symmetry

MDPI

Article

On Neutrosophic Fuzzy Metric Space and Its Topological Properties

Samriddhi Ghosh [1], Sonam [1], Ramakant Bhardwaj [1] and Satyendra Narayan [2,*]

[1] Department of Mathematics, Amity University Kolkata, Newtown, Kolkata 700135, West Bengal, India; samriddhi.ghosh3@s.amity.edu (S.G.); smotla@kol.amity.edu (S.); rbhardwaj@kol.amity.edu (R.B.)
[2] Department of Applied Computing, Sheridan Institute of Technology, Oakville, ON L6H 2L1, Canada
* Correspondence: satyendra.narayan@sheridancollege.ca or narayan.satyendra@gmail.com

Abstract: The present research introduces a novel concept termed "neutrosophic fuzzy metric space", which extends the traditional metric space framework by incorporating the notion of neutrosophic fuzzy sets. A thorough investigation of various structural and topological properties within this newly proposed generalization of metric space has been conducted. Additionally, counterparts of well-known theorems such as the Uniform Convergence Theorem and the Baire Category Theorem have been established for this generalized metric space. Through rigorous analysis, a detailed understanding of its fundamental characteristics has been attained, illuminating its potential applications and theoretical significance.

Keywords: fuzzy sets; neutrosophic sets; neutrosophic fuzzy sets; neutrosophic metric space; neutrosophic fuzzy metric space

Citation: Ghosh, S.; Sonam; Bhardwaj, R.; Narayan, S. On Neutrosophic Fuzzy Metric Space and Its Topological Properties. *Symmetry* **2024**, *16*, 613. https://doi.org/10.3390/sym16050613

Academic Editor: Calogero Vetro

Received: 10 April 2024
Revised: 30 April 2024
Accepted: 2 May 2024
Published: 15 May 2024

1. Introduction

Fuzzy set theory is a mathematical framework that deals with sets whose elements have degrees of membership. This departure from traditional set theory, where an element either belongs or does not belong to a set, allows for more nuanced and flexible modelling, particularly in situations where boundaries between categories are ambiguous or uncertain. Zadeh's [1] seminal paper, "Fuzzy Sets", published in 1965, laid the foundation for this field of study. This whole new field of study has influenced many other scientific fields since that time. Numerous advancements have emerged subsequent to Zadeh's introduction of fuzzy sets in 1965. For instance, Atanassov [2] introduced intuitionistic fuzzy sets, Smarandache [3] proposed neutrosophic sets, and in 2023, Al-Shami and Mhemdi [4] introduced orthopair fuzzy sets. These foundational contributions paved the way for the development of various related concepts, including the introduction of fuzzy metric spaces by Kramosil and Michálek [5] in 1975 and its modification by George and Veeramani [6] in 1994 , intuitionistic fuzzy topological spaces by Coker [7] in 1997, and intuitionistic fuzzy metric spaces by Park [8] in 2004.

The introduction of all these concepts and ideologies has impacted many other research works by several mathematicians, such as: in 1984, Kaleva and Seikkala [9] defined fuzzy metric space as a distance between two points and expressed it as a positive fuzzy number; in 2006, Smarandache [10] gave the notion of neutrosophic sets as a generalization of intuitionistic fuzzy sets; Salama and Alblowi [11] in 2012 extended the concepts of fuzzy topological space and intuitionistic fuzzy topological space to the case of neutrosophic sets; Ejegwa [12], in 2014, introduced some algebraic operations such as modal operator, and normalization to intuitionistic fuzzy sets; and Majumdar [13] conducted an exploration into the practical applications of neutrosophic sets within decision-making contexts. Apart from these, several researchers delved into various generalizations of fuzzy metric space such as intuitionistic fuzzy metric space, soft fuzzy metric space, fuzzy soft metric space, orthogonal fuzzy metric space, bipolar fuzzy metric space, etc. [14–24]. Furthermore, Broumi [25] introduced several novel concepts related to neutrosophic sets, including refined neutrosophic sets, bipolar neutrosophic sets, neutrosophic hesitant sets, multi-valued neutrosophic sets, rough neutrosophic sets, and rough bipolar neutrosophic sets. Additionally, Jeyaraman [26] applied multi-criteria decision-making (MCDM) techniques in conjunction with neutrosophic sets to address a real-life problem involving wind turbines.

In 2020, Kirisci and Şimşek [27] introduced the notion of neutrosophic metric spaces. Further, several fixed-point results have been established in this generalization [28] with some generalizations of neutrosophic fuzzy metric space were introduced by researchers such as orthogonal neutrosophic metric spaces by Ishtiaq et al. [29], neutrosophic 2-metric spaces by Asghar et al. [30], orthogonal neutrosophic 2-metric spaces by Janardhanan et al. [31], neutrosophic pentagonal metric spaces by Mani et al. [32], neutrosophic b-metric space [33], etc. Further, Das et al. [34] introduced the conception of neutrosophic fuzzy sets. It is quite noticeable how the concept of fuzzy sets has been used to establish the concept of fuzzy metric space. In a similar way, the ideology of neutrosophic sets has been used to define the conception of neutrosophic metric space. Keeping these in mind, the primary objective of this research is to elucidate the conceptual framework of neutrosophic fuzzy metric space. The concept of neutrosophic fuzzy sets has been utilised for this purpose, in a similar manner as described above.

The motivation behind introducing neutrosophic fuzzy metric spaces (NFMS) lies in their ability to model uncertainty and indeterminacy in real-world phenomena more accurately than traditional fuzzy or crisp sets. Neutrosophic sets allow for the representation of elements with three components, truth, indeterminacy, and falsehood, providing a more nuanced description of uncertainty. An example that highlights the need for NFMS could be in the field of medical diagnosis. Consider a scenario where a patient presents symptoms that are not clearly indicative of a single disease. Traditional fuzzy sets could represent the likelihood of the patient having a certain disease on a continuum between 0 and 1. However, this approach might not adequately capture the uncertainty associated with the diagnosis, especially if the symptoms are ambiguous or conflicting. In contrast, neutrosophic sets can represent the uncertainty more comprehensively by explicitly accounting for the degree of truth, indeterminacy, and falsehood associated with each symptom and potential diagnosis. NFMS can then be used to define a metric space where the distance between two diagnoses reflects not only their similarity but also the degree of uncertainty or indeterminacy associated with each.

By introducing NFMS, we provide a mathematical framework that aligns more closely with the complex and uncertain nature of real-world phenomena, such as medical diagnosis, decision making under uncertainty, or pattern recognition in ambiguous data sets. This motivates the development of NFMS theory to enhance our ability to model, analyze, and make decisions in uncertain environments more effectively.

The following describes the structure of the paper. Some characteristics and fundamental ideas of neutrosophic fuzzy sets and neutrosophic metric spaces are provided in Section 2. Section 3 introduces the conception of neutrosophic fuzzy metric space, followed by examples of neutrosophic fuzzy metric space. The topological characteristics of the established generalization of metric space have been demonstrated in this section. In this section, we have also presented various results describing distinct properties of the established neutrosophic fuzzy metric space, such as the Hausdorff property, compactness, completeness, and nowhere denseness. Section 4 of this work contains the conclusion. Further, this study can be extended to investigate neutrosophic fuzzy metric space in connection with other concepts such as best proximity point results, optimization theory, approximation theory, etc. Moreover, concepts like soft sets, soft equality, soft lattices, etc., can be connected to the established theory and results [35,36].

2. Preliminaries

In this section, we provide some fundamentally useful definitions for establishing the main results. Here, $]0^-, 1^+[$ is a non-standard unit interval, where the non-standard finite numbers $(1^+) = 1 + \varepsilon$, where "1" is its standard part and ε its non-standard part, and $(0^-) = 0 - \varepsilon$, where "0" is its standard part and ε its non-standard part. Here, 0 and 1 are analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, respectively, and belong to the non-standard unit interval $]0^-, 1^+[$.

Definition 1 ([1])**.** *With respect to a universal set X, a fuzzy set F is characterised by the expression* $F = \{< a, \mu_F(a) >: 0 \le \mu_F(a) \le 1, a \in X\}$*. Here,* $\mu_F(a)$ *signifies the membership grade of a in F.*

Definition 2 ([27])**.** *A neutrosophic set* $\mathbf{N}$ *with respect to a universal set X is declared as* $\mathbf{N} = \{< a, (T_{\mathbf{N}}(a), I_{\mathbf{N}}(a), F_{\mathbf{N}}(a)) >: a \in X,\ T_{\mathbf{N}}(a), I_{\mathbf{N}}(a), F_{\mathbf{N}}(a) \in]0^-, 1^+[\}$*. Here,* $T_{\mathbf{N}}(a), I_{\mathbf{N}}(a)$*, and* $F_{\mathbf{N}}(a)$ *denote the truth, indeterminacy, and falsity membership grades of a in* $\mathbf{N}$*, respectively, and* $]0^-, 1^+[$ *a non-standard unit interval.*

Definition 3 ([34])**.** *A neutrosophic fuzzy set B in a universal set X is defined as*

$$B = \{< x, (\mu_B(x), T_B(x,\mu), I_B(x,\mu), F(x,\mu)) >: x \in X, \mu_B(x) \in [0,1],\\ T_B(x,\mu), I_B(x,\mu),\ F(x,\mu) \in]0^-, 1^+[\}$$

In this context, each membership grade $\mu_B(x)$ *is expressed by a truth, indeterminacy, and falsity membership grade denoted by* $T_B(x,\mu), I_B(x,\mu)$ *and* $F(x,\mu)$*, respectively, and* $]0^-, 1^+[$ *a non-standard unit interval.*

Definition 4 ([6,27])**.** *A function* $\star : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous t-norm (TN) if the following conditions hold* $\forall\ t, s, j, k \in [0,1]$*:*

1. $t \star 1 = t$;
2. $t \le s$ *and* $j \le k$ *implies* $t \star j \le s \star k$;
3. *continuity of* $\star$;
4. *commutativity and associativity of* $\star$.

