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Decomposition Integrals of Set-Valued Functions Based on Fuzzy Measures

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Abstract: The decomposition integrals of set-valued functions with regards to fuzzy measures are introduced in a natural way. These integrals are an extension of the decomposition integral for real-valued functions and include several types of set-valued integrals, such as the Aumann integral based on the classical Lebesgue integral, the set-valued Choquet, pan-, concave and Shilkret integrals of set-valued functions with regard to capacity, etc. Some basic properties are presented and the monotonicity of the integrals in the sense of different types of the preorder relations are shown. By means of the monotonicity, the Chebyshev inequalities of decomposition integrals for set-valued functions are established. As a special case, we show the linearity of concave integrals of set-valued functions in terms of the equivalence relation based on a kind of preorder. The coincidences among the set-valued Choquet, the set-valued pan-integral and the set-valued concave integral are presented.

Keywords: set-valued function; fuzzy measure; decomposition integral; choquet integral; pan-integral; concave integral

MSC: 28C15; 46G12

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1. Introduction

In [1], Even and Lehrer introduced the decomposition integral based on a decomposition system and a capacity. In general, the different decomposition systems induce different decomposition integrals. The well-known nonlinear integrals, the Choquet integral, the concave integral, the pan-integral and the Shilkret integral, are based on the chains of sets, arbitrary finite set systems, finite partitions and singletons, respectively. The decomposition integral forms a general setting for these nonlinear integrals (see also [2,3]). In recent years, decomposition integrals have attracted more and more attention from researchers, and a lot of results have been obtained (see [2–12]).

The above-mentioned decomposition integrals, including the Choquet, pan-, concave and Shilkret integrals, concern real-valued functions and number-valued capacities. As is well known, set-valued integrals for set-valued functions, such as the Aumann integral, the Debreu integral and others which are based on Lebesgue integrals, are extensions of classical integrals for number-valued functions and number-valued measures (see [13–15]). Analogously to the idea of defining classical set-valued integrals, several types of integrals of set-valued functions based on nonlinear integrals have been introduced and discussed, such as set-valued fuzzy integrals [16–18], set-valued Choquet integrals [19–21], Gould-type integrals for multisubmeasures [22,23], Aumann–Pettis–Sugeno integrals [24], etc. More studies on the topics of set-valued functions and integrals of set-valued functions have also appeared in [23,25–30]. In particular, the Choquet integrals of set-valued functions have been deeply investigated (see [19–21,31,32]), and the pan-integrals of set-valued functions have just been introduced and discussed in [33]. Noting that the decomposition integral forms a unified framework for the Choquet integrals, pan-integrals, concave integrals, etc., we naturally want to define a set-valued decomposition integral so that the previous set-valued integrals, such as the set-valued Choquet, the set-valued pan- and the set-valued concave integrals, become some special cases of set-valued decomposition integrals.

In this paper, we will define the decomposition integral of set-valued functions in a natural way (i.e., in a way similar to the integral of Aumann). This integral is an extension of a decomposition integral with respect to a capacity for real-valued functions. Analogously to the previous discussion of the set-valued Choquet and the set-valued pan-integral, we present some basic properties, positive homogeneity, monotonicity in the sense of inclusion relation, etc. We respectively investigate the monotonicity of the set-valued decomposition integral in the sense of two preorders: a kind of preorder on the class of all nonempty sets of $\mathbb{R}^1$ and another kind of preorder on the set of all decomposition systems. By using the monotonicity, we will establish the Chebyshev inequality of the set-valued decomposition integral, and as special cases, Chebyshev's inequality of the set-valued Choquet, pan-, concave and Shilkret integrals is also presented. Under the conditions of subadditivity, submodularity and the (M)-property of fuzzy measures, the coincidences among the set-valued pan-integral, set-valued Choquet integral and set-valued concave integral are shown, respectively.

2. Preliminaries

Let $(\Omega, \mathfrak{A})$ denote a measurable space, i.e., Ω is a nonempty set and $\mathfrak{A}$ is a σ-algebra of subsets of Ω.

2.1. Fuzzy Measures

A set function $\kappa : \mathfrak{A} \to [0, \infty]$ is called a *fuzzy measure* ([34,35]) on $(\Omega, \mathfrak{A})$ if (1) $\kappa(\emptyset) = 0$ and (2) $\kappa(C_1) \leq \kappa(C_2)$ whenever $C_1, C_2 \in \mathfrak{A}$, $C_1 \subseteq C_2$.

A fuzzy measure is also known as "capacity" (in the case of $\kappa(\Omega) = 1$) [36,37], "monotone measure" [6,38], "non-additive measure" [28,29], "non-additive probability", etc.

Let $\mathfrak{M}$ denote the set of all fuzzy measures defined on $(\Omega, \mathfrak{A})$.

A fuzzy measure κ is called *subadditive* [38,39] if $\kappa(S \cup W) \leq \kappa(S) + \kappa(W)$ whenever $S, W \in \mathfrak{A}$; *superadditive* [38,39], if $\mu(S \cup W) \geq \mu(S) + \mu(W)$ whenever $S, W \in \mathfrak{A}$ and $S \cap W = \emptyset$.

2.2. Decomposition Integrals

A *collection* from $\mathfrak{A} \setminus \{\emptyset\}$ is a finite nonempty subset of $\mathfrak{A} \setminus \{\emptyset\}$. A *decomposition system* on $(\Omega, \mathfrak{A})$ is a nonempty set $\mathcal{H}$ of collections from $\mathfrak{A} \setminus \{\emptyset\}$. Let $\mathbb{X}$ denote the set of all decomposition systems on $(X, \mathfrak{A})$.

In [1], Even and Lehrer introduced the decomposition integral (see also [3]).

Let $\mathfrak{F}^+$ denote the set of all non-negative real-valued $\mathfrak{A}$-measurable functions on Ω and χ_E be the characteristic function of $E \in \mathfrak{A}$.

Definition 1 (Even and Lehrer [1])**.** *Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$. The mapping $\int_{\mathcal{H}} \cdot d\kappa : \mathfrak{F}^+ \to \overline{\mathbb{R}}_+$, defined by*

$$\int_{\mathcal{H}} h d\kappa = \sup \Big\{ \sum_{j=1}^{m} d_j \kappa(D_j) : (D_j)_{j=1}^{m} \in \mathcal{H}, \sum_{j=1}^{m} d_j \chi_{D_j} \leq h, d_j \geq 0 \Big\}, \tag{1}$$

is called a decomposition integral with respect to $\mathcal{H}$ and κ.

