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Quadratic American Strangle Options in Light of Two-Sided Optimal Stopping Problems

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Abstract: The aim of this paper is to examine some American-style financial instruments that lead to two-sided optimal hitting problems. We pay particular attention to derivatives that are similar to strangle options but have a quadratic payoff function. We consider these derivatives in light of much more general payoff structures under certain conditions which guarantee that the optimal strategy is an exit from a strip. Closed-form formulas for the optimal boundaries and the fair price are derived when the contract has no maturity constraints. We obtain the form of the optimal boundaries under the finite maturity horizon and approximate them by maximizing the financial utility of the derivative holder. The Crank–Nicolson finite difference method is applied to the pricing problem. The importance of these novel financial instruments is supported by several features that are very useful for financial practice. They combine the characteristics of the power options and the ordinary American straddles. Quadratic strangles are suitable for investors who need to hedge strongly, far from the strike positions. In contrast, the near-the-money positions offer a relatively lower payoff than the ordinary straddles. Note that the usual options pay exactly the overprice; no more, no less. In addition, the quadratic strangles allow investors to hedge the positions below and above the strike together. This is very useful in periods of high volatility when large market movements are expected but their direction is unknown.

Keywords: two-sided optimal stopping problems; American quadratic strangles; optimal boundaries; perpetual options; finite maturity options

MSC: 35R35; 35Q91; 60G44; 91G20

JEL Classification: C41; C61; G12; G13

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1. Introduction

Derivatives are one of the major instruments used to limit financial risk. They exhibit a very large variety—the most popular are options, futures, bonds, swaps, etc. Conventionally speaking, we can recognize two types—European and American. The European options have a previously fixed date at which the transaction is executed. Alternatively, an American-style instrument gives its holder the right to choose when to exercise until maturity. This right makes American derivatives preferable for investors and determines the largest segment of the traded assets they have in the modern financial markets. The main exchanges at which the options are traded are Chicago Board Options Exchange (CBOE, https://www.cboe.com/, accessed on 5 May 2024), Philadelphia Stock Exchange (NASDAQ OMX PHLX, https://www.nasdaq.com/solutions/nasdaq-phlx, accessed on 5 May 2024). Frankfurt-based Eurex Exchange (https://www.eurex.com/ex-en/, accessed on 5 May 2024), Tokyo Stock Exchange (TSE), Taiwan Futures Exchange (TAIFEX), and two electronic platforms—the Boston Options Exchange (BOX) and Miami International

Securities Exchange (MIAX). We have to mention that not all markets offer both types of options. Many kinds of American-style options are available, a fact that has caused the growth in scientific literature devoted to American-style derivatives. In Brennan and Schwartz [1], the authors present a pricing method and apply it to some contracts traded at the real exchanges. The works of Karatzas [2], Kim [3], Jacka [4], Carr et al. [5] present the classical mathematical methods for studying the optimal boundary and the option pricing task. The presentation of the American option price as a sum of the related European one and the premium for the early exercise right is considered too. Some Fourier-based methods are used in Chan [6] and Chan [7]. American options under regime switching are studied in Lu and Putri [8] and Jeon et al. [9].

Other, more exotic American-style derivatives include the American barrier options (Lin and He [10]), American power options (Miao et al. [11] and Lee [12]), American straddles (Alobaidi and Mallier [13], Alobaidi and Mallier [14], Xu and Ye [15], and Goard and AbaOud [16]), American strangles (Qiu [17] and Jeon and Kim [18]), quanto options (Battauz et al. [19]), American capped options (Broadie and Detemple [20] and Detemple and Tian [21]), American better-of options (Gao et al. [22] and Jeon and Kim [23]), derivatives with double continuation regions (Battauz et al. [24] and Battauz and Rotondi [25]), look-back options (Deng [26], Woo and Choe [27], and Zhang et al. [28]), variable annuities (Jeon and Kim [29]), the so-called redeemable shares (Battauz and Rotondi [30]), etc. Some American bond options are examined in Zhang et al. [31].

The American options exhibiting a credit risk are studied in Company et al. [32]. Options working under the assumption that the financial markets are prone to sudden shocks leading to jumps in asset prices are studied in Chan [7], Zhang et al. [33], and Huang et al. [34]. We categorize these sources in Table 1. This variety of American derivatives can be explained by different kinds of financial risks arising from the real markets.

Table 1. Literature categorization.

Topic	Sources
Classical American options	[1–5,35]
American options under regime switching	[8,9]
American options under stochastic volatility	[9]
American options under a credit risk	[32]
American options under jump assumptions	[7,33,34]
American barrier options	[10,36]
Power options	[11,12,37–42]
American straddle options	[13–16]
American strangle options	[17,18,43–45]
American quanto options	[19]
American capped options	[20,21]
American better-of options	[22,23]
Derivatives with double continuation regions	[24,25]
Look-back American options	[26–28]
Variable annuities	[29]
Redeemable shares	[30]
Cancellable American options	[46,47]
American bond options	[31]
Monte Carlo Methods	[48–52]
Fourier-based methods	[6,7,13,43]
Other numerical methods	[36,52–55]

Most of the available financial instruments protect mainly from the so-called linear risks in the sense that the holder would receive the overprice (underprice) of the asset or a proportion of it. Namely, such protection provides the classical option and futures contracts. On the other hand, many risks exhibit a non-linear essence—the holder needs a significantly larger payoff when the underlying asset is deeply above/below the strike. This motivates several authors to consider the so-called power options—see Heynen and

Kat [37], Macovschi and Quittard-Pinon [38], Zhang et al. [39], Nwozo and Fadugba [40], Fadugba and Nwozo [41], Fadugba and Nwozo [42], and Lee [12].

On the other hand, sometimes, especially in the high volatility periods, large movements in both directions are very likely. Therefore, the investors need joint protection against the fall and rise of the assets. This kind of insurance is provided by the straddle and strangle options—we refer to Kang et al. [43], Jeon and Oh [44] Qiu [17], Jeon and Kim [18], and Zaevski [45]. In fact, these options give the right to choose the option's style—call or put. Note that they lead to two-sided optimal hitting problems. This way, the optimal strategy is the first exit of the underlying asset from a strip. If the exit occurs from the lower boundary, then the holder exercises as a put, and, otherwise, as a call.

To combine the non-linearity and the two-sided protection, in this paper, we introduce and examine a specific financial instrument—we name it a *quadratic strangle option*. Its payoff is defined as the square of the corresponding straddle payoff; that is, $(S_T - K)^2$. Moreover, we study these options in a quite general framework considering payment functions that lead to two-sided optimal hitting problems. We state our model under the assumptions of the Black and Scholes [56] model—the underlying asset is driven by geometric Brownian motion. Also, we consider continuously paying dividends on assets using a method presented by Shiryaev et al. [35]. It is based on an additional discount factor instead of the original dividend rate. This approach provides a computational convenience. It is well-known that the empirical stylized facts such as volatility clustering, leverage effect, long-range dependence, sudden market shocks, etc., cannot be captured by the Gaussian returns. Despite these limitations, the outstanding importance of the Black–Scholes model motivates us to work in its framework. On the other hand, our method can be adapted for models driven by other Feller–Markov processes such as exponential Lévy models, stochastic volatility models, regime switching, etc. More precisely, our approach is based on the infinitesimal generators technique. We obtain a condition for the payoff that guarantees that the pricing problem leads namely to a two-sided optimal stopping task. Although the payoff of a quadratic strangle can be presented as a sum of two power payoffs, put and call, this does not mean that we can view it as a portfolio of two power options. This is because the possibility of the option to be exercised as a call influences the put optimal boundary, not only the call one. Of course, the put feature influences the call boundary too. This leads to the fact that the lower quadratic optimal boundary is below the optimal boundary of the related American put option. Also, the upper quadratic boundary is above the call one. As a consequence, the quadratic option has to be more expensive than the portfolio of the related two power options.

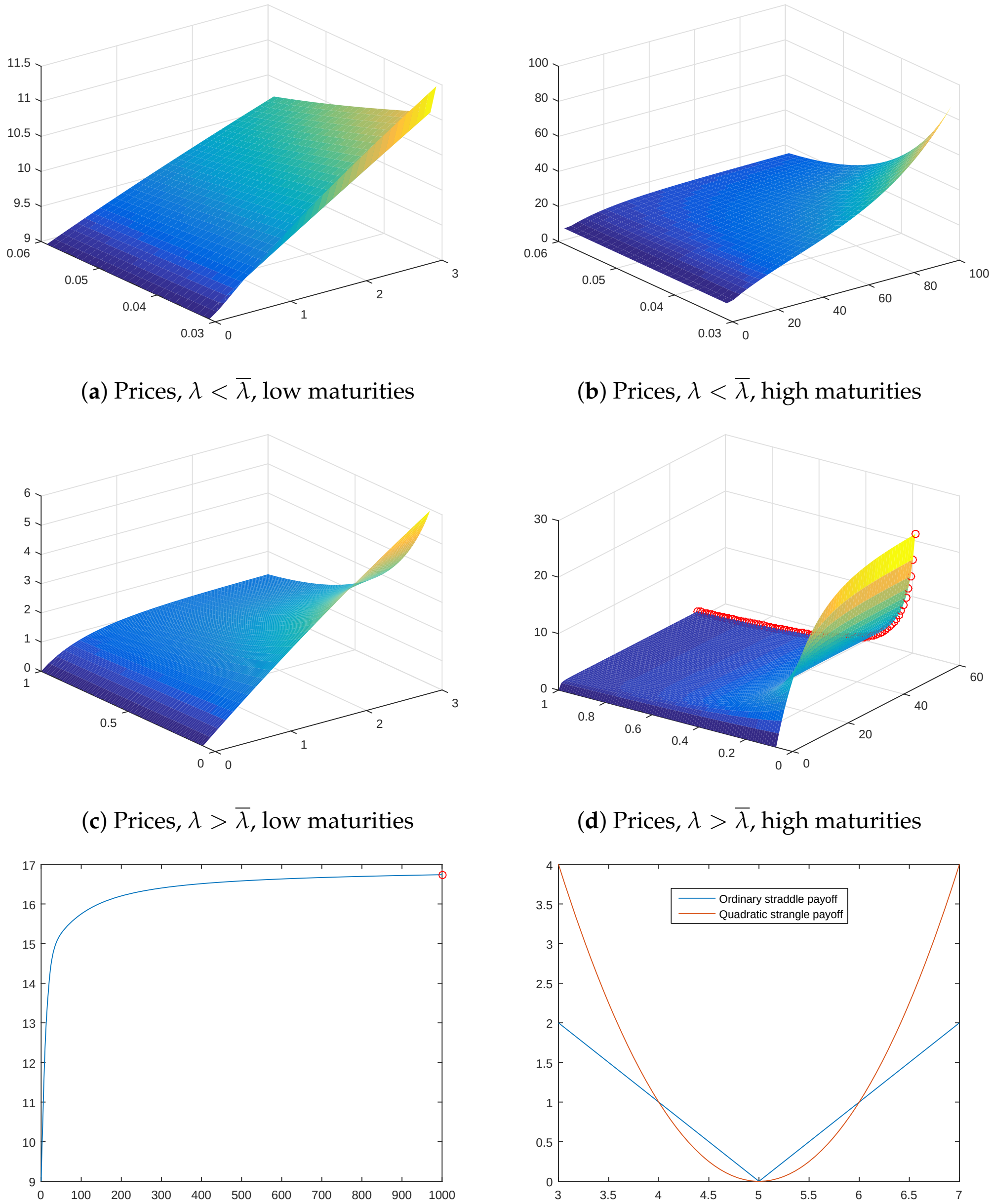
The scheme we use to evaluate such derivatives consists of several steps. First, we derive the values of the optimal boundaries at the maturity. We do this using the variational inequalities arising from the infinitesimal generator. Next, we consider the perpetual instruments. In this case, the optimal boundaries are time-independent because the underlying asset is driven by a Markov process. A system of two equations for the boundaries is obtained. Its solution leads to the fair price. The next task is to approximate the optimal boundaries. A natural assumption is that the holder's strategy maximizes his/her financial utility. So, we divide the time to maturity into several sub-intervals and approximate the boundaries at these nodes as the values that maximize the holder's result. Once we have the optimal boundaries, the free boundary problem that describes the option pricing task turns into a partial differential equation set in a known region. In fact, it is the continuation region—the points that make keeping the derivative preferable. We solve this problem numerically using the Crank–Nicolson finite difference approach. Several alternatives are available. For example, some methods for pricing the usual American options based on Monte Carlo simulations can be enlarged to capture the specifics of the new class of financial instruments. For some such methods, we refer to Broadie and Glasserman [48], Longstaff and Schwartz [49], Rogers [50], Cortazar et al. [51], and Zaevski [52]. Another alternative is the use of the method of binomial trees of Cox et al. [53]. Other numerical methods can be found in Park and Jeon [36] and Madi et al. [54]. We chose the Crank–Nicolson method because it is fast and relatively accurate. We left for further work the comparison with the above-mentioned numerical methods.