Definition 5 ([6,27])**.** *A function* $\diamond : [0,1] \times [0,1] \to [0,1]$ *is said to be a continuous t-conorm (TC) if the following conditions hold* $\forall\ t, s, j, k \in [0,1]$*:*

1. $t \diamond 0 = t$;
2. $t \le s$ *and* $j \le k$ *implies* $t \diamond j \le s \diamond k$;
3. *Continuity of* $\diamond$;
4. *Commutativity and associativity of* $\diamond$.

Definition 6 ([27])**.** *A 6-tuple* $(G, A, M, R, \star, \diamond)$ *is known as a Neutrophic Metric Space (NMS) if G is a non-empty arbitary set,* $\star$ *represents a continuous t-norm,* $\diamond$ *denotes a continuous t-conorm, and A, M, and R are three fuzzy sets defined on* $G^2 \times (0, \infty)$*, subject to the following conditions for all* $p, q, r \in G$ *and* $s, t > 0$*:*

1. $0 \le A(p,q,t) \le 1,\ 0 \le M(p,q,t) \le 1,\ 0 \le R(p,q,t) \le 1$;
2. $A(p,q,t) + M(p,q,t) + R(p,q,t) \le 3$;
3. $A(p,q,t) = A(q,p,t)$;
4. $A(p,q,t) = 1$ *if* $p = q$;
5. $\lim_{t\to\infty} A(p,q,t) = 1$;
6. $A(p,q,t) \star A(q,r,s) \le A(p,r,t+s)$;
7. *Continuity of* $A(p,q,.) : \mathbb{R}^+ \cup 0 \to (0,1]$;
8. $M(p,q,t) = M(q,p,t)$;
9. $M(p,q,t) = 0$ *if* $p = q$;
10. $\lim_{t\to\infty} M(p,q,t) = 0$;
11. $M(p,q,t) \diamond M(q,r,s) \ge M(p,r,t+s)$;
12. *Continuity of* $M(p,q,.) : \mathbb{R}^+ \cup 0 \to (0,1]$;
13. $R(p,q,t) = R(q,p,t)$;

14. $R(p,q,t) = 0$ *if* $p = q$;
15. $\lim_{t\to\infty} R(p,q,t) = 0$;
16. $R(p,q,t) \diamond R(q,r,s) \geq R(p,r,t+s)$;
17. *Continuity of* $R(p,q,.) : \mathbb{R}^+ \cup 0 \to (0,1]$;
18. *For* $t \leq 0$, $A(p,q,t) = 0$, $M(p,q,t) = 1$ *and* $R(p,q,t) = 1$.

In this context, $A(x,y,t)$ *represents the degree of nearness,* $M(x,y,t)$ *stands for the degree of neutralness, and* $R(x,y,t)$ *denotes the degree of non-nearness between* x *and* y *with respect to* t*, respectively.*

3. Neutrosophic Fuzzy Metric Space and Its Topological Properties

This section introduces the concept of neutrosophic fuzzy metric space and explores various topological characteristics of it. Firstly, we present the definition of a neutrosophic fuzzy metric space.

Definition 7. *A 7-tuple* $(G, S, A, M, R, \star, \diamond)$ *is known as a Neutrophic Fuzzy Metric Space (NFMS) if* G *is an arbitary set,* $\star$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm, and* S*,* A*,* M*, and* R *are fuzzy sets on* $G^2 \times (0,\infty)$ *satisfying the following conditions:*
$\forall\ p,q,r \in G$ *and* $t,s > 0$*,*

1. $0 \leq S(p,q,t) \leq 1,\ 0 \leq A(p,q,t) \leq 1,\ 0 \leq M(p,q,t) \leq 1,\ 0 \leq R(p,q,t) \leq 1$;
2. $S(p,q,t) + A(p,q,t) + M(p,q,t) + R(p,q,t) \leq 4$;
3. $S(p,q,t) = S(q,p,t)$;
4. $S(p,q,t) = 1$ *if* $p = q$;
5. $\lim_{t\to\infty} S(p,q,t) = 1$;
6. $S(p,q,t) \star S(q,r,s) \leq S(p,r,t+s)$;
7. $S(p,q,.) : \mathbb{R}^+ \cup \{0\} \to (0,1]$ *is continuous;*
8. $A(p,q,t) = A(q,p,t)$;
9. $A(p,q,t) = 1$ *if* $p = q$;
10. $\lim_{t\to\infty} A(p,q,t) = 1$;
11. $A(p,q,t) \star A(q,r,s) \leq A(p,r,t+s)$;
12. $A(p,q,.) : \mathbb{R}^+ \cup \{0\} \to (0,1]$ *is continuous;*
13. $M(p,q,t) = M(q,p,t)$;
14. $M(p,q,t) = 0$ *if* $p = q$
15. $\lim_{t\to\infty} M(p,q,t) = 0$;
16. $M(p,q,t) \diamond M(q,r,s) \geq M(p,r,t+s)$;
17. $M(p,q,.) : \mathbb{R}^+ \cup \{0\} \to (0,1]$ *is continuous;*
18. $R(p,q,t) = R(q,p,t)$;
19. $R(p,q,t) = 0$ *if* $p = q$;
20. $\lim_{t\to\infty} R(p,q,t) = 0$;
21. $R(p,q,t) \diamond R(q,r,s) \geq R(p,r,t+s)$;
22. $R(p,q,.) : \mathbb{R}^+ \cup \{0\} \to (0,1]$ *is continuous;*
23. *For* $t \leq 0$, $S(p,q,t) = 0$, $A(p,q,t) = 0$, $M(p,q,t) = 1$ *and* $R(p,q,t) = 1$.

In this context, $S(x,y,t)$ *represents the certainity that distance between* x *and* y *is less than* t*,* $A(x,y,t)$ *represents the degree of nearness,* $M(x,y,t)$ *stadns for the degree of neutralness, and* $R(x,y,t)$ *denotes the degree of non-nearness between* x *and* y *with respect to* t*, respectively.*

Example 1. *Let* (G,d) *be a metric space where* $G = (-\infty,\infty)$ *and* $d(x,y) = |x-y|$*. Define the t-norm and t-conorm,* $\star$ *and* $\diamond$ *as* $x \star y = \min\{x,y\}$ *and* $x \diamond y = \max\{x,y\}$*. And let the fuzzy sets* S, A, M, R *on* $G^2 \times (0,\infty)$ *be defined as:*

$$S(x,y,t) = \frac{t + d(x,y)}{t + 2d(x,y)}, A(x,y,t) = \frac{t}{t + d(x,y)}, M(x,y,t) = \frac{d(x,y)}{t + d(x,y)}, R(x,y,t) = \frac{d(x,y)}{t}$$

Note that,
(1) $0 \leq S(x,y,t) \leq 1$*,*
(2) Since $d(x,y) = d(y,x)$*, we have* $S(x,y,t) = S(y,x,t)$*,*

(3) $S(x,y,t) = 1$ *if* $x = y$, *and*
(4) $\lim_{t\to\infty} \frac{t+d(x,y)}{t+2d(x,y)} = 1$, *for all* $x, y \in G$ *and* $t > 0$.
(5) $S(x,y,t) \star S(y,z,s) \leq S(x,z,t+s)$, *for all* $x, y, z \in G$, $s, t > 0$.
Similarlly, all the conditions for A, M, *and* R *can be verified. Therefore,* $(G, S, A, M, R, \star, \diamond)$ *is an NFMS induced by a metric* d, *called the standard neutrosophic fuzzy metric.*

Example 2. *Take* $G = [2.5, 3.4]$, $\star$ *be a TN and* $\diamond$ *be a TC defined as* $p \star q = \min\{0, p+q-1\}$ *and* $p \diamond q = p + q - pq$. *Also,* $\forall\, p, q \in G$, $s \in (0,\infty)$, *we define:*

$$S(p,q,s) = 1 - \frac{|p-q|}{2t}$$

$$A(p,q,s) = \frac{s^3 - |p-q|}{s^3}$$

$$M(p,q,s) = \begin{cases} (q-p)/(q+s) & \text{if } p \leq q \\ (p-q)/(p+s) & \text{if } q \leq p \end{cases}$$

$$R(p,q,s) = \begin{cases} (q^2-p^2)/(q^2+s^2) & \text{if } p \leq q \\ (p^2-q^2)/(p^2+s^2) & \text{if } q \leq p \end{cases}$$

Note that,
(1) $0 \leq S(p,q,s), A(p,q,s), M(p,q,s), R(p,q,s) \leq 1$,
(2) Since $|p-q| = |q-p|$, *we have* $S(p,q,s) = S(q,p,s), A(p,q,s) = A(q,p,s)$, $M(p,q,s) = M(q,p,s), R(p,q,s) = R(q,p,s)$,
(3) $S(p,q,s) = A(p,q,s) = 1,\ M(q,p,s) = R(p,q,s) = 0$, *if* $p = q$,
(4) $\lim_{t\to\infty} 1 - \frac{|p-q|}{2t} = 1, \lim_{t\to\infty} \frac{s^3-|p-q|}{s^3} = 1, \lim_{t\to\infty} M(q,p,s) = 0$, $\lim_{t\to\infty} R(p,q,s) = 0$, *for all* $x, y \in G$ *and* $t > 0$.
(5) $S(p,q,s) \star S(q,r,s) \leq S(p,r,t+s),\ A(p,q,s) \star A(q,r,s) \leq A(p,r,t+s)$
$M(p,q,s) \diamond M(q,r,s) \geq M(p,r,t+s),\ R(p,q,s) \diamond R(q,r,s) \geq R(p,r,t+s)$, *for all* $x, y, z \in G$, $s, t > 0$.
Therefore, $(G, S, A, M, R, \star, \diamond)$ *forms an NFMS.*

Remark 1. *The 7-tuple* $(G, S, A, M, R, \star, \diamond)$ *defined in above Example 1 would not be a NFMS if t-norm* $p \star q = \min\{0, p+q-1\}$ *and t-conorm* $p \diamond q = p + q - pq$.

