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On Uniformly Starlike Functions with Respect to Symmetrical Points Involving the Mittag-Leffler Function and the Lambert Series

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Abstract: The aim of this paper is to define the linear operator based on the generalized Mittag-Leffler function and the Lambert series. By using this operator, we introduce a new subclass of β-uniformly starlike functions $T_{\mathcal{J}}(\alpha_i)$. Further, we obtain coefficient estimates, convex linear combinations, and radii of close-to-convexity, starlikeness, and convexity for functions $f \in T_{\mathcal{J}}(\alpha_i)$. In addition, we investigate the inclusion conditions of the Hadamard product and the integral transform. Finally, we determine the second Hankel inequality for functions belonging to this subclass.

Keywords: uniformly starlike; Hadamard product; Lambert series; Mittag-Leffler function; Hankel determinant

MSC: 30C45; 30C50

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1. Introduction

The term "symmetry" on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ can relate to rotational, reflection, or inversion symmetry, among other kinds of symmetry. The characteristic known as "inversion symmetry" describes how an open unit disk appears when it is inverted with respect to a certain point. When any complex number z in the disk is inverted with respect to the origin, it yields the complex number $-z$ whose inversion is also in the disk, indicating that the open unit disk has inversion symmetry with regard to its center (the origin). The open unit disk, in general, contains a rich set of symmetries that are helpful in several geometric and mathematical situations. Our goal was to investigate other geometric characteristics inside this symmetry area.

If a function maps a disk in the complex plane onto a shape that, with relation to a fixed point on the disk, is star-shaped, it is said to be starlike. Stated differently, a function is said to be starlike if, when subjected to appropriate scaling and rotation, its image is contained inside a star-shaped domain. This domain is created by joining the fixed point to every other point in the domain using straight-line segments. While starlike functions are utilized in geometric function theory and mathematical physics to simulate phenomena like electrostatics [1,2] and fluid flow [3,4], univalent functions are frequently used in geometric function theory to explore conformal mappings and the Riemann mapping theorem.

The one-parameter Mittag-Leffler function $E_\alpha(z)$ for $\alpha \in \mathbb{C}$, with $\Re(\alpha) > 0$ (see [5,6]) is defined as

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, z \in \mathbb{C}$$

Further extension of the Mittag-Leffler function in two-parameters was studied by Wiman [7]. For all $\alpha, \beta \in \mathbb{C}$, with $\Re(\alpha, \beta > 0)$, the two-parameters function $E_{\alpha,\beta}(z)$ is defined as

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},\ z \in \mathbb{C}.$$

In fact, many researchers have worked on the generalization of the Mittag-Leffler function (see [8]). In this study, we confine our attention to the generalization given by Salah and Darus [9], as follows:

$$qF_{\alpha,\ \beta}^{\theta,k} = \sum_{n=0}^{\infty}\prod_{j=1}^{q} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{z^n}{n!} \tag{1}$$

Note that $(\theta)_v$ denotes the familiar Pochhammer symbol, which is defined as

$$(\theta)_v := \frac{\Gamma(\theta+v)}{\Gamma(\theta)} = \begin{cases} 1,\ if\ v=0,\ \theta \in \mathbb{C}\backslash\{0\} \\ \theta(\theta+1)\ldots(\theta+n-1),\ if\ v=n\ \in \mathcal{N},\ \theta\in\mathbb{C}, \end{cases}$$

$$(1)_n = n!,\ n\ \in \mathcal{N}_0,\ \mathcal{N}_0 = \mathcal{N}\cup\{0\},\ \mathcal{N} = \{1,2,3,\ldots\},$$

and

$$\left(q\ \in \mathfrak{N}, j=1,2,3,\ldots q;\ \mathfrak{R}\left\{\theta_j,\ \ \beta_j\right\} > 0, \text{and}\ \mathfrak{R}(\ \alpha_j) > max\{0,\ \mathfrak{R}(\ k_j)-1;\ \mathfrak{R}(\ k_j)\};\ \mathfrak{R}(\ k_j) > 0\right)\cdot$$

In number theory, (see [10–13]), the Lambert series is used for certain problems due to its connection to the well-known arithmetic functions such as

$$\sum_{n=1}^{\infty}\sigma_0(n)x^n = \sum_{n=1}^{\infty}\frac{x^n}{1-x^n}, \tag{2}$$

where $\sigma_0(n) = d(n)$ is the number of positive divisors of $n\cdot$

$$l(z) = \sum_{n=1}^{\infty}\sigma_\alpha(n)x^n = \sum_{n=1}^{\infty}\frac{n^\alpha x^n}{1-x^n}, \tag{3}$$

where $\sigma_\alpha(n)$ is the higher-order sum of divisors function of $n\cdot$

We restrict our attention to the series given by (3). In particular, when $\alpha = 1$, we write $\sigma_1(n) = \sigma(n)\cdot$ Here, $\sigma(n)$ is the sum of divisors function that appears in one of the elementary equivalent statements to the well-known Riemann hypothesis.

We distinguish at the outset between the Lambert series and the Lambert W function, which appears naturally in the solution of a wide range of problems in science and engineering [14].

In 1984, Guy Robin [15] proved that

$$\sigma(n) < e^\gamma n\log\log n + \frac{0.6483n}{\log\log n}, \qquad n\ \geq 3 \tag{4}$$

Moreover, he proved that the Riemann hypothesis is equivalent to

$$\sigma(n) < e^\gamma n\log\log \text{n}, \quad n > 5040, \tag{5}$$

where $\gamma = 0.7721\cdots$, is the Euler–Mascheroni constant.

This article makes no attempt to prove or refute the Robin's inequality (5) or the Riemann hypothesis. For more details, we refer interested readers to the articles listed in the references [16–21].

2. Preliminaries

Let $\mathscr{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \tag{6}$$

and $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent (or one-to-one) functions on $\mathbb{D}$. Let T be the subclass of $\mathscr{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0. \tag{7}$$

The importance of the coefficients given by the power series in (6) emerged in the early stages of the theory of univalent functions.

The focus of this research is to introduce a linear operator to define a new subclass of analytic functions of order α such that $0 \leq \alpha < 1\cdot$ First, it is necessary to recall the two well-known subclasses of starlike and convex functions of order α, as given below:

$$\mathscr{ST}(\alpha) = \left\{ f \in \mathscr{S} : \Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha,\ z \in \mathbb{D} \right\}$$

and

$$\mathscr{C}(\alpha) = \left\{ f \in \mathscr{S} : \Re\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha,\ z \in \mathbb{D} \right\}.$$

Selectively, when $\alpha = 0$, the above classes are reduced to their standard definition and are simply called the starlike and convex functions.

Definition 1. *A function $f \in \mathscr{A}$ of the form (6) is starlike with respect to symmetrical points if*

$$\Re\left\{ \tfrac{2zf'(z)}{f(z) - f(-z)} \right\} \geq 0,\ \ z \in \mathbb{D}.$$

We denote by $\mathscr{STS}$ the class of all such functions.

