ \ + \sum_{k=1}^{3} Q_k\mathbf{U}_k \tag{4A}$$

By taking the square of both sides of the equation, one obtains

$$\begin{aligned}
{m_0}^2 = E^2 &+ \sum_{k=1}^{3} {P_k}^2{\mathbf{\Gamma}_k}^2 + \sum_{k=1}^{3} {Q_k}^2{\mathbf{U}_k}^2 + \sum_{i,\neq j=1}^{3} P_iP_j\{\mathbf{\Gamma}_i,\mathbf{\Gamma}_j\} + \sum_{i,\neq j=1}^{3} Q_iQ_j\{\mathbf{U}_i,\mathbf{U}_j\} \\
&+ \sum_{i,j=1}^{3} P_iQ_j\{\mathbf{U}_i,\mathbf{\Gamma}_j\} + iE\sum_{k=1}^{3}(P_k\{\mathbf{\Theta}_1,\mathbf{\Gamma}_k\} + Q_k\{\mathbf{\Theta}_1,\mathbf{U}_k\})
\end{aligned} \tag{4B}$$

or, equivalently, according to the multiplication rules in Figure 1, one has

$$E^2 = {m_0}^2 + \sum_{k=1}^{3}\left({P_k}^2 + {Q_k}^2\right) - i\mathbf{\Theta}_1\sum_{k=1}^{3} P_kQ_k - iE\sum_{k=1}^{3}(P_k\mathbf{U}_k + Q_k\mathbf{\Gamma}_k)\ \ . \tag{4C}$$

Similarly, for the second assignment, one has

$$\begin{aligned}
m_0 &= iE\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} Q_k\mathbf{V}_k \\
E^2 &= m_0^2 + \sum_{k=1}^{3}\left({P_k}^2 + {Q_k}^2\right) - i\mathbf{\Theta}_2\sum_{k=1}^{3} P_kQ_k - iE\sum_{k=1}^{3}(P_k\mathbf{V}_k + Q_k\mathbf{\Gamma}_k)
\end{aligned} \tag{4D}$$

and for $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_3,\ \mathbf{W}_1,\ \mathbf{W}_2,\ \mathbf{W}_3\}$, one obtains

$$\begin{aligned}
m_0 &= iE\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k, + \sum_{k=1}^{3} Q_k\mathbf{W}_k \\
E^2 &= {m_0}^2 + \sum_{k=1}^{3}\left({P_k}^2 + {Q_k}^2\right)\ - i\mathbf{\Theta}_3\sum_{k=1}^{3} P_kQ_k - iE\sum_{k=1}^{3}(P_k\mathbf{W}_k + Q_k\mathbf{\Gamma}_k).
\end{aligned} \tag{4E}$$

In Equation (4A,D,E), the last two terms of the equations involve operators $\mathbf{W}_k$, which lead to mass–energy oscillations in time for a lepton and quark. Therefore, such results are neither consistent with experimental observations nor in agreement with Einstein's mass–energy relation, which contains additional kinetic energy $\sum_k {Q_k}^2$ due to a particle's internal structural dynamics. Therefore, although the 16-element group of direct-product operators has three 8-element subgroups, the above direct-product model cannot represent three generations of leptons and quarks. In the following sections, we will discuss the non-associative sedenion and octonion algebras, and show that they do not encounter these problems faced by the associative direct-product operator model.

2.3. Non-Associative Octonion Algebra and a Single Fermion Generation

We consider the octonion algebra to avoid the problems faced by the direct-product matrix model. Any element x and its conjugate $\overline{x}$ in the octonion algebra can be expressed in terms of the identity operator $\mathbf{e}_0$ and seven other octonion unit operators

$$x = x_0\mathbf{e}_0 + X, \quad \overline{x} = x_0\mathbf{e}_0 - X, \quad X = \sum_{k=1}^{7} x_k\mathbf{e}_k, \tag{5}$$

where $\mathbf{e}_k$ satisfies the anti-commutative relation $\{\mathbf{e}_i,\ \mathbf{e}_j\} = 0, \quad i \neq j$ for a different pair of indices. These non-associative octonion operators follow the specific multiplication rules in Figure 2.

$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$

Figure 2. The color-coded multiplication table for the eight basis octonions. According to the multiplication rules, other than the identity operator, all seven operators anti-commute with each other; however, the multiplications are not associative. The arrays are color-coded to illustrate domains of cyclic multiplication rules for quaternions. Each color-coded field of quaternion triplets follows cyclic multiplication rules.