We apply this general framework to price the above-defined quadratic strangles. It turns out that there exists a critical value for the additional discount factor above which we have a real two-sided instrument. We separately consider finite and infinite maturities. Closed-form formulas for the optimal boundaries as well as for the quadratic strangle price are derived. Also, we obtain the boundaries' values at the maturity. Having the endpoints, we approximate the whole boundaries via exponents of piece-wise linear functions using some first exit properties of the Brownian motion from a strip. The prices are derived via the above-mentioned finite difference method.

On the other hand, if the discount factor is not above the critical value, then the quadratic strangle leads to a one-sided hitting problem. We apply the approach of Zaevski [55] to evaluate the option. The closed-form formulas are derived for the perpetual modification. There exists a significant difference when the discount factor is equal to the critical value or below it. If it is below, then the early exercise is never optimal for the perpetual quadratic strangle. In contrast, the immediate exercise can be the best strategy for the critical discount rate. In both cases, the finite maturities provide possibilities for early exercising. We validate and confirm all these results using several numerical tests.

The main contributions of the paper aim in several directions. First, we consider a relatively large class of novel American-style financial instruments. We establish general conditions for the payoffs which guarantee that the optimal holder's strategy is the first exit of the underlying asset price from a strip—thus, we need to find the boundaries of this strip. Usually, the American-style derivatives are considered to be free boundary problems for which we need to obtain the set in which they are held as well as to solve the differential task. We provide a novel approach based on identifying the points in the time–price space for which the immediate exercise is the best holder's strategy and these for which keeping the derivative is preferable. Our approach is based on maximizing the holder's financial utility. We separately consider the options without maturity constraints and finite maturity ones. For the perpetual ones, we obtain a two-dimensional system that the boundaries have to solve. Alternatively, we approximate them under the finite maturity horizon. This way, the free boundary differential problem turns into a boundary value problem in a known region. Note that the differential dynamics are driven by the infinitesimal generator of the stochastic process that presents the underlying asset price, whereas the payoff influences the boundary conditions.

Next, we turn to the second main purpose of the paper, namely to examine the above-mentioned quadratic options. Their importance is determined by both of the features they combine. First, the quadratic feature allows the investor to better hedge the positions that are deeply far from the strike. Whereas the usual call option would provide only the overprice above the strike (underprice for a put), the quadratic one provides the square of this amount. On the other hand, the holder of the quadratic option will receive a relatively lower amount if he/she exercises near the strike. The second major importance is related to the possibility for the holder to jointly hedge the positions that are above or below the strike—a very important feature in high volatility periods. These conclusions are supported by the payoff structure—we compare the payoffs of an ordinary straddle option to a quadratic one in Figure 1f.



(**a**) Prices, $\lambda < \overline{\lambda}$, low maturities (**b**) Prices, $\lambda < \overline{\lambda}$, high maturities

(**c**) Prices, $\lambda > \overline{\lambda}$, low maturities (**d**) Prices, $\lambda > \overline{\lambda}$, high maturities

(**e**) Prices, $\lambda = \overline{\lambda}$ (**f**) Payoffs

Figure 1. Price behavior.

We apply the derived theoretical results to these financial instruments. It turns out that two-sided exit problems arise for some values of the parameters, whereas other ones lead to one-sided hitting tasks. We examine both cases in detail —the optimal boundaries are approximated and the options are evaluated under and without maturity restrictions.

The paper is organized as follows. The base we use is presented in Section 2. The shape of the optimal exercise regions is obtained in Section 3. The pricing problem is discussed in Section 4. We examine the quadratic strangle options in Section 5. Some numerical experiments are provided in Section 6.

2. Preliminaries

We assume a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$ that satisfies the usual conditions, i.e., the filtration is right-continuous and complete—see, for example, Dellacherie [57] or Protter [58]. Also, let B_t be a Brownian motion defined in this space. We assume that the probability measure Q is risk-neutral. Suppose that the underlying asset, whose price we denote by S_t, follows the process

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{1}$$

where the constant r is the risk-free rate and σ is the volatility. Let λ be the additional discount factor that we can also view as a dividend rate—see proposition 2.3 from Zaevski [46]. We assume that $r + \lambda > 0$, but we do not impose positiveness for the risk-free rate. Let $\mathcal{A}$ be the infinitesimal generator of process (1) and $\mathcal{B}$ be another related differential operator:

$$\begin{aligned} (\mathcal{A}f)(x) &= rxf'(x) + \frac{\sigma^2 x^2}{2} f''(x) \\ (\mathcal{B}f)(x) &= (\mathcal{A}f)(x) - (r+\lambda)f(x). \end{aligned} \tag{2}$$

The infinitesimal generator $\mathcal{A}$ has this form since process (1) exhibits a diffusion feature. The operator $\mathcal{B}$ arises from discounting at the total rate $r + \lambda$. In fact, the operator $\mathcal{B}$ is related to the famous Black–Scholes equation. Let us denote by t, T, and τ the current time, the maturity, and the time to maturity, respectively. Thus, $t \leq T \leq \infty$ and $\tau = T - t$. We shall parameterize w.r.t. the present time as well as w.r.t. the time to maturity—the difference will be recognized via the used variables. Thus, we shall denote the price of the studied instruments by $P(t, x)$ w.r.t. the current time and by $P(x; \tau)$ w.r.t. the time to maturity.

Let a derivative pay an amount of $N(t, x)$ if its holder exercises at the moment t at the spot price $S_t = x$. Suppose also that the time dependence is presented only by additional discounting at the rate λ. Assuming that the veritable payoff is presented by the twice-differential function $G(x)$, we write

$$N(t, x) = e^{-\lambda t} G(x). \tag{3}$$

We restrict the choice of functions $G(x)$ in a way that guarantees that the pricing problem for the related derivative is a two-sided optimal stopping problem.

Characterization 1. *There exist constants $C \leq D$, such that $(\mathcal{B}G)(x) < 0$ for $x < C$ and $x > D$, and $(\mathcal{B}G)(x) \geq 0$ for $x \in [C, D]$.*

Note that if $C = 0$ or $D = \infty$, then we have one-sided problems.

3. Exercise Regions

Let the set at which the immediate exercise is the holder's optimal strategy be denoted by Υ—it is known as the optimal region. If $(t, x) \in \Upsilon$, then we shall call the point (t, x) optimal. Also, we use the symbol $\overline{\Upsilon}$ for the region in which keeping the derivative leads to a better financial result—the so-called continuation region. We shall mark with a subscript the dependence w.r.t. the current moment or w.r.t. the time to maturity—$\Upsilon_{[t,T]}$ and $\overline{\Upsilon}_{[t,T]}$ or Υ_τ and $\overline{\Upsilon}_\tau$, respectively.

We shall now characterize the shape of the optimal region as well as of the continuation set. It turns out that the optimal region consists of two disjoint subsets—the first one is below some boundary, whereas the second is above another curve. We shall prove that the lower boundary is below the above-defined constant C, whereas the upper one is above D. These boundaries are known as early exercise or optimal boundaries. Thus, all points between them form the continuation set. We shall investigate the features of the optimal boundaries, deriving their endpoints and establishing their behavior. This allows us to approximate them later and, as a consequence, to construct an algorithm for deriving the fair derivative price.

3.1. The Form of the Regions

We shall now investigate the shape of the optimal region. The main goal is to prove that if a point below C is optimal, then all points below it are optimal too. Analogously, it turns out that if an optimal point is above the level D, then all points above it are optimal too. This way, we confirm the above-mentioned feature.

The following statements for the shape of the regions and boundaries hold.

Proposition 1. *If a point (t, x) is optimal, then $(\mathcal{B}G)(x) < 0$.*

Proof. The variational inequality

$$N_t(t,x) + \mathcal{A}N(t,x) - rN(t,x) < 0 \tag{4}$$

has to be satisfied in the optimal region. It is equivalent to $(\mathcal{B}G)(x) < 0$. □

Proposition 2. *If $\tau_1 > \tau_2$ and the point $(x; \tau_1)$ is optimal, then the point $(x; \tau_2)$ is optimal too.*

Proof. The proposition holds because the set of possible strategies for a longer maturity contains all strategies for a shorter one. □

Proposition 3. *Let $(x; \tau)$ be an optimal point. The following two statements hold.*

1. *If $y > x > D$, then the point $(y; \tau)$ is optimal too.*
2. *If $y < x < C$, then the point $(y; \tau)$ is optimal.*

Proof. Note first that if $x \in [C, D]$, then it cannot be optimal, because $(\mathcal{B}G)(x) \geq 0$, and this contradicts Proposition 1. Suppose that $x < C$, $(x; \tau) \in \Upsilon_\tau$, and $y < x$. Let us denote by ζ^x the first hitting moment of the underlying asset to the value x if its initial point is y. Also, let us define ζ as the strategy $\zeta = \overline{\zeta} \wedge \zeta^x \wedge \tau$ for an arbitrary stopping time $\overline{\zeta}$. The strategy ζ is not worse than $\overline{\zeta}$ for the holder because the points $(x; \overline{\tau})$ for all $\overline{\tau} < \tau$ are optimal due to Proposition 2.
We have that $(\mathcal{B}G)(S_u(\omega)) < 0$ for every $u < \zeta(\omega)$ because $S_u(\omega) < x < C$, and thus we can use Characterization 1. Dynkin's formula leads to

$$\begin{aligned} E^y\left[e^{-(r+\lambda)\overline{\zeta}} G\left(S_{\overline{\zeta}}\right)\right] - G(y) &\leq E^y\left[e^{-(r+\lambda)\zeta} G(S_\zeta)\right] - G(y) \\ &= E^y\left[\int\limits_0^\zeta (\mathcal{B}G)(S_u)\right] < 0. \end{aligned} \tag{5}$$

Hence the point $(y; \tau)$ is optimal too. The second statement can be proven analogously using an upper construction. □

Proposition 3 shows that the continuation region is a strip in the time-state space. Also, the optimal set consists of two parts—one below the continuation region and another above it. We shall name them call- and put-optimal sets via an analogy with the usual options. We shall denote them by Υ^c and Υ^p, respectively. Thus, there are two optimal boundaries—one between Υ^p and $\overline{\Upsilon}$ and another between Υ^c and $\overline{\Upsilon}$. We denote these curves by $c(t)$ and $d(t)$, respectively. Hence, we can write $\Upsilon^p = \{(t,x) : t \in \mathbb{R}^+, x \in (0, c(t))\}$ and $\Upsilon^c = \{(t,x) : t \in \mathbb{R}^+, x \in (d(t), \infty)\}$.

