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Full Classification of Finite Singleton Local Rings

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Abstract: The main objective of this article is to classify all finite singleton local rings, which are associative rings characterized by a unique maximal ideal and a distinguished basis consisting of a single element. These rings are associated with four positive integer invariants p, n, s, and t, where p is a prime number. In particular, we aim to classify these rings and count them up to isomorphism while maintaining the same set of invariants. We have found interesting cases of finite singleton local rings with orders of p^6 and p^7 that hold substantial importance in the field of coding theory.

Keywords: local rings; chain rings; isomorphism classes;Galois rings

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1. Introduction

In this article, all rings considered are finite, associative (but generally not necessarily commutative) and have an identity element. One of the main challenges in modern algebra is the problem of describing and classifying finite rings with small orders. If R is a finite ring, its additive group can be decomposed as the direct sum of its p-primary components, denoted as $R(p)$, where p is a prime number, and these components are pairwise orthogonal ideals. Thus, R can be expressed as the direct sum of the rings $R(p)$. In the classification of finite rings, it is therefore sufficient to focus on rings with prime-power order. Consequently, we will consider R to be of this form. In previous studies, all finite rings of orders p, p^2, p^3, p^4, and p^5 were classified up to isomorphism. For orders of p, p^2, and p^3, determining these rings is a straightforward exercise. However, for $|R| = p^4$, a comprehensive list of noncommutative rings was initially compiled in [1,2]. The authors of those papers specifically restrict themselves to the noncommutative case, as commutative rings can be seen as a sum of local rings [3]. Furthermore, the classification of finite local rings when $|R| = p^5$ has already been accomplished by Corbas [4]. To address this problem, researchers emphasized the significant role of local rings, which are rings that satisfy the condition that the set of all zero divisors $J(R)$ forms an ideal, $R/J(R) = F$, where F represents a field. It has been well-established that any finite local ring is associated with positive integers p, n, r, m, and k, called its invariants. A natural class that represents the application of local rings well is the class consisting of chain rings; the radical is principal $J = (\pi)$ (see[5,6]).The aim of this paper is to fully classify and enumerate finite singleton local rings with specific invariants, namely p, n, r, t, and $k = 1$. The paper focuses on investigating these local rings and their properties. Additionally, the paper aims to explore and analyze local rings with specific orders, namely p^6 and p^7. This study helps to gain a deeper understanding of the distinctive characteristics and properties exhibited by these rings and contributes to the broader comprehension of local rings and their applications across various fields, particularly in coding theory [7–14].

The approach used in the literature relies heavily on the following general statement. Given a finite local ring R, $|R| = p^{mr}$ where $R/J(R) = GF(p^r) = F$, the sequence $R = J^0 \supset J \supset J^2 \supset \ldots$ is considered. By letting $s_i = dim_F J^i/J^{i+1}$, it is established that the sum $\sum_{i\geq 0} s_i = m$. As a consequence, it becomes necessary to consider all possible combinations of values s_0, s_1, s_2, s_3, and so on for a given number n and to describe the rings associated with each defined case. The Jacobson radical of a finite ring R is nilpotent, specifically, $J^m = 0$. This naturally leads to the requirement of describing local rings with a nilpotency index of $2, 3$, or 4 for the Jacobson radical.

Prior works [1,4] demonstrate the construction of finite local rings where the order is p^5, and these findings played a crucial role in their classification. Recently, Alkhamees and Alabiad [15] elucidated the structure of finite local rings based on the number k, which is the number (rank) of distinguished bases of R. They completely established the structure of finite local rings with a singleton basis.

In this work, we provide a thorough categorization of local rings on a singleton basis, building upon the findings of our earlier study [15]. With regard to the invariants p, n, r, and t, we specifically concentrate on finite singleton local rings. To begin, Section 2 of the paper presents a restatement of key definitions and notations concerning finite local rings. In Section 3, we construct a generic formula that enables us to compute and classify all potential singleton local rings with fixed invariants p, n, r, and t. This formula enables us to systematically determine the characteristics and properties of these rings. By obtaining a full characterization of these rings, we gain a deeper understanding of their algebraic structure and behavior. Moreover, in Section 4 of the paper, we present particularly interesting results concerning local rings of specific orders, specifically those of p^6 and p^7.

2. Definitions and Notations

In this section, we will discuss some notations and basic facts about finite local rings. These foundational concepts are essential for understanding the subsequent sections of the paper. Suppose R is a finite local ring, and let J denote its Jacobson radical. We will rely on well-established results from the theory of finite rings (see [15–23]).

Firstly, it is well-known that J is the maximal ideal of R. The order of R is $|R| = p^{mr}$, where p is a prime number. The order of J is $p^{(m-1)r}$, and it satisfies the condition $J^m = 0$. The characteristic of R is p^n, $1 \leq n \leq m$, and $R/J \cong GF(p^r)$. When $m = n$, the finite local ring R is commutative, and its Jacobson radical J is generated by the element p. In this case, R can be expressed as

$$R = \mathbb{Z}_{p^n}[a],$$

where a has an additive order of $p^r - 1$. The ring R can also be represented as

$$R \cong \mathbb{Z}_{p^n}[x]/(g(x)),$$

where $g(x)$ is a monic basic polynomial (irreducible modulo p) of degree r over $\mathbb{Z}_{p^n}$. The group of automorphisms $Aut(R)$ of R is a cyclic group of order r. Elements of R can be uniquely expressed as a sum of terms involving $\alpha_i \in (a) \cup \{0\} = \{0, a, a^2, \ldots, a^{p^r-1}\}$,

$$\gamma = \alpha_0 + p\alpha_1 + p^2\alpha_2 + \cdots + p^{n-1}\alpha_{n-2}. \tag{1}$$

The combination of parameters p, n, and r uniquely determines these rings. For a finite local ring R with a characteristic of p^n (where $1 \leq n \leq m$), there exists a coefficient subring S that takes the form $GR(p^n, r)$. This subring is identified as the maximum Galois subring of R. Additionally, if S_0 is another coefficient subring of R, there exists an invertible element x in R such that $S_0 = xSx^{-1}$.

