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Number of Volatility Regimes in the Muscat Securities Market Index in Oman Using Markov-Switching GARCH Models

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Abstract: The predominant approach for studying volatility is through various GARCH specifications, which are widely utilized in model-based analyses. This study focuses on assessing the predictive performance of specific GARCH models, particularly the Markov-Switching GARCH (MS-GARCH). The primary objective is to determine the optimal number of regimes within the MS-GARCH framework that effectively captures the conditional variance of the Muscat Securities Market Index (MSMI). To achieve this, we employ the Akaike Information Criterion (AIC) to compare different MS-GARCH models, estimated via Maximum Likelihood Estimation (MLE). Our findings indicate that the chosen models consistently exhibit at least two regimes across various GARCH specifications. Furthermore, a validation using the Value at Risk (VaR) confirms the accuracy of volatility forecasts generated by the selected models.

Keywords: Markov-Switching GARCH; volatility; conditional variance; value at risk; forecasting; Muscat Securities Market Index



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1. Introduction

Predicting market volatility holds significance within financial economics. Accurate forecasts of forthcoming volatility are essential for risk and asset managers, and various financial stakeholders aiming to mitigate risks and optimize returns. The consequences of the recent financial crisis emphasized the importance of accurate forecasts, especially given the tightening of financial regulations and widespread skepticism surrounding financial markets. Therefore, understanding volatility behavior is imperative not just for regulatory compliance but also for mitigating the potential impact of future crises. This problem will explore empirical methodologies stemming from the ARCH model, short for Autoregressive Conditional Heteroskedasticity, pioneered by Engle [1].

Among the prevalent approaches for modeling volatility, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model introduced by Bollerslev [2], stands out. GARCH models are favored for their relative ease of estimation and diagnostic testing. Additionally, their popularity stems from their capacity to capture volatility series' characteristics, such as nonlinearity, clustering, and asymmetry, as described by Enders [3].

Despite the plethora of GARCH specifications, many models exhibit excessive persistence, reacting sluggishly to market movements. This conditional dependency inherent in GARCH models aids in capturing volatility clustering but compromises adaptability to sudden shifts in stock movements, as noted by Lamoureux and Lastrapes [4].

Volatility series often undergo shifts due to structural changes or altered market expectations. Terms like “increase” and “decrease” denote states characterized by significant return movements, indicative of high-variance regimes. Conversely, periods characterized by the absence of such extreme fluctuations indicate low-variance regimes. Integrating

states or regimes into a GARCH model modifies its mean-reversion behavior to be contingent on the state, leading to fluctuations in the speed at which the variance returns to its long-run average across distinct regimes. Multi-state models offer greater flexibility compared to single-state models, as they account for the average mean reversion of various states, as highlighted by Alexander and Lazar [5].

In 1994, Cai [6] and Hamilton and Susmel [7] introduced Markov-Switching Regression combined with the ARCH model discussed in [8,9]. This led to the SWARCH model, allowing volatility to transition across various regimes with specified probabilities, providing a more flexible approach to volatility estimation. For further information, refer to He et al. [10].

Derived from the SWARCH model, another advancement emerged known as the Markov-Switching GARCH (MS-GARCH) model, introduced by Gray [11] and Klaassen [12]. Research conducted by Marcucci [13] and Ardia [14] showcased the superior short-term forecasting accuracy of MS-GARCH compared to traditional GARCH models when applied to the S&P 100 index. Despite its favorable attributes, there exists a scarcity of literature exploring the full potential and capabilities of MS-GARCH.

Recently De la Torre-Torres et al. [15,16] investigated the application of two-regime Markov-switching models featuring asymmetric, time-varying exponential Generalized Autoregressive Conditional Heteroskedasticity (MS-EGARCH) variances within the framework of random-length Lumber Futures trading. They explored a trading strategy based on a two-regime framework (low volatility with $s = 1$ and high volatility with $s = 2$), where the decision to invest in Lumber Futures or 3-month U.S. Treasury bills (TBills) depended on the probability of being in the high-volatility regime $s = 2$ being less than or equal to 50%.

Tamilselvan and Vali [17] conducted a study on the Muscat Securities Market Index (MSMI), focusing on forecasting stock market volatility based on daily observations between January 2001 and November 2015; they employed GARCH(1,1), EGARCH(1,1), and TGARCH(1,1) models in their analysis. The results reveal a direct relationship between return and risk. Furthermore, their research emphasizes the enduring impact of volatility shocks and identifies substantial evidence of asymmetry in stock returns using asymmetric GARCH models. Their study emphasizes the considerable persistence of volatility, an asymmetrical association between return shocks and adjustments in volatility, and the existence of a leverage effect across all four indices. Consequently, investors are encouraged to develop investment strategies by examining both historical and recent information, as well as forecasting future market movements, to efficiently manage financial risks and capitalize on opportunities in the stock market. Also, many researchers have studied the volatility behavior of the Muscat Securities Market Index (MSMI). In particular, Prabhakaran [18] studied the volatility of the MSM Index by using GARCH(1,1) and EGARCH(1,1). However, Sha et al. [19] directed their attention towards examining the volatility of the stock market involving both Regular and Parallel market players in Muscat Oil and Gas companies, while also evaluating their interrelations. This study employed the GARCH model to gauge the volatility of the Muscat Securities Market, particularly focusing on Oil and Gas companies listed in the MSM. However, to capture more structural breaks in volatility, such as several states depicted in the conditional variance process, and assess the optimal number of volatility regimes exhibited by the Muscat Securities Market Index series, in this paper we introduce the switch in time series, provided by Markov-switching regime MS-GARCH models. In Section 2, we present a different MS-GARCH model, while in Section 3 we delve into the data methodology concerning our time-series analysis of the MSMI. The empirical findings are elaborated upon in Section 4, and Section 5 provides a summary of the study's results.