For the octonion algebra of Figure 2, the fourth element anti-commutes with all other operators except the identity operator. However, in Figure 1, for the direct-product operators, the fourth element a_4, a_8 or a_{12} is commutative with all other operators, except the identity element. This property is essential for the corresponding mass–energy relation to be consistent with Einstein's relativity. Therefore, the model with seven non-associative but anti-commutative octonion operators is the correct model to describe a single generation of leptons or quarks. The octonion model invokes three extra degrees of freedom to represent the internal structural dynamics of a fermion as three momentum components concerning the center-of-mass reference frame. In contrast, the other three anti-commutative operators define the external degrees of freedom as three momentum operators for the particle concerning the laboratory frame.

2.4. Three Octonion Sub-Algebras in Sedenion Algebra and Three Generations of Charged/Neutral Leptons

In the previous section, we explained that the octonion algebra leads to only one fermion generation. To accommodate three generations, one needs to consider a higher-dimensional hypercomplex algebra, namely, the sedenion algebra. The sedenion algebra consists of 16 basis sedenion operators $\{e_k,\ k = 0, 1, 2, \ldots, 15\}$, denoted sequentially.

The multiplication rules for 16 sedenion basis operators are given in Figure 3 and are different from the table for the direct-product operator model shown earlier in Figure 1.

I	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3	Θ_2	V_1	V_2	V_3	Θ_3	W_1	W_2	W_3
Γ_1	$-I$	Γ_3	$-\Gamma_2$	U_1	$-\Theta_1$	$-U_3$	U_2	V_1	$-\Theta_2$	$-V_3$	V_2	$-W_1$	Θ_3	W_3	$-W_2$
Γ_2	$-\Gamma_3$	$-I$	Γ_1	U_2	U_3	$-\Theta_1$	$-U_1$	V_2	V_3	$-\Theta_2$	$-V_1$	$-W_2$	$-W_3$	Θ_3	W_1
Γ_3	Γ_2	$-\Gamma_1$	$-I$	U_3	$-U_2$	U_1	$-\Theta_1$	V_3	$-V_2$	V_1	$-\Theta_2$	$-W_3$	W_2	$-W_1$	Θ_3
Θ_1	$-U_1$	$-U_2$	$-U_3$	$-I$	Γ_1	Γ_2	Γ_3	Θ_3	W_1	W_2	W_3	$-\Theta_2$	$-V_1$	$-V_2$	$-V_3$
U_1	Θ_1	$-U_3$	U_2	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2	W_1	$-\Theta_3$	W_3	$-W_2$	V_1	$-\Theta_2$	V_3	$-V_2$
U_2	U_3	Θ_1	$-U_1$	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$	W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	$-\Theta_2$	V_1
U_3	$-U_2$	U_1	Θ_1	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$	W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	$-\Theta_2$
Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_3$	$-W_1$	$-W_2$	$-W_3$	$-I$	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3
V_1	Θ_2	$-V_3$	V_2	$-W_1$	Θ_3	W_3	$-W_2$	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2	$-U_1$	Θ_1	U_3	$-U_2$
V_2	V_3	Θ_2	$-V_1$	$-W_2$	$-W_3$	Θ_3	W_1	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$	$-U_2$	$-U_3$	Θ_1	U_1
V_3	$-V_2$	V_1	Θ_2	$-W_3$	W_2	$-W_1$	Θ_3	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$	$-U_3$	U_2	$-U_1$	Θ_1
Θ_3	W_1	W_2	W_3	Θ_2	$-V_1$	$-V_2$	$-V_3$	$-\Theta_1$	U_1	U_2	U_3	$-I$	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$
W_1	$-\Theta_3$	W_3	$-W_2$	V_1	Θ_2	V_3	$-V_2$	$-U_1$	$-\Theta_1$	U_3	$-U_2$	Γ_1	$-I$	Γ_3	$-\Gamma_2$
W_2	$-W_3$	$-\Theta_3$	W_1	V_2	$-V_3$	Θ_2	V_1	$-U_2$	$-U_3$	$-\Theta_1$	U_1	Γ_2	$-\Gamma_3$	$-I$	Γ_1
W_3	W_2	$-W_1$	$-\Theta_3$	V_3	V_2	$-V_1$	Θ_2	$-U_3$	U_2	$-U_1$	$-\Theta_1$	Γ_3	Γ_2	$-\Gamma_1$	$-I$

Figure 3. The color-coded multiplication table for 16 basis sedenion operators $\{e_k, k = 0, 1, 2, \ldots, 15\}$. The region algebra contains three distinct types of the octonion sub-algebras, denoted by $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$,$\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ and $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_3, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$. Each color-coded domain contains quaternion triplets following cyclic multiplication rules. Unlike the table in Figure 1 for the associative direct-product matrices, the 4th, 8th and 12th operators in the non-associative sedenion algebra anti-commute with **U**, **V**, **W** operators.