Definition 8. *Consider* $(G, S, A, M, R, \star, \diamond)$ *as an NFMS. For* $\varepsilon \in (0,1)$, $t \in \mathbb{R}^+$, *and* $p \in G$, *denote the set:*

$$O(p,\varepsilon,t) = \{b \in G : S(p,q,t) > 1-\varepsilon,\ A(p,q,t) > 1-\varepsilon,\ M(p,q,t) < \varepsilon,\ R(p,q,t) < \varepsilon\}$$

as an open ball (OB), where p *serves as the center and* ε *as the radius with respect to* t.

Definition 9. *Consider* $(G, S, A, M, R, \star, \diamond)$ *as an NFMS. A subset* H *of* G *is said to be an open set if for each* $p \in H$, *there exists an open ball* $O(p,\varepsilon,t)$ *such that* $O(p,\varepsilon,t) \subset H$.

Theorem 1. *Each OB,* $O(p,\varepsilon,t)$ *within an NFMS constitutes an open set (OS).*

Proof. We consider an open ball $O(p,\varepsilon,t)$ and choose $q \in O(p,\varepsilon,t)$. Then, $S(p,q,t) > 1-\varepsilon,\ A(p,q,t) > 1-\varepsilon,\ M(p,q,t) < \varepsilon,\ R(p,q,t) < \varepsilon$.
There exists $t_0 \in (0,t)$ such that,
$S(p,q,t_0) > 1-\varepsilon$, $A(p,q,t_0) > 1-\varepsilon$, $M(p,q,t_0) < \varepsilon$, $R(p,q,t_0) < \varepsilon$. [Since, $S(p,q,t) > 1-\varepsilon$]
Taking $\varepsilon_0 = S(p,q,t_0)$, $\exists\, \varrho \in (0,1)$ for $\varepsilon_0 > 1-\varepsilon$, so that $\varepsilon_0 > 1-\varrho > 1-\varepsilon$.
For any given ε_0 and ϱ so that $\varepsilon_0 > 1-\varrho$. Then, $\exists\, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in (0,1)$ s.t,
$\varepsilon_0 \star \varepsilon_1 > 1-\varrho,\ \varepsilon_0 \star \varepsilon_2 > 1-\varrho,\ (1-\varepsilon_0) \diamond (1-\varepsilon_3) \leq \varrho,\ (1-\varepsilon_0) \diamond (1-\varepsilon_4) \leq \varrho$.
Choosing $\varepsilon_5 = \max\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ and considering the open ball $O(q, 1-\varepsilon_5, t-t_0)$, our

motive is to show that $O(q, 1-\varepsilon_5, t-t_0) \subset O(p, \varepsilon, t)$.
Now taking $r \in O(q, 1-\varepsilon_5, t-t_0)$ we get,
$S(q, r, t-t_0) > \varepsilon_5,\ A(q, r, t-t_0) > \varepsilon_5,\ M(q, r, t-t_0) < \varepsilon_5,\ R(q, r, t-t_0) < \varepsilon_5.$
Also,
$S(p, r, t) > S(p, q, t_0) \star S(q, r, t-t_0) \geq \varepsilon_0 \star \varepsilon_5 \geq \varepsilon_0 \star \varepsilon_1 \geq 1-\varrho > 1-\varepsilon,$
$A(p, r, t) > A(p, q, t_0) \star A(p, r, t-t_0) \geq \varepsilon_0 \star \varepsilon_5 \geq \varepsilon_0 \star \varepsilon_2 \geq 1-\varrho > 1-\varepsilon,$
$M(p, r, t) > M(p, q, t_0) \diamond M(q, r, t-t_0) \leq (1-\varepsilon_0) \diamond (1-\varepsilon_5) \leq (1-\varepsilon_0) \diamond (1-\varepsilon_3) \leq \varrho < \varepsilon,$
$R(p, r, t) > R(p, q, t_0) \diamond R(q, r, t-t_0) \leq (1-\varepsilon_0) \diamond (1-\varepsilon_5) \leq (1-\varepsilon_0) \diamond (1-\varepsilon_4) \leq \varrho < \varepsilon.$
Hence, $r \in O(p, \varepsilon, t)$ and $O(q, 1-\varepsilon_5, t-t_0) \subset O(p, \varepsilon, t)$. □

Remark 2. *From Definition 9 and Theorem 1, it can be said that,*

$$\tau_N = \{H \subset G : \exists\, t > 0 \text{ and } 0 < \varepsilon < 1 \text{ such that } O(p, \varepsilon, t) \subset H,\ \forall\, p \in H\}$$

is a topology on G. In this case every neutrosophic fuzzy metric $N = (S, A, M, R)$ on G produces a topology τ_N on G which has a base as the family of open sets $\{O(p, \varepsilon, t) : p \in N, \varepsilon \in (0,1),$ and $t > 0\}$.

Theorem 2. *Let $(G, S, A, M, R, \star, \diamond)$ be an NFMS. Then,*
(i) ϕ and G are open sets in $(G, S, A, M, R, \star, \diamond)$.
(ii) The union of any finite, countable, or uncountable family of open sets is open.
(iii) The intersection of a finite family of open sets is open.

Proof. (i) As an empty set contains no points, the necessity that each point in ϕ is the centre of an open ball contained in it is satisfied automatically.

The whole space G is open since every open ball centred at any of its points is contained in G.
(ii) Let $\{M_\alpha : \alpha \in \Lambda\}$ be a family of open sets and $H = \cup_{\alpha \in \Lambda} M_\alpha$. If $H = \phi$ then it is open [by (i)]. So let $H \neq \phi$. Taking $x \in H$, it can be said that $x \in M_\alpha$ for some $\alpha \in \Lambda$. Since M_α is open, $\exists\, r > 0$ such that $O(x, r, t) \subseteq M_\alpha \subseteq H$ (where $t > 0$). Thus, for each $x \in H\ \exists\, r > 0$ such that $O(x, r, t) \subseteq H$. Implies that H is open.
(iii) Let $\{M_i : 1 \leq i \leq n\}$ be a finite family of open sets in G, and let $M = \cap_{i=1}^{n} M_i$. If M is empty, then it is open. (by (i)).
Suppose $M \neq \phi$, and $x \in M$. Then, $x \in M_j,\ j = 1, 2, ..., n$. Since M_j is open, $\exists\, r_j > 0$ such that $O(x, r_j, t) \subseteq M_j$ where $t > 0$ and $j = 1, 2, ..., n$.
Let $r = min\{r_1, r_2, ..., r_n\}$. Then $r > 0$ and $O(x, r, t) \subseteq O(x, r_j, t)$ where $t > 0$ and $j = 1, 2, ..., n$. Therefore, $O(x, r, t)$ centred at x satisfies

$$O(x, r, t) \subseteq \cap_{j=1}^{n} O(x, r_j, t) \subseteq M$$

This completes the proof. □

Theorem 3. *A subset X in a NFMS $(G, S, A, M, R, \star, \diamond)$ is open if it is the union of all open balls contained in X.*

Proof. Let X be open. If $X = \phi$, then there are no open balls contained in it. Thus, the union of all open balls contained in X is the union of an empty class, which is empty and therefore equal to X.
Now if $X \neq \phi$, then since X is open, each of its points is the centre of an open ball entirely contained in X. So X is the union of all open balls contained in it. The converse follows from Theorem 3.1 and Theorem 3.3. □

Definition 10. *Let A be a subset of an NFMS $(G, S, A, M, R, \star, \diamond)$. A point $x \in G$ is an interior point of A if $\exists\, O(x, r, t) \subseteq A$ for some $r, t > 0$. That is, $x \in O(x, r, t) \subseteq A$.*

Theorem 4. *Let A be a subset of an NFMS $(G, S, A, M, R, \star, \diamond)$. Then,*
(i) $Int(A)$ is an open subset of A that contains every open subset of A.
(ii) A is open if $A = Int(A)$.

Proof. Let $x \in Int(A)$ be arbitary. Then, by Definition 3.4, $\exists\, O(x, r, t) \subseteq A$ for some $r, t > 0$. But since $O(x, r, t)$ is an open set [by Theorem 3.1], each point of it is the centre of an open ball contained in $O(x, r, t)$ and consequently also in A. Therefore, each point of $O(x, r, t)$ is an interior point of A. That is, $O(x, r, t) \subseteq Int(A)$. Thus, x is the centre of an open ball contained in $Int(A)$. Since $x \in Int(A)$ is arbitary, it can be said that each $x \in Int(A)$ has the property of being the centre of an open ball contained in $Int(A)$. Hence, $Int(A)$ is open. Now it is to be shown that $Int(A)$ contains every open subset $X \subseteq A$. Let $x \in X$. Since X is open, $\exists\, O(x, r, t) \subseteq X \subseteq A$. So $x \in Int(A)$. This shows that $x \in X \implies x \in Int(A)$. In other words, $X \subseteq Int(A)$.
(ii) is immediate after (i). □

Definition 11. *Let $(G, S, A, M, R, \star, \diamond)$ be an NFMS and let $C(G)$ be the collection of all non empty compact subsets of G. Consider $P, Q \in C(G)$ and $t > 0$.*
Define $H_{\mathfrak{A}}, H_{\mathfrak{B}}, H_{\mathfrak{C}}$ and $H_{\mathfrak{D}} : C(G) \times C(G) \times (0, \infty) \to (0, \infty)$ as follows:

$$H_{\mathfrak{A}}(P, Q, t) = \min\left\{ \inf_{w \in P} \mathfrak{A}(w, Q, t), \inf_{k \in Q} \mathfrak{A}(P, k, t) \right\},$$

$$H_{\mathfrak{B}}(P, Q, t) = \min\left\{ \inf_{w \in P} \mathfrak{B}(w, Q, t), \inf_{k \in Q} \mathfrak{B}(P, k, t) \right\},$$

$$H_{\mathfrak{C}}(P, Q, t) = \max\left\{ \sup_{w \in P} \mathfrak{C}(w, Q, t), \sup_{k \in Q} \mathfrak{C}(P, k, t) \right\},$$

$$H_{\mathfrak{D}}(P, Q, t) = \max\left\{ \sup_{w \in P} \mathfrak{D}(w, Q, t), \sup_{k \in Q} \mathfrak{D}(P, k, t) \right\}.$$

The 6-tuple $(H_{\mathfrak{A}}, H_{\mathfrak{B}}, H_{\mathfrak{C}}, H_{\mathfrak{D}}, \star, \diamond)$ is called Hausdorff NFMS, or shortly HNFMS.