2. Markov-Switching GARCH Models

In [11], Gray introduced the concept of consolidating conditional variances from two regimes at each time step while developing a generalized regime-switching model

for short-term interest rates. This combined conditional variance from a single regime is then utilized as an input for the calculation of the conditional variance at the following time step. To elaborate further, Gray's method entails constructing the conditional variance equation within the framework of the GARCH(1,1) model in a regime-switching context, as outlined below.

In an MS-GARCH(1,1) model featuring dual regimes, the state variable progresses via a Markov chain. Specifically, it employs a first-order Markov chain, where the likelihood of the current state, known as the transition probability, is contingent solely upon the immediately preceding state, which can be mathematically described as follows:

$$\Pr(S_t = j | S_{t-1} = i, S_{t-2} = i_{t-2}, \dots, S_0 = i_0) = \Pr(S_t = j | S_{t-1} = i) = p_{ij}, \quad (1)$$

where $(i_0, i_1, \dots, i_{t-2}, i, j) \in \mathbb{N}^{t+1}$, $(S_t)_{t \in \mathbb{N}}$ is a stochastic process.

In this section, we recall the MS-GARCH model, as introduced by Bollerslev [2]. In what follows, we denote by P_t the price of the stock market index at time t , and by r_t its log return given by

$$r_t = 100 \times \log \left(\frac{P_t}{P_{t-1}} \right). \quad (2)$$

The time index of r_t is then partitioned into two subsets: an in-sample and an out-sample. The total sample period extends from $t = -D + 1, -D + 2, \dots, 0$, with the out-sample ranging from $t = 1, 2, \dots, n$. The period within the sample ranges from 1 January 2000 to 7 May 2018, while the out-sample period spans from 8 May 2014 to 29 September 2022. Moreover, we make the assumption that $\mathbb{E}[r_t] = 0$, and the series (r_t) is devoid of serial correlation.

The MS-GARCH model, as delineated by Ardia et al. [20], is characterized by the following definition:

$$r_t | (S_t = k, \mathcal{T}_{t-1}) \sim f(0, h_{k,t}, \Phi_k),$$

where the function $f(\cdot)$ represents a continuous distribution characterized by a mean of zero and a conditional variance $h_{k,t}$ that switches within regime (k). Φ_k describes the set of all additional parameters of the model and \mathcal{T}_{t-1} stands for the set of accumulated information up to time $t - 1$ that is generated by $\{r_{t-1}, r_{t-2}, \dots\}$. In each model, the conditional variance $h_{k,t}$ can move according to a Markov process $S_t \in \{1, 2, \dots, K\} \subset \mathbb{N}$. We define the matrix of transition for S_t by

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{bmatrix},$$

where the entries p_{ij} of the matrix above are defined as $\Pr(S_t = j | S_{t-1} = i)$ providing the probability to be in state or regime (j) at time t given that the Markov chain was in state (i) at time $t - 1$ and $\sum_{j=1}^K p_{ij} = 1$ for i fixed. When $i = j$, p_{ii} is referred to as the persistence probability within the specified regime (i). Our focus in this work lies on the scenario of three regimes, specifically $S_t \in \{1, 2, 3\}$.

Up to now, the concept underlying MS-GARCH specifications entailed the integration of GARCH structures with parameters that dynamically adjust to accommodate structural breaks in the conditional variance. Nevertheless, this approach brings about an issue of path dependence, wherein the conditional variance at time t is dependent on the complete sequence of regimes S_t ($t = 1, \dots, K$). To circumvent this issue, we refer to the work of Haas et al. [21] and Ardia et al. [20].

Numerous studies have demonstrated that GARCH(1,1) offers more accurate depictions of market volatility dynamics compared to even higher-order ARCH specification [21]. In line with Bollerslev's work [2], the expression for the conditional variance process of GARCH(1,1) can be represented as follows:

$$\begin{cases} r_t = \epsilon_t \sqrt{h_t}, & \epsilon_t \stackrel{i.i.d.}{\sim} f(0,1), \\ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} = \psi(r_{t-1}, h_{t-1}), \end{cases} \quad (3)$$

with $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$ to guarantee a positive variance. The sequence (ϵ_t) consists of independent and identically distributed (i.i.d.) random variables with a mean of zero and a variance of one. A conditional distribution $f(\cdot)$ must be designated.