As can be seen from Figure 3, the sedenion algebra contains three distinct types of the octonion algebra, which are denoted by $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$, $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$ and $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_3, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$. The multiplication tables for these three distinct types of octonion algebra are illustrated in Figure 4.

(A)

I	Γ_1	Γ_2	Γ_3	Θ_1	U_1	U_2	U_3
Γ_1	$-I$	Γ_3	$-\Gamma_2$	U_1	$-\Theta_1$	$-U_3$	U_2
Γ_2	$-\Gamma_3$	$-I$	Γ_1	U_2	U_3	$-\Theta_1$	$-U_1$
Γ_3	Γ_2	$-\Gamma_1$	$-I$	U_3	$-U_2$	U_1	$-\Theta_1$
Θ_1	$-U_1$	$-U_2$	$-U_3$	$-I$	Γ_1	Γ_2	Γ_3
U_1	Θ_1	$-U_3$	U_2	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2
U_2	U_3	Θ_1	$-U_1$	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$
U_3	$-U_2$	U_1	Θ_1	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$

(B)

	e_0	e_1	e_2	e_3	e_8	e_9	e_{10}	e_{11}
e_0	I	Γ_1	Γ_2	Γ_3	Θ_2	V_1	V_2	V_3
e_1	Γ_1	$-I$	Γ_3	$-\Gamma_2$	V_1	$-\Theta_2$	$-V_3$	V_2
e_2	Γ_2	$-\Gamma_3$	$-I$	Γ_1	V_2	U_3	$-\Theta_2$	$-V_1$
e_3	Γ_3	Γ_2	$-\Gamma_1$	$-I$	V_3	$-V_2$	V_1	$-\Theta_2$
e_8	Θ_2	$-V_1$	$-V_2$	$-V_3$	$-I$	Γ_1	Γ_2	Γ_3
e_9	V_1	Θ_2	$-V_3$	V_2	$-\Gamma_1$	$-I$	$-\Gamma_3$	Γ_2
e_{10}	V_2	U_3	Θ_2	$-V_1$	$-\Gamma_2$	Γ_3	$-I$	$-\Gamma_1$
e_{11}	V_3	$-V_2$	V_1	Θ_2	$-\Gamma_3$	$-\Gamma_2$	Γ_1	$-I$

(C)

I	Γ_1	Γ_2	Γ_3	Θ_3	W_1	W_2	W_3
Γ_1	$-I$	Γ_3	$-\Gamma_2$	$-W_1$	Θ_3	W_3	$-W_2$
Γ_2	$-\Gamma_3$	$-I$	Γ_1	$-W_2$	$-W_3$	Θ_3	W_1
Γ_3	Γ_2	$-\Gamma_1$	$-I$	$-W_3$	W_2	$-W_1$	Θ_3
Θ_3	W_1	W_2	W_3	$-I$	$-\Gamma_1$	$-\Gamma_2$	$-\Gamma_3$
W_1	$-\Theta_3$	W_3	$-W_2$	Γ_1	$-I$	Γ_3	$-\Gamma_2$
W_2	$-W_3$	$-\Theta_3$	W_1	Γ_2	$-\Gamma_3$	$-I$	Γ_1
W_3	W_2	$-W_1$	$-\Theta_3$	Γ_3	Γ_2	$-\Gamma_1$	$-I$

Figure 4. (**A**) The multiplication rules for the octonions $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$. (**B**) The multiplication table of $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_2, \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$. (**C**) The table of $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_3, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3\}$. Each color-coded domain of quaternion triplets follows cyclic multiplication rules.