Theorem 5. *Every NFMS possesses the property of being Hausdorff.*

Proof. We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS. Let $p, q \in G$ be different. Then, $S(p, q, t),\ A(p, q, t),\ M(p, q, t),\ R(p, q, t) \in (0, 1)$.
Take $\varepsilon_1 = S(p, q, t), \varepsilon_2 = A(p, q, t), \varepsilon_3 = M(p, q, t), \epsilon_4 = R(p, q, t)$
and $\varepsilon = \max\{\varepsilon_1, \varepsilon_2, 1 - \varepsilon_3, 1 - \varepsilon_4\}$.
Now, taking $\varepsilon_0 \in (\varepsilon, 1), \exists\, \varepsilon_5,\ \varepsilon_6,\ \varepsilon_7,\ \varepsilon_8$ such that,
$\varepsilon_5 \star \varepsilon_5 \geq \varepsilon_0, \varepsilon_6 \star \varepsilon_6 \geq \varepsilon_0, (1 - \varepsilon_7) \diamond (1 - \varepsilon_7) \leq 1 - \varepsilon_0$ and $(1 - \varepsilon_8) \diamond (1 - \varepsilon_8) \leq 1 - \varepsilon_0$.
Taking $\varepsilon_9 = \max\{\varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8\}$ if we consider the open balls $O(p, 1 - \varepsilon_9, \frac{t}{2})$ and $O(q, 1 - \varepsilon_9, \frac{t}{2})$. Then, it is clear that, $O(p, 1 - \varepsilon_9, \frac{t}{2}) \cap O(q, 1 - \varepsilon_9, \frac{t}{2}) = \emptyset$.
Now, if we choose a point $r \in O(p, 1 - \varepsilon_9, \frac{t}{2}) \cap O(q, 1 - \varepsilon_9, \frac{t}{2})$ then,

$$\varepsilon_1 = S(p, q, t) \geq S(p, r, \tfrac{t}{2}) \star S(r, q, \tfrac{t}{2}) \geq \varepsilon_9 \star \varepsilon_9 \geq \varepsilon_5 \star \varepsilon_5 \geq \varepsilon_0 > \varepsilon_1.$$
$$\varepsilon_2 = A(p, q, t) \geq A(p, r, \tfrac{t}{2}) \star A(r, q, \tfrac{t}{2}) \geq \varepsilon_9 \star \varepsilon_9 \geq \varepsilon_6 \star \varepsilon_6 \geq \varepsilon_0 > \varepsilon_2.$$
$$\varepsilon_3 = M(p, q, t) \leq M(p, r, \tfrac{t}{2}) \diamond M(r, q, \tfrac{t}{2}) \leq (1 - \varepsilon_9) \diamond (1 - \varepsilon_9) \leq (1 - \varepsilon_7) \diamond (1 - \varepsilon_7) \leq 1 - \varepsilon_0 < \varepsilon_3.$$
$$\varepsilon_4 = R(p, q, t) \leq R(p, r, \tfrac{t}{2}) \diamond R(r, q, \tfrac{t}{2}) \leq (1 - \varepsilon_9) \diamond (1 - \varepsilon_9) \leq (1 - \varepsilon_8) \diamond (1 - \varepsilon_8) \leq 1 - \varepsilon_0 < \varepsilon_4,$$

which shows a contradiction. Therefore, every NFMS is Hausdorff. □

Definition 12. *We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS. A set $W \subset G$ is known as nuetrosophic bounded (NB) if, there remains a positive real number t and ε in $(0, 1)$ satisfying,*
$S(p, q, t) > 1 - \varepsilon,\ A(p, q, t) > 1 - \varepsilon,\ M(p, q, t) < \varepsilon$ and $R(p, q, t) < \varepsilon\ \forall p, q \in W$.

Definition 13. *Let an NFMS $(G, S, A, M, R, \star, \diamond)$ be given, then*

(1) Consider C_N as an assembly of open sets and $W \subseteq \cup_{U \in C_N} U$. Then, the assembly C_N is said to be an open cover(OC) of W.
(2) A subspace W of G is compact, if every open cover of W has a finite subcover.
(3) A subspace W of G is said to be sequentially compact if each sequence in W possesses a convergent subsequence in W.

Theorem 6. *In an NFMS, every compact subset is neutrosophic bounded.*

Proof. We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS, and W be a compact subset of G. Also consider an open cover $\{O(p, \varepsilon, t) : p \in W\}$ of W where $t > 0$ and $0 < \varepsilon < 1$. W being a compact set, there exists $p_1, p_2, ..., p_n \in W$ in such a way that $W \subseteq \cup_{k=1}^{n} O(p_k, \varepsilon, t)$. Further, for $p, q \in W$ there remains some $k, m \in \{1, 2, 3, ..., n\}$ such that $p \in O(p_k, \varepsilon, t)$ and $q \in O(p_m, \varepsilon, t)$. Then, we can write
$S(p, p_k, t) > 1 - \varepsilon,\ A(p, p_k, t) > 1 - \varepsilon,\ M(p, p_k, t) < \epsilon,\ R(p, p_k, t) < \varepsilon$ and
$S(q, p_m, t) > 1 - \varepsilon,\ A(q, p_m, t) > 1 - \varepsilon,\ M(q, p_m, t) < \varepsilon,\ R(q, p_m, t) < \varepsilon$
Now, take $\alpha = \min\{S(p_k, p_m, t) : 1 \leq k, m \leq n\}$, $\beta = \min\{A(p_k, p_m, t) : 1 \leq k, m \leq n\}$
$\gamma = \max\{M(p_k, p_m, t) : 1 \leq k, m \leq n\}$, $\delta = \max\{R(p_k, p_m, t) : 1 \leq k, m \leq n\}$.
Then, $\alpha, \beta, \gamma, \delta > 0$. From here, we can have some $\varrho_1,\ \varrho_2,\ \varrho_3,\ \varrho_4 \in (0, 1)$ satisfying,

$$\begin{gathered} S(p, q, 3t) \geq S(p, p_k, t) \star S(p_k, p_m, t) \star S(p_m, q, t) \geq (1 - \varepsilon) \star (1 - \varepsilon) \star \alpha > 1 - \varrho_1, \\ A(p, q, 3t) \geq A(p, p_k, t) \star A(p_k, p_m, t) \star A(p_m, q, t) \geq (1 - \varepsilon) \star (1 - \varepsilon) \star \beta > 1 - \varrho_2, \\ M(p, q, 3t) \leq M(p, p_k, t) \diamond M(p_k, p_m, t) \diamond M(p_m, q, t) \leq \varepsilon \diamond \varepsilon \diamond \gamma < \varrho_3 \\ R(p, q, 3t) \leq R(p, p_k, t) \diamond R(p_k, p_m, t) \diamond R(p_m, q, t) \leq \varepsilon \diamond \varepsilon \diamond \delta < \varrho_4 \end{gathered}$$

Taking $\varrho = \max\{\varrho_1,\ \varrho_2,\ \varrho_3,\ \varrho_4\}$ and $t_0 = 3t$, we have $S(p, q, t_0) > 1 - \varrho,\ A(p, q, t_0) > 1 - \varrho,\ M(p, q, t_0) < \varrho,\ R(p, q, t_0) < \varrho\ \forall\ p, q \in W$. Therefore, the set W is Neutrosophic bounded. □

If $(G, S, A, M, R, \star, \diamond)$ is a NFMS produced by a metric d on G and $W \subset G$, then W is Neutrosophic bounded if it is bounded. As a result with Theorems 5 and 6 it can be written that:

Corollary 1. *In an NFMS, each compact set is both closed set and bounded set.*

Theorem 7. *We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS and τ_N to be the topology on G produced by the FM, then $\{p_m\} \to p \in G$ if and only if,*

$$S(p_m, p, s) \to 1, A(p_m, p, s) \to 1,\ M(p_m, p, s) \to 0,\ R(p_m, p, s) \to 0,\ \textit{as } m \to \infty.$$

Proof. We consider $s > 0$. Suppose $p_m \to p$. For any given $\varepsilon \in (0, 1)$, there remains $N \in \mathbb{N}$ such that,

$$p_m \in O(p, \varepsilon, s)\ \forall\ m \geq N.$$

Thus, $1 - S(p_m, p, s) < \varepsilon,\ 1 - A(p_m, p, s) < \varepsilon,\ M(p_m, p, s) < \varepsilon,\ R(p_m, p, s) < \varepsilon$.
In these cases, we express it as follows:

$$S(p_m, p, s) \to 1,\ A(p_m, p, s) \to 1,\ M(p_m, p, s) \to 0,\ R(p_m, p, s) \to 0,\ \textit{as } m \to \infty.$$

Conversely, suppose $S(p_m, p, s) \to 1, A(p_m, p, s) \to 1,\ M(p_m, p, s) \to 0,\ R(p_m, p, s) \to 0$, as $m \to \infty$, for each $s > 0$. Then for any $\varepsilon \in (0, 1)$, there exists $N \in \mathbb{N}$ so that $1 - S(p_m, p, s) < \varepsilon,\ 1 - A(p_m, p, s) < \varepsilon,\ M(p_m, p, s) < \varepsilon,\ R(p_m, p, s) < \varepsilon\ \forall\ N \in \mathbb{N}$. By this, we obtain $S(p_m, p, s) > 1 - \varepsilon,\ A(p_m, p, s) > 1 - \varepsilon,\ M(p_m, p, s) < \varepsilon,\ R(p_m, p, s) < \varepsilon\ \forall\ N \in \mathbb{N}$. Thus, $p_m \in O(p, \varepsilon, s)\ \forall\ m \geq N$. Hence the proof. □

Definition 14. *We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS. Any sequence $\{p_n\}$ in G is said to be a Cauchy sequence if for any positive real numbers ε and t, there subsists $N \in \mathbb{N}$ so that, the following conditions hold:*

$$S(p_n, p_m, t) > 1 - \varepsilon,\ A(p_n, p_m, t) > 1 - \varepsilon,\ M(p_n, p_m, t) < \varepsilon,\ R(p_n, p_m, t) < \varepsilon,$$

for all $n, m \geq N$.