Throughout the subsequent discussions, h_t is presumed to represent a Markov-Switching GARCH(1,1) process described by S_t , ($t = 1, \dots, K$):

$$h_t = \psi(r_{t-1}, h_{t-1}, S_t).$$

We substitute S_t into Equation (3), resulting in the representation of the conditional variance in switching regimes as follows:

$$h_t = \omega_k + \alpha_{1,k} r_{t-1}^2 + \beta_{1,k} h_{t-1}.$$

As described in [11], Gray (1996) employs the information set available at time $t - 1$ to integrate hidden regimes, thus mitigating path-dependence concerns, depicted as follows:

$$h_t = \sum_{k=1}^K p_{k,t} h_{k,t},$$

where, $h_{k,t}$ represents the conditional variance of r_t within regime (k), while $p_{k,t} = \Pr(S_t = k | \zeta_{t-1})$ denotes the probability of being within a particular regime (k) based on information available up to $t - 1$.

Suppose that $S_t \in \{1, 2\}$, in this case, the conditional distribution of r_t involves a switching between distribution $f(h_{k,t})$ within regime (k).

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix} \quad (4)$$

To simplify, let us consider the scenario with only two regimes. The unconditional probability of being in a particular state $S_t = 1$, usually named ergodic probability, is calculated as $\pi_1 = (1 - q) / (2 - p - q)$. Thus, the generalized form of the MS-GARCH(1,1) model featuring two regimes can be represented as follows:

$$r_t | \zeta_{t-1} \sim \begin{cases} f(h_{1,t}) & \text{with transition rate } p_{1,t} \\ f(h_{2,t}) & \text{with transition rate } p_{2,t}. \end{cases} \quad (5)$$

In the equation above, $f(\cdot)$ denotes one of the potential conditional distributions, such as normal (N), Student's (t), or GED. The quantity $h_{i,t}$ represents the conditional variance in the i th regime, defining the distribution. Additionally, $p_{1,t} = \Pr(S_t = 1 | \zeta_{t-1})$ signifies the ex-ante probability, with ζ_{t-1} denoting the information set up to time $t - 1$, encompassing the σ -algebra induced by all observed variables up to that point.

Thus, the MS-GARCH model consists of three key elements: the conditional variance, the regime process, and the conditional distribution. Meanwhile, the conditional mean, often represented as a drift or driftless random walk is represented as follows:

$$r_t = \eta_t \sqrt{h_t} \quad (6)$$

where η_t denotes a process characterized by zero mean and unit variance.

Given the entire trajectory (S_t, S_{t-1}, \dots) the conditional variance of r_t and ζ_{t-1} is denoted by $h_{i,t} = \text{Var}(r_t | S_t = i, \zeta_{t-1})$. It is then expressed as follows:

$$h_{i,t} = \alpha_{0,i} + \alpha_{1,i} r_{t-1}^2 + \beta_{1,i} h_{t-1}. \quad (7)$$

To alleviate the notable influence of negative returns on conditional volatility, often termed the “leverage effect”, we adopt the GJR-GARCH model proposed by Glosten et al. [22]. This model aims to capture asymmetry effects present in our time series, and the conditional variance process can be represented by the following:

$$h_{i,t} = \alpha_{0,i} + (\alpha_{1,i} + \gamma_i \mathbb{1}_{\{r_{t-1} < 0\}}) r_{t-1}^2 + \beta_{1,i} h_{t-1}, \quad (8)$$

where $\mathbb{1}_{\{r_{t-1} < 0\}} = 1$ if $r_{t-1} < 0$ and 0 otherwise. For $i \in \{1, 2\}$, where $\gamma_i \geq 0$, the parameter γ_i serves as the measure of asymmetry in the conditional variance process.

Another critical aspect affecting the effectiveness of our conditional volatility modeling concerns the assumed distribution of innovations (η_t), which must be carefully specified. In our investigation, we concentrate on three distributions: Student’s (t), normal (N), and generalized error distribution (GED). We prioritize skewed distributions to address asymmetry. For the definition of skewed density, we defer to Fernandez and Steel [23], presented as follows:

$$f_{\xi}(z) = \frac{2\sigma_{\xi}}{\xi + \xi^{-1}} f^*(z_{\xi}),$$

where

$$z_{\xi} = \begin{cases} \xi^{-1}(\sigma_{\xi} z + \mu_{\xi}) & \text{if } z \geq -\frac{\mu_{\xi}}{\sigma_{\xi}}, \\ \xi(\sigma_{\xi} z + \mu_{\xi}) & \text{otherwise} \end{cases}$$

with $\mu_{\xi} = m(\xi - \xi^{-1})$, $\sigma_{\xi}^2 = (1 - m^2)(\xi^2 - \xi^{-2}) + 2m^2 - 1$, and $m = \int_0^{\infty} 2t f^*(t) dt$.

The asymmetry degree is captured by the parameter $0 < \xi < \infty$, whereas $f^*(\cdot)$ describes a symmetric density function with a mean of zero and a variance of one.

To derive the one-step-ahead forecast of the MS-GARCH, we aggregate potential expected conditional variances across each state, weighting them by the ex-ante probability denoted as $p_{j,t}$ and represents the likelihood of being in the initial regime at time t ; based on the information available up to time $t - 1$, as outlined in Hamilton [9], one can write

$$p_{j,t} = \Pr(S_t = j | \zeta_{t-1}) = \frac{\sum_{i=1}^2 p_{ij} \frac{f(r_{t-1} | S_{t-1} = i) p_{i,t-1}}{\sum_{k=1}^2 f(r_{t-1} | S_{t-1} = k) p_{k,t-1}}}, \quad (9)$$

where p_{ij} denote the transition probabilities, while f represents the density functions as defined in Equation (5).