Suppose R is a finite local ring, and let S be its coefficient subring. In this context, there exist elements $\pi_1, \pi_2, \ldots, \pi_k$ in the Jacobson radical $J(R)$ and $\sigma_1, \sigma_2, \ldots, \sigma_k$ in $\mathrm{Aut}(S)$ such that

$$R = S \oplus S\pi_1 \oplus \cdots \oplus S\pi_k \tag{2}$$

as $S-$modules, and $\pi_i\, s = s^{\sigma_i}\pi_i$ for each s in S and for all $i = 1, 2, \ldots, k$. Furthermore, it is important to note that the automorphisms $\sigma_1, \sigma_2, \ldots, \sigma_k$ are characterized by R and S. The set $\pi_1, \pi_2, \ldots, \pi_k$ is referred to as a basis of R over S.

The integers p, n, r, k, and m are known as the invariants of R. Using Equation (2), we can express the Jacobson radical $J(R)$ as the direct sum

$$J = pS \oplus S\pi_1 \oplus \cdots \oplus S\pi_k.$$

Let $S[x, \sigma]$ be the skew polynomial ring over S determined by the automorphism σ, whose elements are of the form

$$\sum_i s_i x^i,$$

where $s_i \in S$, and the multiplication satisfies the relation $xs = \sigma(s)x$ for every s in S.

If the elements π_i are defined as $\pi_i = \pi^i$ for $i = 1, 2, \ldots, k-1$, the finite local ring R becomes a chain ring. In this case, the Jacobson radical J of R is equal to $R\pi$. Furthermore, it can be observed that there is a k such that $p \in J^k$. Additionally, there exists an integer t' satisfying $1 \le t' \le k$ and $m = (n-1)k + t'$. Note that

$$\pi^k = p(u_0 + u_1\pi + u_2\pi^2 + \ldots + u_{k-1}\pi^{k-1}),$$

where u_0 is a unit. This means π is a root of $g(x)$ (Eisenstein polynomial) as follows:

$$g(x) = x^k - p\sum_{i=0}^{k-1} u_i x^i. \tag{3}$$

The numbers p, n, r, k, k', and m are referred to as the invariants of R as follows:

$$R \cong S[x, \sigma]/(g(x), p^{n-1}x^{t'}). \tag{4}$$

The group of units $U(R)$ of R is decomposed as follows:

$$U(R) = (a) \otimes H, \tag{5}$$

where $H = 1 + J$ is called the one group. In the case where R is a chain ring, and letting $\pi \in J \setminus J^2$, then

$$\pi^k = p\beta h, \tag{6}$$

where $\beta \in (a) = \Gamma^*(r)$.

Throughout the manuscript, we will maintain the symbols and notations as specified above.

3. Classification of Singleton Local Rings

This section aims to fully classify and characterize finite singleton local rings up to isomorphism with specific invariants. To do so, we first deal with the associated polynomials of such rings. As $k = 1$, let π belong to the Jacobson radical $J(R)$, and let t represent the additive order of π ($p^t\pi = 0$). Henceforth, let R represent a finite singleton local ring with p, n, r, and t. Assuming R possesses a singleton basis denoted by $\{\pi\}$, let $\sigma \in$Aut(S). Moreover, let $g(x)$ be always defined as

$$g(x) = x^2 - p^d\beta h - p^e\beta_1 h_1 x, \tag{7}$$

where $\beta, \beta_1 \in \Gamma^*(r)$ and $h, h_1 \in 1 + pS$. By the results of [15], R has the following structure:

$$R \cong S[x, \sigma]/(g(x), p^t x). \tag{8}$$

Definition 1. *If there exists π in T_R of R such that $\pi^2 = p^d\beta h$, we call R a pure singleton, and if $h = 1$, then R is said to be a very pure singleton.*

Suppose that $p \neq 2$ and $e < t$, then by completing the square in Equation (7), $g(x) = (x - p^e s)^2 - p^q \beta_2 h_2$, where $s \in S$, and $q \geq \min\{d, 2e\}$. Replacing $x - p^e s$ with x, and since $p \neq 2$, we obtain

$$g(x) = x^2 - p^q \beta. \tag{9}$$

Note that if $2e > n$ and $d = n$, then $g(x) = x^2$.