A decomposition integral depends on a decomposition system $\mathcal{H} \in \mathbb{X}$ and a fuzzy measure $\kappa \in \mathfrak{M}$. There are four common decomposition integrals: the Choquet integral, the pan-integral (with respect to the pair of standard addition and multiplication $(+, \cdot)$), the concave integral and the Shilkret integral. They are based on the chains of sets, finite partitions, arbitrary finite set systems and singletons, respectively.

We denote, respectively, by $\mathcal{H}_{Ch}$, $\mathcal{H}_{cav}$ and $\mathcal{H}_{Sh}$ the sets of all finite chains, all families of finite sets and singletons in $\mathfrak{A} \setminus \{\emptyset\}$, and by $\mathcal{H}_{pan}$ the set of all finite measurable partitions of X.

Example 1. *Given $\mu \in \mathfrak{M}$ and let $h \in \mathfrak{F}^+$.*

(i) The decomposition integral $\int_{\mathcal{H}_{Ch}} hd\mu$ is the Choquet integral ([1,36]) of h with respect to μ, i.e.,

$$\int_{\mathcal{H}_{Ch}} h\,d\mu = \int_0^{\infty} \mu\big(\{\omega \in \Omega : h(\omega) \geq s\}\big)\,ds.$$

(ii) The decomposition integral $\int_{\mathcal{H}_{cav}} hd\mu$ is the concave integral of h with respect to μ (Lehrer [1,37,40]).

(iii) The decomposition integral $\int_{\mathcal{H}_{pan}} hd\mu$ is the pan-integral of h with respect to μ, see [38], i.e.,

$$\int_{\mathcal{H}_{pan}} h\,d\mu = \sup_{\mathcal{P} \in \mathcal{H}_{pan}} \Big\{ \sum_{A \in \mathcal{P}} \big[\big(\inf_{\omega \in A} h(\omega)\big) \cdot \mu(A)\big] \Big\}.$$

(iv) The decomposition integral $\int_{\mathcal{H}_{Sh}} hd\mu$ is the Shilkret integral of h with respect to μ, i.e., $\int_{\mathcal{H}_{Sh}} hd\mu = \sup\{s \cdot \mu(\{\omega \in \Omega : h(\omega) \geq s\}) : s \in [0,\infty]\}$ ([6,41]).

For the convenience of discussion, and to not confuse the symbols with set-valued integrals, we still use common symbols, denoting, respectively, by $\int^{Ch} \cdot\, d\mu$, $\int^{cav} \cdot\, d\mu$, $\int^{pan} \cdot\, d\mu$ and $\int^{Sh} \cdot\, d\mu$ the integrals $\int_{\mathcal{H}_{Ch}} \cdot d\mu$, $\int_{\mathcal{H}_{cav}} \cdot d\mu$, $\int_{\mathcal{H}_{pan}} \cdot d\mu$ and $\int_{\mathcal{H}_{Sh}} \cdot d\mu$ of the real-valued integrals.

Let $h \in \mathfrak{F}^+$. h be called $\mathcal{H}$-integrable with respect to κ if $\int_{\mathcal{H}} hd\kappa < \infty$. We denote $I(\mathcal{H},\kappa) = \{h \in \mathfrak{F}^+ : \int_{\mathcal{H}} hd\kappa < \infty\}$.

For more details concerning non-additive measures and integrals, see [1,3,6,34,35,37–40].

2.3. Set-Valued Maps

We recall some basic definitions dealing with set-valued maps [13,15].

We denote $\mathbb{R}^1 = (-\infty,+\infty)$, $\mathbb{R}^1_+ = [0,\infty)$, $\mathfrak{P}(\mathbb{R}^1) = 2^{\mathbb{R}^1} \setminus \{\emptyset\}$, $\mathfrak{P}(\mathbb{R}^1_+) = 2^{\mathbb{R}^1_+} \setminus \{\emptyset\}$, and denote by $\mathcal{C}(\mathbb{R}^1_+)$, $\mathcal{K}(\mathbb{R}^1_+)$ and $\mathcal{K}_C(\mathbb{R}^1_+)$ the families of all nonempty closed, compact and compact convex sets of $\mathbb{R}^1_+$, respectively.

Let $C_1, C_2 \in \mathfrak{P}(\mathbb{R}^1_+), k \in \mathbb{R}^1_+$. The sum of C_1 and C_2 is defined by

$$C_1 + C_2 = \{c_1 + c_2 : c_1 \in C_1, c_2 \in C_2\}$$

and the scalar multiplication of k and C_1 is the set defined by

$$kC_1 = \{kc_1 : c_1 \in C_1\}.$$

The classes $\mathcal{C}(\mathbb{R}^1_+)$, $\mathcal{K}(\mathbb{R}^1_+)$ and $\mathcal{K}_C(\mathbb{R}^1_+)$ are closed under the operations of addition and scalar multiplication, respectively [42].

The preorder of A_1 and A_2, denoted by $A_1 \preceq A_2$, means that for each $s \in A_1$, there is $t \in A_2$ such that $s \leq t$, and for each $q \in A_2$, there is $p \in A_1$ such that $p \leq q$ ([18], see also [16,19,21]). Moreover, we define a relation "$\approx$" on $\mathfrak{P}(\mathbb{R}^1_+)$: $A_1 \approx A_2$ iff $A_1 \preceq A_2$ and $A_2 \preceq A_1$. The relation "$\approx$" is an equivalence relation.

A set-valued function is a mapping $F : \Omega \to \mathfrak{P}(\mathbb{R}^1_+)$. We denote

$$F^{-1}(A) \triangleq \{\omega \in \Omega : F(\omega) \cap A \neq \emptyset\},$$

where $A \in \mathfrak{P}(\mathbb{R}^1)$. The set-valued function F is called measurable with respect to $\mathfrak{A}$ (short for measurable) if, for all closed subsets $C \in \mathcal{C}(\mathbb{R}^1_+)$, $F^{-1}(C) \in \mathfrak{A}$ (see [13]). The set-valued function F is said to be closed-valued if its values are closed subsets of $\mathbb{R}^1_+$, i.e., $F : \Omega \to \mathcal{C}(\mathbb{R}^1_+)$. We denote by $\mathfrak{R}[\Omega]$ the class of all closed-valued functions defined on Ω. For more discussions of the measurability of closed-valued functions, refer to [13,15].

Let G be a set-valued function defined on Ω and $g \in \mathfrak{F}^+$. If for all $\omega \in \Omega$, $g(\omega) \in G(\omega)$ holds, then g is called a *measurable selection of* G. If $G \in \mathfrak{R}[\Omega]$, then G has at least one measurable selection ([15]).