We examine here the mass–energy relation according to the octonion algebra of $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_1,\ \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$ that represents the basis octonion operators $\{e_0,\ e_1,\ e_2,\ e_3,\ e_5,\ e_6,\ e_7,\ e_8\}$ sequentially. We can generalize Dirac's equation and utilize

$$
\begin{aligned}
&1)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} A_k\mathbf{U}_k \\
&iE\ \mathbf{I} = -m_0\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{\Theta}_1\mathbf{\Gamma}_k, + \sum_{k=1}^{3} A_k\mathbf{\Theta}_1\mathbf{U}_k = -m_0\mathbf{\Theta}_1 - \sum_{k=1}^{3} P_k\mathbf{U}_k + \sum_{k=1}^{3} A_k\mathbf{\Gamma}_k \\
&-E^2 = \left(-m_0\mathbf{\Theta}_1 + \sum_{k=1}^{3} A_k\mathbf{\Gamma}_k - \sum_{k=1}^{3} P_k\mathbf{U}_k,\right)^2 \\
&\quad = {m_0}^2{\mathbf{\Theta}_1}^2 + \sum_{k=1}^{3} {A_k}^2{\mathbf{\Gamma}_k}^2 + {P_k}^2{\mathbf{U}_k}^2 \\
&\quad + \sum_{i,\neq j}^{3} A_iA_j\{\mathbf{\Gamma}_i,\ \mathbf{\Gamma}_j\} + \sum_{\sum_{i,\neq j}^{3} P_i} P_iP_j\{\mathbf{U}_i,\ \mathbf{U}_j\} - m_0P\{\mathbf{\Theta}_1,\ \mathbf{\Gamma}_k\} + m_0\sum_{k=1}^{3} P_k\{\mathbf{\Theta}_1,\ \mathbf{U}_k\} \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \qquad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} {A_k}^2.
\end{aligned}
\tag{6A}
$$

The above result is consistent with Einstein's mass–energy relation, indicating that the octonion model gives rise to a single generation of fermions with an internal structural dynamic. Similarly, for $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_2,\ \mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3\}$ and $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_3,\ \mathbf{W}_1,\ \mathbf{W}_2,\ \mathbf{W}_3\}$, one can obtain

$$
\begin{aligned}
&2)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} B_k\mathbf{V}_k \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \qquad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} {B_k}^2,
\end{aligned}
\tag{6B}
$$

and

$$
\begin{aligned}
&3)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} C_k\mathbf{W}_k \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \qquad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} {C_k}^2.
\end{aligned}
\tag{6C}
$$

The three above equations reproduce Einstein's mass–energy relation, e.g., $E^2 = {m_{0,eff}}^2 + \sum_k {P_k}^2,\quad {m_{0,eff}}^2 \equiv {m_0}^2 + \sum_k {Q_k}^2,\quad Q_k = A_k,\ B_k,\ C_k$ for a particle with an effective rest mass $m_{0,eff}$, which contains the kinetic energy of its internal structural dynamics. Equation (6A–C) can represent three generations of charged leptons, namely the electron, the muon and the tau. For three generations of neutral leptons, i.e., the corresponding neutrino for each generation of leptons, we use the following assignments:

$$
\begin{aligned}
&1)\quad m_0\ \mathbf{I} = iE\ i\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} A_k(\mathbf{V}_k + i\mathbf{W}_k) \\
&iE\ \mathbf{I} = -m_0\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{U}_k + \sum_{k=1}^{3} A_k(\mathbf{W}_k - i\mathbf{V}_k) \\
&-E^2 = {m_0}^2{\mathbf{\Theta}_1}^2 + {P_k}^2{\mathbf{U}_k}^2 + \sum_{k=1}^{3} {A_k}^2(\mathbf{W}_k - i\mathbf{V}_k)^2 \\
&\quad - m_0\sum_{k=1}^{3} A_k\{\mathbf{\Theta}_1, \mathbf{W}_k - i\mathbf{V}_k\} + \sum_{i,j=1}^{3} P_i\{\mathbf{U}_i, \mathbf{W}_j - i\mathbf{V}_j\} \\
&E^2 = {m_0}^2 + \sum_{k=1}^{3} {P_k}^2
\end{aligned}
\tag{7A}
$$

$$
\begin{aligned}
&2)\quad m_0\,\mathbf{I} = iE\, i\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k \mathbf{\Gamma}_k + \sum_{k=1}^{3} B_k(\mathbf{W}_k + i\mathbf{U}_k)\\
&iE\,\mathbf{I} = -m_0\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k \mathbf{V}_k + \sum_{k=1}^{3} B_k(\mathbf{U}_k - i\mathbf{W}_k)\\
&E^2 = {m_0}^2 + \sum_{k=1}^{3} {P_k}^2
\end{aligned}
\tag{7B}
$$

$$
\begin{aligned}
&3)\quad m_0\,\mathbf{I} = iE\,\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k \mathbf{\Gamma}_k + \sum_{k=1}^{3} C_k(\mathbf{U}_k + i\mathbf{V}_k)\\
&iE\,\mathbf{I} = -m_0\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k \mathbf{W}_k + \sum_{k=1}^{3} C_k(\mathbf{V}_k - i\mathbf{U}_k)\\
&E^2 = {m_0}^2 + \sum_{k=1}^{3} {P_k}^2.
\end{aligned}
\tag{7C}
$$