Proposition 1. *Let $p \neq 2$. Then, every commutative singleton local ring is very pure. In particular, $g(x) = x^2$ if $2e > n$ and $d = n$.*

Lemma 1. *Suppose that R is a local ring with its structure given by (8). Then, $n \geq d \geq 1$ and $t \geq e \geq 1$.*

Proof. Suppose that $d = 0$. Then, $\pi^2 = \beta h + p^e \beta_1 h_1 \pi$, and thus π^2 will be a unit that is a contradiction, since $\pi^2 \in J$. Now, assume that $e = 0$. This leads to $\pi^2 = p^d \beta h + \beta_1 h_1 \pi$, and so $(\pi - \beta_1 h_1)\pi = p^d \beta h$. As $(\pi - \beta_1 h_1)$ is a unit in R, thus $\pi \in (p) = pR$, which is impossible. Thus, each $e \geq 1$ and $d \geq 1$. $\square$

Theorem 1. *If $r = 1$ and $n = m - 1$, then*

$$R = S[x]/(x^2 - p^{m-2}\beta, p^t x). \tag{10}$$

Proof. Since $\mathrm{char}(R) = p^{m-1}$, then $S = \mathbb{Z}_{p^{m-1}}$. As $| R |= p^m$ and $R = S \oplus S\pi_1 \oplus \cdots \oplus S\pi_k$, then there is only $1 \leq i \leq k$ such that $R = S \oplus S\pi_i$. This means R is a singleton ring. As $t = m - n = 1$, then $p\pi = 0$. Also, $d + t \geq m - 1$, which leads to $d = m - 2$. Thus,

$$\pi^2 = p^{m-2}\beta. \quad \square \tag{11}$$

Lemma 2. *Suppose that R is a singleton local ring.*

(i) *If $g(x) = x^2 - p^d \beta h$, then σ is of order 2.*
(ii) *When $g(x) = x^2 - p^d \beta h - p^e \beta_1 h_1 x$, then R is commutative.*
(iii) *In the case when $g(x) = x^2$, then $o(\sigma)$ divides r.*

Proof. (i) Assume that R is a pure singleton local with $g(x) = x^2 - p^\beta h$. As $\pi^2 a = \sigma^2(a)\pi^2$, then $\sigma^2(a) = a$. This means that $o(\sigma) = 2$. For (ii), direct calculation will lead to $\sigma^2(a) = \sigma(a)$, and thus $\sigma = id_S$. The final claim is proved. $\square$

Remark 1. *This remark characterizes $g(x)$ based on aforementioned results.*

(1) *If $p \neq 2$, then*

$$g(x) = \begin{cases} x^2 - p^q \beta, & q \geq min\{d, 2e\} \\ x^2, & \text{if } d = n \text{ and } e = t \text{ or } 2e \geq n. \end{cases}$$

(2) *If $p = 2$,*

$$g(x) = x^2 - 2^d h - 2^e \beta h_1 x.$$

The following proposition presents a relation between changes of distinguished bases of R.

Proposition 2. *If $\theta \in T_R$, then*

$$\theta = \begin{cases} \alpha\pi, & \text{if } p \neq 2, \\ \gamma\pi, & \text{if } p = 2, \end{cases}$$

where $\gamma \in U(S)$ and $\alpha \in \Gamma^*(r)$.

Proof. Suppose that $\theta \in T_R$, which means that $J = pS \oplus \theta S$. Then, there are $a, b \in S$ such that $\theta = pa + b\pi$. As $p^t\theta = 0$, then $\theta = p^{n-t}a + b\pi$. Conversely, if $b \in (p)$, this will lead to $\theta \in (p)$, i.e., $R = S$, which cannot be, as R is a singleton. Thus, $b \in U(S)$, the group of units of S. In the noncommutative case, we apply the same reasoning as in Lemma (1), $p^{n-t}a = 0$. Thus, $\theta = \gamma\pi$, where $b = \gamma$, and by Lemma (2), R is very pure, since $p \neq 2$. Therefore, $\gamma = \alpha \in \Gamma^*(r)$. Next, we assume that $p \neq 2$ and R is commutative,

$$\begin{aligned} p^q\beta' = \theta^2 &= (p^{n-t}a + b\pi)^2 \\ &= p^{2n-t}a^2 - 2p^{n-t}ab\pi + b^2\pi^2 \\ &= p^{2n-t}a^2 - 2p^{n-t}ab\pi + b^2p^q\beta. \end{aligned}$$

Thus, $2p^{n-t}ab = 0$. Since b and 2 are units in S, $p^{n-t}a = 0$. Moreover, $b^2\beta = \beta'$, and this gives $b = \alpha \in (a)$. Therefore, the result follows. Finally, if $p = 2$, then $\theta = b^{-1}(p^{n-t}a' + \pi)$. Replacing $p^{n-t}a' + \pi$ with π gives $\theta = \gamma\pi$, where $\gamma = b^{-1} \in U(s)$. □

Theorem 2. *Suppose that R and T are two local singleton rings with p, n, r, t, d, and e. Then, $T \cong R$ if and only if there exist $\alpha \in \Gamma^*(r)$, $\omega \in 1 + pS$, and $\tau \in Aut(S)$.*

(1) If $p = 2$,

$$\begin{cases} \begin{cases} \tau(\beta_1){\beta_1'}^{-1} = \alpha^2, & \\ \tau(h_1) = \omega^2 h_1' & \textit{mod } p^{n-e}, \textit{ if } d \geq e \quad (R \textit{ is commutative}) \\ \tau(h) = \omega^2 h' & \textit{mod } p^{n-d}, \textit{ otherwise} \end{cases} \\ \\ \tau(h) = \omega^f h' \quad \textit{mod } p^{n-d} \qquad (R \textit{ is noncommutative}), \end{cases}$$

where $x^2 - p^d h$ and $x^2 - p^d h'$ are the associated polynomials in the noncommutative case, and $g(x) = x^2 - 2^d h - 2^e\beta h_1 x$ and $h(x) = x^2 - 2^d h' - 2^e\beta' h_1' x$ are in the commutative case for R and T, respectively.

(2) If $p \neq 2$,

$$\tau(\beta)\beta'^{-1} = \alpha^f, \textit{ and } f = \begin{cases} 2, & \textit{if } R \textit{ is commutative} \\ p^{\frac{r}{2}} + 1, & \textit{if } R \textit{ is noncommutative,} \end{cases}$$

where $x^2 - p^q\beta$ and $x^2 - p^q\beta'$ are the associated polynomials of R and T, respectively.