Definition 2. *A function $f \in \mathscr{A}$ of the form (6) is β-uniformly starlike of order α, if*

$$\Re\left\{ \tfrac{zf'(z)}{f(z)} - \alpha \right\} \geq \beta \left| \tfrac{zf'(z)}{f(z)} - 1 \right|, \qquad (0 \leq \alpha < 1, \beta \geq 0; z \in \mathbb{D}).$$

We denote by $\mathscr{UST}(\alpha,\ \beta)$the class of all such functions.

Definition 3. *A function $f \in \mathscr{A}$ of the form (6) isβ-uniformly convex of order α, if*

$$\Re\left\{ 1 + \tfrac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| \tfrac{zf''(z)}{f'(z)} \right|,\ (0 \leq \alpha < 1, \beta \geq 0; z \in \mathbb{D}).$$

We denote by $\mathscr{CU}(\alpha, \beta)$, the class of all such functions.

In particular, the classes $\mathscr{U}CV := \mathscr{U}CV(1,\ 0)$, $\mathscr{UST} := \mathscr{UST}(1,\ 0)$, $\beta - \mathscr{U}CV := \mathscr{U}CV(\beta,\ 0)$, and $\beta - \mathscr{UST} := \mathscr{UST}(\beta,\ 0)$ are introduced by Goodman [22,23] (see also, William Ma and David Minda [24] and Kanas and Wisniowska [25]). Furthermore, $\mathscr{UST}(1,\ 0) \subset \mathscr{STS}\cdot$ In other words, every uniformly starlike function is starlike with respect to symmetrical points.

Note that $f(z) \in \mathscr{U}CV(\alpha, \beta) \Longleftrightarrow zf'(z) \in \mathscr{UST}(\alpha, \beta)$.

The class of β-starlike functions of order α is an extension of the relatively more-well-known class of β-starlike functions investigated by Kanas et al. [26,27] (for further details, refer to Refs. [28–30]).

New subclasses of analytic functions have been introduced for various applications, such as fractional calculus and quantum calculus, by involving some special functions, such as the Mittag-Leffler and Faber polynomial functions [31–33]. The most common concern in such studies is the inclusion conditions. Alternatively, it means that for a given new subclass, $\mathscr{H}$, we seek a set of useful conditions on the sequence $\{a_n\}$ that are both necessary and sufficient for $f(z)$ to be a member of $\mathscr{H}$.

By following the same pattern, this study attempts to apply the Lambert series which has not been so yet considered in the theory of univalent functions. Consequently, this may lead to relevant studies if one considers extending the Lambert series, whose the coefficients are the sum of divisors function to other subclasses of analytic functions. Hence, we can investigate various topics such as Hankel determinants, subordination properties, and Fekete–Szegö inequalities. Furthermore, these results can be extended to multivalent functions and meromorphic functions. In addition, by using the two Robin's inequalities, one of which is analogous to the Riemann hypothesis, we can extend the resulting conclusions of some parts of this work and derive further findings. We can also obtain additional forms of the Mittag-Leffler function, including the exponential function, if we take into account certain values of the parameters in the generalized Mittag-Leffler function given by (1) and then study various special cases.

Here, we recall the definition of the Hadamard product (convolution): For a given function $f \in \mathscr{A}$ of the form (6) and $g \in \mathscr{A}$ of the form

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \qquad z \in \mathbb{D}, \tag{8}$$

the convolution $(*)$ of the two functions f and g is obtained as follows:

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{D}. \tag{9}$$

Subsequently, we utilize the Lambert series $\mathscr{L}(z)$, whose coefficients are the sum of divisors function $\sigma(n)$. The mathematical form is

$$\mathscr{L}(z) = \sum_{n=1}^{\infty} \frac{n\, z^n}{1 - z^n} = \sum_{n=1}^{\infty} \sigma(n) z^n = z + \sum_{n=2}^{\infty} \sigma(n) z^n, \in \mathbb{D}.$$

In addition, since $qF^{\theta,k}_{\alpha,\,\beta}$ does not belong to the class $\mathscr{A}$, we consider some normalization by introducing

$$q\mathbb{F}^{\theta,k}_{\alpha,\,\beta} = \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \left(qF^{\theta,k}_{\alpha,\,\beta} - 1 \right) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{z^n}{n!} \tag{10}$$

For a function $f \in \mathscr{A}$ of the form (7), we define the linear operator $\mathcal{J}(\mathscr{L}, f)(z) : \mathscr{A} \longrightarrow \mathscr{A}$ as follows:

$$\mathcal{J}(\mathscr{L}, \mathbb{F})(z) := (\mathbb{F} * \mathscr{L})(z) = z + \sum_{n=2}^{\infty} \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{n!} a_n z^n, \in \mathbb{D}.$$

The above linear operator leads us to propose a definition in the following manner:

Definition 4. *A function $f \in \mathscr{A}$ of the form (6) is said to be in the class $\mathcal{UST}_{\mathcal{J}}(\alpha)$ if the function f satisfies the following condition:*

$$\Re\left\{ \frac{z(\mathcal{J}(\mathscr{L}, \mathbb{F})(z))'}{\mathcal{J}(\mathscr{L}, \mathbb{F})(z)} - \alpha \right\} \geq \left| \frac{z(\mathcal{J}(\mathscr{L}, \mathbb{F})(z))'}{\mathcal{J}(\mathscr{L}, \mathbb{F})(z)} - 1 \right|, \ 0 \leq \alpha < 1, \ z \in \mathbb{D}. \tag{11}$$

Finally, we consider functions with negative coefficients $f \in \mathrm{T}$, similarly to the condition (11), and simply write: $\mathrm{T}_{\mathcal{J}}(\alpha) = \mathcal{U}\mathcal{S}\mathcal{T}_{\mathcal{J}}(\alpha) \bigcap \mathrm{T}\cdot$ Based on Definition 3 and the subclass $\mathrm{T}_{\mathcal{J}}(\alpha)$, the analytic characterization of the function f reduces to the following definition.

Definition 5. *A function $f \in \mathscr{A}$ of the form (7) is said to be in the class $T_{\mathcal{J}}(\alpha)$ if the function f satisfies the condition (11).*

3. Characterization Property

In this section, we discuss the characterization properties of the members that belong to the new family of analytic functions. The characterization properties include a couple of theorems related to the inclusion of functions, consequent corollaries, and a closure theorem.

Theorem 1. *A function $f \in \mathscr{A}$ of the form (7) is said to be in the class $T_{\mathcal{J}}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \tau(n)\left(\tfrac{2n-1-\alpha}{1-\alpha}\right)|a_n| \leq 1, \tag{12}$$

where

$$\tau(n) = \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{\sigma(n)}{n!}.$$

The result is sharp.