Owing to the absence of $\sum_k {Q_k}^2$ in the above results for three neutrino generations, one has a vanishingly small rest mass $E^2 \approx \sum_k {P_k}^2$ if m_0 is close to zero. Unlike the cases for the charged leptons, one has $E^2 = {m_{0,eff}}^2 + \sum_k {P_k}^2$, where ${m_{0,eff}}^2 = {m_0}^2 + \sum_k {Q_k}^2 \approx \sum_k {Q_k}^2$, even if m_0 is close to zero. According to the above sedenion model, their masses could be close to zero and are much smaller than those of their corresponding leptons. The experimental observations of flavor mixing and the mass oscillations among three generations of neutrinos are believed to be induced by the symmetry-breaking mechanism of the sedenion algebra. It will be shown later in Section 2.6 that the sedenion algebra can be linked to SU(5). The symmetry breaking of SU(5) into SU(3)⊗SU(2)⊗U(1) might be caused by the flavor mixing and mass oscillations of neutrinos.

2.5. Sedenion Algebra and Three Generations of Quarks

Here, we propose the following assignments of the sedenion operators for three quark generations:

$$
\begin{aligned}
&1)\quad m_0\,\mathbf{I} = iE\, i\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k \mathbf{\Gamma}_k + \sum_{k=1}^{3} (B_k\mathbf{V}_k + C_k\mathbf{W}_k)\\
&iE\,\mathbf{I} = -m_0\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k \mathbf{U}_k + \sum_{k=1}^{3} (B_k\mathbf{W}_k - C_k\mathbf{V}_k)\\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \quad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3}\left({B_k}^2 + {C_k}^2\right)
\end{aligned}
\tag{8A}
$$

$$
\begin{aligned}
&2)\quad m_0\,\mathbf{I} = iE\, i\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k \mathbf{\Gamma}_k + \sum_{k=1}^{3} (C_k\mathbf{W}_k + A_k\mathbf{U}_k)\\
&iE\,\mathbf{I} = -m_0\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k \mathbf{U}_k + \sum_{k=1}^{3} (C_k\mathbf{U}_k - A_k\mathbf{W}_k)\\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \quad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3}\left({A_k}^2 + {C_k}^2\right)
\end{aligned}
\tag{8B}
$$

$$
\begin{aligned}
&3)\quad im_0\,\mathbf{I} = iE\, i\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k \mathbf{\Gamma}_k + \sum_{k=1}^{3} (A_k\mathbf{U}_k + B_k\mathbf{V}_k)\\
&iE\,\mathbf{I} = -m_0\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k \mathbf{U}_k + \sum_{k=1}^{3} (-A_k\mathbf{V}_k + B_k\mathbf{U}_k)\\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \quad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3}\left({A_k}^2 + {B_k}^2\right).
\end{aligned}
\tag{8C}
$$

One could also make different assignments to the generalized energy and momentum operators for the other three heavier quark generations.

$$\begin{aligned}
&1)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} (A_k\mathbf{U}_k + B_k\mathbf{V}_k + B_k\mathbf{W}_k) \\
&iE\mathbf{I} = i\mathbf{m}_0\mathbf{\Theta}_1 + \sum_{k=1}^{3} P_k\mathbf{U}_k + \sum_{k=1}^{3} (A_k\mathbf{\Gamma}_k + B_k\mathbf{W}_k - B_k\mathbf{V}_k) \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \quad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} \left({A_k}^2 + 2{B_k}^2\right),
\end{aligned} \tag{9A}$$

$$\begin{aligned}
&2)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_2 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} (C_k\mathbf{U}_k + B_k\mathbf{V}_k + A_k\mathbf{W}_k) \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \qquad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} \left({B_k}^2 + 2{C_k}^2\right),
\end{aligned} \tag{9B}$$

$$\begin{aligned}
&3)\quad m_0\mathbf{I} = iE\mathbf{\Theta}_3 + \sum_{k=1}^{3} P_k\mathbf{\Gamma}_k + \sum_{k=1}^{3} (A_k\mathbf{U}_k + A_k\mathbf{V}_k + C_k\mathbf{W}_k) \\
&E^2 = {m_{0,eff}}^2 + \sum_{k=1}^{3} {P_k}^2, \qquad {m_{0,eff}}^2 = {m_0}^2 + \sum_{k=1}^{3} \left({C_k}^2 + 2{A_k}^2\right).
\end{aligned} \tag{9C}$$