Proof. Suppose $T \cong R$, and let ϕ be the isomorphism. First, let $p = 2$. Thus, when R is commutative,

$$\begin{aligned} 2^d\tau(h) - 2^e\tau(\beta)\tau h_1\theta = \phi(\pi^2) &= (\phi(\pi))^2 \\ &= \alpha^2\omega^2\theta \\ &= \alpha^2\omega^2(2^d h' - 2^e\beta' h_1'\theta), \end{aligned}$$

where $\alpha \in \Gamma^*(r)$ and $\omega \in 1 + pS$. By comparing sides, we obtain the equations

$$\begin{aligned} &\tau(\beta) = \alpha^2\beta', \\ &\tau(h) = \omega^2 h' \bmod p^{n-d}, \\ &\tau(h_1) = \omega^2 h_1' \bmod p^{n-d}. \end{aligned}$$

In the case where R is noncommutative, $g(x) = p^d h$. Thus, in the same way, we conclude that

$$\tau(h) = \omega^f h',$$

where $f = p^{\frac{r}{2}} + 1$. Next, we suppose that $p \neq 2$. By Remark (1), $g(x) = x^2 - p^q\beta$, and hence

$$\begin{aligned} p^q\tau(\beta) = \phi(\pi^2) &= (\phi(\pi))^2 \\ &= \alpha^2\theta^2 = p^q\alpha^2\beta'. \end{aligned}$$

Thus,

$$\tau(\beta) = \alpha^2\beta'.$$

For the converse, assume that there are $\alpha \in \Gamma^*(r)$, $\omega \in 1 + pS$, and $\tau \in \mathrm{Aut}(S)$ such that the relations are satisfied. When R is commutative with $p = 2$, then $\tau(\beta_1) = \alpha^2\beta_1'$, $p^e\tau(h_1) = p^e\omega^2h_1'$, and $p^d\tau(h) = p^d\omega^2h'$. If we consider the correspondence ϕ from R into T defined by

$$\phi(s_1 + s_2\pi) = \tau(s_1) + \tau(s_2)\alpha\omega\theta,$$

where s_1 and s_2 in S, we have to show that $\phi(\pi^2) = (\phi(\pi))^2$. Note that

$$\begin{aligned} \phi(\pi^2) = \phi(p^dh + p^e\beta_1h_1\pi) =& p^d\tau(h) + p^e\tau(\beta_1)\tau(h_1)\theta \\ =& p^d\alpha^2\omega^2h' + p^e\alpha^2\beta_1'\omega^2h_1'\theta \\ =& (\alpha\omega)^2\left(p^dh' + p^e\beta_1'h_1'\theta\right) \\ =& (\alpha\omega)^2\theta^2 \\ =& (\alpha\omega\theta)^2 \\ =& (\phi(\pi))^2. \end{aligned}$$

Therefore, ϕ is an isomorphism, and $R \cong T$. The case in which R is noncommutative with $p = 2$ is similar to the previous discussion. Finally, suppose $p \neq 2$. In this case, there are τ and α such that $\tau(\beta) = \alpha^f\beta'$. Consider the correspondence $\phi : R \to T$, defined by $\phi(s_1 + s_2\pi) = \tau(s_1) + \tau(s_2)\alpha\theta$. Observe

$$\begin{aligned} \phi(\pi^2) = \phi(p^q\beta) =& p^q\tau(\beta) \\ =& p^q\alpha^f\beta' \\ =& (\alpha\theta)^2 \\ =& (\phi(\pi))^2. \end{aligned}$$

This means that $R \cong T$. $\square$

Theorem 3. *The number of isomorphic classes of finite singleton local rings and of p, n, r, t, d, and e invariants is*

$$N(p,n,r,t,d,e) = \begin{cases} \frac{1}{r}\left(\sum_{i=0}^{r-1}(p^i - 1, z)\right)\left(\sum_{i=0}^{r-1} p^{(i,r)d_o - 1}\right), & \text{if } p = 2, \\ \frac{1}{r}\left(\sum_{i=0}^{r-1}(p^i - 1, z)\right), & \text{if } p \neq 2, \end{cases}$$

where $d_0 = min\{t_0, n-d, n-e\}$ and

$$z = \begin{cases} (p^r - 1, f), & \text{if } p \neq 2, \\ 2, & \text{if } p = 2. \end{cases}$$

Proof. First, consider $p = 2$. We use results of Theorem (2). Note that $\omega \in 1 + pS$, and then $\omega^2 \in (1 + pS)^2$. Suppose that $(1 + pS)^2 = 1 + p^{t_0}S$ for some $t_0 \leq n$. Also note that, when $n \leq 2$, $t_0 = n = 2$. This means that the relations $\tau(h) = \omega^2h'$ and $\tau(h_1) = \omega^2h_1'$ can be reduced to

$$\begin{aligned} \tau(h) &= \omega^2h' \bmod p^{d_0}, \\ \tau(h_1) &= \omega^2h_1' \bmod p^{d_0}, \end{aligned}$$

where $d_0 = \min\{t_0, n-d, n-e\}$. Now, identify $1 + p^{d_0}S$ with $(\Gamma^*(r))^{d_0}$ and replace $(a)/(a^f)$ with $\mathbb{Z}_z$, where $z = (p^r - 1, f)$. Consider the action of $\mathrm{Aut}(S) = (\rho)$ over $\mathbb{Z}_z \times (\Gamma^*(r))^{d_0-1}$ defined by