Proof. To prove the assertion in (12), it is sufficient to show that

$$\left|\frac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right| \leq \Re\left\{\frac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - \alpha\right\}.$$

After adding and subtracting 1 from the right-hand side, we obtain

$$\left|\frac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right| \leq \Re\left\{\frac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right\} + (1-\alpha),$$

that is

$$\begin{aligned}\left|\tfrac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right| - \Re\left\{\tfrac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right\} &\leq 2\left|\tfrac{z(\mathcal{J}(\mathscr{L},\mathbb{F})(z))'}{\mathcal{J}(\mathscr{L},\mathbb{F})(z)} - 1\right| \\ &\leq \frac{\sum\limits_{n=2}^{\infty}(n-1)\tau(n)|a_n|}{1-\sum\limits_{n=2}^{\infty}\tau(n)|a_n|}.\end{aligned}$$

The above expression is bounded by $(1-\alpha)$, thus proving our assertion. Conversely, let us assume that $f \in \mathrm{T}_{\mathcal{J}}(\alpha)$, then (12) yields

$$\frac{1-\sum\limits_{n=2}^{\infty} n\tau(n)|a_n|z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\tau(n)|a_n|z^{n-1}} - \alpha \geq \frac{1-\sum\limits_{n=2}^{\infty}(n-1)\tau(n)|a_n|z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\tau(n)|a_n|z^{n-1}}.$$

Letting $z \to 1$ along the real axis results in the inequality

$$\sum_{n=2}^{\infty}(2n-1-\alpha)\tau(n)|a_n| \leq 1-\alpha.$$

Finally, the result is sharp with extremal function f given by

$$f(z) = z - \frac{1-\alpha}{\tau(n)(2n-1-\alpha)}\, z^n$$

□

Corollary 1. *Let a function f defined by (7) belong to the class $T_{\mathcal{J}}(\alpha)$, then,*

$$|a_n| \le \frac{1}{\tau(n)} \cdot \frac{1-\alpha}{2n-1-\alpha}, \qquad n \ge 2.$$

Next, we obtain lower bounds for the coefficients a_n using Robin's inequalities in (4) and (5), the latter of which we simply refer to as the Riemann hypothesis.

Corollary 2. *Let a function f defined by (7) belong to the class $T_{\mathcal{J}}(\alpha)$. If*

$$|a_n| = \frac{1}{\tau(n)} \cdot \frac{1-\alpha}{2n-1-\alpha},$$

then,

$$|a_n| > \frac{1-\alpha}{2n-1-\alpha} \cdot \frac{(n-1)!\log\log n}{e^{\gamma}(\log\log n)^2 + 0.6483} \cdot \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}} \frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}, \; n \ge 3.$$

Proof. The proof follows from Corollary 1 and inequality (4). □

Corollary 3. *Let a function f defined by (7) belong to the class $T_{\mathcal{J}}(\alpha)$. Assuming that the Riemann hypothesis is true, and*

$$|a_n| = \frac{1}{\tau(n)} \cdot \frac{1-\alpha}{2n-1-\alpha},$$

then,

$$|a_n| > \frac{1-\alpha}{2n-1-\alpha} \cdot \frac{(n-1)!}{e^{\gamma}\log\log n} \cdot \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j n}}{(\theta_j)_{k_j n}} \frac{(\alpha_j)_{k_j}}{(\beta_j)_{\alpha_j}}, \; n > 5040.$$

Proof. The proof follows from Corollary 1 and inequality (5). □

Example 1. *This is a special case; if $q = \theta_1 = \beta_1 = \alpha_1 = k_1 = 1$, then*

$$qF^{\theta,k}_{\alpha,\,\beta} = 1F^{1,1}_{1,1} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp(z),$$

and we obtain the following special cases of the previous results:

1. *The condition of Theorem 1 reduces to* $\sum\limits_{n=2}^{\infty} \left(\frac{2n-1-\alpha}{1-\alpha}\right) \frac{\sigma(n)}{n!} |a_n| \le 1$;
2. *The coefficients bound in a Corollary 1 become* $|a_n| \le \frac{n!}{\sigma(n)} \cdot \frac{1-\alpha}{2n-1-\alpha}, n \ge 2$;
3. *If p is a prime number, then* $|a_p| \le \frac{p!}{p+1} \cdot \frac{1-\alpha}{2p-1-\alpha}$;
4. *If m is a perfect number, then* $|a_m| \le \frac{m!}{2m} \cdot \frac{1-\alpha}{2m-1-\alpha}$;
5. *The extremal function$f$ is given by* $f(z) = z - \frac{1-\alpha}{2n-1-\alpha} \cdot \frac{n!}{\sigma(n)}\, z^n$.

Similarly, the lower bounds in Corollaries 2 and 3, respectively, will be given by

Example 2. *If* $q = \theta_1 = \beta_1 = \alpha_1 = k_1 = 1$, *and*

$$|a_n| = \frac{n!}{\sigma(n)} \cdot \frac{1-\alpha}{2n-1-\alpha},$$

then,

$$|a_n| > \frac{1-\alpha}{2n-1-\alpha} \cdot \frac{(n-1)!\log\log n}{e^{\gamma}(\log\log n)^2 + 0.6483}, \; n \geq 3.$$

Example 3. *Under the same conditions of Example 2, assuming the Riemann hypothesis yields*

$$|a_n| > \frac{1-\alpha}{2n-1-\alpha} \cdot \frac{(n-1)!}{e^{\gamma} loglogn}\;, n > 5040.$$

Theorem 2. *Let a function f defined by (7) and $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ be in the class $T_{\mathcal{J}}(\alpha)$, then, the function h states that*

$$h(z) = (1-\beta)f(z) + \beta g(z) = z - \sum_{n=2}^{\infty} c_n z^n \quad , \tag{13}$$

where $c_n = (1-\beta)a_n + \beta b_n$, $0 \leq \beta \leq 1$ also belongs to the class $\mathrm{T}_{\mathcal{J}}(\alpha)$.

Proof. The result follows easily upon using (12) and (13). □

Next, we define the following functions $f_i(z), (i = 1, 2, 3, \ldots, m)$ of the form

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n, \qquad a_{n,i} \geq 0, \qquad z \in \mathbb{D}. \tag{14}$$

Theorem 3. *Let the functions $f_i(z), (i = 1, 2, 3, \ldots, m)$ defined by (14) be in the classes $T_{\mathcal{J}}(\alpha_i)$, then, the function h defined by*

$$h(Z) = Z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{i=1}^{m} a_{n,i} \right) z^n$$

belongs to the class $\mathrm{T}_{\mathcal{J}}(\alpha)$ for $\alpha = \min_{1 \leq i \leq m} \{\alpha_i\}$, with $0 \leq \alpha_i < 1$.