According to the above operator assignments, there are six possible types of assignments for the quarks, namely, the assignments in Equation (8A–C) could be related to the lighter three-member family of the up, charm and top quarks, while the assignments in Equation (9A–C) could be linked to the heavier three-member family of the bottom, strange and bottom quarks. Thus, the sedenion algebra is shown to lead naturally to six types of quarks of the Standard Model.

2.6. Mapping Octonions to SU(3) Generators and Sedenion to SU(5) Generators

In this section, we discuss the mapping of the octonion operators to 8 SU(3)'s generators and the mapping of sedenion operators to 24 SU(5) generators. Each type of octonion operator can be shown to be related to Clifford algebra C↕(6) [20]. Here, we define three pairs of fermion creation and annihilation operators, which satisfy the anti-commutation relations as

$$\begin{aligned}
&\alpha_1 = (-e_6 + ie_5)/2, \quad \alpha_2 = (-e_3 + ie_1)/2, \quad \alpha_3 = (-e_7 + ie_2)/2 \\
&\{\alpha_i, \alpha_j\} = \left\{\alpha_i^+, \alpha_j^+\right\} = 0 \quad , \quad \left\{\alpha_i, \alpha_j^+\right\} = \delta_{ij}
\end{aligned} \tag{10A}$$

For the first lepton/quark generation, these eight basis octonion operators are denoted by $\{\mathbf{I}, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \mathbf{\Gamma}_3, \mathbf{\Theta}_1, \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3\}$

$$\begin{aligned}
&\alpha_1 = (-\mathbf{U}_2 + i\mathbf{U}_1)/2, \ \alpha_2 = (-\mathbf{\Gamma}_3 + i\mathbf{\Gamma}_1)/2, \ \alpha_3 = (-\mathbf{U}_3 + i\mathbf{\Gamma}_2)/2 \\
&\alpha_1^+ = (\mathbf{U}_2 + i\mathbf{U}_1)/2, \ \alpha_2^+ = (\mathbf{\Gamma}_3 + i\mathbf{\Gamma}_1)/2, \ \alpha_3^+ = (\mathbf{U}_3 + i\mathbf{\Gamma}_2)/2.
\end{aligned} \tag{10B}$$

One can define a tensor product $|i\rangle\langle j| \equiv \alpha_i{}^+\alpha_j$ to construct the following eight SU(3) generators, which are related to Gell-Mann's lambda matrices $\mathbf{\Lambda}_k$ as

$$\begin{aligned}
&\mathbf{\Lambda}_1 = |2\rangle\langle 1| + |1\rangle\langle 2| = i\ (\mathbf{U}_3 - \mathbf{U}_2)/2 \\
&\mathbf{\Lambda}_2 = -i|1\rangle\langle 2| + i|2\rangle\langle 1| = -i\ (\mathbf{U}_1 - \mathbf{\Theta}_1)/2 \\
&\mathbf{\Lambda}_3 = |1\rangle\langle 1| - |2\rangle\langle 2| = i(\mathbf{\Gamma}_3 - \mathbf{\Gamma}_2)/2 \\
&\mathbf{\Lambda}_4 = |1\rangle\langle 3| + |3\rangle\langle 1| = \ i(\mathbf{\Theta}_1 - \mathbf{\Gamma}_2)/2 \\
&\mathbf{\Lambda}_5 = -i|1\rangle\langle 3| + i|3\rangle\langle 1| = -i\ (\mathbf{\Gamma}_1 + \mathbf{U}_3)/2 \\
&\mathbf{\Lambda}_6 = |2\rangle\langle 3| + |3\rangle\langle 2| = -i\ (\mathbf{\Gamma}_1 + \mathbf{U}_2)/4 \\
&\mathbf{\Lambda}_7 = -i|2\rangle\langle 3| + i|3\rangle\langle 2| = -i(\mathbf{\Theta}_1 - \mathbf{\Gamma}_3)/2 \\
&\mathbf{\Lambda}_8 = (|1\rangle\langle 1| + |2\rangle\langle 2| - 2|3\rangle\langle 3|)/\sqrt{3} = i(\mathbf{\Gamma}_3 + \mathbf{\Gamma}_2 - 2\mathbf{U}_2)/2.
\end{aligned} \tag{10C}$$