$$\rho^i(a, \alpha) = (p^i a, \alpha^{p^i}).$$

The above-mentioned relation is an equivalence relation, and Theorem (2) demonstrates that the number of classes of singleton local rings corresponds to the number of equivalence classes. The number of elements fixed by ρ^i is $(p^i - 1, z)[(p^i - 1, p^r - 1) + 1]^{d_0-1}$, but

$$(p^i - 1, z)[(p^i - 1, p^r - 1) + 1]^{d_0-1} = (p^i - 1, z)p^{(i,r)d_0-1}.$$

The Burnside lemma suggests that

$$N(2, n, r, t, d, e) = \frac{1}{r}\left(\sum_{i=0}^{r-1}(p^i - 1, z)\right)\left(\sum_{i=0}^{r-1} p^{(i,r)d_o-1}\right).$$

Secondly, when $p \neq 2$, in this case, $\mathrm{Aut}(S)$ acts only on the set $\mathbb{Z}_z$ by the same action. Similarly, we get

$$N(p, n, r, t, d, e) = \frac{1}{r}\left(\sum_{i=0}^{r-1}(p^i - 1, z)\right). \tag{12}$$

□

Corollary 1. *Assume $g(x) = x^2$. Then,*

$$N(p, n, r, t) = r. \tag{13}$$

Corollary 2. *Suppose that $g(x) = x^2 - p^e\beta_1 h_1 x$. Then,*

$$N(2, n, r, t, e) = \frac{1}{r}\left(\sum_{i=0}^{r-1} p^{(i,r)n-e-1}\right). \tag{14}$$

Remark 2. *Note that*

$$\frac{1}{r}\sum_{i=0}^{r-1}(p^i - 1, z) = \sum_{c|z}\frac{\phi(c)}{\tau(c)}, \tag{15}$$

where ϕ is the Euler function, and $\tau(c)$ is the order of p in $\mathbb{Z}_c$. The last formula has been derived in [15].

4. Categorizing Singleton Local Rings of Orders p^6 and p^7

In ring theory, a fundamental technique involves the characterization and classification of all finite local rings of certain orders and with the same invariant properties. In this section, we undertake the classification of such rings with a singleton basis of order P^i, where $1 \leqslant i \leqslant 7$. It is worth noting that, compared to chain rings, the classification of singleton local rings poses a greater challenge. Based on results in the previous section, we list all local rings of orders p^i, where $i = 6, 7$ up to isomorphism. In what follows, we need to address the notions

$$\begin{cases} \pi^2 = p^d\beta h + p^e\beta_1 h_1 \pi; \\ \beta, \beta_1 \in \Gamma^*(s) \text{ and } h, h_1 \in 1 + pS \\ m \leq 2n, q \geq \min\{d, 2e\}; \\ t = m - n,\ n - t \leq d \leq n \text{ and } 1 \leq e \leq t; \\ 2 \leq l \leq m; J^l = 0 \text{ and } J^{l-1} \neq 0; \\ z = (p^r - 1, f), d_0 = \min\{t_0, n - d, n - e\} \end{cases} \quad (*).$$

Next, we will need the following remarks.

Remark 3. *The number of very pure singleton local rings with invariants p, n, r, and t (up to isomorphism) is*

$$N(p,n,r,t,d) = \begin{cases} 1, & \text{if } p = 2; \\ 2, & \text{if } p \neq 2. \end{cases}$$

Remark 4. *The number of pure singleton local rings (up to isomorphism) is*

$$N(p,n,r,t,d) = \begin{cases} \frac{1}{r}\sum_{i=0}^{r-1} p^{(i,r)n-d-1}, & \text{if } p = 2; \\ \frac{1}{r}\sum_{i=0}^{r-1} (p^i - 1, z), & \text{if } p \neq 2. \end{cases}$$

4.1. Singleton Local Rings of Order Less than p^6

The classification of rings with orders less than p^6 has been accomplished using other methods [1,4]. However, our approach presents a notable distinction, as it offers a more effective and straightforward means of classifying singleton local rings with the invariants p, n, r, t, d, and e. Furthermore, the technique proposed in this article exhibits versatility, as it can be extended to rings of higher orders. This is exemplified in the subsequent subsections, where we apply the approach to investigate rings of order p^6 and p^7.

4.1.1. Local Rings of Order p

In this case, $n = m = 1$, and $r = 1$. Thus, there is a unique finite local ring F_p that is not a singleton.

4.1.2. Local Rings of Order p^2

Since $mr = 2$, singleton local rings will occur when $m = 2$, $r = 1$ (commutative), and $n = 1$. In this case, we have $t = m - n = 1$. This implies that $p\pi = 0$ and $\pi^2 = 0$. Therefore, there is only one singleton local ring satisfying these conditions:

$$R = F_p[x]/(x^2).$$

4.1.3. Local Rings of Order p^3

The only possibility for a singleton local ring with an order of p^3 to occur is when $m = 3$, $n = 2$, and $r = 1$. In this case, we have $t = 1$, which implies $p\pi = 0$. Furthermore, we have $d = 1 = e = t$, and thus $\pi^2 = p\beta h = p\beta$ due to $n = 2$. As a result, the construction of R is as follows:

$$\begin{aligned} R_1 &\cong \mathbb{Z}_{p^2}[x]/(x^2, px), \\ R_2 &= \mathbb{Z}_{p^2}[x]/(x^2 - p\beta, px), \quad \text{if } p \neq 2, \\ R_3 &= \mathbb{Z}_{2^2}[x]/(x^2 - 2, 2x), \qquad \text{if } p = 2. \end{aligned}$$

There is only one ring of the form R_1 by Corollary (1). Additionally, Remark (3) states that there are two rings of type R_2 and one ring of type R_3. It is noteworthy that these rings are of the chain type.