Proof. Since $f_i \in \mathrm{T}_{\mathcal{J}}(\alpha_i), (i = 1, 2, 3, \ldots, m)$, by applying Theorem 1, we observe that

$$\sum_{n=2}^{\infty} \tau(n)(2n-1-\alpha) \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) = \frac{1}{m} \sum_{i=1}^{m} \left(\sum_{n=2}^{\infty} \tau(n)(2n-1-\alpha) a_{n,i} \right).$$

□

Theorem 1. *Again entails thathis a member of $T_{\mathcal{J}}(\alpha)$.*

4. Results Involving Convolution

This section discusses the convolutional results of two functions, $f_1(z) \in \mathrm{T}_{\mathcal{J}}(\alpha)$ and $f_2(z) \in \mathrm{T}_{\mathcal{J}}(\beta)$. Apart from presenting several theorems, some useful corollaries are also deduced.

Theorem 4. *For two functions $f_i(z)$, $(i = 1, 2)$ defined by (14), let $f_1(z) \in T_{\mathcal{J}}(\alpha)$ and $f_2(z) \in T_{\mathcal{J}}(\beta)$. Then, $f_1 * f_2 \in T_{\mathcal{J}}(\xi)$, where*

$$\xi \leq 1 - \frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\tau(n) - (1-\alpha)(1-\beta)}, \;\; n \geq 2. \tag{15}$$

Proof. In view of Theorem 1, it suffices to prove that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \tau(n) a_{n,1} a_{n,2} \leq 1. \ n \ \geq 2$$

It follows from Theorem 1 and the Cauchy–Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{\sqrt{2n-1-\alpha} \cdot \sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}} \sqrt{\tau(n)} \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{16}$$

Thus, it suffices to find ξ such that

$$\sum_{n=2}^{\infty} \frac{2n-1-\xi}{1-\xi} \sqrt{\tau(n)} a_{n,1} a_{n,2} \leq \sum_{n=2}^{\infty} \frac{\sqrt{2n-1-\alpha} \cdot \sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}} \sqrt{\tau(n)} \sqrt{a_{n,1} a_{n,2}} \leq 1,$$

or

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{\sqrt{2n-1-\alpha} \cdot \sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}} \cdot \frac{1-\xi}{2n-1-\xi}.$$

By virtue of (12), it suffices to find ξ such that

$$\frac{\sqrt{(1-\alpha)(1-\beta)}}{\sqrt{2n-1-\alpha}\sqrt{2n-1-\beta}\sqrt{\tau(n)}} \leq \frac{\sqrt{2n-1-\alpha}\sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}} \cdot \frac{1-\xi}{2n-1-\xi},$$

which concedes the assertion of our theorem. □

Again, by using the inequalities (4) and (5), we establish the next two results. For brevity, we use $\phi(n)$ and $\Phi(n)$ in the forthcoming results, as indicated below.

$$\phi(n) = \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{1}{(n-1)!} \left(e^{\gamma} \log\log n + \frac{0.6483}{\log\log n} \right),$$

$$\Phi(n) = \prod_{j=1}^{q} \frac{(\beta_j)_{\alpha_j}}{(\alpha_j)_{k_j}} \frac{(\theta_j)_{k_j n}}{(\beta_j)_{\alpha_j n}} \cdot \frac{e^{\gamma} \log\log n}{(n-1)!}.$$

Corollary 4. *For two functions $f_i(z), (i = 1, 2)$ defined by (14), let $f_1(z) \in T_{\mathcal{J}}(\alpha)$ and $f_2(z) \in T_{\mathcal{J}}(\beta)$. Then, $f_1 * f_2 \in T_{\mathcal{J}}(\xi)$, where*

$$\xi \leq 1 - \frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\phi(n) - (1-\alpha)(1-\beta)}, \ n \geq 3 \ .$$

Corollary 5. *For two functions $f_i(z), (i = 1, 2)$ defined by (14), let $f_1(z) \in T_{\mathcal{J}}(\alpha)$ and $f_2(z) \in T_{\mathcal{J}}(\beta)$. If the Riemann hypothesis holds true, then, $f_1 * f_2 \in T_{\mathcal{J}}(\xi)$, where*

$$\xi \leq 1 - \frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\Phi(n) - (1-\alpha)(1-\beta)}, \ n > 5040.$$

Corollary 6. *Let the functions $f_j(z), (j = 1, 2)$ defined by (14) belong to the class $T_{\mathcal{J}}(\alpha)$. Then, $(f_1 * f_2)(z) \in T_{\mathcal{J}}(\delta)$, where*

$$\delta \leq 1 - \frac{2(n-1)(1-\alpha)^2}{(2n-1-\alpha)^2 \phi(n) - (1-\alpha)^2}, \ n \geq 2.$$

Proof. The result is established if we replace $\beta = \alpha$ in Theorem 4. □

Similarly, by using (4) and (5), we deduce two more corollaries, as shown below.

Corollary 7. *Let the functions $f_j(z),(j=1,2)$ defined by (13) belong to the class $T_{\mathcal{J}}(\alpha)$. Then, $(f_1 * f_2)(z) \in T_{\mathcal{J}}(\delta)$, where*

$$\delta \le 1 - \frac{2(n-1)(1-\alpha)^2}{(2n-1-\alpha)^2\phi(n) - (1-\alpha)^2},\ n \ \geq 3.$$

Corollary 8. *Let the functions $f_j(z),(j=1,2)$ defined by (13) belong to the class $T_{\mathcal{J}}(\alpha)$, assuming that the Riemann hypothesis is true, then, $(f_1 * f_2)(z) \in T_{\mathcal{J}}(\delta)$, where*

$$\delta \le 1 - \frac{2(n-1)(1-\alpha)^2}{(2n-1-\alpha)^2\Phi(n) - (1-\alpha)^2},\ n > 5040.$$

Theorem 5. *Let the function f defined by (7) belong to the class $T_{\mathcal{J}}(\alpha)$, and let $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n$, for $|b_n| \le 1$. Then, $(f * g)(z) \in T_{\mathcal{J}}(\alpha)$.*

Proof. Using the convolution property and the concept defined by the left-hand side of (12), we construct the following relation:

$$\begin{aligned} \sum_{n=2}^{\infty} \tau(n)(2n-1-\alpha)|a_n b_n| &= \sum_{n=2}^{\infty} \tau(n)(2n-1-\alpha)|a_n||b_n| \\ &\le \sum_{n=2}^{\infty} \tau(n)(2n-1-\alpha)|a_n| \le 1-\alpha. \end{aligned}$$

Hence, it follows that $(f * g)(z) \in \mathrm{T}_{\mathcal{J}}(\alpha)$. □

Corollary 9. *Let the function f defined by (7) belong to the class $T_{\mathcal{J}}(\alpha)$. Furthermore, let $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n$ for $0 \le b_n \le 1$, then, $(f * g)(z) \in T_{\mathcal{J}}(\alpha)$.*