In the following, we present the operations and orders of set-valued functions on $\mathfrak{R}[\Omega]$ ([15]). Let $F, H \in \mathfrak{R}[\Omega]$, $k \in \mathbb{R}^1_+$.

(1) $(kH)(\omega) \triangleq kH(\omega)$ for any $\omega \in \Omega$;
(2) $(F + H)(\omega) \triangleq F(\omega) + H(\omega)$ for any $\omega \in \Omega$;
(3) $F \preceq H$ iff $(F)(\omega) \preceq H(\omega)$ for any $\omega \in \Omega$.
(4) $F \subseteq H$ iff $(F)(\omega) \subseteq H(\omega)$ for any $\omega \in \Omega$.

Let $F \in \mathfrak{R}[\Omega]$, $A \in \mathfrak{A}$. We define

$$(\chi_A F)(\omega) \triangleq \chi_A(\omega)F(\omega) = \begin{cases} F(\omega) & \text{if } \omega \in A, \\ \{0\} & \text{if } \omega \in \Omega \setminus A, \end{cases}$$

then $\chi_A F \in \mathfrak{R}[\Omega]$, and for any $S, T \in \mathfrak{A}$, $S \subseteq T$, it holds that $\chi_S F \preceq \chi_T F$, i.e., $\chi_S(\omega)F(\omega) \preceq \chi_T(\omega)F(\omega)$ for every $\omega \in \Omega$.

3. Set-Valued Decomposition Integrals of Set-Valued Functions

In this section, we define the decomposition integrals of set-valued functions with respect to fuzzy measures and present some of their properties.

3.1. Definition of Set-Valued Decomposition Integrals

Definition 2. *Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, let $G \in \mathfrak{R}[\Omega]$. The set-valued decomposition integral of G with respect to the fuzzy measure κ on Ω (short for decomposition integral of G) is defined by*

$$\int_{\mathcal{H}} G d\kappa = \left\{ \int_{\mathcal{H}} g d\kappa : g \in S_{\mathcal{H}}(G, \kappa) \right\}, \tag{2}$$

where $S_{\mathcal{H}}(G, \kappa) = \{g : g \in I(\mathcal{H}, \kappa) \text{ and } g(\omega) \in G(\omega) \text{ on } \Omega\}$.

When we take $\mathcal{H} = \mathcal{H}_{Ch}$ and $\mathcal{H}_{pan}$, respectively, then the set-valued decomposition integral $\int_{\mathcal{H}} \cdot d\kappa : \mathfrak{R}[\Omega] \to \mathfrak{P}(\mathbb{R}^1_+)$ goes back to the set-valued Choquet integral [19,21,32] and the set-valued pan-integral [33], respectively. In the following section, we will discuss the set-valued concave integral.

Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, the set-valued function G is called *$\mathcal{H}$-integrable with respect to κ* if $\int_{\mathcal{H}} G d\kappa \neq \emptyset$; it is *$\mathcal{H}$-integrably bounded with respect to κ* if there is $l \in I(\mathcal{H}, \kappa)$, i.e., $l \in \mathfrak{F}^+$ and $\int_{\mathcal{H}} l d\kappa < \infty$, such that

$$\phi_G(\omega) \triangleq \sup\{s : s \in G(\omega)\} \leq l(\omega)$$

holds for all $\omega \in \Omega$.

Note that ϕ_G is a measurable function on $(\Omega, \mathfrak{A})$ [15], and for every $\omega \in \Omega$, $\phi_G(\omega) \in G(\omega)$, $G(\omega)$ is a nonempty closed set; therefore, ϕ_G is a measurable selection of G and $\phi_G \in I(\mathcal{H}, \kappa)$.

3.2. Basic Properties of Set-Valued Decomposition Integrals

The set-valued decomposition integral is positive-homogeneous for any $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$.

Proposition 1. *Let $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, and $G \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrable with respect to κ. Then, for all $k \geq 0$,*

$$\int_{\mathcal{H}} kG d\kappa = k \int_{\mathcal{H}} G d\kappa. \tag{3}$$

Proposition 2. *Let $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, and $G \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrable with respect to κ. Then, for any $A \in \mathfrak{A}$,*

$$\int_{\mathcal{H}} \chi_A G d\kappa = \left\{ \int_{\mathcal{H}} g\chi_A d\kappa : g \in S_{\mathcal{H}}(G,\kappa) \right\}, \tag{4}$$

where $S_{\mathcal{H}}(G,\kappa) = \{g : g \in I(\mathcal{H},\kappa) \text{ and } g(\omega) \in G(\omega) \text{ on } \Omega\}$.

Proposition 3. *Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, let $G \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrably bounded with respect to κ. Then,*

(1) G is $\mathcal{H}$-integrable with respect to κ;

(2) For every $\omega \in \Omega$, $G(\omega)$ is a bounded closed set of $\mathbb{R}^1_+$, i.e., $G(\omega) \in \mathcal{K}(\mathbb{R}^1_+)$;

(3) For any measurable selection f of G, g is $\mathcal{H}$-integrable with respect to κ, i.e., $g \in I(\mathcal{H},\kappa)$. In particular, $\phi_G \in I(\mathcal{H},\kappa)$;

(4) $\int_{\mathcal{H}} G d\kappa$ is a bounded set in $\mathbb{R}^1_+$;

(5) There exists a sequence, $\{g_n : n \in \mathbb{N}\} \subset S_{\mathcal{H}}(G,\kappa)$, such that $G(\omega) = cl\{g_n(\omega) : n \in \mathbb{N}\}$ for every $\omega \in \Omega$ (i.e., a representation of F by measurable selections).

Proof. (1) $G \in \mathfrak{R}[\Omega]$ implies that there is a measurable selection g of G, i.e., $g \in \mathfrak{F}^+$ and $g(\omega) \in G(\omega)$ for all $\omega \in \Omega$. G is $\mathcal{H}$-integrably bounded with respect to κ, so there is a $l \in \mathfrak{F}^+$ and $\int_{\mathcal{H}} l d\kappa < \infty$, such that $\phi_G(\omega) \leq l(\omega)$ holds for all $\omega \in \Omega$. This implies that $g(\omega) \leq l(\omega)$, and hence $\int_{\mathcal{H}} g d\kappa < \infty$. Therefore, $\int_{\mathcal{H}} G d\kappa \neq \emptyset$ from $\int_{\mathcal{H}} g d\kappa \in \int_{\mathcal{H}} G d\kappa$, i.e., G is $\mathcal{H}$-integrable with respect to κ.