4.1.4. Local Rings of Order p^4

Let R be a finite local ring with $|R| = p^4$. Since $mr = 4$, then we will consider two cases. The case when $r = 4$ and $m = 1$ will lead to $n = 1$, and hence $R = F_{p^4}$, which is not a local ring with a singleton basis.

Case a. If $r = 1$ and $m = 4$, thus, in this case, $n = 2$ or 3.

Case a1. If $n = 2$, then $t = 2$, $d = 1, 2$, and $e = 1, 2$. Therefore,

$$R_1 \cong \mathbb{Z}_{p^2}[x]/(x^2),$$
$$\begin{cases} R_2 \cong \mathbb{Z}_{p^2}[x]/(x^2 - p\beta), \\ R_3 \cong \mathbb{Z}_{p^2}[x]/(x^2 - p\beta x), \end{cases} \quad (p \neq 2)$$
$$\begin{cases} R_4 \cong \mathbb{Z}_{2^2}[x]/(x^2 - 2), \\ R_5 \cong \mathbb{Z}_{2^2}[x]/(x^2 - 2 - 2x), \\ R_6 \cong \mathbb{Z}_{2^2}[x]/(x^2 - 2x). \end{cases} \quad (p = 2)$$

According to Corollary (1), there is a unique ring of the form R_1. Remark (3) indicates that there are two rings of the form R_2, and it is worth noting that $R_3 \cong R_2$ since $q = 1$. Furthermore, when $p = 2$, there are three rings that satisfy the given conditions.

Case a2. When $n = 3$, then $d = 2, 3$, and $t = 1 = e$. Thus,

$$\begin{cases} R_1 \cong \mathbb{Z}_{p^3}[x]/(x^2, px), & \text{if } d = 3 \\ R_2 \cong \mathbb{Z}_{p^3}[x]/(x^2 - p^2\beta, px), & \text{if } d = 2 \text{ and } p \neq 2 \\ R_3 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4, 2x). \end{cases}$$

There exists a single ring classified as R_1, accompanied by two rings classified as R_2 and one ring classified as R_3.

Case b. If $r = 2$, then $m = 2$. In this case, $n = 1$. Thus, the local rings are given by

$$\begin{cases} R_1 \cong \mathbb{F}_{p^2}[x]/(x^2), \\ R_2 \cong \mathbb{F}_{p^2}[x, \sigma]/(x^2). \end{cases}$$

These are the only singleton local rings up to isomorphism.

4.1.5. Local Rings of Order p^5

If the order of the ring R is $\mid R \mid = p^5$, we can have two possible cases: either $r = 5$ and $m = 1$ or $r = 1$ and $m = 5$. However, the first case does not result in a singleton ring. Therefore, we consider the case where $m = 5$ and n can be either 3 or 4.

Case a: Let us assume $n = 3$, which implies $t = 2$. In this case, e can take values of 1 or 2, and d can take values of 1, 2, or 3.

Case a1. Considering the subcase where $d = 1$, the associated polynomial is given by $g(x) = x^2 - p\beta h - p^e \beta_1 x$. Hence,

$$R_1 \cong \mathbb{Z}_{p^3}[x]/(x^2 - p\beta, p^2 x) \text{ if } p \neq 2,$$
$$\begin{cases} R_2 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h, 4x), \\ R_3 \cong \mathbb{Z}_{p^3}[x]/(x^2 - 2h - 2x, 4x). \end{cases} \quad (p = 2)$$

This implies that, according to Remark (4), there are two rings of type R_2 and, according to Theorem (3), there are two rings of type R_3 when $p = 2$. Additionally, Remark (3) suggests that there are two rings of the R_1 form.

Case a2. Now consider the option where d takes the values of 2 or 3. In this case, the construction and properties of the rings can be further explored and analyzed as follows:

$$R_1 \cong \mathbb{Z}_{p^3}[x]/(x^2, p^2 x) \text{ if } d = 3, e = 2,$$
$$R_2 \cong \mathbb{Z}_{p^3}[x]/(x^2 - p^2\beta, p^2 x), \text{ if } p \neq 2, d = 2, e = 1, 2 (q = 2)$$
$$\begin{cases} R_3 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4 - 2x, 4x), \\ R_4 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4, 4x), \\ R_5 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2x, 4x). \end{cases} \quad (p = 2)$$

Based on Remark (3), there are two rings of the form R_2. Therefore, the number of singleton local rings when $p = 2$ is three.

Case b. Let us consider the case when $n = 4$, which implies $t = 1$. In this scenario, we have $e = 1$, and d can take values of 3 or 4. Therefore, further investigation can be done to study the construction and properties of the rings in this case.

$$\begin{aligned} R_1 &\cong \mathbb{Z}_{p^4}[x]/(x^2, px), && \text{if } d = 4 \\ R_2 &\cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta, px) && \text{if } d = 3, p \neq 2, \\ R_3 &\cong \mathbb{Z}_{2^4}[x]/(x^2 - 8, 2x) && \text{if } d = 3, p = 2, \end{aligned}$$

The class of R_2 rings consists of two distinct local rings up to isomorphism. Additionally, there is one ring each of types R_1 and R_3.

4.2. Local Rings of Order p^6

We are classifying singleton local rings with an order of p^6; that is, $mr = 6$. We explore various values for r and m.