Now, we consider the following:

Theorem 6. *Let the functions $f_j(z),(j=1,2)$ defined by (14) belong to the class $T_{\mathcal{J}}(\alpha)$. Then, the function h, defined by $h(z) = z - \sum\limits_{n=2}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n$, belongs to the class $T_{\mathcal{J}}(\mu)$, where*

$$\mu \le 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2\tau(n) - 2(1-\alpha)^2}, \qquad n \ge 2. \tag{17}$$

Proof. In view of Theorem 1, it suffices to show that

$$\sum_{n=2}^{\infty} \tau(n)\frac{2n-1-\mu}{1-\mu}\left(a_{n,1}^2 + a_{n,2}^2\right) \le 1. \tag{18}$$

From (11) and Theorem 1, we find that

$$\sum_{n=2}^{\infty}\left[\tau(n)\frac{2n-1-\alpha}{1-\alpha}\right]^2 a_{n,j}^2 \le \left[\sum_{n=2}^{\infty}\tau(n)\frac{2n-1-\alpha}{1-\alpha}\ a_{n,j}\right]^2, \tag{19}$$

which yields

$$\sum_{n=2}^{\infty}\tfrac{1}{2}\left[\tau(n)\tfrac{2n-1-\alpha}{1-\alpha}\right]^2\left(a_{n,1}^2 + a_{n,2}^2\right) \le 1\ . \tag{20}$$

Upon comparing the inequalities (19) and (20), it is evident that the inequality (17) is satisfied if

$$\tau(n)\frac{2n-1-\mu}{1-\mu} \leq \frac{1}{2}\left[\tau(n)\frac{2n-1-\alpha}{1-\alpha}\right]^2,\ n\ \geq 2,$$

that is, if

$$\mu \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2\tau(n) - 2(1-\alpha)^2}.$$

This completes the proof. □

Using (4) and (5), respectively, we can readily prove the next two inequalities.

Corollary 10. *Let the functions $f_j(z), (j=1,2)$ defined by (14) belong to the class $T_{\mathcal{J}}(\alpha)$. Then, the function h defined by $h(z) = z - \sum_{n=2}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n$ belongs to the class $T_{\mathcal{J}}(\mu)$, where*

$$\mu \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2\phi(n)-2(1-\alpha)^2}, \qquad n \geq 3.$$

Corollary 11. *Let the functions $f_j(z), (j=1,2)$ defined by (14) belong to the class $T_{\mathcal{J}}(\alpha)$ and let us assume the Riemann hypothesis is true, then, the function h defined by $h(z) = z - \sum_{n=2}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n$ belongs to the class $T_{\mathcal{J}}(\mu)$, where*

$$\mu \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2\Phi(n)-2(1-\alpha)^2}, \qquad n > 5040.$$

5. The Integral Transform of Class $T_{\mathcal{J}}(\alpha)$

To convert class $T_{\mathcal{J}}(\alpha)$ into integral form, we define the following integral transform:

$$V_\mu(f)(z) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt,$$

where $\mu(t)$ is a real valued, non-negative, and normalized weight function such that $\int_0^1 \mu(t)dt = 1$.

The special case of $\mu(t)$ is $\mu(t) = \frac{(c+1)^\delta}{\mu(\delta)}t^c\left(\log\frac{1}{t}\right)^{\delta-1},\ c > -1, \delta \geq 0$, which yields the Komatu operator.

Theorem 7. *Let $f \in T_{\mathcal{J}}(\alpha)$, then, $V_\mu(f) \in T_{\mathcal{J}}(\alpha)$.*

Proof. By definition, we have,

$$\begin{aligned}
V_\mu(f) &= \frac{(c+1)^\delta}{\mu(\delta)}\int_0^1 (-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}a_n z^n t^{n-1}\right)dt \\
&= \frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)}\lim_{r\to 0^+}\left[\int_r^1 t^c(\log t)^{\delta-1}\left(z - \sum_{n=2}^{\infty}a_n z^n t^{n-1}\right)dt\right].
\end{aligned}$$

By applying basic mathematical principles, we derive the following expression:

$$V_\mu(f)(z) = z - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \tau(n) \cdot \frac{2n-1-\alpha}{1-\alpha} \left(\frac{c+1}{c+n}\right)^{\delta} |a_n| \leq 1. \tag{21}$$

Conversely, $f \in \mathrm{T}_{\mathcal{J}}(\alpha)$ if and only if,

$$\sum_{n=2}^{\infty} \tau(n) \cdot \frac{2n-1-\alpha}{1-\alpha} |a_n| \leq 1.$$

This shows that $\frac{c+1}{c+n} < 1$, and, hence, Equation (21) holds. Thus, the proof is evident. □

Next, we derive the radii of starlikeness and convexity of $V_\mu(f)$

Theorem 8. *Let $f \in T_{\mathcal{J}}(\alpha)$, then, $V_\mu(f)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \cdot \frac{(1-\xi)(2n-1-\alpha)}{(n-\xi)(1-\alpha)} \tau(n) \right]^{\frac{1}{n-1}}.$$

The result is sharp with extremal function $f(z)$ given in the proof of Theorem 1.

Proof. It is sufficiently fair to confirm that $\left| \frac{z\left(V_\mu(f)(z)\right)'}{V_\mu(f)(z)} - 1 \right| < 1 - \xi$.

Considering the left-hand side of the above inequality, we write

$$\begin{aligned} \left| \frac{z\left(V_\mu(f)(z)\right)'}{V_\mu(f)(z)} - 1 \right| &= \left| \frac{\sum_{n=2}^{\infty} (1-n) \left(\frac{c+1}{c+n}\right)^{\delta} |a_n| z^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} |a_n| z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} (n-1) \left(\frac{c+1}{c+n}\right)^{\delta} |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} |a_n| |z|^{n-1}}. \end{aligned}$$

The last expression is less than $1 - \xi$ as

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^{\delta} \frac{(1-\xi)(2n-1-\alpha)}{(n-\xi)(1-\alpha)} \tau(n).$$

This completes the proof. □

Utilizing inequalities (4) and (5) again, we receive the following

Corollary 12. *Let $f \in T_{\mathcal{J}}(\alpha)$. Then, $V_\mu(f)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \cdot \frac{(1-\xi)(2n-1-\alpha)}{(n-\xi)(1-\alpha)} \phi(n) \right]^{\frac{1}{n-1}}, \ n \geq 3.$$

Corollary 13. *Let $f \in T_{\mathcal{J}}(\alpha)$ and let us assume that the Riemann hypothesis is true. Then, $V_\mu(f)$ is starlike of order $0 \leq \xi < 1$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \cdot \frac{(1-\xi)(2n-1-\alpha)}{(n-\xi)(1-\alpha)} \Phi(n) \right]^{\frac{1}{n-1}}, \ n > 5040.$$

Finally, for this section, we have:

Theorem 9. *If $f \in T_{\mathcal{J}}(\alpha)$, then, $V_\mu(f)$ is convex of order $0 \le \gamma < 1$, in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\gamma)(2n-1-\alpha)}{n(n-\gamma)(1-\alpha)} \tau(n) \right]^{\frac{1}{n-1}}.$$

The result is sharp with extremal function $f(z)$ given in the proof of Theorem 1.

