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Operational Matrix of New Shifted Wavelet Functions for Solving Optimal Control Problem

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Abstract: This paper is devoted to proposing an approximate numerical algorithm based on the use of the state parameterization technique in order to find the solution to the optimal control problem (OCP). An explicit formula for new shifted wavelet (NSW) functions is constructed. A new formula that expresses the first-order derivative of the NSW in terms of their original NSW is established. The development of our suggested numerical algorithms begins with the extraction of a new operational matrix of derivative from this derivative formula. The expansion's convergence study is performed in detail, and some illustrative examples of OCP are displayed. The proposed algorithm is compared with the exact one and some other methods in the literature. This confirms the accuracy and the high efficiency of the presented algorithm.

Keywords: new shifted wavelet functions; optimal control problem; convergence criteria; error analysis

MSC: 49M99

1. Introduction

The dynamics in some mathematical models are represented by a system of ordinary differential equations for a set of dependent functions $x(t)$ when an independent procedure set controls such systems $u(t)$. In this case, the aim is to determine $u(t)$ that optimizes the dynamical system, and this problem is called optimal control. Numerous studies have focused on the approximate solutions of optimal control problems, which can be found in many fields [1–5]. The two general techniques, which are indirect and direct, are used for the approximate solution of optimal control problems. An indirect method transforms the original optimal control problem to a boundary value problem, which can be solved either analytically or numerically.

Direct methods are more suitable techniques and can be quickly and simply utilized to a new optimal control problem. Optimal control in natural methods is seen as a standard optimization problem by searching for the control function $u(t)$ that optimizes the performance index. Different algorithms were used for solving optimal control problems, including the indirect modified pseudospectral method [6], a direct Chebyshev cardinal functions method [7], Cauchy discretization technique [8], the synthesized optimal control technique [9], Legendre functions method [10], Evolutionary Algorithm-Control Input Range Estimation [11], a hybrid of block-pulse function, and orthonormal Taylor polynomials [12]. (See [13–17] for some other articles exploring various optimal control problems.) Wavelet functions have important roles in numerical analysis for solving optimal control problems [18–21]. In particular, the Chebyshev wavelets families are widely applied in contributions to the field of approximation theory. For example, the authors in [22] employed the Boubaker wavelets together with the operation matrix of derivative in order to solve the singular initial value problem. The collocation method is presented in [23] based on the second kind of Chebyshev wavelets for solving calculus of variation problems. The use of the operational matrices of derivatives and integrals has been highlighted in the

field of numerical analysis [24]. This utilization gives special algorithms to obtain accurate approximate solutions of many types of differential and integral equations with flexible computations. An operational matrix of derivatives is extracted based on choosing suitable basis functions in terms of celebrated special functions and expressing the first derivative of these basis functions in terms of their original types.

Motivated by the above discussion, we are mainly interested in presenting new shifted wavelet functions with some important properties. A novel state parameterization method is suggested to solve the optimal control problem. Such a method is used together with NSW as a basis function to parameterize the states variables. The proposed technique is constructed to simultaneously reach the accuracy and efficiency. The rest of the work is organized as follows: Section 2 provides the definition of NSW. In Section 3, the convergence of the NSW is studied. The general exact formula of the NSW differentiation operational matrix is generated in Section 4, and then the suggested algorithm to solve the optimal control problem is illustrated in Section 5. Section six discusses the application of the NSW by considering various examples in the optimal control. Simulation results are also given in Section 7, followed by concluding remarks that are summarized in Section 8.

2. The New Shifted Wavelet Functions

The expression for the special polynomials $M_m(t)$ in the interval $[-1, 1]$ can be defined as below:

$$M_0(t) = 2, \ M_1(t) = t, \ M_2(t) = t^2 - 2, \ \dots$$

The general recurrence relation for obtaining $M_m(t)$, $m = 2, 3, 4, \dots$ is given by:

$$M_{m+1}(t) = tM_{m-1}(t) - M_m(t), m = 1, 2, 3, 4, \dots,$$

with the given initial conditions $M_0(t)$ and $M_1(t)$.

Sometimes, it is convenient to use the half-interval $[0, 1]$ instead of the interval $[-1, 1]$. In this case, the term shifted is defined and indicated by $Ms_m(t)$. In this work, $Ms_m(t)$ are defined as $Ms_m(t) = M_m(2t - 1)$.

Wavelet functions have been used successfully in scientific and engineering fields. The special new shifted wavelet functions can be defined as below:

$$Q_{nm}(t) = \begin{cases} \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} Ms_m\left(2^k t - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le t \le \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

where $n = 1, 2, \dots, 2^k$, k can be assumed to be any positive integer, m is the degree of the shifted polynomials, and t denotes the time for $m = 0, 1, \dots, M$.