Case a. Let us assume $r = 1$. In this instance, the value of n ranges from 3 to 5. Thus, we consider different values of n.

Case a1. When $n = 3$, we have $t = 3$, resulting in values for d of $1, 2$, and 3, and values for e of $1, 2$, and 3.

$$\begin{aligned} R_1 &\cong \mathbb{Z}_{p^3}[x]/(x^2) && \text{if } d = e = 3, \\ R_2 &\cong \mathbb{Z}_{p^3}[x]/(x^2 - p\beta) && \text{if } d = 1, \\ R_3 &\cong \mathbb{Z}_{p^3}[x]/(x^2 - p^2\beta) && \text{if } d = 2, \\ R_4 &\cong \mathbb{Z}_{p^3}[x]/(x^2 - p^2\beta x) && \text{if } d = 3, \end{aligned}$$

$$\left\{ \begin{array}{ll} R_5 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h), & \\ R_6 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h - 2h_1 x), & \\ R_7 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2h - 4x), & \\ R_8 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4), & (p = 2) \\ R_9 \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4 - 2hx), & \\ R_{10} \cong \mathbb{Z}_{2^3}[x]/(x^2 - 4 - 4x), & \\ R_{11} \cong \mathbb{Z}_{2^3}[x]/(x^2 - 2hx) & \end{array} \right.$$

Under these conditions, we observe a unique ring of type R_1, two rings of type R_2 (chain rings), and two rings of the form $R_3 \cong R_4$. When considering the classification of singleton local rings with $p = 2$, we identify a total of 12 distinct rings. Specifically, there are two copies each of types R_5, R_6, R_7, R_9, and R_{11}.

Case a2. Suppose $n = 4$, which implies $t = 2$. Consequently, we have values for d of $2, 3$, and 4, and values for e of 1 and 2.

$$\begin{aligned} R_1 &\cong \mathbb{Z}_{p^4}[x]/(x^2, p^2x) && \text{if } d = 4, e = 2, \\ R_2 &\cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta, p^2x) && \text{if } d = 2, p \neq 2, \\ R_3 &\cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta - p\beta_1 x, p^2x), && \text{if } d = 3, e = 1 \\ R_4 &\cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta) && \text{if } d = 3, e = 2, \end{aligned}$$

$$\left\{ \begin{array}{ll} R_5 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4h, 4x), & \\ R_6 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4h - 2x, 4x), & (p = 2) \\ R_7 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 2h, 4x) & \end{array} \right.$$

In this case, we have a unique ring of type R_1, two rings each of types R_2 and R_3 (with $q = 2$), and two rings of type R_4. Furthermore, when considering $p = 2$, there are two copies each of types R_5, R_6, and R_7.

Case a3. We assume $n = 5$. As the results of Theorem (1), we get

$$\begin{aligned} &R_1 \cong \mathbb{Z}_{p^5}[x]/(x^2, px), \text{ if } d = 5, \\ &R_2 \cong \mathbb{Z}_{p^5}[x]/(x^2 - p^4\beta, px), \text{ if } d = 4, p \neq 2, \\ &R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 16, 2x), \text{ if } d = 4, p = 2. \end{aligned}$$

With these specifications, we find four singleton local rings with the same invariants.

Case b. Now, let us consider the option where $r = 2$ and $m = 3$. This implies that $n = 2$, resulting in d taking values of 1 and 2, while t and e are both equal to 1. Therefore,

$$\begin{aligned} &R_1 \cong \mathbb{Z}_{p^2}[x,\sigma]/(x^2, px), \text{ if } d = 2 \\ &R_2 \cong \mathbb{Z}_{p^2}[x,\sigma]/(x^2 - p\beta, px), \text{ if } d = 1, p \neq 2, \\ &R_3 \cong \mathbb{Z}_{2^2}[x,\sigma]/(x^2 - 2, 2x), \text{ if } p = 2. \end{aligned}$$

Based on the previously mentioned results, we find two rings of type R_1, two rings of type R_2, and one ring of type R_1.

Case c. Let $r = 3$. Consequently, $m = 2$, which leads to $n = 1$. In this case, $d = t = 1 = e$. As a result, the construction of R is as follows:

$$R \cong \mathbb{Z}_p[x, \sigma^i]/(x^2), \tag{16}$$

where $i = 1, 2, 3$. Thus, we have only three singleton locals of these rings based on Corollary (1).

4.3. Local Rings of Order p^7

In this subsection, we examine the scenario where $r = 1$, indicating commutative rings. Hence, we consider the case where $m = 7$, leading to n taking values of $4, 5, 6$. Depending on the specific values of n, we investigate the following cases.

Case a. When $n = 4$, and since $t = 3$, we consider different values for d (specifically, $d = 1, 2, 3, 4$) and e (specifically, $e = 1, 2, 3$). Within this case, we further divide it into subcases based on the values of d.

Case a1. If $d = 1$, the classification of such rings is as follows:

$$\begin{aligned} &R_1 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p\beta, p^3x), \text{ if } p \neq 2, \\ &\left\{ \begin{array}{l} R_2 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 2h - 2h_1x, 8x), \\ R_3 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 2h - 4x, 8x), \qquad (p = 2) \\ R_4 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 2h, 8x). \end{array} \right. \end{aligned}$$

Based on our findings, there exist two distinct rings of R_1. In the case where $p = 2$, we observe a total of six classes of these rings, divided into two classes for each type.