(2) For a given $\omega \in \Omega$, it follows from $\phi_G(\omega) \leq l(\omega) < \infty$ that $G(\omega)$ is a bounded closed set of $\mathbb{R}^1_+$.

(3) From $g(\omega) \leq \phi_G(\omega) \leq l(\omega)$ for all $\omega \in \Omega$, and $l \in I(\mathcal{H},\kappa)$, then $f \in I(\mathcal{H},\kappa)$.

(4) For any $r \in \int_{\mathcal{H}} G d\kappa$, there is $h \in I(\mathcal{H},\kappa)$ and $h(\omega) \in G(\omega)$ on Ω, such that $r = \int_{\mathcal{H}} h d\kappa \leq \int_{\mathcal{H}} l d\kappa \triangleq M < \infty$. This shows that $\int_{\mathcal{H}} G d\kappa$ is a bounded set in $\mathbb{R}^1_+$.

(5) For $G \in \mathfrak{R}[\Omega]$, there is a sequence $\{g_n : n \in \mathbb{N}\}$ of measurable selection of G such that $G(\omega) = cl\{g_n(\omega) : n \in \mathbb{N}\}$ holds for every $\omega \in \Omega$ (see [15]). This implies that for every $n = 1, 2, \ldots$, $g_n(\omega) \in G(\omega)$ holds for every $\omega \in \Omega$. Since $G \in \mathfrak{R}[\Omega]$ is $\mathcal{H}$-integrably bounded with respect to κ, and based on the above (3), then for every $n = 1, 2, \ldots$, $g_n \in S_{\mathcal{H}}(G,\kappa)$. □

3.3. Monotonicity of Set-Valued Decomposition Integrals

In this subsection, we present several versions of monotonicity of set-valued decomposition integrals.

The following result is clarified by Definition 2.

Proposition 4. *Let $G, H \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrable with respect to κ. Then,*

$$G \subseteq H \text{ implies } \int_{\mathcal{H}} G d\kappa \subseteq \int_{\mathcal{H}} H d\kappa. \tag{5}$$

The following is a version of monotonicity of set-valued decomposition integrals with respect to the preorder relation "$\preceq$".

Proposition 5. *Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, let $G, H \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrably bounded with respect to κ. Then*

$$G \preceq H \text{ implies } \int_{\mathcal{H}} G d\kappa \preceq \int_{\mathcal{H}} H d\kappa. \tag{6}$$

Proof. Suppose $G \preceq H$. For any $s \in \int_{\mathcal{H}} G d\kappa$, we prove that there is $t \in \int_{\mathcal{H}} H d\kappa$ such that $s \leq t$. From the definition of set-valued decomposition integrals (Definition 2), there is $g \in I(\mathcal{H}, \kappa)$ and $g(\omega) \in G(\omega)$ on Ω such that $s = \int_{\mathcal{H}} g d\kappa$. Thus, by the condition $G \preceq H$, for every $\omega \in \Omega$, there is $r_0(\omega) \in H(\omega)$ such that $g(\omega) \leq r_0(\omega)$. Since H is $\mathcal{H}$-integrably bounded with respect to κ, then the function

$$\phi_H(\omega) = \sup\{r : r \in H(\omega)\}$$

is a measurable selection H, i.e., for every $\omega \in \Omega$, $\phi_H(\omega) \in H(\omega)$ and $\phi_H \in I(\mathcal{H}, \kappa)$. We take $t = \int_{\mathcal{H}} \phi_H d\kappa$, then $t \in \int_{\mathcal{H}} H d\kappa$. Noting that for every $\omega \in \Omega$, $g(\omega) \leq r_0(\omega) \leq \phi_H(\omega)$, then

$$s = \int_{\mathcal{H}} g d\kappa \leq \int_{\mathcal{H}} \phi_H d\kappa = t.$$

Similarly, we can prove that for any $q \in \int_{\mathcal{H}} H d\kappa$, there is $p \in \int_{\mathcal{H}} G d\kappa$ such that $p \leq q$. We obtain $\int_{\mathcal{H}} G d\kappa \preceq \int_{\mathcal{H}} H d\kappa$. $\square$

The following is an immediate consequence of Proposition 5.

Corollary 1. *Given $\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$, let $G, H \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrably bounded with respect to κ. Then*

$$G \approx H \text{ implies } \int_{\mathcal{H}} G d\kappa \approx \int_{\mathcal{H}} H d\kappa. \tag{7}$$

There are two kinds of relations for decomposition systems from $\mathbb{X}$: "$\subseteq$"—the standard-set inclusion relation; and "$\sqsubseteq$"—the preorder relation (see [3]: for $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$, "$\mathcal{G} \sqsubseteq \mathcal{H}$" means that for each $(A_i)_{i=1}^n \in \mathcal{G}$, there is $(B_j)_{j=1}^m \in \mathcal{H}$ such that $\{A\}_{i=1}^n \subseteq \{B\}_{j=1}^m$). Moreover, we define a relation "$\approx_{\sqsubseteq}$": for $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$, $\mathcal{G} \approx_{\sqsubseteq} \mathcal{H}$ if $\mathcal{G} \sqsubseteq \mathcal{H}$ and $\mathcal{H} \sqsubseteq \mathcal{G}$. The relation "$\approx_{\sqsubseteq}$" is an equivalence relation on the space $\mathbb{X}$.

Obviously, for any $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$, $\mathcal{G} \subseteq \mathcal{H}$ implies $\mathcal{G} \sqsubseteq \mathcal{H}$, but not vice-versa. If $\mathcal{G} \subseteq \mathcal{H}$ or $\mathcal{G} \sqsubseteq \mathcal{H}$, then for any $(\kappa, h) \in \mathcal{M} \times \mathfrak{F}^+$, it holds that $\int_{\mathcal{G}} h d\kappa \leq \int_{\mathcal{H}} h d\kappa$ [3] (see also [8]).

For the relations "$\sqsubseteq$", "$\subseteq$" and "$\preceq$", we present some results in the following.