Proof. The proof is evident from the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike. □

Corollary 14. *If $f \in T_{\mathcal{J}}(\alpha)$,then, $V_\mu(f)$is convex of order $0 \le \xi < 1$, in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)(2n-1-\alpha)}{n(n-\xi)(1-\alpha)} \phi(n) \right]^{\frac{1}{n-1}}, \; n \ge 3.$$

Corollary 15. *If $f \in T_{\mathcal{J}}(\alpha)$ and if the Riemann hypothesis is true, then, $V_\mu(f)$ is convex of order $0 \le \xi < 1$, in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1} \right)^\delta \frac{(1-\xi)(2n-1-\alpha)}{n(n-\xi)(1-\alpha)} \Phi(n) \right]^{\frac{1}{n-1}}, \; n > 5040.$$

6. Second Hankel Determinant

We derive the second Hankel determinant inequality for a function $f \in \mathrm{T}_{\mathcal{J}}(\alpha)$. First, we recall the definition of the Hankel determinant of a locally univalent analytic function $f(z)$ for $s \ge 1$, $n \ge 1$, (See [34])

$$H_s(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+s-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+s} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+s-1} & a_{n+s} & \dots & a_{n+2s-2} \end{vmatrix}.$$

Here, we consider the second Hankel determinant of f the case when $q = 2$ and $n = 2$, i.e., $H_2(2) = a_2a_4 - a_3^2$.

Lemma 1. ([35]). *Let $\mathscr{P}$ (Carathèodory class of functions) be the class of all analytic functions $p(z)$ of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} C_n z^n, \tag{22}$$

satisfying $\Re(p(z)) > 0 (z \in \mathbb{D})$ and $p(0) = 1$. Then,

$$|C_n| \le 2 (n = 1, 2, 3, \dots).$$

This inequality is sharp for each n. In particular, equality holds for all n for the function

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n$$

Lemma 2. ([36]). *If the function $p \in \mathscr{P}$, then*

$$2C_2 = C_1^2 + (4 - C_1^2)x$$
$$4C_3 = C_1^3 + 2C_1x(4 - C_1^2) - C_1x^2(4 - C_1^2) + 2(4 - C_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \le 1$ and $|z| \le 1$.

Theorem 10. *Let $f(z)$ given by (7) be in the class $T_{\mathcal{J}}(\alpha)$.*

(1) If

$$\begin{aligned}
&\mathscr{M} + 3P_1^2\tau(3)^2 - \left(2P_1\mathcal{S} - \tfrac{9}{4}P_1^2\right)\tau(2)\tau(4) \le 0, \\
&\mathcal{N} + \mathscr{M} + \tau(3)^2P_1^2 + \left(\tfrac{13}{4}P_1^2 - \mathcal{S}^2 - 2P_1\mathcal{S}\right) \le 0,
\end{aligned}$$

then the second Hankel determinant satisfies the inequality;

$$\left|a_2a_4 - a_3^2\right| \le \frac{P_1^2}{4\tau(3)^2}.$$

(2) If

$$\begin{aligned}
&\mathscr{M} + 3P_1^2\tau(3)^2 - \left(2P_1\mathcal{S} - \tfrac{9}{4}P_1^2\right)\tau(2)\tau(4) \ge 0, \\
&\Big[2\mathcal{N} - \mathscr{M} - 7\tau(3)^2P_1^2 + 2 \\
&\times\left(P_1^2 - \mathcal{S}^2 + P_1\mathcal{S} - \tfrac{9}{8}P_1^2\right)\tau(2)\tau(4)\Big] \ge 0
\end{aligned}$$

or the conditions

$$\begin{aligned}
&\mathscr{M} + 3P_1^2\tau(3)^2 - \left(2P_1\mathcal{S} - \tfrac{9}{4}P_1^2\right)\tau(2)\tau(4) \le 0, \\
&\mathcal{N} + \mathscr{M} + \tau(3)^2P_1^2 + \left[\tfrac{13}{4}P_1^2 - \mathcal{S}^2 - 2P_1\mathcal{S}\right] \ge 0.
\end{aligned}$$

Then the second Hankel determinant satisfies the inequality

$$\left|a_2a_4 - a_3^2\right| \le \frac{\mathcal{N} + \mathscr{M} + \tau(3)^2P_1^2 + \left(\tfrac{7}{2}P_1^2 - \mathcal{S}^2 - 2P_1\mathcal{S}\right)\tau(2)\tau(4)}{\tau(2)\tau(4)\tau(3)^2}.$$

(3) If

$$\begin{aligned}
&\mathscr{M} + 3P_1^2\tau(3)^2 - \left(2P_1\mathcal{S} - \tfrac{9}{4}P_1^2\right)\tau(2)\tau(4) > 0, \\
&\big[2\mathcal{N} - \mathscr{M} - 7\phi_3^2P_1^2 + 2 \\
&\times\left(P_1^2 - \mathcal{S}^2 + P_1\mathcal{S} - \tfrac{9}{8}P_1^2\right)\tau(2)\tau(4)\big] \le 0,
\end{aligned}$$

then the second Hankel determinant satisfies the inequality

$$\begin{aligned}
\left|a_2a_4 - a_3^2\right| \le \quad & \tfrac{1}{16\tau(2)\tau(4)\tau(3)^2}\Big[4\tau(2)\tau(4)P_1^2 \\
& - \tfrac{\left[4\mathscr{M} + 12P_1^2\tau(3)^2 - \left(8P_1\mathcal{S} - 9P_1^2\right)\tau(2)\tau(4)\right]^2}{\left[\mathcal{N} - 2\tau(3)^2P_1^2 + \left(P_1^2 - \mathcal{S}^2\right)\tau(2)\tau(4)\right]}\Big],
\end{aligned}$$

where$\mathcal{N}$, $\mathscr{M}$, and $\mathcal{S}$ are given by

$$\begin{aligned}
\mathcal{N} &= P_1\tau(3)^2\left[P_1^3 + (P_3 - 2P_2 + P_1) + (P_2 - P_1)P_1\right], \\
\mathscr{M} &= P_1\tau(3)^2\left[P_1^2 + 2(P_2 - P_1)\right] \\
\mathcal{S} &= P_1^2 + (P_2 - P_1).
\end{aligned} \tag{23}$$

The result is sharp with extremal function $f(z)$ given in the proof of Theorem 1.