This yields the calculation for the one-step-ahead forecast:

$$\begin{aligned} \hat{h}_{T,T+1} &= \Pr(S_{T+1} = 1 | \zeta_T) (\alpha_{0,1} + \alpha_{1,1} r_T^2 + \beta_{1,1} h_{1,T}) \\ &\quad + \Pr(S_{T+1} = 2 | \zeta_T) (\alpha_{0,2} + \alpha_{1,2} r_T^2 + \beta_{1,2} h_{2,T}). \end{aligned} \quad (10)$$

According to Marcucci’s regime-switching GARCH [13], at time $T - 1$, the forecast for volatility h steps ahead can be computed as follows:

$$\hat{h}_{T,T+h} = \sum_{m=1}^h \hat{h}_{T,T+m} = \sum_{m=1}^h \sum_{i=1}^2 \Pr(S_{T+m} = i | \zeta_{T-1}) \hat{h}_{i,T,T+m}. \quad (11)$$

In this context, $\hat{h}_{T,T+h}$ denotes the combined volatility forecast at time T for the following h steps, and $\hat{h}_{i,T,T+m}$ indicates the m -step-ahead volatility forecast in regime i at time T , which can be computed iteratively.

$$\hat{h}_{i,T,T+m} = \alpha_{0,i} + (\alpha_{1,i} + \beta_{1,i})\mathbb{E}_T[h_{i,T,T+m-1}|S_{T+m}], \quad (12)$$

where $\hat{h}_{i,T,T} = h_{i,T}$ and \mathbb{E}_T stands for the conditional expectation given in the information up to time T (ζ_T).

The collection of log-likelihood functions is expressed as follows:

$$\ell = \sum_{-D+1+w}^{T+w} \log[p_{1,t}f(r_t|S_t = 1) + (1 - p_{1,t})f(r_t|S_t = 2)], \quad (13)$$

where w takes values from the set $\{0, 1, \dots, n\}$, D represents the duration of trading days included in the in-sample analysis, and $f(r_t|S_t = i)$, as defined by Marcucci [13], denotes the conditional distribution given regime i occurring at time t .

3. Data and Methodology

This paper focuses on estimating a multi-regime GARCH model using data from the MSMI index. The dataset consists of daily rate of return information, derived from intra-daily extreme values of stock returns and closing prices obtained from an investment platform's historical data. The total dataset spans from 1 January 2000 to 29 November 2022, encompassing 5394 observations from the MSMI, accounting for various holidays. Recall that the rate of return is given (2). As anticipated, the volatility fluctuates throughout the period, displaying clusters of volatility where significant changes in the index are often succeeded by further significant changes, while small changes are typically followed by small changes (refer to Figures 1 and 2). Additionally, we present the correlation between the magnitude of fluctuations in log returns and the evolution of the stock market index. The time index of r_t is within the set $\{1, 2, \dots, n\}$.

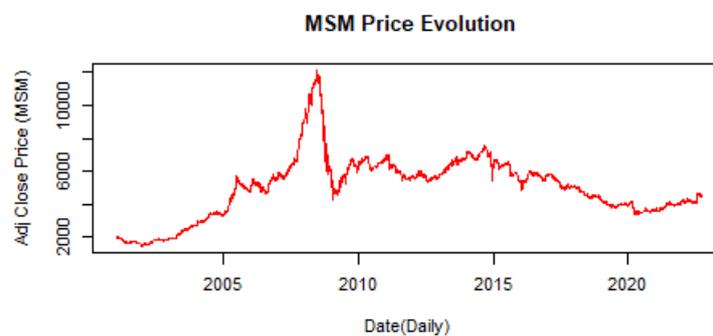


Figure 1. Illustration of the evolution of the close price index for MSM.

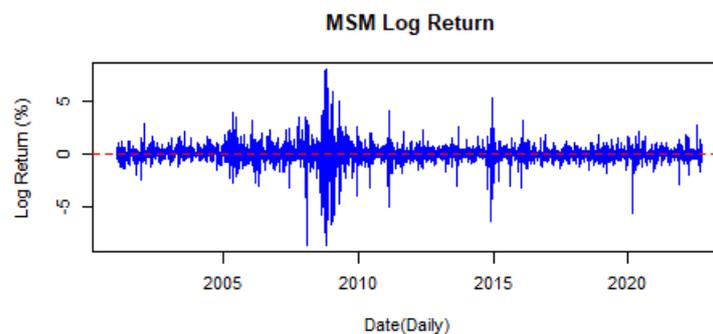


Figure 2. Illustration of the evolution of logarithmic return (%) for the MSM index.