Proposition 6. *Let $\kappa \in \mathfrak{M}$ be fixed, let $(\mathcal{G}, \mathcal{H}) \in \mathbb{X} \times \mathbb{X}$ and $F \in \mathfrak{R}[\Omega]$ be both $\mathcal{G}$-integrably bounded and $\mathcal{H}$-integrably bounded with respect to κ. Then*

$$\mathcal{G} \sqsubseteq \mathcal{H} \text{ implies } \int_{\mathcal{G}} F d\kappa \preceq \int_{\mathcal{H}} F d\kappa. \tag{8}$$

Proof. For any $s \in \int_{\mathcal{G}} F d\kappa$, there is $f \in I(\mathcal{G}, \kappa)$ and $f(\omega) \in F(\omega)$ on Ω, such that $s = \int_{\mathcal{G}} f d\kappa$. The condition $\mathcal{G} \sqsubseteq \mathcal{H}$ implies that $\int_{\mathcal{G}} f d\kappa \leq \int_{\mathcal{H}} f d\kappa$ ([3]). Since $F \in \mathfrak{R}[\Omega]$ is $\mathcal{H}$-integrably bounded with respect to κ and $f(\omega) \in F(\omega)$, it follows from Proposition 3(3) that $f \in I(\mathcal{H}, \kappa)$. Denote $b = \int_{\mathcal{H}} f d\kappa$, then $b \in \int_{\mathcal{H}} F d\kappa$ and $a \leq b$.

It is similar to prove that for any $q \in \int_{\mathcal{H}} F d\kappa$, there is $p \in \int_{\mathcal{G}} F d\kappa$ such that $p \leq q$. $\square$

Corollary 2. *Under the assumption of Proposition 6, the following statement is true:*

$$\mathcal{G} \approx_{\sqsubseteq} \mathcal{H} \text{ implies } \int_{\mathcal{G}} F d\kappa \approx \int_{\mathcal{H}} F d\kappa. \tag{9}$$

From Proposition 6 and noting that $\mathcal{G} \subseteq \mathcal{H}$ implies $\mathcal{G} \sqsubseteq \mathcal{H}$ ([3]), we obtain the following result.

Corollary 3. *Under the assumption of Proposition 6, the following statement is true:*

$$\mathcal{G} \subseteq \mathcal{H} \text{ implies } \int_{\mathcal{G}} F d\kappa \preceq \int_{\mathcal{H}} F d\kappa.$$

Note that since $\mathcal{H}_{Sh} \subseteq \mathcal{H}_{Ch} \subseteq \mathcal{H}_{cav}$ and $\mathcal{H}_{Sh} \subseteq \mathcal{H}_{pan} \subseteq \mathcal{H}_{cav}$, we have the following result.

Proposition 7. *Let $\kappa \in \mathfrak{M}$ be fixed and let $F \in \mathfrak{R}[\Omega]$ be $\mathcal{H}_{Sh}$-, $\mathcal{H}_{Ch}$-, $\mathcal{H}_{pan}$- and $\mathcal{H}_{cav}$-integrable with respect to κ, respectively. Then*

$$\int_{\mathcal{H}_{Sh}} F d\kappa \preceq \int_{\mathcal{H}_{Ch}} F d\kappa \preceq \int_{\mathcal{H}_{cav}} F d\kappa$$

and

$$\int_{\mathcal{H}_{Sh}} F d\kappa \preceq \int_{\mathcal{H}_{pan}} F d\kappa \preceq \int_{\mathcal{H}_{cav}} F d\kappa.$$

Remark 1. *Note that the above discussions only concern two kinds of relations for decomposition systems from $\mathbb{X}$: "$\subseteq$" and "$\sqsubseteq$". There are also other preorders on the class of decomposition systems, see, e.g., [3,43,44]. Similarly, we can discuss the monotonicity of set-valued decomposition integrals in the sense of these preorders.*

3.4. Chebyshev's Inequality of Set-Valued Decomposition Integrals

The Chebyshev inequality is an important inequality in classical measures and integrals [45], which is stated as follows: for any $\xi \in \mathfrak{F}^+$ and $s > 0$, it holds that

$$m\Big(\{\omega \in \Omega \mid \xi(\omega) \geq s\}\Big) \leq \frac{1}{s} \int \xi dm, \tag{10}$$

where m is a σ-additive measure and the integral on the right side is the Lebesgue integral.

In [5], Kang and Li established Chebyshev's inequality for decomposition integrals as follows:

Proposition 8 (Kang and Li [5])**.** *Let $\mathcal{H} \in \mathbb{X}$ be complete. Then, for any $(\mu, h) \in \mathfrak{M} \times \mathfrak{F}^+$ and $c > 0$, it holds that*

$$\mu\Big(\{\omega \in \Omega |\ h(\omega) \geq c\}\Big) \leq \frac{1}{c} \int_{\mathcal{H}} h d\mu. \tag{11}$$

Now, we extend the above result to the case of set-valued decomposition integrals. We present a version of Chebyshev's inequality for set-valued decomposition integrals.

We introduce some notations that will be used in establishing the Chebyshev inequality.

Let $A, B \in \mathfrak{P}(\mathbb{R}^1_+)$ and $k \in \mathbb{R}^1_+$. When $A = \{a\}$ is a single point set, we use the notation "$a \preceq B$" to denote the order relation $\{a\} \preceq B$ (i.e., $a \preceq b$ for any $b \in B$). We denote as $I_k(\omega) = \{k\}$ all $\omega \in \Omega$ (when $k = 1$, denote as $I(\omega) = I_1(\omega) = \{1\}$ for short). We use the notation "$k \preceq H$" to denote the order relation $I_k \preceq H$ (i.e., for any $\omega \in \Omega$, $I_k(\omega) \preceq H(\omega)$), and for given $\omega \in \Omega$, the notation "$k \preceq H(\omega)$" denotes $\{k\} \preceq H(\omega)$ (i.e., $k \leq b$ holds for any $b \in H(\omega)$).

The complete decomposition system [3] plays an important role in the discussion of the decomposition integrals. Let $\mathcal{H} \in \mathbb{X}$. The decomposition system $\mathcal{H}$ is called *complete* [3], if for each $A \in \mathfrak{A} \setminus \{\emptyset\}$ there exists $(A_i)_{i=1}^k \in \mathcal{H}$ such that

Proposition 9. *$\mathcal{H} \in \mathbb{X}$ and $\kappa \in \mathfrak{M}$ are finite, and let $l(\omega) \equiv 1$ be $\mathcal{H}$-integrable with respect to κ. If $\mathcal{H}$ is complete, then for any $A \in \mathfrak{A}$ with $\kappa(A) < \infty$, it holds that*

$$\kappa(A) \preceq \int_{\mathcal{H}} \chi_A I d\kappa, \tag{12}$$

where $I(\omega) = \{1\}$ for all $\omega \in \Omega$.