Another consequence of Proposition 2 describes the optimal boundaries.

Corollary 1. *The boundary $c(\tau)$ decreases w.r.t. the time to maturity, whereas $d(\tau)$ increases.*

3.2. Initial Boundary Values

Further, we derive the optimal boundary values when the time to maturity tends to zero.

Proposition 4. *The values of the optimal boundaries when the time to maturity is zero are C and D.*

Proof. Let us consider the value of the lower boundary—we denote it by $\overline{C}$. We have that $\overline{C} \leq C$ due to Characterization 1 and Proposition 1. Suppose that $\overline{C} < C$. Hence, there exists an $x < C$ such that the points (t, x) belong to the continuation region near the maturity. Therefore,

$$\begin{aligned} 0 &< \lim_{t \to T} \frac{P(t,x) - N(t,x)}{T-t} \\ &= -\lim_{t \to T} \frac{P(T,x) - P(t,x)}{T-t} + \lim_{t \to T} \frac{N(T,x) - N(t,x)}{T-t} \\ &= \mathcal{A}P(T,x) - rP(T,x) + N_t(T,x) \\ &= (\mathcal{B}G)(x) < 0. \end{aligned} \tag{6}$$

The contradiction leads to $\overline{C} = C$. Similar arguments show that the upper boundary is namely D at the maturity. □

3.3. Perpetual Boundary Values

After we establish that the initial values of the optimal boundaries are namely the constants C and D, we turn to the perpetual ones. To find them, we prove two propositions that characterize the price of derivatives that expire at a certain level. The first proposition is about levels below C, whereas the second one is for the levels above D. Once we have these statements, we can obtain a two-dimensional system that the optimal boundaries have to solve. Note that they are time-independent constants because the holder is not threatened by time expiring.

We shall denote by A and B the perpetual values of the lower and upper boundaries, respectively. We have $A < C \leq D < B$. Suppose that the initial asset value is x and it belongs to the continuation region, i.e., $A < x < B$. We search for these values of A and B that maximize the financial result of the strategy of the first exit from the strip (A, B). Let ζ_A and ζ_B be the first hitting moments of the underlying asset to the levels A and B and $\zeta = \zeta^A \wedge \zeta^B$. These stopping times can be viewed as the first hitting times of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ to the values

$$\begin{aligned} \tilde{A} &= \frac{\ln A - \ln x}{\sigma} < 0 \\ \tilde{B} &= \frac{\ln B - \ln x}{\sigma} > 0. \end{aligned} \tag{7}$$

We shall use a result reported in Borodin and Salminen [59] as Equations (3.0.5 a & b).

Lemma 1. *Let ζ be the first exit of a Brownian motion with drift μ from a strip (a, b). Then, we have*

$$E\left[e^{-y\zeta} I_{\zeta = \zeta^A}\right] = e^{\mu a} \frac{\sinh\left(b\sqrt{2y + \mu^2}\right)}{\sinh\left((b-a)\sqrt{2y + \mu^2}\right)} \tag{8}$$

$$E\left[e^{-y\zeta} I_{\zeta = \zeta^B}\right] = e^{\mu b} \frac{\sinh\left(-a\sqrt{2y + \mu^2}\right)}{\sinh\left((b-a)\sqrt{2y + \mu^2}\right)}. \tag{9}$$

Let the constants p and q be defined as

$$\begin{aligned} p &:= 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \\ q &:= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \frac{r}{\sigma^2} - \frac{1}{2}. \end{aligned} \tag{10}$$

Thus, the strategy ζ leads to the financial result

$$\begin{aligned} f(A,B;x) &= E^x\left[e^{-(r+\lambda)\zeta}G(S_\zeta)\right] \\ &= G(A)E^x\left[e^{-(r+\lambda)\zeta^B}I_{\zeta^B\le\zeta^A}\right] + G(B)E^x\left[e^{-(r+\lambda)\zeta^A}I_{\zeta^A<\zeta^B}\right] \\ &= G(A)e^{\psi\tilde{A}}\frac{\sinh\left(\frac{\sigma p\tilde{B}}{2}\right)}{\sinh\left(\frac{\sigma p(\tilde{B}-\tilde{A})}{2}\right)} + G(B)e^{\psi\tilde{B}}\frac{\sinh\left(-\frac{\sigma p\tilde{A}}{2}\right)}{\sinh\left(\frac{\sigma p(\tilde{B}-\tilde{A})}{2}\right)} \\ &= G(A)e^{(p-q)(\ln x-\ln A)}\frac{e^{p(\ln B-\ln x)}-1}{e^{p(\ln B-\ln A)}-1} + G(B)e^{(q)(\ln B-\ln x)}\frac{e^{p(\ln x-\ln A)}-1}{e^{p(\ln B-\ln A)}-1} \\ &= G(A)\left(\frac{x}{A}\right)^{p-q}\frac{\left(\frac{B}{x}\right)^p-1}{\left(\frac{B}{A}\right)^p-1} + G(B)\left(\frac{B}{x}\right)^q\frac{\left(\frac{x}{A}\right)^p-1}{\left(\frac{B}{A}\right)^p-1} \\ &= G(A)\left(\frac{A}{x}\right)^q\frac{B^p-x^p}{B^p-A^p} + G(B)\left(\frac{B}{x}\right)^q\frac{x^p-A^p}{B^p-A^p}. \end{aligned} \tag{11}$$

We can write the derivatives of Function (11) w.r.t. A and B as

$$\begin{aligned} f_A(A,B;x) &= \frac{A^{q-1}(B^p-x^p)}{x^q(B^p-A^p)^2}\left[\begin{array}{c} (G'(A)A+G(A)q)(B^p-A^p) \\ -A^{p-q}p(G(B)B^q-G(A)A^q) \end{array}\right] \\ f_B(A,B;x) &= \frac{B^{q-1}(x^p-A^p)}{x^q(B^p-A^p)^2}\left[\begin{array}{c} (G'(B)B+G(B)q)(B^p-A^p) \\ -B^{p-q}p(G(B)B^q-G(A)A^q) \end{array}\right]. \end{aligned} \tag{12}$$

Let us first fix the value of the boundary $A < C$. The following proposition for the price function $f(A,B;x)$ holds.

Proposition 5. *Let the boundary A be fixed and let us examine price (11) as a function of B—we write $f(B;x)$.*

1. *If for some $x > A$ the function $f(B;x)$ has a local maximum or minimum at a point $\overline{B} > x$, then it has the same local extremum at the point $\overline{B}$ for all $x \in \left(A,\overline{B}\right)$.*
2. *If for some $x > A$ the function $f(B;x)$ has a local maximum at a point $\overline{B} > x$, then $(\mathcal{B}G)\left(\overline{B}\right) < 0$.*
3. *If for some $x > A$ the function $f(B;x)$ has a local maximum at a point $\overline{B} > x$, then $\overline{B} > D$.*
4. *If the function $f(B;x)$ has a local minimum at a point $\overline{B}$, then $(\mathcal{B}G)\left(\overline{B}\right) > 0$.*
5. *The function $f(B;x)$ has no more than one local maximum.*

Proof. Let us denote by $\zeta^{A,B}$ the first exit of the underlying asset from the strip (A,B).

1. This statement holds due to the form of the derivative $f_B(B;x)$ given in Equations (12)—its root w.r.t. B is independent of x.

2. Suppose that the function $f(B;x)$ has a local maximum at a point $\overline{B}$. Therefore, there exist small enough but positive constants ϵ and δ such that $f(\overline{B};x) > f(\overline{B}+\epsilon;x) > f(\overline{B}-\delta;x)$. Using Dynkin's formula, we derive

$$\begin{aligned} G(\overline{B}) &= \lim_{x\uparrow\overline{B}} f(\overline{B};x) > \lim_{x\uparrow\overline{B}} f(B+\epsilon;x) = f(B+\epsilon;\overline{B}) \\ &= E^{\overline{B}}\left[e^{-(r+\lambda)\zeta^{A,B+\epsilon}} G\left(S_{\zeta^{A,B+\epsilon}}\right)\right] \\ &= G(\overline{B}) + E^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} (\mathcal{B}G)(S_u)du\right] \end{aligned} \tag{13}$$

and, therefore,

$$E^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} (\mathcal{B}G)(S_u)du\right] < 0. \tag{14}$$

Inequality (14) leads to

$$\begin{aligned} 0 &> E^{\overline{B}}\left[E^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} (\mathcal{B}G)(S_u)du \,\middle|\, \mathcal{F}_{\zeta^{\overline{B}-\delta,\overline{B}+\epsilon}}\right]\right] \\ &= E^{\overline{B}}\left[\int_0^{\zeta^{\overline{B}-\delta,\overline{B}+\epsilon}} (\mathcal{B}G)(S_u)du\right] + E^{\overline{B}-\delta}\left[\int_0^{\zeta^{A,\overline{B}+\epsilon}} (\mathcal{B}G)(S_u)du\right]. \end{aligned} \tag{15}$$

On the other hand, similarly to inequality (14), we can prove

$$E^{\overline{B}-\delta}\left[\int_0^{\zeta^{A,\overline{B}+\epsilon}} (\mathcal{B}G)(S_u)du\right] > 0. \tag{16}$$

Combining inequalities (15) and (16), we conclude that $(\mathcal{B}G)(x)$ has to be negative in a small enough neighborhood of $\overline{B}$. Hence, $(\mathcal{B}G)(\overline{B}) < 0$.

3. Suppose the opposite, i.e., that the function $f(B;x)$ has a local maximum at a point $\overline{B}$ such that $D > \overline{B} > x$. The previous statement shows that $\overline{B} \leq C$. Hence, there exists a small enough constant ϵ such that $f(\overline{B};x) > G(\overline{B}-\epsilon)$. Using Dynkin's formula again, we see that

$$\begin{aligned} G(\overline{B}-\epsilon) &< f(\overline{B};\overline{B}-\epsilon) = E^{\overline{B}-\epsilon}\left[e^{-(r+\lambda)\zeta^{A,\overline{B}}} G\left(S_{\zeta^{A,\overline{B}}}\right)\right] \\ &= G(\overline{B}-\epsilon) + E^{\overline{B}-\epsilon}\left[\int_0^{\zeta^{A,\overline{B}}} (\mathcal{B}G)(S_u)du\right] < G(\overline{B}-\epsilon). \end{aligned} \tag{17}$$

The last equation is true due to $\overline{B} \leq C$ and Characterization 1. The contradiction proves the third statement.