Moreover, $(G, S, A, M, R, \star, \diamond)$ is said to be a complete NFMS if each Cauchy sequence converges with respect to the topology τ_N in G.

Theorem 8. *We consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS. Under the assumption that each Cauchy sequence of G possesses a converging subsequence, the NFMS $(G, S, A, M, R, \star, \diamond)$ is considered complete.*

Proof. We consider $\{p_n\}$ as a Cauchy sequence in G and assume a subsequence $\{p_{i_n}\}$ of $\{p_n\}$ coverges to a point $p \in G$. We complete the proof by showing $p_n \to p$. Let $t > 0$ and $0 < \nu < 1$. Taking $\varepsilon \in (0, 1)$ so that $(1 - \varepsilon) \star (1 - \varepsilon) \geq 1 - \nu$, $\epsilon \diamond \epsilon \leq \nu$. By the Cauchyness of $\{a_n\}$, there exists $N \in \mathbb{N}$ so that $\forall\ m, n \geq N$,

$$S(p_m, p_n, \frac{t}{2}) > 1 - \varepsilon,\ A(p_m, p_n, \frac{t}{2}) > 1 - \varepsilon,\ M(p_m, p_n, \frac{t}{2}) < \varepsilon,\ R(p_m, p_n, \frac{t}{2}) < \varepsilon.$$

Since $p_n \to p$, there remains $i \in \mathbb{N}$ satisfying $S(p_i, p, \frac{t}{2}) > 1 - \varepsilon,\ A(p_i, p, \frac{t}{2}) > 1 - \varepsilon,\ M(p_i, p, \frac{t}{2}) < \varepsilon,\ R(p_i, p, \frac{t}{2}) < \varepsilon, \forall\ i > N$. Then, for $n \geq N$,

$$\begin{gathered} S(p_n, p, t) \geq S(p_n, p_i, \tfrac{t}{2}) \star S(p_i, p, \tfrac{t}{2}) > (1 - \varepsilon) \star (1 - \varepsilon) \geq 1 - \nu, \\ A(p_n, p, t) \geq A(p_n, p_i, \tfrac{t}{2}) \star A(p_i, p, \tfrac{t}{2}) > (1 - \varepsilon) \star (1 - \varepsilon) \geq 1 - \nu, \\ M(p_n, p, t) \leq M(p_n, p_i, \tfrac{t}{2}) \diamond M(p_i, p, \tfrac{t}{2}) > \varepsilon \diamond \varepsilon \leq \nu, \\ R(p_n, p, t) \leq R(p_n, p_i, \tfrac{t}{2}) \diamond R(p_i, p, \tfrac{t}{2}) > \varepsilon \diamond \varepsilon \leq \nu, \end{gathered}$$

Thus, we have $a_n \to a$. Therefore, NFMS $(G, S, A, M, R, \star, \diamond)$ is complete. □

Theorem 9. *Consider $(G, S, A, M, R, \star, \diamond)$ as an NFMS and W be a subset of G with the subspace NFM $(S_W, A_W, M_W, R_W) = (S|_{W^2 \times \infty}, A|_{W^2 \times \infty}, M|_{W^2 \times \infty}, R|_{W^2 \times \infty})$. Then $(W, S_W, A_W, M_W, R_W, \star, \diamond)$ is complete if and only if $W \subset G$ is closed.*

Proof. Let $W \subset G$ is closed. Suppose $\{p_n\}$ be Cauchy in $(W, S_W, A_W, M_W, R_W, \star, \diamond)$. Since $\{p_n\}$ is Cauchy in G, there exists $p \in G$ such that $p_n \to p$. It is clear that, $p \in \overline{W} = W$ and so $\{p_n\}$ converges in W.

In contrary, let $(W, S_W, A_W, M_W, R_W, \star, \diamond)$ be complete. Also, suppose W is open. If we choose a point $p \in \overline{W}/W$, then there exists $\{p_n\}$ of points belonging to W converging to p and thus, $\{p_n\}$ is Cauchy. This gives, for $n, m \geq N$, each $\nu \in (0, 1)$ and $t > 0$, $\exists\ N \in \mathbb{N}$ satisfying

$$S(p_n, p_m, t) > 1 - \nu,\ A(p_n, p_m, t) > 1 - \nu,\ M(p_n, p_m, t) < \nu,\ R(p_n, p_m, t) < \nu.$$

Now, since $\{p_n\}$ is in W, we can write $S(p_n, p_m, t) = S_W(p_n, p_m, t),\ A(p_n, p_m, t) = A_W(p_n, p_m, t),\ M(p_n, p_m, t) = M_W(p_n, p_m, t),\ R(p_n, p_m, t) = R_W(p_n, p_m, t)$. Therefore, $\{p_n\}$ is Cauchy in W. Since $(G, S, A, M, R, \star, \diamond)$ is complete, $\exists\ q \in W$ such that $p_n \to q$. Hence, $\exists\ n \in \mathbb{N}$ such that $S_W(q, p_n, t) > 1 - \nu,\ A_W(q, p_n, t) > 1 - \nu,\ M_W(q, p_n, t) < \nu,\ R_W(q, p_n, t) < \nu$ for all $n \geq N$, each $\nu \in (0, 1)$ and $t > 0$. Since $\{p_n\}$ is in W and $q \in W$, we can write $S(q, p_n, t) = S_W(q, p_n, t),\ A(q, p_n, t) = A_W(q, p_n, t),\ M(q, p_n, t) = M_W(q, p_n, t),\ R(q, p_n, t) = R_W(q, p_n, t)$. This concludes that $\{p_n\}$ converges both to p and q in $(G, S, A, M, R, \star, \diamond)$. Since $q \in W$ and $p \notin W$, this implies $p \neq q$. This contradiction leads to the desired result. □

Lemma 1. *Let $(G, S, A, M, R, \star, \diamond)$ be a NFMS. If $t > 0$ and $0 < \varepsilon_1, \varepsilon_2 < 1$ satisfy $(1-\varepsilon_2) \star (1-\varepsilon_2) \geq (1-\varepsilon_1)$, $\varepsilon_2 \diamond \varepsilon_2 \leq \varepsilon_1$, then $\overline{O(p, \varepsilon_2, \frac{t}{2})} \subset O(p, \varepsilon_1, t)$.*

Proof. We take $q \in \overline{O(p, \varepsilon_2, \frac{t}{2})}$ and $O(q, \varepsilon_2, \frac{t}{2})$ be an open ball with radius ε_2 and centered at q. Since $O(p, \varepsilon_2, \frac{t}{2}) \cap O(q, \varepsilon_2, \frac{t}{2}) \neq \emptyset$, $\exists\, r \in O(p, \varepsilon_2, \frac{t}{2}) \cap O(q, \varepsilon_2, \frac{t}{2})$. Then we obtain,

$$S(p,q,t) \geq S(p,r,\tfrac{t}{2}) \star S(q,r,\tfrac{t}{2}) > (1-\varepsilon_2) \star (1-\varepsilon_2) \geq 1-\varepsilon_1,$$
$$A(p,q,t) \geq A(p,r,\tfrac{t}{2}) \star A(q,r,\tfrac{t}{2}) > (1-\varepsilon_2) \star (1-\varepsilon_2) \geq 1-\varepsilon_1,$$
$$M(p,q,t) \leq M(p,r,\tfrac{t}{2}) \star M(q,r,\tfrac{t}{2}) < \varepsilon_2 \star \varepsilon_2 \leq \varepsilon_1,$$
$$R(p,q,t) \leq R(p,r,\tfrac{t}{2}) \star R(q,r,\tfrac{t}{2}) < \varepsilon_2 \star \varepsilon_2 \leq \varepsilon_1.$$

Hence, $q \in O(p, \varepsilon_1, t)$ and thus $\overline{O(p, \varepsilon_2, \frac{t}{2})} \subset O(p, \varepsilon_1, t)$. □

Theorem 10. *For an NFMS $(G, S, A, M, R, \star, \diamond)$, a subset $W \subset G$ is nowhere dense if and only if each non-empty OS in G consists of an OB such that the closure of the OB and W are disjoint.*

Proof. We consider $\psi \neq \emptyset$, $\psi \subset G$ as open. Then, there exists a non-empty OS Δ such that $\Delta \subset \psi$, $\Delta \cap \overline{W} \neq \emptyset$. If we take $p \in \Delta$, then $\exists\ 0 < \varepsilon_1 < 1$ and $t \in \mathbb{R}^+$ such that $O(p, \varepsilon_1, t) \subset \Delta$. Now, choose $0 < \varepsilon_2 < 1$ such that $(1-\varepsilon_2) \star (1-\varepsilon_2) \geq 1-\varepsilon_1$ and $\varepsilon_2 \diamond \varepsilon_2 \leq \varepsilon_1$. By Lemma 1, we obtain $\overline{O(p, \varepsilon_2, \frac{t}{2})} \subset O(p, \varepsilon_1, t)$. In this scenario, we can assert that $O(p, \varepsilon_2, \frac{t}{2}) \subset \psi$ and $\overline{O(p, \varepsilon_2, \frac{t}{2})} \cap W = \emptyset$.