Proof. In [5], it is shown that $\mathcal{H}$ is complete if and only if

$$\int_{\mathcal{H}} \chi_A d\kappa \geq \kappa(A)$$

holds for any $(\kappa, A) \in \mathfrak{M} \times \mathfrak{A}$. From Proposition 2, we have $\int_{\mathcal{H}} \chi_A I d\kappa = \{ \int_{\mathcal{H}} \chi_A d\kappa \}$, and hence

$$\{\kappa(A)\} \preceq \left\{ \int_{\mathcal{H}} \chi_A d\kappa \right\} = \int_{\mathcal{H}} \chi_A I d\kappa,$$

which is the inequality (12). □

Proposition 10 (Chebyshev's inequality). *Let $\mathcal{H} \in \mathbb{X}$ be complete and $\kappa \in \mathfrak{M}$. Then, for any $F \in \mathfrak{R}[\Omega]$ which is $\mathcal{H}$-integrably bounded with respect to κ, and $t > 0$, it holds that*

$$\kappa\Big(\{\omega \in \Omega \mid t \preceq F(\omega)\}\Big) \preceq \frac{1}{t} \int_{\mathcal{H}} F d\kappa, \tag{13}$$

where $t \preceq F(\omega)$ means that for a given $\omega \in \Omega$, $\{t\} \preceq F(\omega)$.

Proof. Denote $C_t = \{\omega \in \Omega \mid t \preceq H(\omega)\}$. We have

$$t\chi_{C_t} I = \chi_{C_t} I_t \preceq F,$$

and hence $\chi_{C_t} I \preceq \frac{1}{t} F$. From Propositions 1, 5 and 9, we obtain

$$\kappa(C_t) \preceq \int_{\mathcal{H}} \chi_{C_t} I \preceq \int_{\mathcal{H}} \frac{1}{t} F = \frac{1}{t} \int_{\mathcal{H}} F.$$

This is the inequality (13). □

Note that the decomposition systems $\mathcal{H}_{Ch}, \mathcal{H}_{cav}, \mathcal{H}_{pan}$ and $\mathcal{H}_{Sh}$ are all complete. As a special result of Proposition 10, we obtain the following corollary.

Corollary 4. *Let $\kappa \in \mathfrak{M}$ be finite and $t > 0$. If $F \in \mathfrak{R}[\Omega]$ is $\mathcal{H}_*$-integrably bounded with respect to κ, then*

$$\kappa\Big(\{\omega \in \Omega \mid t \preceq F(\omega)\}\Big) \preceq \frac{1}{t} \int_{\mathcal{H}_*} F d\kappa, \tag{14}$$

where $\mathcal{H}_*$ stands for $\mathcal{H}_{Ch}$, $\mathcal{H}_{pan}$, $\mathcal{H}_{cav}$ or $\mathcal{H}_{Sh}$, respectively.

Remark 2. *In [16] (see also [21]), Guo and Zhang proposed the set-valued fuzzy measure. A set-valued function $\pi : \mathfrak{A} \to \mathfrak{P}(\mathbb{R}^1_+)$ is called a set-valued fuzzy measure on $\mathfrak{A}$ if it holds that*

(1) $\pi(\emptyset) = \{0\}$;

(2) For any $S, W \in \mathfrak{A}$, $S \subseteq W$ implies $\pi(S) \preceq \pi(W)$.

Let $\kappa \in \mathfrak{M}$ and $F \in \mathfrak{R}[\Omega]$ be $\mathcal{H}$-integrably bounded with respect to κ. Define

$$\pi^{\mathcal{H}}_F(A) = \int_{\mathcal{H}} \chi_A F d\kappa, \quad A \in \mathfrak{A}.$$

Then, $\pi_F^{\mathcal{H}} : \mathfrak{A} \to \mathfrak{P}(\mathbb{R}_+^1)$ *is a set-valued fuzzy measure on* $\mathfrak{A}$*. In fact,* $\pi_F^{\mathcal{H}}(\emptyset) = \{0\}$*. Note that for any* $A, B \in \mathfrak{A}$*,* $A \subseteq B$ *implies* $\chi_A F \preceq \chi_B F$*. Therefore,* $A \subseteq B$ *implies*

$$\int_{\mathcal{H}} \chi_A F d\kappa \preceq \int_{\mathcal{H}} \chi_B F d\kappa$$

from Proposition 5, i.e., $\pi_F^{\mathcal{H}}(A) \preceq \pi_F^{\mathcal{H}}(B)$*.*

4. Concave Integrals

In this section, we recall the concave integral ([37,40], see also [6]), and show some special properties of set-valued concave integrals.

Let $\kappa \in \mathfrak{M}$ be fixed and let $h \in \mathfrak{F}^+$.

The *concave integral* of h on Ω with respect to κ is defined by

$$\int^{cav} h d\kappa = \sup\left\{ \sum_{j=1}^{n} t_j \kappa(T_j) : \sum_{j=1}^{n} t_j \chi_{T_j} \le h, (T_j)_{j=1}^n \in \mathcal{H}_{cav}, t_j \ge 0, n \in \mathbb{N} \right\}.$$

If $\int^{cav} h d\kappa < \infty$, then h is called concave-integrable.

The concave integral $\int^{cav} \cdot d\kappa$, as a functional from $\mathfrak{F}^+$ to $\mathbb{R}_+^1$, is positive-homogeneous and concave (Lehrer and Teper [37]).

Proposition 11. *Let* $\kappa \in \mathfrak{M}$ *be fixed, For any* $g, h \in \mathfrak{F}^+, t \in [0,1]$*, we have*

$$\int^{cav} [tg + (1-t)h] d\kappa \ge t \int^{cav} g d\kappa + (1-t) \int^{cav} h d\kappa.$$

When κ is subadditive, the concave integral $\int^{cav} \cdot d\kappa$ is positive linear on $\mathfrak{F}^+$ (Ouyang et al. [46]).

Proposition 12. *Let* $\kappa \in \mathfrak{M}$ *be subadditive,* $g, h \in \mathfrak{F}^+, t \in [0,1]$*. Then*

$$\int^{cav} [tg + (1-t)h] d\kappa = t \int^{cav} g d\kappa + (1-t) \int^{cav} h d\kappa.$$

For more basic properties of concave integrals, see [3,37,40,47–49].