4. Some similar arguments to those presented for the second statement prove the desired result.

5. If there exist two local maxima in the interval (D,∞), then there exists a local minimum in this interval. The previous statement means that the value of $(\mathcal{B}G)(u)$ is positive in it, which is impossible due to Characterization 1.

□

Similar arguments support the analog of Proposition 5 when the upper boundary $B > D$ in Function (11) is fixed. We now consider (11) as a function of the lower boundary A, $f(A; x)$.

Proposition 6.

1. *If the function $f(A; x)$ has a local extremum at a point $\overline{A} < x$ for some $x < B$, then it has the same local extremum at the same point $\overline{A}$ for all $x \in (\overline{A}, B)$.*
2. *If the function $f(A; x)$ has a local maximum at a point $\overline{A}$, then $(\mathcal{B}G)(\overline{A}) < 0$.*
3. *If the function $f(A; x)$ has a local maximum at a point $\overline{A}$, then $\overline{A} < C$.*
4. *If the function $f(A; x)$ has a local minimum at a point $\overline{A}$, then $(\mathcal{B}G)(\overline{A}) > 0$.*
5. *The function $f(A; x)$ has no more than one local maximum.*

Propositions 5 and 6 and derivatives (12) show that we can derive the optimal boundaries as the solution to the following system:

$$\begin{aligned} &(G'(A)A + G(A)q)(B^p - A^p) - A^{p-q}p(G(B)B^q - G(A)A^q) = 0 \\ &(G'(B)B + G(B)q)(B^p - A^p) - B^{p-q}p(G(B)B^q - G(A)A^q) = 0 \end{aligned} \tag{18}$$

in the domain of $A < C$ and $B > D$. Note that if the solution exists, then it is unique. The system (18) can be rewritten as

$$\begin{aligned} \left(\frac{A}{B}\right)^q &= \frac{G'(B)B - (p-q)G(B)}{G'(A)A - (p-q)G(A)} \\ \left(\frac{A}{B}\right)^{p-q} &= \frac{G'(A)A + qG(A)}{G'(B)B + qG(B)}. \end{aligned} \tag{19}$$

Once we derive the solution to system (19) as the pair (A, B), we obtain the derivative's price, estimating it in Formula (12).

3.4. Finite Maturities

Now, we present the algorithm for approximating the optimal boundaries when the maturity is finite. Suppose that we have the time grid $0 \equiv t_0 < t_1 < t_2 < \ldots < t_n \equiv T$ and two continuous piecewise linear functions w.r.t. it $a(t) < b(t)$:

$$\begin{aligned} a(t) &= \sum_{i=1}^{n} a_i(t) I_{t \in [t_{i-1}, t_i]} \equiv \sum_{i=1}^{n} (a_{1,i}t + a_{2,i}) I_{t \in [t_{i-1}, t_i]} \\ b(t) &= \sum_{i=1}^{n} b_i(t) I_{t \in [t_{i-1}, t_i]} \equiv \sum_{i=1}^{n} (b_{1,i}t + b_{2,i}) I_{t \in [t_{i-1}, t_i]}, \end{aligned} \tag{20}$$

$a_i(t_i) = a_{i+1}(t_i)$ and $b_i(t_i) = b_{i+1}(t_i)$, $i = 1, 2, \ldots, n-1$. Also, let $a(0) < 0 < b(t)$. We shall approximate the optimal boundaries as exponents of such functions—$c(t) = \exp(a(t))$ and $d(t) = \exp(b(t))$. We denote the values at the grid nodes by $A_i = c(t_i)$ and $B_i = d(t_i)$, $i = 0, 1, \ldots, n$. The derivative price can be written as

$$P(x, t_1, \ldots t_n; A_1, \ldots, A_n; B_1, \ldots, B_n) = E^x\left[e^{-(r+\lambda)\zeta} G(S_{\zeta \wedge T})\right]. \tag{21}$$

Remark 1. *Of course, the expectation in Formula (21) cannot be derived in a closed form for every choice of the payoff $G(\cdot)$. Alternatively, it can be found via the Kolmogorov backward equation or using some Monte Carlo simulations. On the other hand, a large part of the useful payoff functions admit closed-form formulas—see, for example, Section 5.*

We shall apply the following backward algorithm.

1. The boundaries at the maturities are the constants C and D. Thus, $A_n = C$ and $B_n = D$.
2. Suppose that we have obtained all values $A_m, A_{m+1}, \ldots, A_n$ and $B_m, B_{m+1}, \ldots, B_n$ for some $m \leq n$.
3. We approximate the lower boundary A_{m-1} in the following way. For fixed constants $A < x \leq A_m$, we define $B(x, A)$ as the maximizer of

$$P(x; 0, t_m - t_{m-1}, \ldots t_n - t_{m-1}; A, A_m, \ldots, A_n; B, B_m, \ldots, B_n) \tag{22}$$

amongst all $B > B_m$. We now consider (22) as a function of A and we denote by $A(x)$ the maximizer of

$$P(x; 0, t_m - t_{m-1}, \ldots t_n - t_{m-1}; A, A_m, \ldots, A_n; B(x, A), B_m, \ldots, B_n), \tag{23}$$

that we find having in mind Remark 1. We approximate A_{m-1} as the largest x for which $x = A(x)$—it is the largest optimal value in the lower segment for the underlying asset.
4. Analogously, we obtain B_{m-1}. Let it, for the fixed constants $x < B$, $A(x, B)$, be the maximizer of function (22) w.r.t. the variable A. Also, let $B(x)$ maximize

$$P(x; 0, t_m - t_{m-1}, \ldots t_n - t_{m-1}; A(x, B), A_m, \ldots, A_n; B, B_m, \ldots, B_n) \tag{24}$$

amongst all $B > x$. Thus, we approximate the boundary B_{m-1} as the smallest x for which $x = B(x)$.

4. Pricing

Once we approximate the optimal boundaries, we can view the derivative's evaluation as a boundary value problem (BVP) in the region $(t, x) \in \{(0, T) \times (c(t), d(t))\}$. If the initial point is outside, then the price is $e^{-\lambda t} G(x)$. We can write the BVP as

$$\begin{aligned} & P_t(t, x) + rxP_x(t, x) + \frac{1}{2}\sigma^2 x^2 P_{xx}(t, x) - rP(t, x) = 0 \\ & P(t, c(t)) = e^{-\lambda t} G(x), \ \ t \in (0, T) \\ & P(t, d(t)) = e^{-\lambda t} G(x), \ \ t \in (0, T) \\ & P(T, x) = e^{-\lambda T} G(x), \ \ x \in (C, D). \end{aligned} \tag{25}$$

We present below the Crank–Nicolson finite difference approach applied to BVP (25).

1. We divide the time-state space as $T \equiv t_1 > t_2 > \ldots > t_M \equiv 0$ and $0 \equiv x_1 < x_2 < \ldots < x_N \equiv c(0)$. We denote the price at the (m, n)-th node by $P(m, n)$. The values $P(1, n)$ and $P(M, n)$ are the prices at the maturity and in the initial moment, respectively, since we work backward.
2. We approximate the boundaries $c(t)$ and $d(t)$ at $\overline{M} << M$ using the results of Section 3.4. Next, we interpolate the whole boundaries using cubic splines.
3. The terminal condition can be written as

$$P(1, n) = e^{-\lambda T} G(x_n). \tag{26}$$

4. Let us denote by l_m the highest n such that $x_{l_m} < A_m$. Analogously, let k_m be the lowest n such that $x_{k_m} > B_m$.
5. The lower and upper boundary conditions appear as

$$P(m, n) = e^{-\lambda t_m} G(x_n) \ \forall \ m \ and \ n \leq l_m \ or \ n \geq k_m. \tag{27}$$

6. We obtain the values of $P(m, n)$ via the Crank–Nicolson scheme, iteratively using the already-derived $P(i, n)$ for all n and $i < m$. The derivatives can be approximated

via Appendix A, Formula (A1). Thus, the BVP (25) can be written as Equation (A2). Rearranging w.r.t. $n \in \{l_m + 1, \dots, k_m - 1\}$, we find a linear system for $P(m, n)$, namely Equations (A3)–(A5).

Finally, we formulate a useful proposition that gives the time dependence of the price.

Proposition 7. *If we mark the dependence on the maturity, then the following relation holds:*

$$P(t, T, x) = e^{-\lambda t} P(0, T - t, x). \tag{28}$$

Proof. The proof is very similar to proposition 2.1 of Zaevski [47] and we omit it. □

Remark 2. *Note that Proposition 7 still holds if one of the boundaries vanishes or if the option is European-style.*

5. Quadratic Options

Before defining and investigating the quadratic options, we shall summarize the above-presented approach.

We turn now to a specific class of American-style derivatives—we name them quadratic strangle options. The payoff is defined by the function

$$G(x) = (x - K)^2. \tag{29}$$

We shall call the constant K strike and we assume that it is positive. At the end of this section, we briefly discuss what changes if $K \leq 0$. First, we derive the price of the corresponding European option.

Proposition 8. *The price of the European-style quadratic option at the initial moment is*

$$P(S_0) = S_0^2 e^{\left(r+\sigma^2-\lambda\right)T} - 2KS_0 e^{-\lambda T} + K^2 e^{-(r+\lambda)T}. \tag{30}$$

Proof. Using the risk-neutral pricing principle, we derive

$$\begin{aligned} P(S_0) &= E\left[e^{-(r+\lambda)T}(S_T - K)^2\right] \\ &= e^{\left(r+\sigma^2-\lambda\right)T} E\left[e^{-\left(2r+\sigma^2\right)T} S_T^2\right] - 2Ke^{-\lambda T} E\left[e^{-rT} S_T\right] + K^2 e^{-(r+\lambda)T}. \end{aligned} \tag{31}$$

We finish the proof having in mind that $e^{-\left(2r+\sigma^2\right)T} S_T^2$ and $e^{-rT} S_T$ are martingales. □

We are ready to apply Algorithm 1 to the pricing task for the American quadratic strangles. The value of the operator $\mathcal{B}$ applied to function (29) is

$$(\mathcal{B}G)(x) = x^2\left(r + \sigma^2 - \lambda\right) + 2\lambda K x - (r + \lambda)K^2. \tag{32}$$

The discriminant of this quadratic function is positive since it can be written as $K^2\left(r^2 + \sigma^2(r + \lambda)\right) > 0$. We now have to separately examine the cases w.r.t. the sign of $\left(r + \sigma^2 - \lambda\right)$ or, equivalently, w.r.t. the position of the discount rate λ to the value of $r + \sigma^2$.

We need the following lemmas before continuing the financial analysis. They are necessary to distinguish what kind of optimal stopping problem describes the option pricing task—one- or two-sided. The first one is related to the Brownian motion hitting a boundary, whereas the second one leads to an exit from a strip. Next, we examine these cases separately since they lead to different mathematical tasks.