We have shown above that the eight lambda matrices, as the SU(3) generators, can be constructed by $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_1,\ \mathbf{U}_1,\ \mathbf{U}_2,\ \mathbf{U}_3\}$ because the sedenion algebra consists of three octonion sub-algebras, similar to other octonion basis sets, e.g., $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3,\ \mathbf{\Theta}_2,\ \mathbf{V}_1,\ \mathbf{V}_2,\ \mathbf{V}_3\}$ to make two other lambda-matrix generators. Together with the U-type, V-type and W-type octonion algebras, which are the three distinct sub-algebras of the sedenion algebra, we can build altogether 24 generators for SU(5).

In Equation (10) we use octonion basis operators for the construction of three pairs of creation and annihilation operators which could be employed to construct eight 3×3 lambda matrix generators for the SU(3) symmetry group. Here, by extending the use of three pairs of creation and annihilation operators for the octonion algebra, we can now define five pairs of fermion creation and annihilation operators as

$$\begin{aligned}
&\alpha_1 = (-e_6 + ie_5)/2, \quad \alpha_2 = (-e_3 + ie_1)/2, \quad \alpha_3 = (-e_7 + ie_2)/2 \\
&\alpha_4 = (-e_{14} + ie_{13})/2, \quad \alpha_5 = (-e_{11} + ie_9)/2 \\
&\{\alpha_i,\ \alpha_j\} = \left\{\alpha_i^+,\ \alpha_j^+\right\} = 0, \quad \left\{\alpha_i,\ \alpha_j^+\right\} = \delta_{ij},
\end{aligned} \tag{11A}$$

or, equivalently,

$$\begin{aligned}
&\alpha_1 = (-\mathbf{U}_2 + i\mathbf{U}_1)/2, \ \alpha_2 = (-\mathbf{\Gamma}_3 + i\mathbf{\Gamma}_1)/2, \ \alpha_3 = (-\mathbf{U}_3 + i\mathbf{\Gamma}_2)/2 \\
&\alpha_4 = (-\mathbf{V}_2 + i\mathbf{V}_1)/2, \quad \alpha_5 = (-\mathbf{W}_2 + i\mathbf{W}_1)/2.
\end{aligned} \tag{11B}$$

Similar to the construction of eight 3×3 lambda matrices in Equation (10C) for the SU(3) generators from three pairs of creation and annihilation operators, from five pairs of fermionic creation and annihilation operators, we could construct a total of twenty-four generalized lambda matrices as the SU(5) generators. Put simply, there are eight 3×3 matrix generators in SU(3). The number eight equates to the square of three minus one by excluding an identity matrix, whereas in SU(5), there are twenty-four 5×5 matrix generators, which equal the square of five minus one by excluding an identity matrix. Similar to the automorphic relationship between the octonion algebra and SU(3), in this work, we establish the relationship between the sedenion algebra and SU(5). We have shown in Equation (10C) the explicit assignments of the SU(3) generators from pairs of the octonion basis operators. Similar pair assignments of sedenion operators to the twenty-four lambda generators for SU(5) are quite lengthy but straightforward, and will not be given here. Using the five pairs of fermionic creation and annihilation operators, one can construct ten pairs of 5×5 SU(5) generator matrices as $|m\rangle\langle n| + |n\rangle\langle m|, \quad -i|m\rangle\langle n| + i|n\rangle\langle m|$, for $m \neq n = 1,\ 2,\ \ldots,\ 5$. With the same indices, one can make four diagonal but orthogonal 5×5 matrices. Therefore, among these twenty-four SU(5) generators represented by 5×5 generalized lambda matrices, there are four diagonal matrices and twenty off-diagonal matrices. The SU(5) symmetry plays an essential role in the GUT (grand unification theory) [27,28], which has been advocated to unify the electromagnetic, weak and strong interactions of elementary particles. These three types of the eight-element octonion algebra are not independent of each other because they contain the same quaternion algebra $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3\}$. The cooling of the universe after the Big Bang plays a vital role in the symmetry breaking of SU(5) to become SU(3)⊗SU(2)⊗U(1), and in the breakdown of the sedenion algebra into a direct product of octonion and quaternion algebras. Such a breakdown results in the flavor mixing of neutrinos and their mass oscillations.