Proof. By using the properties of Carathèodory functions, we can rewrite the definition of $T_{\mathcal{J}}(\alpha)$. Setting

$$p(z) = \frac{z(\mathcal{J}(\mathscr{L}, \mathbb{F})(z))'}{\mathcal{J}(\mathscr{L}, \mathbb{F})(z)},$$

and we write

$$\Re(p(z)) > \beta|p(z) - 1| + \alpha, \ z \in \mathbb{D},$$

or, $p(z) \prec p_{\beta,\alpha}$, where

$$p_{\beta,\,\alpha} = 1 + P_1 z + P_2 z^2 + \ldots$$

is a function with positive real part, which maps the unit disk onto a domain $\Omega_{\beta,\alpha}$ described by the inequality $\Re(w) > \beta|w-1| + \alpha$.

Let $\mathrm{T}_{\mathcal{J}}(\alpha)$, then there exists a Schwarz function $w, w(0) = 1$, $|w(z)| < 1$ for $z \in \mathbb{D}$, such that

$$\frac{z(\mathcal{J}(\mathcal{L},\mathbb{F})(z))'}{\mathcal{J}(\mathcal{L},\mathbb{F})(z)} = p_{\beta,\alpha}(w(z))$$

Further, let

$$p_0(z) = \frac{1+w(z)}{1-w(z)} = 1 + C_1 z + C_2 z^2 + \cdots \tag{24}$$

or, equivalently

$$w(z) = \frac{p_0(z)-1}{p_0(z)+1} = \frac{1}{2}\left[C_1 z + \left(C_2 - \frac{C_1^2}{2}\right)z^2 + \cdots\right]$$

where the function p_0 is analytic in the unit disk and has a positive real part, by using the Taylor expansion of $p_{\beta,\alpha}$ and w, we obtain

$$\begin{aligned} p_{\beta,\alpha}(w(z)) = \quad & 1 + \tfrac{P_1 C_1}{2} z + \left(\tfrac{P_1 C_2}{2} + \tfrac{(P_2 - P_1)C_1^2}{4}\right) z^2 \\ & + \left(\tfrac{P_1 C_3 + (P_2 - P_1) C_1 C_2}{2} + \tfrac{(P_1 + P_3) C_1^3}{8} - \tfrac{P_2 C_1^3}{4}\right) z^3 + \cdots . \end{aligned} \tag{25}$$

Now, we have

$$\frac{z(\mathcal{J}(\mathcal{L},\mathbb{F})(z))'}{\mathcal{J}(\mathcal{L},\mathbb{F})(z)} = 1 + \tau(2)a_2 z + \left(2\tau(3)a_3 - \tau(2)^2 a_2^2\right)z^2 + \left(3\tau(4)a_4 - 3\tau(2)a_2\tau(3)a_3 + \tau(2)^2 a_2^2\right)z^3 \tag{26}$$

By equating the last two Equations (25) and (26), we get

$$a_2 = \frac{P_1 C_1}{2\tau(2)} \tag{27}$$

$$a_3 = \frac{1}{4\tau(3)}\left[P_1^2 C_1^2 + 2P_1 C_2 + C_1^2(P_2 - P_1)\right] \tag{28}$$

$$\begin{aligned} a_4 = \quad & \tfrac{C_1^3\left[P_1^3 + (P_3 - 2P_2 + P_1) + (P_2 - P_1)P_1\right]}{8\tau(4)} \\ & + \tfrac{2C_1 C_2\left[P_1^2 + 2(P_2 - P_1)\right]}{8\tau(4)} + \tfrac{4P_1 C_3}{8\tau(4)} \end{aligned} \tag{29}$$

Therefore,

$$H_2(2) = a_2 a_4 - a_3^2 = \frac{\begin{array}{c} C^4\mathcal{N} + 2C^2 C_2 \mathcal{M} + 4C_3 C P_1^2 \tau(3)^3 \\ -\tau(2)\tau(4)\left[2C_2 P_1 + C^2\mathcal{S}\right]^2 \end{array}}{16\tau(2)\tau(4)\tau(3)^2}$$

where $C = C_1 > 0$ and $\mathcal{N}$, $\mathcal{M}$, $\mathcal{S}$ are given by (23). Now by applying Lemma 2, we obtain

$$\begin{aligned} H_2(2) = & \tfrac{C^4\left[\mathcal{N} + \mathcal{M} + \tau(3)^2 P_1^2 - \tau(2)\tau(4)P_1^2 - \tau(2)\tau(4)\mathcal{S}^2 - 2\tau(2)\tau(4)P_1\mathcal{S}\right]}{16\ \tau(2)\tau(4)\tau(3)^2} \\ & + \tfrac{\tau C^2\left(4 - C^2\right)\left[\mathcal{M} + 2P_1^2\tau(3)^2 - 2\tau(2)\tau(4)P_1\mathcal{S} - 2\tau(2)\tau(4)P_1^2\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \tfrac{\tau^2\left(4 - C^2\right)\left[C^2 P_1^2\tau(3)^2 + \tau(2)\tau(4)P_1^2\right]}{16\ \tau(2)\tau(4)\tau(3)^2} \\ & + \tfrac{2C\left(4 - C^2\right)\phi_3^2 P_1^2\left(1 - |\tau|^2\right)\mathcal{S}}{16\tau(2)\tau(4)\tau(3)^2} \end{aligned} \tag{30}$$

Now, we may assume, without restriction, that $C \in [0,2]$. Since $p(z) \in \mathcal{P}$, so $|C_1| \leq 2$. We set $\rho = |\tau|$, where $-1 \leq \rho \leq 1$ and applying triangle inequality on $H_2(2)$ for all $|z| \leq 1$, we obtain

$$|H_2(2)| \leq \Upsilon(\rho, C) = v\left(t_1\rho^2 + t_2\rho + t_3\right), \tag{31}$$

where

$$\begin{aligned} t_1 &= \left(4 - C^2\right)\left\{C^2 P_1^2 \tau(3)^2 + \tau(2)\tau(4)P_1^2 - 2C\tau(3)^2 P_1^2\right\} \\ &= \left(4 - C^2\right)P_1^2\left[\tau(3)^2 C(C-2)\tau(2)\tau(4)\right] \end{aligned} \tag{32}$$

$$t_2 = C^2\left(4 - C^2\right)\left|\mathscr{M} + 2P_1^2\tau(3)^2 - 2\tau(2)\tau(4)P_1\mathcal{S} - 2\tau(2)\tau(4)P_1^2\right| \tag{33}$$

$$t_3 = C^4\left|\mathscr{N} + \mathscr{M} + \tau(3)^2P_1^2 - \tau(2)\tau(4)P_1^2 - \tau(2)\tau(4)\mathcal{S}^2 - 2\tau(2)\tau(4)P_1\mathcal{S}\right| + 2C\left(4 - C^2\right)\tau(3)^2P_1^2 \tag{34}$$

$$v = \frac{1}{16\tau(2)\tau(4)\tau(3)^2}$$

Differentiating (32) with respect to ρ we get

$$\frac{\partial Y(\rho, C)}{\partial \rho} = v(2t_1\rho + t_2)$$

The inequality $t_2 \geq 0$ is obvious; $t_1 \geq 0$ such that

$$t_1 = \left(4 - C^2\right)P_1^2\left[\tau(3)^2 C(C-2) + \tau(2)\tau(4)\right].$$