Algorithm 1 Approach for pricing the studied financial instruments.

Our approach is based on the following steps.

1. We apply the related to the infinitesimal generator differential operator $\mathcal{B}$, defined in Formula (2), to the payoff $G(x)$. Thus, we obtain the function $g(\cdot)$ as $g(x) = (\mathcal{B}G)(x)$.
2. We check whether the function $g(x)$ satisfies Characterization 1, which guarantees that the American pricing task leads to a two-sided optimal stopping problem.
3. If the conditions of Characterization 1 are satisfied, then we obtain the constants $C < D$ such that $g(x) < 0$ for $x \in (0, C) \cup (D, \infty)$, and $g(x) > 0$ for $x \in (C, D)$.
4. We set the initial values of the optimal boundary to C and D, i.e., $c(0) = C$ and $d(0) = D$.
5. We obtain the boundary values at infinity, $c(\infty)$ and $d(\infty)$, as the solution to the two-dimensional system established in (19).
6. The perpetual price is obtained via Formula (11) using $A = c(\infty)$ and $B = d(\infty)$.
7. Once we have the endpoints of the functions $c(\cdot)$ and $d(\cdot)$, we approximate the whole boundaries, applying the algorithm provided in Section 3.4.
8. When the optimal boundaries $c(\cdot)$ and $d(\cdot)$ are known, the free boundary differential task that describes the pricing problem turns into a boundary value problem (25).
9. We solve it numerically, using the Crank–Nicolson finite difference approach presented in Section 4.

Lemma 2. *The constants $\lambda - r - \sigma^2$ and $p - q - 2$ have the same signs (the constants p and q are defined by Equation (10)). Moreover, if $\lambda = r + \sigma^2$, then $q = 2\frac{r}{\sigma^2} + 1$.*

Proof. Suppose first that $\lambda > r + \sigma^2$. The inequality $p - q - 2 > 0$ is equivalent to

$$\sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}} > \left(\frac{r}{\sigma^2} + \frac{1}{2}\right) + 1. \tag{33}$$

Statement (33) holds when the right-hand side is negative. If it is positive, then we reach the desired result by rising at the second power.

In contrast, if $r + \sigma^2 \geq \lambda$, then $\frac{r}{\sigma^2} + \frac{3}{2} > 0$, thus we can raise on the second power.

The second part holds due to the presentation

$$\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2} = \left(\frac{r}{\sigma^2} + \frac{3}{2}\right)^2 \tag{34}$$

when $\lambda = r + \sigma^2$. We finish the proof having in mind $\frac{r}{\sigma^2} + \frac{3}{2} > \frac{r}{\sigma^2} + 1 = \frac{\lambda}{\sigma^2} > 0$. □

Lemma 3. *Let $0 < a < 1$, $0 < \epsilon < n$, and $0 < \delta < m$. The following inequality holds:*

$$(m + \delta)\left(1 - a^{n-\epsilon}\right) + (m - \delta)\left(1 - a^{n+\epsilon}\right) < 2m(1 - a^n). \tag{35}$$

Proof. Inequality (35) is equivalent to

$$\delta a^{n-\epsilon}\left(1 - a^{2\epsilon}\right) + m a^{n-\epsilon}(1 - a^{\epsilon})^2 > 0. \tag{36}$$

□

5.1. Negative Quadratic Coefficient

Suppose first that $\lambda > r + \sigma^2$. Characterization 1 holds, because the vertex of quadratic function (32) is at the positive point $\frac{\lambda K}{\lambda - r - \sigma^2}$ and $(\mathcal{B}G)(0) = -(r + \lambda)K^2 < 0$. Under this assumption, we have a two-sided optimal stopping problem related to the first exit of the Brownian motion from a strip. First, we shall consider quadratic options without maturity constraints. We transform the two-dimensional system (19) that the optimal boundaries solve into a polynomial-style equation, and then we prove the uniqueness of its solution.

We can easily solve this equation numerically. As a consequence, we derive the option price. After that, we examine finite maturity options, adapting the general approach presented in Section 3.4 to the quadratic options. This way, the free boundary task turns into a boundary value problem in a known region, to which we apply the Crank–Nicolson finite difference scheme, presented in Section 4.

The constants C and D that determine the initial values of the optimal boundaries are the roots of (32), namely

$$\{C, D\} = K\frac{\lambda \mp \sqrt{r^2 + \sigma^2(r+\lambda)}}{\lambda - r - \sigma^2}. \tag{37}$$

Equation (18), which determines the perpetual boundaries, turns into

$$\begin{aligned} \left(\frac{A}{B}\right)^q &= \frac{B-K}{A-K}\,\frac{(p-q)(B-K)-2B}{(p-q)(A-K)-2A} \\ \left(\frac{A}{B}\right)^{p-q} &= \frac{A-K}{B-K}\,\frac{2A+q(A-K)}{2B+q(B-K)}. \end{aligned} \tag{38}$$

We shall use the notations $a = \frac{A}{B}$ and $x = \frac{K}{B}$. Note that $a < x < 1$ and $\frac{x}{a} > 1$. System (38) can be rewritten as

$$\begin{aligned} g(x) &:= x^2(p-q)(1-a^q) - 2x(p-q-1)\left(1-a^{q+1}\right) + (p-q-2)\left(1-a^{q+2}\right) = 0 \\ h\left(\frac{x}{a}\right) &:= \left(\frac{x}{a}\right)^2 q(1-a^{p-q}) - 2\left(\frac{x}{a}\right)(q+1)\left(1-a^{p-q-1}\right) + (q+2)\left(1-a^{p-q-2}\right) = 0. \end{aligned} \tag{39}$$

The following proposition stands.

Proposition 9. *System (39) has a unique solution such that $\{x, a\} \in (0,1)$ and $\frac{x}{a} > 1$.*

Proof. Let us use the notation $y = \frac{x}{a}$. We first consider the function $g(x)$. We have that $g(0) > 0$ and $g(1) < 0$ due to Lemmas 2 and 3. Hence, the equation $g(x) = 0$ has two roots such that $0 < x_1 < 1 < x_2$. Analogously, the roots of the equation $h(y) = 0$ satisfy the same order—$0 < y_1 < 1 < y_2$. Marking the dependence on the variable a, we need to prove that there exists a unique value $a \in (0,1)$ such that $m(a) = 0$ for $m(a) = x_1(a) - ay_2(a)$. The existence follows the inequalities

$$\begin{aligned} m(0) &= 1 - \frac{2}{p-q} > 0 \\ \lim_{a\to 1} m(a) &= -2\frac{\sqrt{(p-q-1)^2(q+1)^2 - q(p-q)(p-q-2)(q+2)}}{q(p-q)} < 0. \end{aligned}$$

To prove the uniqueness, we rewrite system (39) as

$$\begin{aligned} a^{q+2} &= \frac{f(ay)}{f(y)} \\ a^{p-q-2} &= \frac{F(y)}{F(ay)}. \end{aligned} \tag{40}$$

for

$$\begin{aligned} f(y) &= y^2(p-q) - 2y(p-q-1) + (p-q-2) \\ F(y) &= y^2 q - 2y(q+1) + q + 2. \end{aligned} \tag{41}$$

Let us consider the function $f(y)$. Its roots are $1 - \frac{2}{p-q}$ and 1. We use a notation w.r.t. the variable y, i.e., the pair $\{a_1(y), y\}$ which solves the first equation from (40). We need the values $y > 1$ and $ay = x < 1$. Therefore, $f(y) > 0$. Hence, $f(ya_1(y)) > 0$, too, or, equivalently, $a_1(y)y < 1 - \frac{2}{p-q}$. Hence, $f(y)$ increases w.r.t. y, but $f(ay)$ decreases for a fixed a in the domain $ay < 1$. Therefore, $a_1(y)$ decreases. If we consider that the second equation from (40) w.r.t. y, i.e., $\{a_2(y), y\}$, solves it, we can conclude that $a_2(y)$ is an increasing function because the roots of function $F(y)$ are 1 and $1 + \frac{2}{q}$. Hence, the equation $a_1(y) = a_2(y)$ has no more than one root. This finishes the proof. $\square$

Proposition 9 shows that the equation $x_1(a) = ay_2(a)$ has a unique solution. Hence, we derive the perpetual optimal boundaries via the following theorem.

Theorem 1. *Let $\overline{a}$ be the solution to*

$$\begin{aligned}&\frac{(p-q-1)(1-a^{q+1}) - \sqrt{(p-q-1)^2(1-a^{q+1})^2 - (p-q)(p-q-2)(1-a^q)(1-a^{q+2})}}{(p-q)(1-a^q)}\\ &= a\frac{(q+1)(1-a^{p-q-1}) + \sqrt{(q+1)^2(1-a^{p-q-1})^2 - q(q+2)(1-a^{p-q})(1-a^{p-q-2})}}{q(1-a^{p-q})}\end{aligned} \tag{42}$$

in the interval $(0,1)$ and $\overline{x}$ be defined as

$$\overline{x} = \frac{(p-q-1)\left(1-\overline{a}^{q+1}\right) - \sqrt{(p-q-1)^2\left(1-\overline{a}^{q+1}\right)^2 - (p-q)(p-q-2)(1-\overline{a}^q)(1-\overline{a}^{q+2})}}{(p-q)(1-\overline{a}^q)}. \tag{43}$$

The optimal boundaries of a perpetual quadratic strangle are $\overline{A} = \frac{\overline{a}}{\overline{x}}K$ and $\overline{B} = \frac{K}{\overline{x}}$. The derivative price is

$$f(\overline{A}, \overline{B}; x) = (x - \overline{A})^2\left(\frac{\overline{A}}{x}\right)^q \frac{\overline{B}^p - x^p}{\overline{B}^p - \overline{A}^p} + (x - \overline{B})^2\left(\frac{\overline{B}}{x}\right)^q \frac{x^p - \overline{A}^p}{\overline{B}^p - \overline{A}^p}. \tag{44}$$