On the contrary, presume that W is not nowhere dense. Hence, $int(\overline{W}) \neq \emptyset$, indicating the existence of an OS $\psi \neq \phi$ such that $\psi \subset \overline{W}$. Assume $O(p, \varepsilon_1, t)$ as an OB, so as $O(p, \varepsilon_1, t) \subset \psi$. Consequently, $\overline{O(p, \varepsilon_2, t)} \cap W \neq \emptyset$, contradicts the assumption. □

Now, we present the Baire Category Theorem in NFMS.

Theorem 11. *In a complete NFMS, the intersection of countably many dense open sets is dense.*

Proof. Consider a sequence of dense open subsets $\{\psi_n : n \in \mathbb{N}\}$ in a complete NFMS $(G, S, A, M, R, \star, \diamond)$. Then, it is to be proved that the intersection $\bigcap_{n \in \mathbb{N}} \psi_n$ is dense within G. Consider Δ as a non-empty open set of G. As $\psi_1 \in G$ is dense, $\Delta \cap \psi_1 \neq \emptyset$. Let $p_1 \in \Delta \cap \psi_1$. Now, $\Delta \cap \psi_1$ being open, $\exists\ \varepsilon_1 \in (0,1)$, $t_1 > 0$ so that $O(p_1, \varepsilon_1, t_1) \subset \Delta \cap \psi_1$. Choose $\varepsilon_1^* < \varepsilon_1$ and $t_1^* = \min\{t_1, 1\}$ such that $\overline{O(p_1, \varepsilon_1^*, t_1^*)} \subset \Delta \cap \psi_1$. As $\psi_2 \in G$ is dense, $O(p_1, \varepsilon_1^*, t_1^*) \cap \psi_2 \neq \emptyset$. Choose $p_2 \in O(p_1, \varepsilon_1^*, t_1^*) \cap \psi_2$. As $O(p_1, \varepsilon_1^*, t_1^*) \cap \psi_2$ is open, $\exists\ \varepsilon_2 \in (0, 1/2)$ and $t_2 > 0$ so that $O(p_2, \varepsilon_2, t_2) \subset O(p_1, \epsilon_1^*, t_1^*) \cap \psi_2$. Choose $\varepsilon_2^* < \varepsilon_2$ and $t_2^* = \min\{t_2, 1/2\}$ so that $\overline{O(p_2, \varepsilon_2^*, t_2^*)} \subset O(p_1, \varepsilon_1^*, t_2^*) \cap \psi_2$. By proceeding in this manner, we establish a sequence $\{p_n\}$ in G along with another sequence $\{t_n^*\}$ where $t_n^* \in (0, 1/n)$, and

$$\overline{O(p_n, \varepsilon_n^*, t_n^*)} \subset O(p_{n-1}, \varepsilon_{n-1}^*, t_{n-1}^*) \cap \psi_n$$

Next, we demonstrate that $\{p_n\}$ is Cauchy. For any $t,\ \nu > 0$, we select $N \in \mathbb{N}$ such that $\frac{1}{N} < t$ and $\frac{1}{N} < \nu$. Thus, for $n \geq N$ and $m \geq n$, we have

$$S(p_n, p_m, t) \geq S(p_n, p_m, 1/n) \geq 1 - \tfrac{1}{n} > 1 - \nu,$$
$$A(p_n, p_m, t) \geq A(p_n, p_m, 1/n) \geq 1 - \tfrac{1}{n} > 1 - \nu,$$
$$M(p_n, p_m, t) \leq M(p_n, p_m, 1/n) \leq \tfrac{1}{n} \leq \nu,$$
$$R(p_n, p_m, t) \leq R(p_n, p_m, 1/n) \leq \tfrac{1}{n} \leq \nu.$$

This shows that $\{p_n\}$ is Cauchy. G being complete, there exists $p \in G$ so that $\{p_n\} \to p$. As $a_k \in O(p_n, \varepsilon_n^*, t_n^*)\ \forall\ k \geq n$, we obtain $p \in \overline{O(p_n, \varepsilon_n^*, t_n^*)}$. Hence, it can be written that $a \in \overline{O(a_n, \epsilon_n^*, t_n^*)} \subset O(a_{n-1}, \epsilon_{n-1}^*, t_{n-1}^*) \cap \psi_n, \forall\ n$. Then, $\Delta \cap (\cap_{n \in \mathbb{N}} \psi_n) \neq \emptyset$. Then $\cap_{n \in \mathbb{N}} \psi_n$ is dense in G. □

Definition 15. *Consider an NFMS* $(G, S, A, M, R, \star, \diamond)$*. We define a collection* $(\mathfrak{D}_m)$*, where* $m \in \mathbb{N}$*, to possess neutrosophic diameter zero (NDZ) if for every* $\varepsilon \in (0,1)$ *and* $t \in \mathbb{R}^+$*, there exists* $N \in \mathbb{N}$ *such that* $S(p,q,t) > 1-\varepsilon$*,* $A(p,q,t) > 1-\varepsilon$*,* $M(p,q,t) < \varepsilon$*, and* $R(p,q,t) < \varepsilon$ *hold for all* $p, q \in \mathfrak{D}_N$*.*

Theorem 12. *The NFMS* $(G, S, A, M, R, \star, \diamond)$ *is complete if and only if each sequence* $(\mathfrak{D}_m)_{m\in\mathbb{N}}$ *of nested non-empty closed sets, each possessing NDZ, has a nonempty intersection.*

Proof. Assuming the given condition is satisfied, i.e., every sequence $(\mathfrak{D}_m)_{m\in\mathbb{N}}$ of nested nonempty closed sets, each possessing NDZ, has a nonempty intersection. It remains to be demonstrated that $(G, S, A, M, R, \star, \diamond)$ is complete. Consider the Cauchy sequence $\{p_m\} \in G$. Define $\mu_m = a_k : k \geq m$ and $\mathfrak{D}_m = \overline{\mu_m}$. It can then be asserted that $(\mathfrak{D}_m)$ possesses NDZ. Given $0 < \varrho < 1$ and $t > 0$, choose $\varepsilon \in (0,1)$ such that $(1-\varepsilon) \star (1-\varepsilon) \star (1-\varepsilon) > 1-\varrho$ and $\varepsilon \diamond \varepsilon \diamond \varepsilon < \varrho$. As $\{p_m\}$ is Cauchy, there exists a natural number N satisfying

$$S(p_m, q_n, \frac{t}{3}) > 1-\varepsilon,\ A(p_m, q_n, \frac{t}{3}) > 1-\varepsilon,\ M(a_n, a_m, \frac{t}{3}) < \varepsilon,\ R(p_m, q_n, \frac{t}{3}) < \varepsilon,\ \forall\, m, n \geq N.$$

So, we obtain $S(p,q,\frac{t}{3}) > 1-\epsilon,\ A(p,q,\frac{t}{3}) > 1-\epsilon,\ M(p,q,\frac{t}{3}) < \epsilon,\ R(p,q,\frac{t}{3}) < \varepsilon,\ \forall\, m,n \geq \mu_{\mathcal{N}}$ [where $\mu_{\mathcal{N}} = (S, A, M, R)$, the NFM on G].

Select $p, q \in \mathfrak{D}_N$. Then, sequences $\{p_m^*\}$ and $\{q_m^*\}$ exist such that $p_m^* \to p$ and $q_m^* \to q$. Hence, for sufficiently large m, $p_m^* \in O(p, \varepsilon, \frac{t}{3})$ and $q_m^* \in O(q, \varepsilon, \frac{t}{3})$. Consequently, we obtain

$$\begin{gathered}
S(p,q,t) \geq S(p,p_m^*,\tfrac{t}{3}) \star S(p_m^*,q_m^*,\tfrac{t}{3}) \star S(q_m^*,q,\tfrac{t}{3}) > (1-\varepsilon) \star (1-\varepsilon) \star (1-\varepsilon) > 1-\varrho, \\
A(p,q,t) \geq A(p,p_m^*,\tfrac{t}{3}) \star A(p_m^*,q_m^*,\tfrac{t}{3}) \star A(q_m^*,q,\tfrac{t}{3}) > (1-\varepsilon) \star (1-\varepsilon) \star (1-\varepsilon) > 1-\varrho, \\
M(p,q,t) \leq M(p,p_m^*,\tfrac{t}{3}) \diamond M(p_m^*,q_m^*,\tfrac{t}{3}) \diamond M(q_m^*,q,\tfrac{t}{3}) < \varepsilon \diamond \varepsilon \diamond \varepsilon < \varrho, \\
R(p,q,t) \leq R(p,p_m^*,\tfrac{t}{3}) \diamond R(p_m^*,q_m^*,\tfrac{t}{3}) \diamond R(q_m^*,q,\tfrac{t}{3}) < \varepsilon \diamond \varepsilon \diamond \varepsilon < \varrho.
\end{gathered}$$

Therefore, $S(p,q,t) > 1-\varrho$, $A(p,q,t) > 1-\varrho$, $M(p,q,t) < \varrho$, $R(p,q,t) < \varrho$ $(\forall\, p, q \in \mathfrak{D}_N)$. This implies that $(\mathfrak{D}_N)$ possesses NDZ, hence, by the assumption, $\bigcap_{m\in\mathbb{N}} \mathfrak{D}_m$ is non-empty. Let $p \in \cap_{m\in\mathbb{N}}\mathfrak{D}_m$. Then, for ε in $(0,1)$ and t, a positive real number, $\exists\, N_1 \in \mathbb{N}$ such that $S(p_m, p, t) > 1-\varepsilon,\ A(p_m, p, t) > 1-\varepsilon,\ M(p_m, p, t) < \varepsilon,\ R(p_m, p, t) < \varepsilon$ $(\forall\, n \geq N_1)$. That is, $S(p_m, p, t) \to 1,\ A(p_m, p, t) \to 1,\ M(p_m, p, t) \to 0,\ R(p_m, p, t) \to 0$ for each $t > 0$ as $m \to \infty$. Hence, $p_m \to p$, which means $(G, S, A, M, R, \star, \diamond)$ is complete.