From Propositions 11 and 12 and similar to the proof of Proposition 4.2 in [33], it is not difficult to obtain the corresponding result for set-valued concave integrals as follows:

Proposition 13. *Given* $\kappa \in \mathfrak{M}$*, let* $G, H \in \mathfrak{R}[\Omega]$ *be* $\mathcal{H}_{cav}$*-integrably bounded with respect to* κ *and* $t \in [0,1]$*. Then*

$$t \int_{\mathcal{H}_{cav}} G d\kappa + (1-t) \int_{\mathcal{H}_{cav}} H d\kappa \;\preceq\; \int_{\mathcal{H}_{cav}} [tG + (1-t)H] d\kappa. \tag{15}$$

Furthermore, if κ *is subadditive, then*

$$\int_{\mathcal{H}_{cav}} [tG + (1-t)H] d\kappa \;\approx\; t \int_{\mathcal{H}_{cav}} G d\kappa + (1-t) \int_{\mathcal{H}_{cav}} H d\kappa, \tag{16}$$

in particular,

$$\int_{\mathcal{H}_{cav}} [G + H] d\kappa \;\approx\; \int_{\mathcal{H}_{cav}} G d\kappa + \int_{\mathcal{H}_{cav}} H d\kappa. \tag{17}$$

Proposition 14. *Let* $\kappa \in \mathcal{M}$ *be subadditive and* G *be concave integrable. Then* $\int_{\mathcal{H}_{cav}} G d\kappa$ *is convex whenever* G *is convex-valued (i.e., for every* $\omega \in \Omega$*,* $G(\omega)$ *is a convex set of* $\mathbb{R}_+^1$*).*

Proof. Suppose that $a, b \in \int_{\mathcal{H}_{cav}} G\, d\kappa, t \in [0,1]$. Then there are g_1, $g_2 \in S_{\mathcal{H}_{cav}}(G,\kappa)$ such that $a = \int_{\mathcal{H}_{cav}} g_1 d\kappa$ and $b = \int_{\mathcal{H}_{cav}} g_2 d\kappa$. These imply that $g_1(\omega), g_2(\omega) \in G(\omega)$ for every $\omega \in \Omega$ and g_1, $g_2 \in I(\mathcal{H}_{cav},\kappa)$. Since G is convex-valued, $\big(tg_1(\omega) + (1-t)g_2(\omega)\big) \in G(\omega), \omega \in \Omega$. Thus, it follows from the subadditivity of κ and Proposition 12 that

$$\int^{cav} [tg_1 + (1-t)g_2]d\kappa = t\int^{cav} g_1 + (1-t)\int^{cav} g_2 d\kappa < \infty.$$

Therefore, $tg_1 + (1-t)g_2 \in S_{\mathcal{H}_{cav}}(G,\kappa)$, and hence

$$ta + (1-t)b = \int^{cav} [tg_1 + (1-t)g_2]d\kappa \in \int^{cav} G d\kappa.$$

This shows that $\int_{\mathcal{H}_{cav}} G d\kappa$ is convex. □

5. Relationships of the Set-Valued Choquet, the Set-Valued Pan-Integral and the Set-Valued Concave Integral

In this section we discuss the coincidences among three kinds of set-valued decomposition integrals: the set-valued Choquet, the set-valued pan-integral and the set-valued concave integral.

We recall two concepts which play important roles in the discussion of concidences of the pan-, Choquet and concave integrals (see [37,49]). Let $\kappa \in \mathfrak{M}$. κ is said to have *(M)-property*, if for any $V, W \in \mathfrak{A}$ and $V \subset W$, there is $T \in \mathfrak{A}$ such that $T \subset V$, and $\kappa(T) = \kappa(V)$ and $\kappa(W) = \kappa(T) + \kappa(W \setminus T)$ (Mesiar et al. [49]). κ is called *submodular* (or *concave*) if $\kappa(S \cup W) + \kappa(S \cap W) \le \kappa(S) + \kappa(W)$ holds for any $S, W \in \mathfrak{A}$.

We recall the following results ([37,48,49], see also [47,49]).

Proposition 15. *Let $\kappa \in \mathfrak{M}$ be fixed.*

(1) κ is submodular if and only if

$$\int^{Ch} g d\kappa = \int^{cav} g d\kappa \tag{18}$$

for all $g \in \mathfrak{F}^+$ ([37]).

(2) If κ is subadditive, then

$$\int^{cav} g d\kappa = \int^{pan} g d\kappa \tag{19}$$

holds for all $g \in \mathfrak{F}^+$ ([48]).

(3) If κ has (M)-property, then

$$\int^{pan} g d\kappa = \int^{Ch} g d\kappa \tag{20}$$

holds for all $g \in \mathfrak{F}^+$ ([49]).

In the following, we show, respectively, the equivalence among the set-valued pan-integral, the set-valued Choquet integral and the set-valued concave integral.

Proposition 16. *Let $\kappa \in \mathfrak{M}$ be fixed. If κ is submodular, then*

$$\int_{\mathcal{H}_{Ch}} G d\kappa = \int_{\mathcal{H}_{cav}} G d\kappa \tag{21}$$

holds for any $G \in \mathfrak{R}[\Omega]$ that is $\mathcal{H}_{Ch}$-integrable and $\mathcal{H}_{cav}$-integrable with respect to κ.

Proof. Suppose $a \in \int_{\mathcal{H}_{Ch}} G d\kappa$. Then there is $g \in S_{\mathcal{H}_{Ch}}(G,\kappa)$ such that $a = \int^{Ch} g d\kappa \in \int_{\mathcal{H}_{Ch}} G d\kappa$. Note that $\int^{Ch} g d\kappa < \infty$ and $g(\omega) \in G(\omega)$ on Ω, from Proposition 15, then $\int^{cav} g d\kappa = \int^{Ch} g d\kappa < \infty$ and hence $g \in I_{\mathcal{H}_{cav}}(\kappa)$, and $g(\omega) \in G(\omega)$ on Ω, i.e., $g \in S_{\mathcal{H}_{cav}}(G)$. Therefore $a = \int^{Ch} g d\kappa = \int^{cav} g d\kappa \in \int_{\mathcal{H}_{cav}} G d\kappa$, which implies

$$\int_{\mathcal{H}_{Ch}} G d\kappa \subseteq \int_{\mathcal{H}_{cav}} G d\kappa.$$

Similarly, we can obtain the converse relationship. □

Similarly, we can obtain the coincidences of the set-valued concave integral and the set-valued pan-integral, and of the set-valued pan-integral and the set-valued Choquet integral.

Proposition 17. *Let $\kappa \in \mathfrak{M}$ be fixed. If κ is subadditive, then*

$$\int_{\mathcal{H}_{cav}} G d\kappa = \int_{\mathcal{H}_{pan}} G d\kappa \tag{22}$$

holds for any set-valued random variables $G \in \mathfrak{R}[\Omega]$ that are $\mathcal{H}_{cav}$-integrable and $\mathcal{H}_{pan}$-integrable with respect to κ.