Suppose now that the maturity is finite. We shall use the algorithm presented in Section 3.4. To do this, we need to derive the expectation in Formula (21). We have

$$\begin{aligned}
P(x, T; c(t), d(t)) &= E^x\left[e^{-(r+\lambda)\zeta^A}\left(S_{\zeta^A} - K\right)^2 I_{\zeta^A=\zeta, \zeta<T}\right]\\
&+ E^x\left[e^{-(r+\lambda)\zeta^B}\left(S_{\zeta^B} - K\right)^2 I_{\zeta^B=\zeta,\zeta<T}\right] + E^x\left[e^{-(r+\lambda)T}(S_T - K)^2 I_{T\le\zeta}\right]\\
&= K^2\sum_{i=1}^n\left(E\left[e^{-(r+\lambda)\zeta^A} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right] + E\left[e^{-(r+\lambda)\zeta^B} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&- 2Kx\sum_{i=1}^n\left(e^{\sigma a_{2,i}} E\left[e^{-\psi_{1,i}\zeta^A} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right] + e^{\sigma b_{2,i}} E\left[e^{-\psi_{2,i}\zeta^B} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&+ x^2\sum_{i=1}^n\left(e^{2\sigma a_{2,i}} E\left[e^{-\eta_{1,i}\zeta^A} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right] + e^{2\sigma b_{2,i}} E\left[e^{-\eta_{2,i}\zeta^B} I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&+ K^2 e^{-(r+\lambda)T} Q(v_1 < B_T < v_2, T \le \zeta) - 2Kxe^{-\psi_3 T} E\left[e^{\sigma B_T} I_{v_1<B_T<v_2,T\le\zeta}\right]\\
&+ x^2 e^{-\psi_4 T} E\left[e^{2\sigma B_T} I_{v_1<B_T<v_2,T\le\zeta}\right],
\end{aligned} \tag{45}$$

where

$$
\begin{aligned}
\psi_{1,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma a_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma a_{1,i} \\
\psi_{2,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma b_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma b_{1,i} \\
\eta_{1,i} &= (r+\lambda) - 2\left(\left(r - \frac{\sigma^2}{2}\right) - \sigma a_{1,i}\right) = \lambda + \sigma^2 - r - 2\sigma a_{1,i} \\
\eta_{2,i} &= (r+\lambda) - 2\left(\left(r - \frac{\sigma^2}{2}\right) - \sigma b_{1,i}\right) = \lambda + \sigma^2 - r - 2\sigma b_{1,i} \\
\psi_3 &= \lambda + \frac{\sigma^2}{2} \\
\psi_4 &= \lambda + \sigma^2 - r \\
v_1 &= \frac{1}{\sigma}\ln\left(\frac{C}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T \\
v_2 &= \frac{1}{\sigma}\ln\left(\frac{D}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T.
\end{aligned}
\tag{46}
$$

The expectations in statement (45) can be derived through the results of Zaevski [60]. Note that for some values of the parameters, we need to use the analytic continuation of the erf-function—see, for example, Abramowitz and Stegun [61]. Based on Formulas (45) and (46), we approximate the optimal boundaries using the algorithm established in Section 3.4. Once we have these approximations, we apply the Crank–Nicolson finite difference approach presented in Section 4 to evaluate the derivative.

5.2. Non-Negative Quadratic Coefficient

Suppose now that $\lambda \le r + \sigma^2$. Let the constant C be defined as

$$C = \tfrac{r+\lambda}{2\lambda}K \quad if\ \lambda = r + \sigma^2 \tag{47}$$

$$C = \frac{\sqrt{r^2+\sigma^2(r+\lambda)}-\lambda}{r+\sigma^2-\lambda}K \quad if\ \lambda < r + \sigma^2. \tag{48}$$

We have that $(\mathcal{B}G)(x) < 0$ for $x < C$ and $(\mathcal{B}G)(x) > 0$ when $x > C$, where $(\mathcal{B}G)$ is given by Equation (32). Hence, we have a one-sided optimal stopping problem. It is put-style in the sense that the optimal points are below the exercise boundary. We shall use a method established in Zaevski [55] to examine such kinds of financial instruments. First, we shall consider the perpetual options deriving the optimal boundary and, as a consequence, the fair option price. After that, we shall approximate the optimal boundary for the finite maturity options and shall apply the finite-difference approach presented in Section 4 to the pricing task.

Propositions 5 and 6 from Zaevski [55] show that the optimal boundary starts from the value C and decreases to its perpetual level. We shall derive it now. Suppose that $S_0 = x$ is a large enough initial point for the underlying asset. Suppose also that the holder exercises when the underlying asset reaches the value $A < x$. This stopping time, which we denote by ζ^A, is the first hitting moment of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ to the value

$$\tilde{A} = \frac{\ln A - \ln x}{\sigma} < 0. \tag{49}$$

Using the equation reported in Borodin and Salminen [59] (p. 223), (2.0.1),

$$E\left[e^{-y\zeta^A} I_{\zeta^A<\infty}\right] = e^{\left(\sqrt{\psi^2+2y}+\psi\right)\tilde{A}}, \tag{50}$$

we derive, for the financial result of this strategy,

$$\begin{aligned}P(x;A) &= E^x\left[e^{-(r+\lambda)\zeta^c}\left(S_{\zeta^A}-K\right)^2 I_{\zeta^A<\infty}\right] + \lim_{T\to\infty} E^x\left[e^{-(r+\lambda)T}(S_T-K)^2 I_{T<\zeta^A}\right]\\ &= (A-K)^2 E^x\left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A<\infty}\right] + \lim_{T\to\infty} e^{-(r+\lambda)T} E^x\left[(S_T-K)^2 I_{T<\zeta^A}\right]\\ &= (A-K)^2\left(\frac{A}{x}\right)^q + K^2 \lim_{T\to\infty} e^{-(r+\lambda)T} Q\left(T<\zeta^A\right)\\ &- 2Kx \lim_{T\to\infty} e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T} E^x\left[e^{\sigma B_T} I_{T<\zeta^A}\right] + x^2 \lim_{T\to\infty} e^{-(\lambda+\sigma^2-r)T} E^x\left[e^{2\sigma B_T} I_{T<\zeta^A}\right].\end{aligned} \tag{51}$$

We now need the following statement for the expectations above—its proof can be found in theorem 3.2 from Zaevski [62]:

$$E^x\left[e^{\theta B_T} I_{T<\zeta}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} 1-N\left(\frac{-\psi T+\tilde{A}-T\theta}{\sqrt{T}}\right)\\ -e^{2\tilde{A}(\theta+\psi)}\left(1-N\left(\frac{-\psi T+\tilde{A}-T\theta-2\tilde{A}}{\sqrt{T}}\right)\right)\end{array}\right]. \tag{52}$$

The first limit in price function (51) is zero. The second one is zero, too, when $\lambda>0$, and finite when $\lambda=0$. Also, the third limit is the infinity when $\lambda<r+\sigma^2$ and, therefore, $P(x;A)=\infty$. This means that optimal exercising is never optimal, and, hence, $c(\infty)=0$.

Suppose now that $\lambda=r+\sigma^2$. Using Lemma 2, we see that the third limit is

$$\left(1-\left(\frac{A}{x}\right)^{q+2}\right).$$

Thus, the price function (51) turns into

$$\begin{aligned}P(x;A) &= (A-K)^2\left(\frac{A}{x}\right)^q + x^2\left(1-\left(\frac{A}{x}\right)^{q+2}\right)\\ &= \frac{K^2A^q-2KA^{q+1}}{x^q}+x^2.\end{aligned} \tag{53}$$

Some calculations show that function (53) achieves its maximum for

$$\overline{A} = \frac{q}{2(q+1)}K. \tag{54}$$

We formulate these results in the following theorem.

Theorem 2. *If $\lambda<r+\sigma^2$, then the early exercise is never optimal for a perpetual quadratic strangle. Its price is infinitely large.*
If $\lambda=r+\sigma^2$, then all points below $\overline{A}$, given by Formula (54), are optimal. The price is $(x-K)^2$ when the initial asset value $S_0=x$ is below $\overline{A}$, and it is given by Equation (53) for $A=\overline{A}$.

Next, we briefly discuss an approach for approximating the optimal boundary when the maturity is finite—for more details, see Zaevski [55]. Note that the early exercise can be optimal even if $\lambda<r+\sigma^2$ under the finite maturity horizon. We shall again use a

one-sided approximation that is similar to those presented in Section 5.1. Pricing Formula (45) now turns into

$$
\begin{aligned}
P(x,T;c(t)) &= E^x\left[e^{-(r+\lambda)\zeta^A}\left(S_{\zeta^A}-K\right)^2 I_{\zeta^A<T}\right] + E^x\left[e^{-(r+\lambda)T}(S_T-K)^2 I_{T\le\zeta^A}\right] \\
&= K^2\sum_{i=1}^{n} E\left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right] - 2Kx\sum_{i=1}^{n} e^{\sigma a_{2,i}} E\left[e^{-\psi_{1,i}\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right] \\
&+ x^2\sum_{i=1}^{n} e^{2\sigma a_{2,i}} E\left[e^{-\eta_{1,i}\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right] + K^2 e^{-(r+\lambda)T} Q\left(v_1<B_T, T\le\zeta^A\right) \\
&- 2Kxe^{-\psi_3 T} E\left[e^{\sigma B_T} I_{v_1<B_T,T\le\zeta^A}\right] + x^2 e^{-\psi_4 T} E\left[e^{2\sigma B_T} I_{v_1<B_T,T\le\zeta^A}\right],
\end{aligned}
\tag{55}
$$

where the constants $\psi_{1,i}$, $\eta_{1,i}$, v_1, ψ_3, and ψ_4 are given again in (46). We derive the expectations in Formula (55) using the results of Zaevski [62]. The algorithm for approximating the boundary is again backward. The initial value is given by Formulas (47) or (48). Let us consider the price (55) as a function of the initial asset price and the boundary level at the current moment, namely $P(x,A)$. We denote by $A(x)$ the maximizer of the function $P(x,A)$ for a fixed x. Our approximation is the largest value of x for which $x = A(x)$, i.e., the largest initial value for which the immediate exercise is the optimal strategy.

Once we approximate the optimal boundary, we again use the Crank–Nicolson finite difference scheme to price the quadratic strangle. We require a small modification to the method presented in Section 4 since the continuation region is open above. We introduce a large enough auxiliary upper boundary at which we approximate the function as the price of the European-style option. To do this, we use Proposition 8 together with Proposition 7. Some numerical tests show that an appropriate level for this auxiliary boundary is $10K$.