Similar to the construction of eight of 3×3 lambda matrices in Equation (10C), for the SU(3) generators from three pairs of creation and annihilation operators, from pairs of five fermionic creation and annihilation operators, one can build a total of twenty-four generalized lambda matrices as the SU(5) generators. Using the five pairs of the fermionic creation and annihilation operators, one can construct ten pairs of 5×5 off-diagonal SU(5) generator matrices as $|m\rangle\langle n| + |n\rangle\langle m|,\ -i|m\rangle\langle n| + i|n\rangle\langle m|$, for $m \neq n = 1,\ 2,\ \ldots,\ 5$. With the same indices, one can build four off-diagonal but orthogonal 5×5 matrices. Therefore, among these twenty-four SU(5) generators represented by 5×5 generalized lambda matrices, there are four diagonal matrices and twenty off-diagonal matrices. The SU(5) symmetry plays an essential role in the GUT (grand unification theory) [27,28] and it has been advocated to unify the electromagnetic, weak and strong interactions of elementary particles. These three types of the eight-element octonion algebra are not independent of each other because they contain the exact quaternion algebra $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3\}$.

3. Conclusions

In this work, we have presented a theoretical analysis of higher-dimensional models beyond the conventional Dirac equation, which was proposed for a point-like fermion. Dirac utilized four anti-commutative matrices for the first derivatives of a particle's space and time coordinates. The extension to higher-dimensional models is necessary because the traditional Dirac theory cannot explain the origin of the observed three generations of leptons and quarks in the Standard Model. In this study, we first consider a 16-element group of 4×4 matrices as direct products of quaternion operators, i.e., three 2×2 Pauli matrices and an identity matrix. This model contains three extra degrees of freedom to incorporate a particle's internal structure. This 16-element group has three 8-element subgroups, possessing a closure property for multiplications among 8 elements of each subset. However, this direct-product matrix model with associative operators could not reproduce Einstein's mass–energy relation, unlike the model with 16 sedenion basis operators. The significant difference between the two models is whether the operator multiplication is associative or not, as illustrated by the differences in the multiplication tables of Figures 1 and 2. The 8-element octonion algebra is shown to be able to lead to a single lepton generation. However, we need a 16-element sedenion algebra to encompass precisely three generations of leptons and quarks. We have shown from the color-coded table arrays in Figure 2 that the sedenion algebra contains three distinct octonion algebras, with the U-, V- and W-type operators, each type of octonion sub-algebra corresponding to a single generation. We have also shown that in Equation (7A–C), the effective neutrinos' rest mass could be vanishingly small in comparison to that in Equation (6A–C) for the counterpart charged leptons. We have also provided operator assignments in Equations (8A–C) and (9A–C) to represent three generations of lighter and heavier quarks. Moreover, we have shown that by adequately pairing up the octonion basis operators, one can construct eight 3×3 Gell-Mann lambda matrices as the eight generators for SU(3). For the U-, V- and W-type octonion sub-algebra of the sedenion algebra, we could pair up the sedenion basis operators to construct a total of twenty-four 5×5 matrices to represent the 24 generators of SU(5). These three types of the eight-element octonion algebra contain the same quaternions $\{\mathbf{I},\ \mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2,\ \mathbf{\Gamma}_3\}$. It is commonly believed that soon after the Big Bang of the universe, the almost instant cooling process played an important role in the symmetry breaking for SU(5) to become SU(3)$\otimes$SU(2)$\otimes$U(1) of the present Standard Model situation, and breaking down sedenion algebra into the direct-product algebra of octonions and quaternions results in neutrinos' flavor mixing and mass-oscillation behavior. The main purpose of this work is to point out the interesting relationship between sedenion algebra and SU(5). Although, in Section 2.4, we explain qualitatively why neutrinos have vanishingly small rest masses as compared to those of their counterpart charged leptons, we do not intend to solve all the puzzles faced by the Standard Model such as neutrinos' mass oscillations. More studies are needed to address such an issue, which might involve combining this sedenion model for SU(5) with the Higgs symmetry-breaking mechanism [29]. Such a challenging task is not the primary interest of this work. The phenomenon of flavor mixing and oscillations among neutrinos is an interesting and important issue. However, it is beyond the scope of this work, and it awaits further investigation from other experts in this field of particle physics. Through this work, by linking the sedenion algebra to SU(5) of GUT, which can be reduced to SU(3)$\otimes$SU(2)$\otimes$U(1), we hope to offer a potential avenue toward the development of an improved theory beyond the Standard Model.

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