Conversely, assume $(G, S, A, M, R, \star, \diamond)$ is complete. Consider a nested sequence $(\mathfrak{D}_m)_{m\in\mathbb{N}}$ of nonempty closed sets having NDZ. For each $m \in \mathbb{N}$, let p_m be a point in $\mathfrak{D}_m$. Our aim is to demonstrate that $\{p_m\}$ is a Cauchy sequence. Since $(\mathfrak{D}_m)$ possesses NDZ, for $t \in \mathbb{R}^+$ and $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that $S(p,q,t) > 1-\varepsilon$, $A(p,q,t) > 1-\varepsilon$, $M(p,q,t) < \varepsilon$, $R(p,q,t) < \varepsilon$ hold for all $p, q \in \mathfrak{D}_N$. Due to the fact that $(\mathfrak{D}_m)$ is nested, it can be written that $S(p_m, p_n, t) > 1-\varepsilon,\ A(p_m, p_n, t) > 1-\varepsilon,\ M(p_m, p_n, t) < \varepsilon,\ R(p_m, p_n, t) < \varepsilon,\ \forall\, m, n \geq N$. Thus, $\{p_m\}$ forms a Cauchy sequence. As $(G, S, A, M, R, \star, \diamond)$ is complete, $\{p_m\}$ converges to some $p \in G$. This implies that $p \in \overline{\mathfrak{D}_m} = \mathfrak{D}_m$, for all m, and therefore, p belongs to $\bigcap_{m\in\mathbb{N}} \mathfrak{D}_m$. $\square$

Theorem 13. *Every separable NFMS possesses second countability.*

Proof. Let $(G, S, A, M, R, \star, \diamond)$ be separable. Consider $W = \{p_j : j \in \mathbb{N}\}, W \subset G$ as countable dense. Define a collection $\mathfrak{B} = \{O(p_k, 1/m, 1/m) : k, m \in \mathbb{N}\}$. It is evident that $\mathfrak{B}$ is countable. Our objective is to demonstrate that $\mathfrak{B}$ serves as a basis for the assembly of all OS within G. Take ψ as an open set in G containing p. Then, $\exists\, t \in \mathbb{R}^+,\ \varepsilon \in (0,1)$ s.t $O(p, \varepsilon, t) \subset \psi$. Due to this fact, $\varepsilon \in (0,1)$, $\exists\, 0 < \varrho < 1$ s.t $(1-\varrho) \star (1-\varrho) > 1-\varepsilon$ and $\varrho \diamond \varrho < \varepsilon$. Chosse $t \in \mathbb{N}$ so that $1/t < \min\{\varrho, t/2\}$. As $W \subset G$ is dense, $\exists\, p_k \in G$ such that $p \in O(p, 1/t, 1/t)$. If $q \in O(p_k, 1/t, 1/t)$, we can write

$$S(p,q,t) \geq S(p,p_k,t/2) \star S(q,p_k,t/2) \geq S(p,p_k,1/t) \star S(q,p_k,1/t) \geq (1-1/t) \star (1-1/t) \geq (1-\varrho) \star (1-\varrho) > 1-\varepsilon,$$
$$A(p,q,t) \geq A(p,p_k,t/2) \star A(q,p_k,t/2) \geq A(p,p_k,1/t) \star A(q,p_k,1/t) \geq (1-1/t) \star (1-1/t) \geq (1-\varrho) \star (1-\varrho) > 1-\varepsilon,$$
$$M(p,q,t) \leq M(p,p_k,t/2) \diamond M(q,p_k,t/2) \leq M(p,p_k,1/t) \diamond M(q,p_k,1/t) \leq 1/t \diamond 1/t \leq \varrho \diamond \varrho < \varepsilon,$$
$$R(p,q,t) \leq R(p,p_k,t/2) \diamond R(q,p_k,t/2) \leq R(p,p_k,1/t) \diamond R(q,p_k,1/t) \leq 1/t \diamond 1/t \leq \varrho \diamond \varrho < \varepsilon.$$

Then $q \in O(p,\varepsilon,t) \subset \psi$, thus $\mathfrak{B}$ forms a basis. □

Remark 3. *It is worth noting that second countability implies separability, and second countability is an inherited property. Hence, it follows that every subspace of a separable NFMS is also separable.*

Definition 16. *Consider a set $W \neq \emptyset$ and a NFMS $(G,S,A,M,R,\star,\diamond)$. We say that the sequence of functions $(\mathfrak{f}_m): W \to G$ uniformly converges to a function $\mathfrak{f}: W \to G$ if corresponding to any $t>0$, $0<\varepsilon<1$, there remains a natural number N satisfying*

$$S(\mathfrak{f}_n(p),\mathfrak{f}(p),t) > 1-\varepsilon,\ \ A(\mathfrak{f}_m(p),\mathfrak{f}(p),t) > 1-\varepsilon,\ M(\mathfrak{f}_m(p),\mathfrak{f}(p),t) < \varepsilon,\ R(\mathfrak{f}_m(p),\mathfrak{f}(p),t) < \varepsilon,$$

for all $n \geq N$ and $p \in W$.

Now we present uniform convergence theorem for NFMS.

Theorem 14. *Let W be a topological space, $(G,S,A,M,R,\star,\diamond)$ be a NFMS, and $\mathfrak{f}_n : W \to G$ be a sequence of continuous functions. If $(\mathfrak{f}_n)$ converges uniformly to $f : W \to G$ then $\mathfrak{f}$ is considered continuous.*

Proof. Assume μ as an OS in G, $p_0 \in \mathfrak{f}^{-1}(\mu)$. Since μ is open, $\exists\ t \in \mathbb{R}^+$, $\varepsilon \in (0,1)$ so that $O(\mathfrak{f}(p_0),\mathfrak{f}(p),t) \subset \mu$. Since $\varepsilon \in (0,1)$, we select $\varrho \in (0,1)$ such that $(1-\varrho) \star (1-\varrho) \star (1-\varrho) > 1-\varepsilon$ and $\varrho \diamond \varrho \diamond \varrho < \varepsilon$. Given that (f_n) is uniformly convergent to $\mathfrak{f}$,for any $t>0$ and $0<\varrho<1\ \exists\ N \in \mathbb{N}$ so that $S(\mathfrak{f}_n(p),\mathfrak{f}(p),\frac{t}{3}) > 1-\varrho$, $A(\mathfrak{f}_n(p),\mathfrak{f}(p),\frac{t}{3}) > 1-\varrho$, $M(\mathfrak{f}_n(p),\mathfrak{f}(p),\frac{t}{3}) < \varrho$, $R(\mathfrak{f}_n(p),\mathfrak{f}(p),\frac{t}{3}) < \varrho$ for all $n \geq N$, $p \in G$. Since $\mathfrak{f}_n$ are continuous, we have for each n that there remains δ, a neighborhood of p_0, so as $\mathfrak{f}_n(\delta) \subset O(\mathfrak{f}_n(p_0),\varrho,\frac{t}{3})$. Therefore, $S(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),\frac{t}{3}) > 1-\varrho$, $A(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),\frac{t}{3}) > 1-\varrho$, $M(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),\frac{t}{3}) < \varrho$, $R(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),\frac{t}{3}) < \varrho\ \forall\ p \in \delta$. Now,

$$S(\mathfrak{f}(p),\mathfrak{f}(p_0),t) \geq S(\mathfrak{f}(p),\mathfrak{f}_n(p),t/3) \star S(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),t/3) \star S(\mathfrak{f}_n(p_0),\mathfrak{f}(p_0),t/3) \geq (1-\varrho) \star (1-\varrho) \star (1-\varrho) > 1-\varepsilon,$$
$$A(\mathfrak{f}(p),\mathfrak{f}(p_0),t) \geq A(\mathfrak{f}(p),\mathfrak{f}_n(p),t/3) \star A(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),t/3) \star A(\mathfrak{f}_n(p_0),\mathfrak{f}(p_0),t/3) \geq (1-\varrho) \star (1-\varrho) \star (1-\varrho) > 1-\varepsilon,$$
$$M(\mathfrak{f}(p),\mathfrak{f}(p_0),t) \leq M(\mathfrak{f}(p),\mathfrak{f}_n(p),t/3) \diamond M(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),t/3) \diamond M(\mathfrak{f}_n(p_0),\mathfrak{f}(p_0),t/3) \leq \varrho \diamond \varrho \diamond \varrho < \varepsilon,$$
$$R(\mathfrak{f}(p),\mathfrak{f}(p_0),t) \leq R(\mathfrak{f}(p),\mathfrak{f}_n(p),t/3) \diamond R(\mathfrak{f}_n(p),\mathfrak{f}_n(p_0),t/3) \diamond R(\mathfrak{f}_n(p_0),\mathfrak{f}(p_0),t/3) \leq \varrho \diamond \varrho \diamond \varrho < \varepsilon.$$

Implies that $\mathfrak{f}(p) \in O(\mathfrak{f}(p_0),\varepsilon,t) \subset \mu\ \forall\ p \in \delta$. Therefore, $\mathfrak{f}(\delta) \subset \mu$ and thus, $\mathfrak{f}$ is continuous. □

4. Conclusions

This study presents an exploration into the realm of neutrosophic fuzzy metric spaces, introducing an innovative extension of traditional metric spaces tailored to accommodate the intricacies of real-world phenomena characterized by uncertainty and imprecision. The motivation behind this work stems from the recognition of the limitations of classical metric spaces in capturing the nuances inherent in complex systems modelled by neutrosophic fuzzy sets. By introducing and investigating the properties of neutrosophic fuzzy metric spaces, we aim to provide a robust theoretical framework capable of addressing these challenges. Our examination of the structural and topological aspects of neutrosophic fuzzy metric spaces has yielded valuable insights into their fundamental characteristics. Through rigorous analysis, we have elucidated key properties such as completeness, continuity, and convergence within this novel framework. Moreover, the establishment of counterparts to well-known theorems such as the Uniform Convergence Theorem and the Baire Category Theorem underscores the theoretical richness and depth of neutrosophic fuzzy metric spaces.