5.3. Negative Strikes

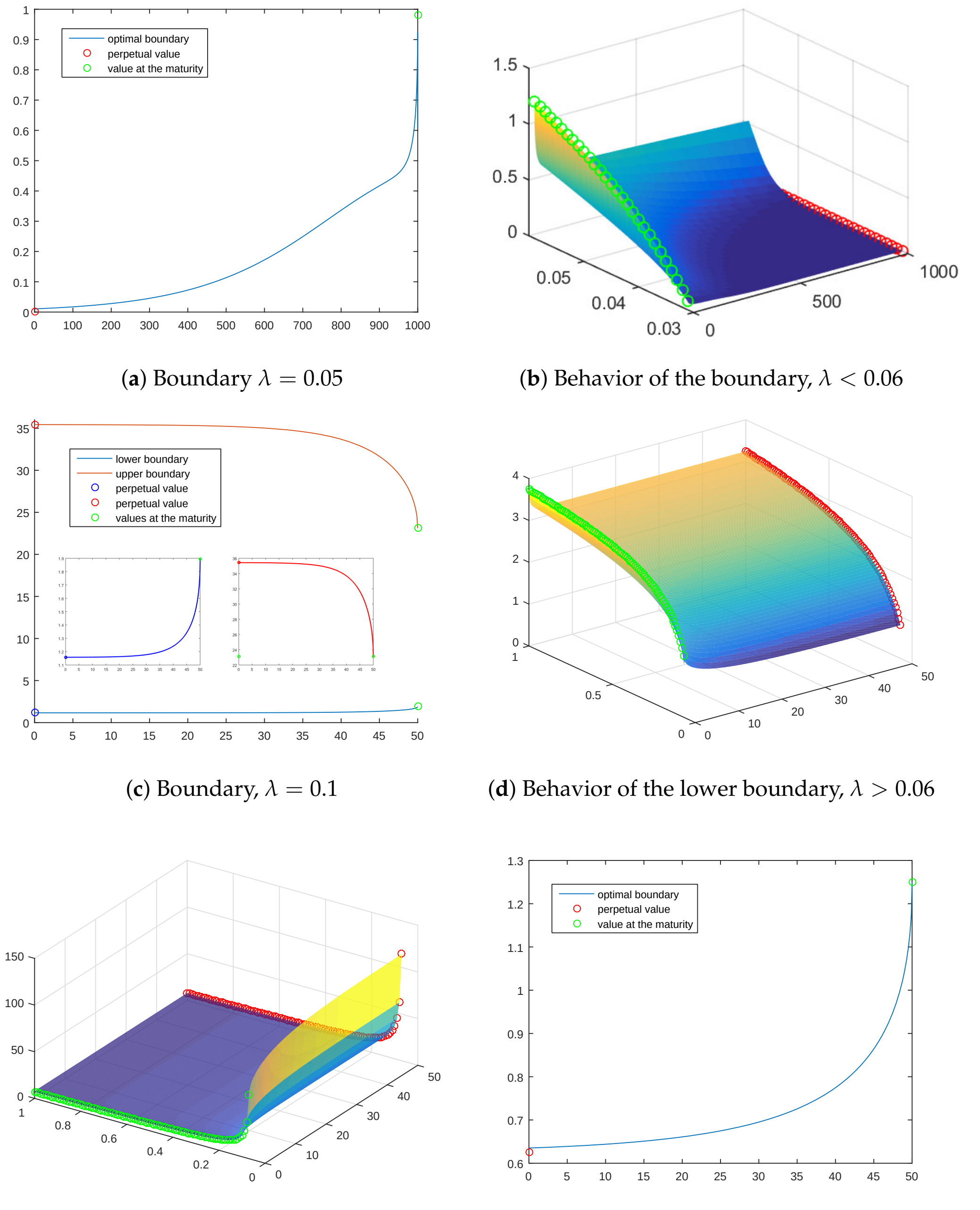
Suppose now that $K \le 0$. If $\lambda < r+\sigma^2$, then function (32) has two roots, and $x_1 < x_2$, such that $(\mathcal{B}G)(x) < 0$ for $x \in (0,x_2)$ and $(\mathcal{B}G)(x) > 0$ when $x \in (x_2,\infty)$. Hence, we have a one-sided problem, and the results presented in Section 5.2 still hold.

If $\lambda > r+\sigma^2$, then function (32) has two negative roots. Also, if $\lambda = r+\sigma^2$, then function (32) has a unique, also negative, root. In both cases, $(\mathcal{B}G)(x) < 0$ for all $x > 0$. Hence, the immediate exercise is never optimal.

6. Numerical Results

We further provide some numerical results related to the above-defined strangle quadratic options. Let the risk-free rate be $r = -0.03$, the volatility be $\sigma = 0.3$, and the strike be $K = 5$. The critical value for the additional discount factor that determines the type of the quadratic option is $\overline{\lambda} = r+\sigma^2 = 0.06$. This way, we have a two-sided optimal stopping problem for $\lambda > 0.06$ and a one-sided task for $\lambda \le 0.06$.

Let us first consider $\lambda < 0.06$. We present in Figure 2a the behavior of the optimal boundary w.r.t. the initial time. We use the discount rate $\lambda = 0.05$. We can see that for large maturities, the optimal boundary tends to zero. This confirms the result derived in Section 5.2—the early exercise is never optimal for the perpetual options when $\lambda < \overline{\lambda}$. This corresponds to $c(\infty) = 0$. The value at the maturity can be derived through Formula (48), and it is $c(0) = 0.9808$—we mark it with a green circle. The behavior of the optimal boundary w.r.t. the discount factor λ and the time to maturity τ is presented in Figure 2d.



(**a**) Boundary $\lambda = 0.05$

(**b**) Behavior of the boundary, $\lambda < 0.06$

(**c**) Boundary, $\lambda = 0.1$

(**d**) Behavior of the lower boundary, $\lambda > 0.06$

(**e**) Behavior of the upper boundary, $\lambda > 0.06$

(**f**) Critical value, $\lambda = 0.06$

Figure 2. Exercise boundaries.

If the discount factor takes its critical value, $\lambda = \overline{\lambda} = 0.06$, then we again have a one-sided optimal hitting problem. The main difference is that early exercising can be optimal even for the perpetual derivatives. The optimal boundary is presented in Figure 2f. The perpetual value is marked with a red circle, and it is $c(\infty) = 0.6250$, due to Equation (54). We can see that the optimal boundary tends namely to this level. Also, the maturity value is obtained via Formula (47), and it is $c(0) = 1.2500$—we again mark it with a green circle.

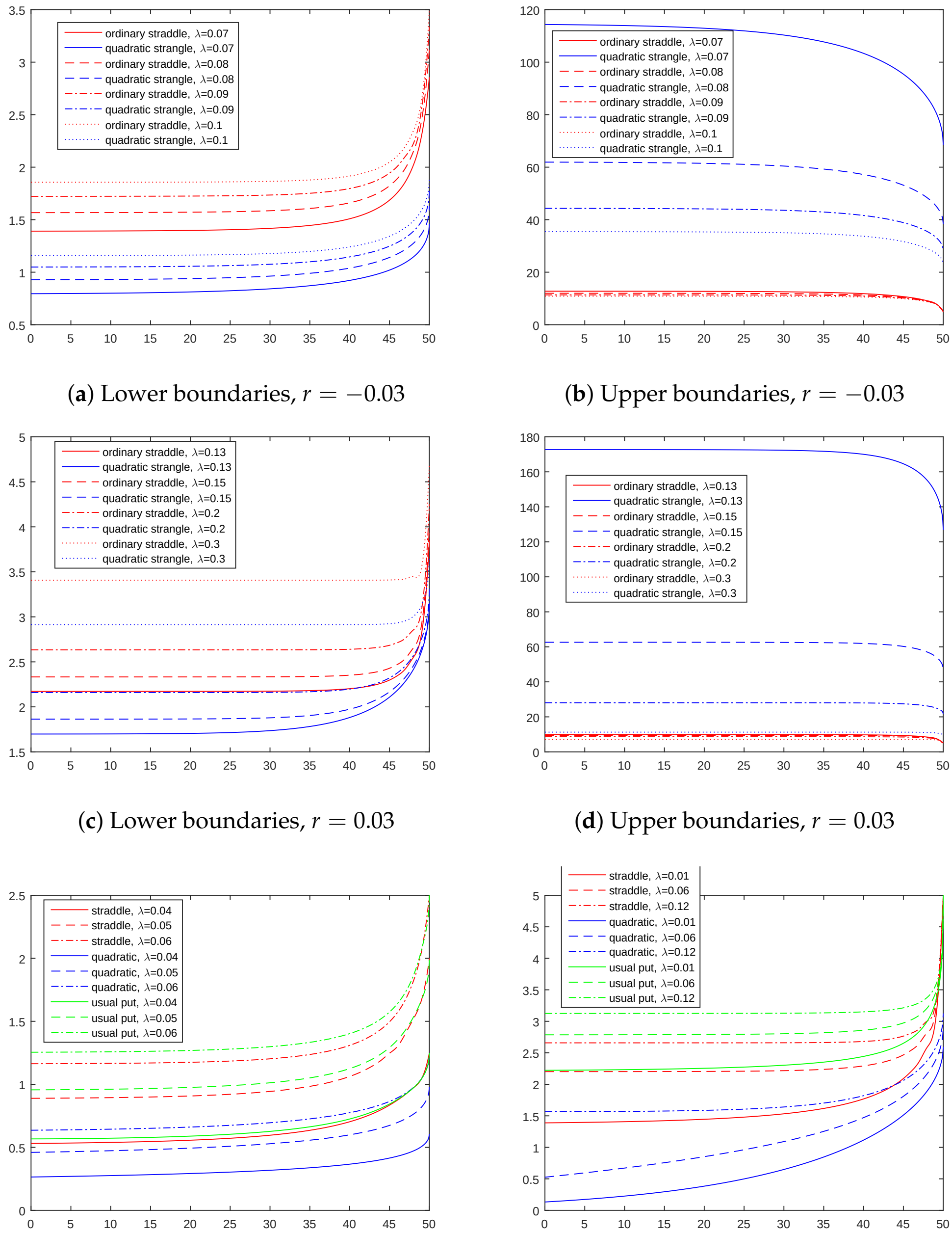
If $\lambda > \overline{\lambda}$, then we have a two-sided optimal hitting problem. Both optimal boundaries for $\lambda = 0.1$ can be seen in Figure 2c—the lower boundary is presented with a blue color, whereas the upper one is in red. The parameterization is w.r.t. the initial time. The perpetual values are obtained via Theorem 1, and they are $c(\infty) = 1.1576$ and $d(\infty) = 35.4366$—we mark them with circles (blue or red). The values at the maturity are obtained through statement (37), and they are $c(0) = 1.8934$ and $d(0) = 23.1066$—we present them with green circles. The behavior of both boundaries w.r.t. the discount factor λ and the time to maturity τ can be seen in Figure 2d,e—the first for the lower boundary and the second for the upper one. The initial points are marked again with the color green and they are obtained via Formula (37). The perpetual boundaries are plotted in red—they are derived through Theorem 1. We can see that both boundaries tend to their perpetual levels when the time to maturity tends to infinity.

The price behavior w.r.t. the time to maturity τ and the discount factor λ can be seen in Figure 1. Particularly, Figure 1a,b present the prices when the discount factor is less than the critical value, $\lambda < \overline{\lambda}$. The maturities are relatively short for Figure 1a, $\tau \in [0, 3]$, and long for Figure 1b, $\tau \in [0, 1000]$. The initial asset price is assumed to be $S_0 = 2$. Again, we have a one-sided hitting problem. We can see that the price tends to infinity when $\tau \to \infty$—this confirms the first result of Theorem 2. We can also observe that the lower the discount factor, the faster this convergence occurs.

We consider discount rates larger than the critical value, $\lambda > \overline{\lambda}$, for Figure 1c (short maturities) and Figure 1d (long maturities). We now have an exit problem from a strip. The initial value of the underlying asset is assumed to be $S_0 = 5$, i.e., at-the-money. We mark with red points the perpetual prices calculated through Theorem 1. We can see that for a fixed discount factor, the price increases from zero (since the quadratic strangle is at-the-money) to the perpetual value.

Next, we consider the critical discount factor, i.e., $\lambda = \overline{\lambda} = 0.06$. We again have a one-sided hitting problem. Differently from the case of $\lambda < \overline{\lambda}$, Theorem 2 shows that the perpetual price is finite, and it is given by Formula (53) for $\overline{A} = c(\infty) = 0.6250$, derived via (54). The current parameters lead to a price of 16.7238.