In this empirical section, we utilize MS-GARCH(1,1) models to estimate the volatility of the log return r_t . To address the fat tails characteristic of financial returns, we investigate three different distributions for the innovations: normal (N), Student's (t), and generalized error distribution (GED). To evaluate the performance of the models, we compare the forecasts produced by various MS-GARCH specifications against the "true" volatility. However, identifying the "true" daily volatility presents challenges. Thus, this study utilizes a measure of the "true volatility". The traditional volatility estimator used is referred to as the Close-Close Volatility Estimator, calculated as follows:

$$\hat{\sigma}_t = \sqrt{252 \times \frac{1}{n} \sum_{k=1}^n r_{t-k}^2} \text{ for } t > n. \quad (14)$$

This represents the primary historical volatility estimator, favored for its simplicity and widespread use. Pérez-Cruz et al. [24] suggested the approximation with $n = 5$.

Figures 1 and 2 displays a graphical representation of our series, allowing us to assess volatility clustering. We observe alternating periods of low and high fluctuations. Additionally, we present the correlation between the magnitude of fluctuations in log returns and the evolution of the stock market index.

Table 1 presents summary statistics of daily log returns for the MSMI. The mean log return is close to zero at 0.015%, supporting the assumption of a zero mean. The standard deviation, approximately unity at 0.853%, indicates considerable volatility (an assumption to be confirmed). The skewness coefficient exhibits a significant negative value, suggesting a leftward spread in the distribution's tail. Additionally, the excess kurtosis exceeds the normal distribution's value of 0, indicating heavier tails in the distributions. The LM-Statistic test confirms the presence of the ARCH effect in all series, rejecting the null hypothesis of "no ARCH effect". Additionally, The Jarque-Bera (JB-Statistic) test rejects the null hypothesis of "normality", indicating that the distributions are not normal.

Table 1. Summary of return data statistics.

Statistic	MSMI
Mean (%)	0.015
Median (%)	0.019
Minimum (%)	−8.038
Maximum (%)	−8.038
Std.Dev. (%)	0.853
Skewness	−1.02
Kurtosis	21.425
JB-Statistic	10,419.2
JB p -value	<0.01
LM (12)	1786
LM p -value	<0.01

4. Estimation and Identification of the Number of Regimes

In our study, we have introduced the standard GARCH model alongside the multi-regime MS-GARCH model, aimed at enhancing the ability to capture the persistence of conditional volatility in the stock market index. To accommodate these complex models, we employed the Maximum Likelihood (ML) approach as outlined by Marcucci [13]. The adequacy of our models was assessed using the Akaike Information Criterion (AIC) to identify the most suitable model.

In this section, we conduct an analysis of log-return results for the MSMI. For our analysis, we fitted all 18 models using historical data spanning from 1 January 2000 to 29 November 2022. Throughout this study, we utilized the R package developed by Ardia et al. [20] to estimate the parameters and AICs to identify the optimal number of regimes (comparison between 1, 2, and 3 regimes).

4.1. Identification of the Number of Regimes

Before commencing model fitting, we pre-processed our series using the AR(1) model, chosen based on the AIC, to ensure the absence of correlation between log-return observations r_t . Table 2 presents the AIC values for the different models. Ardia [14] demonstrated numerous advantages of using the AIC for selecting the most suitable model to provide a more accurate description of stochastic volatility.

Table 2. Akaike Information Criterion.

		MSMI		
		k = 1	k = 2	k = 3
S-GARCH	Sk-N	9211.09	8669.49	8613.02
	Sk-STD	8604.86	8579.34	8581.85
	Sk-GED	8640.25	8570.44	8579.71
GJR-GARCH	Sk-N	9210.49	8671.46	8619.43
	Sk-STD	8606.17	8601.82	8596.48
	Sk-GED	8640.82	8572.95	8636.4

Next, we compare the two-regime models with different distributions (skewed normal, skewed Student's, and skewed GED), starting with their AIC values. For the MSMI, the skewed GED with two regimes provides more adequacy for standard GARCH. Also, the two-regime GJR-GARCH with skewed distribution again offers better adequacy. Thus, regarding the smallest value of AIC (8570.44), the optimal specification for describing MSMI log returns appears to be the two-regime standard GARCH model with a skewed generalized error distribution.

4.2. Estimation of the Tentative Model

In the previous section, we established that the log-returns of the MSMI data under consideration were tentatively characterized by the two-regime MS-GARCH model with a skewed GED. Table 3 presents parameter estimates for a given regime (k), including the parameters of the standard GARCH(1,1) model ($\alpha_{0,k}, \alpha_{1,k}, \beta_{1,k}$), and $\Phi^{(k)} \equiv (\eta_k, \zeta_k)$, where η_k and ζ_k represent the parameters of the skewed GED, representing the tail and asymmetry, respectively. Additionally, the transition matrix elements $p_{ij} = \Pr(S_t = j | S_{t-1} = i)$ are provided, where p_{kk} represents the persistence probability in the k th regime. The results indicate that all estimated parameters are statistically significant.

Also, in Table 4 below we present some additional proprieties as unconditional volatility defined for each regime (for regime i : $\frac{\alpha_{0,i}}{1 - \alpha_{1,i} - \beta_{1,i}}$).

We observe that the first regime exhibits a low unconditional variance of 0.42%, while the second regime demonstrates a significantly higher unconditional variance of 2.98%.

From our analysis, we can infer the unconditional probabilities of the regimes.