The inspiration for the introduction of Neutrosophic Fuzzy Metric Spaces draws from the pioneering work of M. Kirisci in 2020, where the concept was first proposed and its topological properties were established. This framework offers a versatile framework wherein fixed points for various types of mappings can be determined, presenting applications across diverse domains such as boundary value problems, nonlinear differential and integral equations, optimization theory, variational inequality, and complementarity problems. Furthermore, NFMS equips us with essential tools to tackle challenges in mathematical analysis and game theory effectively.

Looking ahead, several avenues for future research and application emerge from our findings. Firstly, further exploration of the properties and structures of neutrosophic fuzzy metric spaces promises to deepen our understanding of their mathematical underpinnings. Investigations into additional properties, such as compactness and connectedness, could uncover new insights and lead to the development of more sophisticated mathematical tools and techniques. Additionally, establishing connections between neutrosophic fuzzy metric spaces and other areas of mathematics, such as functional analysis and differential equations, holds the potential to broaden the scope of their applicability and facilitate interdisciplinary collaborations. Beyond the realm of pure mathematics, neutrosophic fuzzy metric spaces offer a powerful framework for modelling and analyzing uncertain and imprecise data, making them particularly relevant in fields such as computer science, engineering, and decision making. Future applications could include the development of robust algorithms for data analysis, optimization techniques for complex systems, and decision support systems for uncertain environments. By bridging the gap between theoretical advancements and real-world challenges, we aim to contribute to the advancement of knowledge and the development of innovative solutions to complex problems.

Author Contributions: Conceptualization, S.G., S., R.B., and S.N.; methodology, S.G., S. and R.B.; validation, S.G., S., R.B., and S.N.; formal analysis, S.G., S., R.B., and S.N.; investigation, S.G., S., R.B., and S.N.; writing—original draft preparation, S.G., S.; writing—review and editing, S., R.B.; visualization and supervision, S.G., S., R.B. and S.N. All authors have read and agreed to the published of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data was used for the research described in the article.

Acknowledgments: Authors are thankful to the editors and reviewers for their insightful remarks and suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control.* **1965**, *8*, 338–353. [CrossRef]
2. Atanassov, K. Intuitionistic fuzzy sets. *Int. J. Bioautomation* **2016**, *20*, 1.
3. Smarandache, F. *A Unifying Field in Logics: Neutrosophic Logic*; Philosophy, American Research Press: New York, NY, USA, 1999; pp. 1–141.
4. Al-shami, T.M.; Mhemdi, A. Generalized frame for orthopair fuzzy sets:(m, n)-fuzzy sets and their applications to multi-criteria decision-making methods. *Information* **2023**, *14*, 56. [CrossRef]
5. Kramosil, I.; Michálek, J. Fuzzy metrics and statistical metric spaces. *Kybernetika* **1975**, *11*, 336–344.
6. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Sets Syst.* **1994**, *64*, 395–399. [CrossRef]
7. Çoker, D. An introduction to intuitionistic fuzzy topological spaces. *Fuzzy Sets Syst.* **1997**, *88*, 81–89. [CrossRef]
8. Park, J.H. Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2004**, *22*, 1039–1046. [CrossRef]

9. Kaleva, O.; Seikkala, S. On fuzzy metric spaces. *Fuzzy Sets Syst.* **1984**, *12*, 215–229. [CrossRef]
10. Smarandache, F. Neutrosophic set-a generalization of the intuitionistic fuzzy set. In Proceedings of the IEEE International Conference on Granular Computing, Atlanta, GA, USA, 10–12 May 2006.
11. Salama, A.A.; Alblowi, S.A. Neutrosophic set and neutrosophic topological spaces. *IOSR J. Math.* **2012**, *3*, 31–35. [CrossRef]
12. Ejegwa, P.A.; Akowe, S.O.; Otene, P.M.; Ikyule, J.M. An overview on intuitionistic fuzzy sets. *Int. J. Sci. Technol. Res* **2014**, *3*, 142–145.
13. Majumdar, P. Neutrosophic sets and its applications to decision making. In *Computational Intelligence for Big Data Analysis: Frontier Advances and Applications*; Springer International Publishing: Cham, Switzerland, 2015; pp. 97–115.
14. Alaca, C.; Turkoglu, D.; Yildiz, C. Fixed points in intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2006**, *29*, 1073–1078. [CrossRef]
15. Nădăban, S. Fuzzy b-metric spaces. *Int. J. Comput. Commun. Control.* **2016**, *11*, 273–281. [CrossRef]
16. Sonam; Bhardwaj, R.; Narayan, S. Fixed point results in soft fuzzy metric spaces. *Mathematics* **2023**, *11*, 3189. [CrossRef]
17. Sonam. Some fixed point results in soft fuzzy metric spaces via altering soft distance. *Adv. Math. Sci. Appl.* **2024**, *33*, 189–200.
18. Gregori, V.; Miñana, J.J.; Miravet, D. Fuzzy partial metric spaces. *Int. J. Gen. Syst.* **2019**, *48*, 260–279. [CrossRef]
19. Gregori, V.; Romaguera, S. Fuzzy quasi-metric spaces. *Appl. Gen. Topol.* **2004**, *5*, 129–136. [CrossRef]
20. Sonam; Rathore, V.; Pal, A.; Bhardwaj, R.; Narayan, S. Fixed-Point Results for Mappings Satisfying Implicit Relation in Orthogonal Fuzzy Metric Spaces. *Adv. Fuzzy Syst.* **2023**, *2023*, 5037401. [CrossRef]
21. Zararsız, Z.; Riaz, M. Bipolar fuzzy metric spaces with application. *Comput. Appl. Math.* **2022**, *41*, 49. [CrossRef]
22. Vetro, C. Fixed points in weak non-Archimedean fuzzy metric spaces. *Fuzzy Sets Syst.* **2011**,*162*, 84–90. [CrossRef]
23. Sonam; Chauhan, C.S.; Bharadwaj, R.; Narayan, S. Fixed point results in soft rectangular b-metric space. *Nonlinear Funct. Anal. Appl.* **2023**, *28*, 753–774.
24. Sonam; Bhardwaj, R. Existence and Uniqueness of Solutions of Nonlinear Integral Equations through Results in Fuzzy Bipolar Metric Spaces. *J. Nonlinear Model. Anal.* 2024, *in press*.
25. Broumi, S.; Bakali, A.; Talea, M.; Smarandache, F.; Uluçay, V.; Sahin, M.; Dey, A.; Dhar, M.; Tan, R.P.; Bahnasse, A.; et al. Neutrosophic sets: An overview. In *On Neutrosophic Theory and Applications II*; Pons, Ed.; EU: Brussels, Belgium, 2018; pp. 403–434. ISBN 978-1-59973-559-7
26. Jeyaraman, M.; Jeyanthi, V.; Mangayarkkarasi, A.N.; Smarandache, F. Some new structures in neutrosophic metric spaces. *Neutrosophic Sets Syst.* **2021**, *42*, 49–64.
27. Kirişci, M.; Şimşek, N. Neutrosophic metric spaces. *Math. Sci.* **2020**, *14*, 241–248. [CrossRef]
28. Sowndrarajan, S.; Jeyaraman, M.; Smarandache, F. Fixed point results for contraction theorems in neutrosophic metric spaces. *Neutrosophic Sets Syst.* **2020**, *36*, 308–318.
29. Ishtiaq, U.; Javed, K.; Uddin, F.; Sen, M.D.L.; Ahmed, K.; Ali, M.U. Fixed point results in orthogonal neutrosophic metric spaces. *Complexity* **2021**, *2021*, 2809657. [CrossRef]
30. Asghar, A.; Hussain, A.; Ahmad, K.; Ishtiaq, U.; Sulami, H.A.; Hussain, N. On neutrosophic 2-metric spaces with application. *J. Funct. Spaces* **2023**, *2023*, 9057107. [CrossRef]
31. Janardhanan, G.; Mani, G.; Ege, O.; Varadharajan, V.; George, R. Orthogonal neutrosophic 2-metric spaces. *J. Inequalities Appl.* **2023**, *2023*, 112. [CrossRef]
32. Mani, G.; Subbarayan, P.; Mitrović, Z.D.; Aloqaily, A.; Mlaiki, N. Solving Some Integral and Fractional Differential Equations via Neutrosophic Pentagonal Metric Space. *Axioms* **2023**, *2*, 758. [CrossRef]
33. Saeed, M.; Ishtiaq, U.; Kattan, D.A.; Ahmad, K.; Sessa, S. New Fixed Point Results in Neutrosophic b-Metric Spaces with Application. *Int. J. Anal. Appl.* **2023**, *21*, 73. [CrossRef]
34. Das, S.; Roy, B.K.; Kar, M.B.; Kar, S.; Pamučar, D. Neutrosophic fuzzy set and its application in decision making. *J. Ambient. Intell. Humaniz. Comput.* **2020**, *11*, 5017–5029. [CrossRef]
35. Abbas, M.; Ali, B.; Romaguera, S. On Generalized Soft Equality and Soft Lattice Structure. *Filomat* **2014**, *28*, 1191–1203. [CrossRef]
36. Ali, B.; Saleem, N.; Sundus, N.; Khaleeq, S.; Saeed, M.; George, R. A Contribution to the Theory of Soft Sets via Generalized Relaxed Operations. *Mathematics* **2022**, *10*, 2636. [CrossRef]