Proposition 18 ([33])**.** *Let $\kappa \in \mathfrak{M}$ be fixed. If κ has (M)-property, then*

$$\int_{\mathcal{H}_{pan}} G d\kappa = \int_{\mathcal{H}_{Ch}} G d\kappa \tag{23}$$

holds for any set-valued random variables $G \in \mathfrak{R}[\Omega]$ that are $\mathcal{H}_{pan}$-integrable and $\mathcal{H}_{Ch}$-integrable with respect to κ.

In Section 3.3, we have shown that

$$\int_{\mathcal{H}_{Sh}} F d\kappa \preceq \int_{\mathcal{H}_{Ch}} F d\kappa \preceq \int_{\mathcal{H}_{cav}} F d\kappa$$

and

$$\int_{\mathcal{H}_{Sh}} F d\kappa \preceq \int_{\mathcal{H}_{pan}} F d\kappa \preceq \int_{\mathcal{H}_{cav}} F d\kappa,$$

where $\kappa \in \mathfrak{M}$ is fixed and $F \in \mathfrak{R}[\Omega]$ is $\mathcal{H}_{Sh}$-, $\mathcal{H}_{Ch}$-, $\mathcal{H}_{pan}$- and $\mathcal{H}_{cav}$-integrable with respect to κ, respectively.

In general, for some $\mu \in \mathfrak{M}$, $\int_{\mathcal{H}_{pan}} F d\mu$ and $\int_{\mathcal{H}_{Ch}} F d\mu$ are incomparable.

We recall the relationships between the Choquet integral and the pan-integral (see [38]).

Proposition 19. *Let $\kappa \in \mathfrak{M}$ be fixed.*

(1) If κ is subadditive, then for all $g \in \mathfrak{F}^+$, $\int^{Ch} g d\kappa \leq \int^{pan} g d\kappa$.

(2) If κ is superadditive, then for all $g \in \mathfrak{F}^+$, $\int^{pan} g d\kappa \leq \int^{Ch} g d\kappa$.

The following result is an extension of Proposition 19 in the case of the set-valued pan-integral and the set-valued Choquet integral.

Proposition 20. *Let $\kappa \in \mathfrak{M}$ be fixed and $G \in \mathfrak{R}[\Omega]$.*

(1) If κ is subadditive and G is $\mathcal{H}_{pan}$-integrably bounded with respect to κ, then

$$\int_{\mathcal{H}_{Ch}} G d\kappa \preceq \int_{\mathcal{H}_{pan}} G d\kappa. \tag{24}$$

(2) *If κ is superadditive and G is $\mathcal{H}_{Ch}$-integrably bounded with respect to κ, then*

$$\int_{\mathcal{H}_{pan}} G d\kappa \preceq \int_{\mathcal{H}_{Ch}} G d\kappa. \tag{25}$$

Proof. (1) was proved in [33]. We prove (2).

Suppose $s \in \int_{\mathcal{H}_{pan}} G d\kappa$. Then there is $g \in S_{\mathcal{H}_{pan}}(G, \kappa)$ such that $s = \int^{pan} g d\kappa < \infty$ and $g(\omega) \in G(\omega)$ on Ω. Since G is $\mathcal{H}_{Ch}$-integrably bounded with respect to κ, from Proposition 3, $g(\omega) \in G(\omega)$ implies g is Choquet-integrable with respect to κ, i.e., $\int^{Ch} g d\kappa < \infty$. Take $t = \int^{Ch} g d\kappa$, then $t \in \int_{\mathcal{H}_{Ch}} G d\kappa$ and from Proposition 19 we have $s = \int^{pan} g d\kappa \leq \int^{Ch} g d\kappa = t$.

On the other hand, suppose $q \in \int^{\mathcal{H}_{Ch}} G d\kappa$. Then there exists $h \in S_{\mathcal{H}_{Ch}}(G, \kappa)$ such that $q = \int^{Ch} h d\kappa < \infty$ and $h(\omega) \in G(\omega)$ on Ω. Note that κ is superadditive, and hence $\int^{pan} h d\kappa \leq \int^{Ch} h d\kappa < \infty$. We take $p = \int^{pan} h d\kappa$, then $p \in \int_{\mathcal{H}_{pan}} G d\kappa$ and $p \leq q$.

The inequality (25) is shown. $\square$

6. Remarks

(1) In [50], Stupňanová introduced a special type of decomposition integral, the *PC-integral*, based on the so-called PC decomposition system in which the collection includes pairwise disjoint sets and chains of sets. The PC-integral locates between the concave integral and the Choquet integral, and also between the concave integral and the pan-integral. We can consider the PC-integrals of set-valued functions in terms of Stupňanová's work and obtain some special properties.

(2) In [51], Mesiar et al. introduced a new type of decomposition integral by using a family of decomposition integrals based on the collections relating to partitions and maximal chains of sets. This new integral extends the Lebesgue integral, and it is different from those well-known decomposition integrals, such as the Choquet, concave, pan- and Shilkret integrals and the PC-integral. As a special case of Definition 2, we can obtain the set-valued integrals in terms of Mesiar's work.

(3) Note that Šeliga introduced the decomposition integrals for interval-valued functions and dealt with some basic properties of special set-based functions, see [12,52].

7. Conclusions

We have introduced the decomposition integral of set-valued functions and shown some basic properties. The interesting results are the monotonicity of the integrals in the sense of the preorder relations "$\sqsubseteq$" and "$\preceq$" and the inclusion relation "$\subseteq$" (Propositions 4–6, and Corollaries 2 and 3), and Chebyshev's inequality for decomposition integrals of set-valued functions (Proposition 10 and Corollary 4). The relationships among three types of important set-valued decomposition integrals—set-valued Choquet integral, the set-valued pan-integral and the set-valued concave integral—have been shown (Propositions 16–18 and 20).

As we have seen, the set-valued decomposition integral is an extension of the decomposition integral for real-valued functions, and it unifies the previous set-valued integral schemes, including the Aumann integral [14], the set-valued Choquet integral ([19,21]), the set-valued pan-integral ([33]) and the set-valued concave integral, etc.

In further research, we will focus on the study of the convergence of decomposition integrals of set-valued functions.

As is well known, the set-valued integral is very applicable in several mathematical fields, especially in control theory, mathematical economics, statistics, etc. We expect the decomposition integrals of set-valued functions to be a useful tool in these fields.

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