For $K = 2$ (the number of regimes), these probabilities are computed as follows:

$$\pi_1 = \frac{1 - p_{22}}{2 - p_{11} - p_{22}} \quad \text{and} \quad \pi_2 = \frac{1 - p_{11}}{2 - p_{11} - p_{22}},$$

ensuring that $\pi_1 + \pi_2 = 1$. In the case of the MSMI, the unconditional probabilities are found to be approximately 82% for the first regime and 18% for the second regime. This suggests greater stability in the first regime compared to the second.

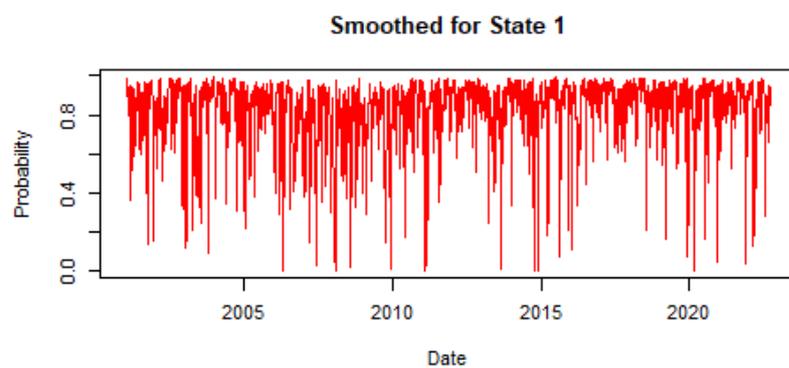
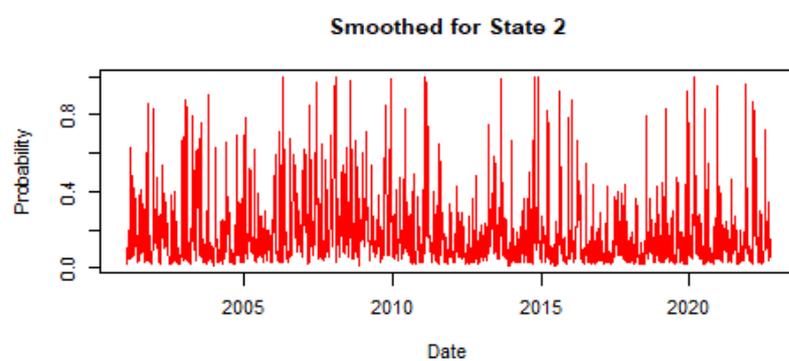
These observations are illustrated by the smoothed probabilities graph, depicting the quantity $\Pr(S_t = 1 | \zeta_{t-1})$, as shown in Figures 3 and 4.

Table 3. Selected model parameters.

Parameter	Mean	MSMI	
		Std.Dev.	p-Value
$\alpha_{0,1}$	0.008	0.0008	<0.001
$\alpha_{1,1}$	0.12	0.0135	<0.001
$\beta_{1,1}$	0.834	0.0043	<0.001
η_1	1.292	0.0394	<0.001
ξ_1	1.053	0.0198	<0.001
$\alpha_{0,2}$	0.303	0.0563	<0.001
$\alpha_{1,2}$	0.663	0.1	<0.001
$\beta_{1,2}$	0.856	0.004	<0.001
η_2	0.856	0.0235	<0.001
ξ_2	0.715	0.0174	<0.001
p_{11}	0.95	0.0202	<0.001
p_{12}	-	-	-
p_{21}	0.23	0.0056	<0.001
p_{22}	-	-	-
$V_{per}^{(1)}$	0.954	-	-
$V_{per}^{(2)}$	0.966	-	-

Table 4. Selected specifications.

MSMI	Specification ($\hat{\Phi}$)			Unconditional Probabilities (π_k)		Unconditional Volatility (%)	
	Class	Dist	Regimes	Regime 1	Regime 2	Regime 1	Regime 2
S-GARCH	Sk-GED	K = 2	0.822	0.178	0.423	2.98	

**Figure 3.** Smoothed probability for the state 1.**Figure 4.** Smoothed probability for the state 2.

4.3. Backtesting of the Selected Models

Our chosen regime-switching models demonstrated significant flexibility in capturing volatility persistence for the MSMI. To further assess the efficacy of these models, we now turn to an out-of-sample analysis.

The out-of-sample period spans from 8 May 2018 to 29 September 2022, comprising approximately 1079 log-return observations for the MSMI. To ensure robustness, a reliable model should precisely predict the Value at Risk (VaR) for a predetermined coverage level. To achieve this, we utilize a broad window, leveraging a family of models capable of accommodating time-varying parameters. This approach enhances the accuracy of forecasting the one-ahead Value at Risk at the 5% coverage level, utilizing the models selected earlier.

Throughout this study, we utilized the R package developed by Ardia et al. [20] to compute p -values for various back-testing hypothesis tests. These tests are crucial for ensuring the accurate conditional coverage of the Value at Risk (VaR). The tests employed in this study include the Unconditional Coverage (UC) test proposed by Kupiec [25], which examines the number of VaR violations (or hits), defined as $I_t(\alpha) = 1$ if $r_t < \text{VaR}_t(\alpha)$ and zero otherwise. Additionally, we utilize the Conditional Coverage (CC) test by Christoffersen [26] and the Dynamic Quantile (DQ) test by Engle and Manganelli [27]. These tests consider the number of violations and require that the violation variable ($I_t(\alpha)$) be independently distributed. These evaluations align with the regulatory requirements set forth by the Basel Committee on Banking Supervision [28,29] regarding the internal validation of VaR models.