Case a2. If $d = 2$, thus

$$\begin{aligned} &\left\{ \begin{array}{l} R_1 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta, p^3x), \\ R_2 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta - p^2\beta_1x, p^3x), \qquad (p \neq 2), \\ R_3 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta - p\beta_1h_1x, p^3x), \end{array} \right. \\ &\left\{ \begin{array}{l} R_2 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4h, 8x), \\ R_3 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4h - 4x, 8x), \qquad (p = 2) \\ R_4 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4h - 2h_1x, 8x). \end{array} \right. \end{aligned}$$

Considering these specifications, we identify two classes of rings belonging to type R_1. It should be noted that R_1 is isomorphic to both R_2 and R_3 due to the shared value of $q = 2$. Additionally, in the case where $p = 2$, there are a total of six copies of singleton rings.

Case a3. Letting $d = 3$, we list all possible singleton local rings:

$$\begin{cases} R_1 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta, p^3x), \\ R_2 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta - p^2\beta_1 x, p^3x), \quad (p \neq 2) \\ R_3 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta - p\beta_1 h_1 x, p^3x), \end{cases}$$
$$\begin{cases} R_4 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 8, 8x), \\ R_5 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 8 - 4x, 8x), \quad (p = 2) \\ R_6 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 8 - 2hx, 8x). \end{cases}$$

As $p \neq 2$, there are two rings of type R_1. There are two rings of the form R_3. Note that $R_2 \cong R_1$ and $R_3 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^3\beta - p\beta_1 h_1 x, p^3x) \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta, p^3x)$. When $p = 2$, there are four of such rings up to isomorphism.

Case a4. If $d = 4$, we classify all such rings as follows:

$$R_1 \cong \mathbb{Z}_{p^4}[x]/(x^2, p^3x),$$
$$\begin{cases} R_2 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p^2\beta x, p^3x), \\ R_3 \cong \mathbb{Z}_{p^4}[x]/(x^2 - p\beta h x, p^3x), \end{cases} (p \neq 2)$$
$$\begin{cases} R_4 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 4x, 8x), \\ R_5 \cong \mathbb{Z}_{2^4}[x]/(x^2 - 2hx, 8x). \end{cases} (p = 2)$$

Case b. Now, let us consider the case where $n = 5$. Similar to the previous reasoning, we have $t = 2$, d can take values of $3, 4, 5$, and e can be 1 or 2. Based on these values, we construct the following structures.

Case b1: Assuming $d = 3$, we list all possible singleton local rings:

$$\begin{cases} R_1 \cong \mathbb{Z}_{p^5}[x]/(x^2 - p^3\beta, p^2x), \\ R_2 \cong \mathbb{Z}_{p^5}[x]/(x^2 - p^3\beta - p\beta_1 x, p^2x), \end{cases} (p \neq 2)$$
$$\begin{cases} R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 8, 4x), \\ R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 8 - 2x, 4x). \end{cases} (p = 2)$$

In this subcase, we identify two classes each for the R_1 and R_2 types of singleton local rings, satisfying the given conditions. In the case where $p = 2$, there are two rings that fall into this category.

Case b2. When $d = 4$, we proceed to classify all rings that satisfy this condition:

$$\begin{cases} R_1 \cong \mathbb{Z}_{p^5}[x]/(x^2 - p^4\beta, p^2x), \\ R_2 = \mathbb{Z}_{p^5}[x]/(x^2 - p^4\beta - p\beta_1 x, p^2x), \end{cases} (p \neq 2)$$
$$\begin{cases} R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 16, 4x), \\ R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 16 - 2x, 4x). \end{cases} (p = 2)$$

Our results reveal the existence of two rings for each type R_1 and R_2. Moreover, in the case where $p = 2$, there are two distinct classes of these rings.

Case b3. Now, let us consider the scenario where $d = 5$. We undertake the classification of all rings that fulfill this criterion:

$$R_1 \cong \mathbb{Z}_{p^5}[x]/(x^2, p^2x),$$
$$R_2 \cong \mathbb{Z}_{p^5}[x]/(x^2 - p\beta_1 x, p^2x), \text{if } p \neq 2$$
$$R_3 \cong \mathbb{Z}_{2^5}[x]/(x^2 - 2x, 4x).$$

When $p \neq 2$, there are two distinct classes of local rings R_1 that satisfy the given specifications. Additionally, for the R_2 type, there are two rings with $q = 2$. In the case where $p = 2$, there are also two rings that meet the criteria.

Case c. Lastly, we consider $n = 6$. According to Theorem (1),

$$R_1 \cong \mathbb{Z}_{p^6}[x]/(x^2 - p^5\beta, px), \quad \text{if } p \neq 2,$$
$$R_2 \cong \mathbb{Z}_{2^6}[x]/(x^2 - 32, 2x), \quad \text{if } p = 2.$$

Under the given conditions, we observe two distinct classes of rings belonging to the R_1 type. Additionally, there is one ring in the R_2 category.

5.Conclusions

The realization that nonlinear codes over finite fields can be connected to linear codes over finite local rings via the Gray maps has significantly enhanced the importance of finite local rings in the study of codes over finite rings. This is primarily due to their characteristic of having unique maximal ideals. As a result, this study opened the door for investigating finite local rings in general. In this article, we have successfully classified up to isomorphism all finite local rings with a singleton basis, i.e., $k = 1$. The classification process took into account six positive integer invariants p, n, s, t, d, and e. As a result, all finite local rings with a singleton basis of orders p^6 and p^7 have been thoroughly classified and counted up to isomorphism. However, the task becomes more challenging when $k \geq 2$, i.e., non-singleton local rings, which complicates the classification of finite local rings. This limitation indicates the necessity for alternative approaches or techniques to address this general case.

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