One can simply show that $\frac{\partial Y(\rho,C)}{\partial \rho} > 0$ for $\rho > 0$, hence, $\Upsilon(\rho, C)$ is an increasing function and, thus, the upper bound for $\Upsilon(\rho, C)$ corresponds to $\rho = 1$ and

$$maxY(\rho) = Y(1, C) = G(C),$$

$$\begin{aligned} G(C) = \; & \frac{C^4\left[\mathscr{N} + \mathscr{M} + \tau(3)^2P_1^2 - \tau(2)\tau(4)P_1^2 -\tau(2)\tau(4)\mathcal{S}^2 - 2\tau(2)\tau(4)P_1\mathcal{S}\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \frac{C^2\left(4 - C^2\right)\left[\mathscr{M} + 2P_1^2\tau(3)^2 - 2\tau(2)\tau(4)P_1\mathcal{S} -2\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \frac{\left[C^2\left(4 - C^2\right)P_1^2\tau(3)^2 + \left(4 - C^2\right)\tau(2)\tau(4)P_1^2\right]}{16\tau(2)\tau(4)\tau(3)^2}. \end{aligned} \tag{35}$$

We simplify as

$$\begin{aligned} G(C) = \; & \frac{C^4\left[\mathscr{N} - 2\tau(3)^2P_1^2 + \left(P_1^2 - \mathcal{S}^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \frac{C^2\left[4\mathscr{M} + 12P_1^2\tau(3)^2 - \left(8P_1\mathcal{S} - 9P_1^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \frac{\left[4\tau(2)\tau(4)P_1^2\right]}{16\tau(2)\tau(4)\tau(3)^2}, \end{aligned} \tag{36}$$

with the first and second derivatives given, respectively:

$$\begin{aligned} G'(C) = \; & \frac{4C^3\left[\mathscr{N} - 2\tau(3)^2P_1^2 + \left(P_1^2 - \mathcal{S}^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} \\ & + \frac{2C\left[4\mathscr{M} + 12P_1^2\tau(3)^2 - \left(8P_1\mathcal{S} - 9P_1^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} \end{aligned}$$

$$G''(C) = \frac{12C^2\left[\mathcal{N}-2\tau(3)^2P_1^2+\left(P_1^2-\mathcal{S}^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2} + \frac{2\left[4\mathcal{M}+12P_1^2\tau(3)^2-\left(8P_1\mathcal{S}-9P_1^2\right)\tau(2)\tau(4)\right]}{16\tau(2)\tau(4)\tau(3)^2}$$

Solving $G'(C) = 0$ shows that the $\max G(C)$ occurs at $C = 0$.
Using (36) we write

$$G(C) = \frac{1}{v}\left(\ell_1\tau^2 + \ell_2\tau + \ell_3\right)$$

where

$$\begin{aligned} \ell_1 &= \mathcal{N} - 2\tau(3)^2P_1^2 + \left(P_1^2 - \mathcal{S}^2\right)\tau(2)\tau(4), \\ \ell_2 &= 4\mathcal{M} + 12P_1^2\tau(3)^2 - \left(8P_1\mathcal{S} - 9P_1^2\right)\tau(2)\tau(4), \\ \ell_3 &= 4\tau(2)\tau(4)P_1^2 \end{aligned}$$

and $C^2 = \tau$. Since

$$\max_{0\le\tau\le4}\left(\ell_1\tau^2 + \ell_2\tau + \ell_3\right) = \begin{cases} \ell_3,\ \ell_2 \ge 0,\ \ell_1 \ge -\frac{\ell_2}{8}(\text{or})\ \ell_2 \le 0, \ell_1 \ge -\frac{\ell_2}{4}, \\ 16\ell_1 + 4\ell_2 + \ell_3, \ell_2 > 0, \ell_1 \le -\frac{\ell_2}{8} \\ \frac{4\ell_1\ell_3-\ell_2^2}{4\ell_1} \end{cases}$$

$$\max_{0\le\tau\le4}\left(\ell_1\tau^2 + \ell_2\tau + \ell_3\right) \quad \ell_2 \le 0,\ \ell_1 \le -\frac{\ell_2}{4}$$

$$= \begin{cases} \ell_3, & \ell_2 \le 0,\ \ell_1 \le -\frac{\ell_2}{4}, \\ 16\ell_1 + 4\ell_2 + \ell_3, & \ell_2 \ge 0, \ell_1 \ge -\frac{\ell_2}{8}\ or\ \ell_2 \le 0,\ \ell_1 \ge -\frac{\ell_2}{4}, \\ \frac{4\ell_1\ell_3-\ell_2^2}{4\ell_1}, & \ell_2 > 0, \ell_1 \le -\frac{\ell_2}{8}, \end{cases}$$

Finally, we have

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{16\tau(2)\tau(4)\tau(3)^2} \times \begin{cases} \ell_3, & \ell_2 \le 0,\ \ell_1 \le -\frac{\ell_2}{4}, \\ 16\ell_1 + 4\ell_2 + \ell_3, & \ell_2 \ge 0, \ell_1 \ge -\frac{\ell_2}{8} or\ \ell_2 \le 0,\ \ell_1 \ge -\frac{\ell_2}{4}. \\ \frac{4\ell_1\ell_3-\ell_2^2}{4\ell_1}, & \ell_2 > 0, \ell_1 \le -\frac{\ell_2}{8}, \end{cases}$$

Which, after simple calculations completes the proof of Theorem 10. □

7. Conclusions

In this article, we introduce a new subclass of uniformly starlike functions by utilizing the Lambert series, with coefficients derived from the arithmetic function σ(n). Consequently, we explore the characteristics of the proposed subclass. Furthermore, we discuss several relevant topics, including the Hadamard product, integral transform, and radii of starlikeness and convexity. In addition, we extended some findings by incorporating Robin's inequalities and the Riemann hypothesis. Thus, applying the Lambert series to additional subclasses of analytic functions may lead to significant research outcomes. Consequently, we can conduct research on various subjects, including Fekete–Szegö inequalities and subordination characteristics. Furthermore, multivalent functions and meromorphic functions can be included in the scope of these conclusions.

Generally, if we apply the same methodology as this study and take into account the Lambert series, whose coefficients are the higher-order sum of divisors function $\sigma_\alpha(n)$, and if we investigate various special cases of the Mittag-Leffler function, we can also get more intriguing results.

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