The results, as presented in Table 5, highlight the effectiveness of selected models in accurately predicting VaR at the 5% risk level.

Table 5. Results of Value at Risk (VaR) backtesting at the 95% confidence level.

	Unconditional Coverage		Conditional Coverage		Dynamic Quantile	
	uc LRstat	uc LRp	cc LRstat	cc LRp	DQstat	DQp
MSMI	0.172	0.667	1.320	0.516	5.839	0.558

The findings from the Unconditional Coverage (UC), Conditional Coverage (CC), and Dynamic Quantile (DQ) tests suggest that the null hypothesis, indicating accurate forecasting of one-ahead VaR at the 5% coverage level, is supported (i.e., p -value > 0.05).

Additionally, a visualization of the backtest results is presented in Figure 5 below, demonstrating the models' ability to capture significant breaks in log returns.

Additionally, we graph the historical volatility alongside the estimates of the two regimes using MS-GARCH with the skewed GED (refer to Figure 6). The red line represents the volatility derived from the historical volatility estimator.

Finally, after exploring the performance of volatility forecasting within the class of MS-GARCH models by the backtesting method, under the same assumptions of the estimated models we provide a one-ahead volatility forecast for 120 future annualized volatility starting from 5395 since we have 5394 observations in the dataset (see Figure 7). We observe that we are still in regime 1, which is characterized by a low volatility.

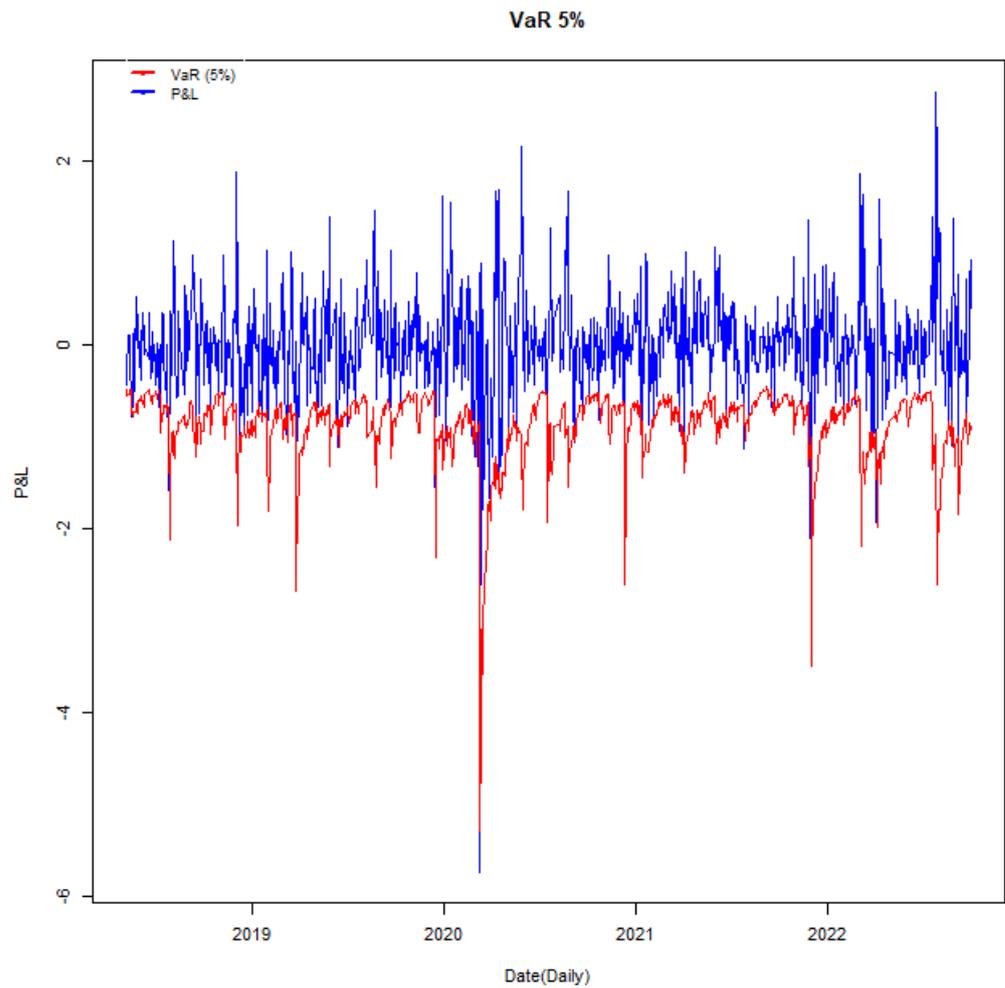


Figure 5. Analysis of Value at Risk for the stock market index using the MS–GARCH model.