We provide a comparison between the optimal boundaries of the quadratic strangle options and the usual straddles. We consider two values for the risk-free rate—one positive and one negative. Suppose first that $r = -0.03$. As we mentioned above, the critical value for the discount factor at which the quadratic coefficient in (32) changes its sign is $\overline{\lambda} = r + \sigma^2 = 0.06$. We have a two-sided optimal stopping problem when $\lambda > \overline{\lambda}$—we consider the discount rate to be amongst $\lambda \in \{0.07, 0.08, 0.09, 0.1\}$. The optimal boundaries are presented in Figure 3a,b—the lower ones in Figure 3a and the upper ones in Figure 3b. We can see that the lower boundary of a quadratic strangle is below the corresponding boundary of the related straddle. The opposite is true for the upper boundaries—the boundary of the quadratic strangle is above the related straddle one. We can also see that for both style options, the smaller the discount factor is, the larger the lower optimal boundary is, and the smaller the upper one is. Note that in the asymptotic case $\lambda \to \infty$, both boundaries tend to the strike, thus the optimal set turns to the singleton $\{K\}$. Another observation we made is that the upper optimal boundary for the quadratic strangle increases relatively faster w.r.t. λ than the lower one decreases. Moreover, the upper boundary tends to infinity for discount rates near the critical value $\overline{\lambda}$. This is due to the fact that when $\lambda \leq \overline{\lambda}$, the quadratic strangle leads to a one-sided first hitting task. Note that it is put-style in the sense that the optimal points are below the optimal boundary. Thus, the upper optimal boundary for the quadratic strangles vanishes. Next, we examine the tasks considering values of $\lambda \in \{0.04, 0.05, 0.06\}$. Note that we need $\lambda > 0.03$ since $r + \lambda > 0$. We present, in Figure 3e, the lower boundary of the usual straddle options as well as the unique quadratic one. We have to mention that it tends to zero for large maturities when $\lambda < \overline{\lambda}$. In contrast, its limit is positive when $\lambda = \overline{\lambda}$. This is in accordance with Theorem 2. In addition, we present the optimal boundary of the related ordinary American put. We can observe that the boundaries of the ordinary put and the straddle are relatively near. However, the put boundary is above the straddle one due to the right of the straddle being exercised as a call. In contrast, the boundary of the quadratic option is significantly below the put and straddle ones. This means that the quadratic option is the most expensive, the usual American is the cheapest, and the straddle is between them.



(**a**) Lower boundaries, $r = -0.03$ (**b**) Upper boundaries, $r = -0.03$

(**c**) Lower boundaries, $r = 0.03$ (**d**) Upper boundaries, $r = 0.03$

(**e**) Lower boundaries, $r = -0.03$ (**f**) Lower boundaries, $r = 0.03$

Figure 3. The lower boundary of the usual American straddle vs. quadratic strangle, $\sigma = 0.3$, $K = 5$.

Let us now consider a positive value for the risk-free rate, $r = 0.03$. Thus, the critical value for the discount factor λ is $\overline{\lambda} = r + \sigma^2 = 0.12$. We again separately examine the values above and below it. Suppose that λ is amongst $\lambda \in \{0.13, 0.15, 0.2, 0.3\}$. Both boundaries are presented in Figure 3c,d. All conclusions made for the case $r = -0.03$ are valid for the positive interest rate too. Particularly, the upper boundary vanishes for $\lambda \leq \overline{\lambda}$. Thus, we consider $\lambda \in \{0.01, 0.06, 0.12\}$. Note that when $r > 0$, the restriction for the additional discount factor is only $\lambda \geq 0$. In Figure 3f, we present the lower boundaries of the ordinary straddles, the unique ones of the quadratic strangles, and the boundaries for the usual American puts. We have to mention that the last one is the strike at maturity when $r \geq 0$, and it is below the strike for negative short rates. All conclusions made for the negative risk-free values are valid again. In addition, we can observe that the optimal quadratic boundaries converge faster to their perpetual values—see, again, Theorem 2.

Some particular quadratic strangle prices are given in Table 2. The risk-free rate is varied amongst $r \in \{-0.04, -0.03, -0.02, -0.01, 0, 0.01\}$, the additional discount factor amongst $\lambda \in \{0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$, and the initial asset price amongst $S_0 \in \{\$1, \$2, \$3, \$4\}$. The time to maturity is assumed to be one year. The first part of the table presents the critical values for the additional discount factor, namely $\overline{\lambda}$, as well as the optimal boundaries. If we have a one-sided problem, equivalently to $\lambda \leq \overline{\lambda}$, then only one boundary (the lower one) exists. Otherwise, both boundaries are displayed. The rest of the table is devoted to the prices themselves. We approximate the boundaries at $\overline{M} = 10$ points using a two-step algorithm presented in Section 3.4 or its modification for one-sided problems given at the end of Section 5.2. We use $M = 500$ time- and $N = 400$ space-nodes for the finite difference grid.

Table 2. Option prices.

Boundaries	$r = -0.04$	$r = -0.03$	$r = -0.02$	$r = -0.01$	$r = 0$	$r = 0.01$
$\overline{\lambda}$	0.05	0.06	0.07	0.08	0.09	0.1
$\lambda = 0.05$	0.4181	0.8175	1.1790	1.4933	1.7601	1.9841
$\lambda = 0.06$	0.7072; 70.5159	1.0428	1.3493	1.6204	1.8551	2.0559
$\lambda = 0.07$	0.9265; 40.3703	1.2182; 81.4564	1.4863	1.7261	1.9366	2.1191
$\lambda = 0.08$	1.1020; 30.1705	1.3613; 45.5616	1.6007; 92.5220	1.8165	2.0079	2.1756
$\lambda = 0.09$	1.2476; 24.9852	1.4817; 33.4892	1.6986; 50.8648	1.8953; 103.6539	2.0712	2.2266
$\lambda = 0.1$	1.3715; 21.8199	1.5854; 27.3888	1.7841; 36.9010	1.9653; 56.2409	2.1281; 114.8156	2.2730
$S_0 = 1$	$r = -0.04$	$r = -0.03$	$r = -0.02$	$r = -0.01$	$r = 0$	$r = 0.01$
$\lambda = 0.05$	16.2544	16.0533	16.0000	16.0000	16.0000	16.0000
$\lambda = 0.06$	16.0933	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda = 0.07$	16.0081	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda = 0.08$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda = 0.09$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda = 0.1$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$S_0 = 2$	$r = -0.04$	$r = -0.03$	$r = -0.02$	$r = -0.01$	$r = 0$	$r = 0.01$
$\lambda = 0.05$	9.7559	9.5432	9.3402	9.1639	9.0457	9.0005
$\lambda = 0.06$	9.6353	9.4518	9.2609	9.1095	9.0210	9.0000
$\lambda = 0.07$	9.5377	9.3451	9.1925	9.0680	9.0062	9.0000
$\lambda = 0.08$	9.4453	9.2650	9.1218	9.0377	9.0000	9.0000
$\lambda = 0.09$	9.3596	9.1966	9.0764	9.0106	9.0000	9.0000
$\lambda = 0.1$	9.2825	9.1394	9.0429	9.0012	9.0000	9.0000
$S_0 = 3$	$r = -0.04$	$r = -0.03$	$r = -0.02$	$r = -0.01$	$r = 0$	$r = 0.01$
$\lambda = 0.05$	5.2581	5.0915	4.9284	4.7704	4.6205	4.4833
$\lambda = 0.06$	5.1706	5.0409	4.8802	4.7257	4.5809	4.4495
$\lambda = 0.07$	5.1156	4.9655	4.8334	4.6829	4.5434	4.4179
$\lambda = 0.08$	5.0636	4.9136	4.7704	4.6421	4.5080	4.3882
$\lambda = 0.09$	5.0129	4.8650	4.7241	4.5931	4.4746	4.3603
$\lambda = 0.1$	4.9634	4.8183	4.6813	4.5543	4.4406	4.3341
$S_0 = 4$	$r = -0.04$	$r = -0.03$	$r = -0.02$	$r = -0.01$	$r = 0$	$r = 0.01$
$\lambda = 0.05$	2.7604	2.6605	2.5645	2.4727	2.3854	2.3030
$\lambda = 0.06$	2.6860	2.6339	2.5390	2.4484	2.3623	2.2814
$\lambda = 0.07$	2.6546	2.5740	2.5139	2.4244	2.3397	2.2603
$\lambda = 0.08$	2.6266	2.5443	2.4675	2.4009	2.3175	2.2397
$\lambda = 0.09$	2.5997	2.5177	2.4400	2.3670	2.2959	2.2195
$\lambda = 0.1$	2.5734	2.4921	2.4151	2.3424	2.2740	2.1998

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Appendix A. Finite Difference Terms

$$
\begin{aligned}
P_t &= \frac{F(m-1,n)-P(m,n)}{\Delta t} \\
P &= \frac{P(m-1,n)+P(m,n)}{2} \\
P_x &= \frac{P(m-1,n)-P(m-1,n-1)+P(m,n)-P(m,n-1)}{2\Delta x} \\
P_{xx} &= \frac{P(m-1,n+1)-2P(m-1,n)+P(m-1,n-1)}{2(\Delta x)^2} \\
&\quad + \frac{P(m,n+1)-2P(m,n)+P(m,n-1)}{2(\Delta x)^2}.
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
0 =& \frac{P(m-1,n)-P(m,n)}{\Delta t} + \\
&+ \frac{1}{2} r x_n \frac{P(m-1,n)-P(m-1,n-1)+P(m,n)-P(m,n-1)}{\Delta x} \\
&+ \frac{1}{4}\sigma^2 x_n^2 \begin{pmatrix} \frac{P(m-1,n+1)-2F(m-1,n)+P(m-1,n-1)}{(\Delta x)^2} \\ + \frac{P(m,n+1)-2P(m,n)+P(m,n-1)}{(\Delta x)^2} \end{pmatrix} \\
&- \frac{1}{2} r (P(m-1,n)+P(m,n)).
\end{aligned} \tag{A2}
$$

- If $n = l_m + 1$, then

$$
\begin{aligned}
& P(m,n)\left(\frac{1}{\Delta t} - \frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r\right) \\
& - P(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = P(m-1,n-1)\left(-\frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right) \\
& + P(m-1,n)\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{r x_n}{\Delta x} - \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r\right) \\
& + P(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& - P(m,l_m)\left(\frac{1}{2}\frac{r x_n}{\Delta x} - \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right).
\end{aligned} \tag{A3}
$$

- If $l_m + 1 < n < k_m - 1$, then

$$
\begin{aligned}
& P(m,n-1)\left(\frac{1}{2}\frac{r x_n}{\Delta x} - \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right) \\
& + P(m,n)\left(\frac{1}{\Delta t} - \frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r\right) \\
& - P(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = P(m-1,n-1)\left(-\frac{1}{2}\frac{r x_n}{\Delta x} + \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)2}\right) \\
& + P(m-1,n)\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{r x_n}{\Delta x} - \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r\right) \\
& + P(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{A4}
$$

- If $n = k_m - 1$, then

$$\begin{aligned}
&P(m,n-1)\left(\frac{1}{2}\frac{rx_n}{\Delta x} - \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right) \\
&+ P(m,n)\left(\frac{1}{\Delta t} - \frac{1}{2}\frac{rx_n}{\Delta x} + \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2}r\right) \\
&= P(m-1,n-1)\left(-\frac{1}{2}\frac{rx_n}{\Delta x} + \frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}\right) \\
&+ P(m-1,n)\left(\frac{1}{\Delta t} + \frac{1}{2}\frac{rx_n}{\Delta x} - \frac{1}{2}\frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2}r\right) \\
&+ P(m-1,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
&+ P(m,n+1)\frac{1}{4}\frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{A5}$$

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