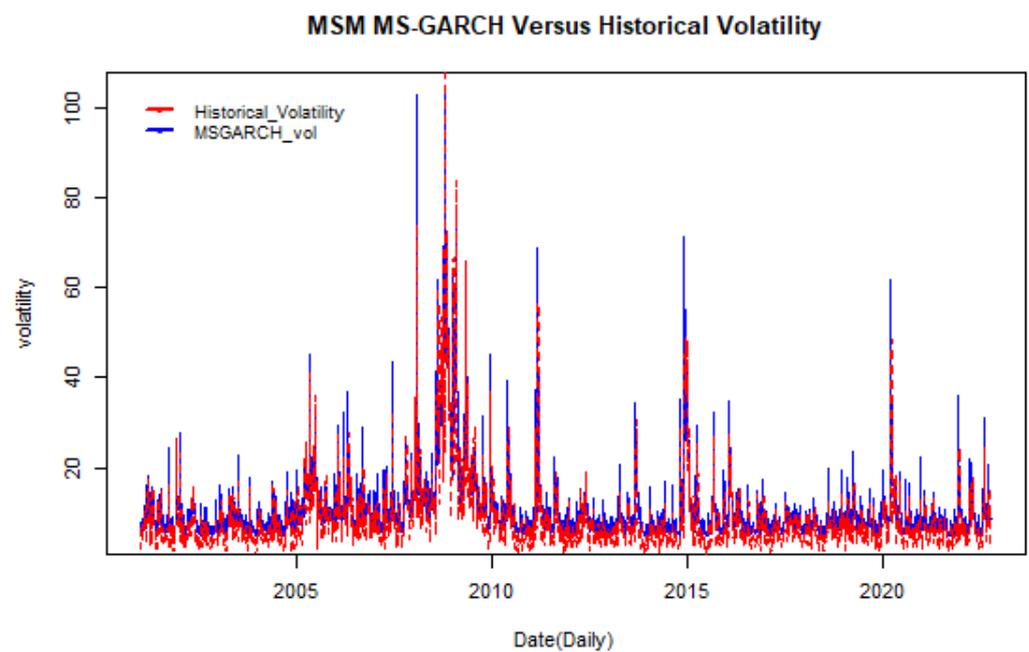


Figure 6. Historical volatility versus MS-GARCH estimated volatility.

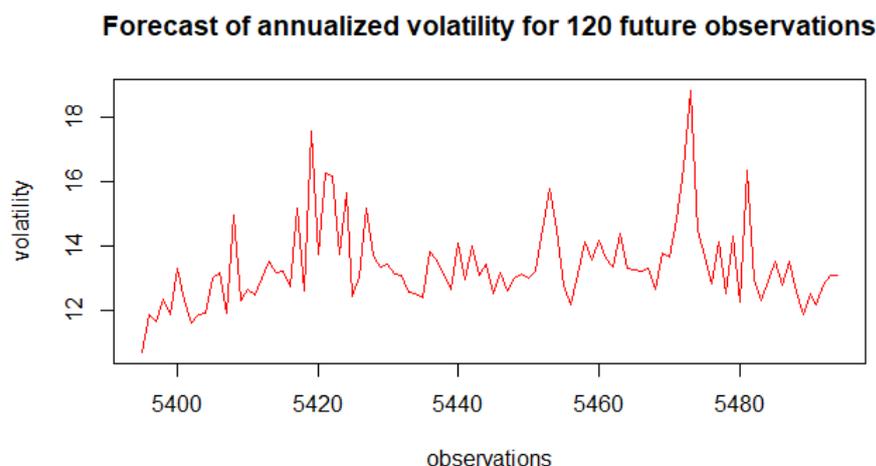


Figure 7. One-ahead forecast for annualized volatility using MS-GARCH models.

5. Conclusions

In this study, we conducted an analysis of stock market indices, particularly focusing on the MSMI (Oman), utilizing their daily log returns spanning from January 2001 to September 2022, encompassing a dataset of 21 years. The aim was to investigate the optimal number of regimes using two categories of GARCH models: the standard GARCH(1,1) model and the asymmetric GJR-GARCH(1,1) model. These models incorporated different skewed conditional distributions (normal, Student's t , and GED), with all parameters permitted to transition across a designated number of regimes.

In the analysis of the empirical data, we used the Maximum Likelihood approach to estimate approximately 18 models. We compared these models based on the Akaike Information Criterion (AIC), which evaluates the balance between model fitting quality and complexity. Model estimation stability was ensured by testing different seeds, with our judgment determining model convergence.

For the MSMI, the GED distribution with two regimes showed greater adequacy for the standard GARCH model. Furthermore, the two-regime GJR-GARCH model with skewed distribution demonstrated even better adequacy. Consequently, based on the smallest AIC value, the most suitable specification for describing MSMI log returns was identified as the two-regime standard GARCH model with a skewed GED. This suggests that the stock market index exhibits two regime specifications: one characterized by low volatility and the other by high conditional variance with persistent volatility.

Finally, we assessed the validity of the selected models through out-of-sample analysis, utilizing statistical tests such as the Unconditional Coverage (UC), Conditional Coverage (CC), and Dynamic Quantile (DQ) tests, aligned with Basel Committee requirements. We also evaluated the models' ability to predict MSMI volatility.

An area of interest for future research is to explore the application of these results using a Bayesian approach, considering prior distributions.

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