Here, $Ms_m(t)$ are called the shifted special polynomials of order m, which are orthogonal with respect to the weight function $w(t)$, and which satisfy the following recursive formula:

$$Ms_m(t) = (2t - 1)Ms_{m-1}(t) - Ms_{m-2}(t), \ m = 2, 3, 4, \dots, \tag{2}$$

with initial conditions:

$$Ms_0(t) = 2, \ Ms_1(t) = 2t - 1. \tag{3}$$

3. Convergence Analysis of New Wavelet Functions

A function approximation $f \in C^2[0, 1)$, with $\left|\ddot{f}(t)\right| \le L, L > 0$ may by expanded in terms of new shifted wavelets as below:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} Q_{nm}(t). \tag{4}$$

where:

$$c_{nm} = \langle f(t), \ Q(t) \rangle. \tag{5}$$

In Equation (5), the symbol $\langle .,. \rangle$ is denoted the inner product operator on Hilbert space over the interval $[0,1]$.

If the infinite series in Equation (4) is truncated, then the solution $f(t)$ can be rewritten in matrix form as below:

$$f(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{nm} Q_{nm}(t) = C^T \Phi(t), \tag{6}$$

where $\Phi(t)$ and C are matrices of $2^{k-1}M \times 1$ dimensions, given by:

$$C = \begin{bmatrix} c_{1,0} \ c_{1,1} \ \cdots \ c_{1,M} \ c_{2,0} \cdots c_{2,M} \ c_{(2^k-1),0} \ \cdots c_{(2^k-1),M} \end{bmatrix}$$

and:

$$\Phi(t) = \begin{bmatrix} Q_{1,0} \ Q_{1,1} \cdots \ Q_{1,M} \ Q_{2,0} \cdots Q_{2,M} \ Q_{(2^k-1),0} \ \cdots Q_{(2^k-1),M} \end{bmatrix}^T$$

Note that both k and n are integer numbers, and m is the degree of shifted polynomials. Now, we state and prove a theorem in order to ensure the convergence of the new shifted wavelet expansion of a function.

Theorem 1. *Assume that a function $f(t) \in L^2_w([0,1])$ where $w(t) = \frac{1}{\sqrt{1-t}}$, $t \neq \pm 1$, with bounded second derivative $\left| \ddot{f}(t) \right| \leq L$, $L > 0$, f can be expanded as an infinite series of the new shifted wavelets (1), then c_{nm} in (4) converges uniformly to f, i.e., c_{nm} satisfy the inequality:*

$$|c_{nm}| \leq L \frac{1}{n^{\frac{3}{2}}} 2^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{(m^2 - 1)} \right). \tag{7}$$

Proof. Let:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} Q_{nm}(t).$$

It follows that for $k = 1, 2, 3, \ldots; n = 1, 2, \ldots, 2^k, m = 0, 1, \ldots, M$.

$$c_{nm} = \langle f(t), Q(t) \rangle = \int_0^1 f(t) Q_{nm}(t) w_k(t) dt.$$
$$= \int_0^{\frac{n-1}{2^{k-1}}} f(t) Q_{nm}(t) w_k(t) dt$$
$$+ \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) Q_{nm}(t) w_k(t) dt +$$
$$\int_{\frac{n}{2^{k-1}}}^{1} f(t) Q_{nm}(t) w_k(t) dt.$$

Using Equation (1), one can obtain:

$$c_{nm} = \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t) \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} M s_m \left(2^k t - 2n + 1 \right) w \left(2^k t - 2n + 1 \right) dt.$$

If $m > 1$, by substituting:

$$2^k t - 2n + 1 = \cos\theta, \ t = \frac{\cos\theta + 2n - 1}{2^k}, \ dt = \frac{-\sin\theta}{2^k} d\theta.$$

$$c_{nm} = \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} \int_0^{\pi} f \left(\frac{\cos\theta + 2n - 1}{2^k} \right) 2\cos m\theta \sqrt{\frac{1}{1 - \cos^2\theta}} \frac{-\sin\theta}{2^k} d\theta.$$

$$c_{nm} = \frac{-2^{\frac{(k+1)}{2}}}{\sqrt{\pi}} \int_0^{\pi} f\left(\frac{cos\theta + 2n - 1}{2^k}\right) cos\, m\theta\, d\theta.$$

By using method of integration by parts, let:

$$\int_0^{\pi} u dv = uv - \int_0^{\pi} v du, u = f\left(\frac{cos\theta + 2n - 1}{2^k}\right), \; du = \dot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right)\left(\frac{-sin\theta}{2^k}\right), \; dv = cos\, m\theta\, d\theta, \; v = \frac{sin\, m\theta}{m}, m \neq 1.$$

$$c_{nm} = \frac{-2^{\frac{(k+1)}{2}}}{\sqrt{\pi}} f\left(\frac{cos\theta + 2n - 1}{2^k}\right)\left(\frac{sin\, m\theta}{m}\right)\Bigg]_0^{\pi} - \frac{2^{\frac{(k+1)}{2}}}{m2^k\sqrt{\pi}} \int_0^{\pi} \dot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right) sin\, m\theta\, sin\theta\, d\theta.$$

Using again the method of integration by parts, let:

$$u = \dot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right), du = \ddot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right)\left(\frac{-sin\theta}{2^k}\right)d\theta, \; dv = sin\, m\theta\, sin\theta\, d\theta,$$

$$v = \left(\frac{sin(m-1)\theta}{m-1} - \frac{sin(m+1)\theta}{m+1}\right).$$

$$c_{nm} = -\frac{2^{\frac{(k+1)}{2}}}{m2^k\sqrt{\pi}} \dot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right)\left(-\frac{sin(m+1)\theta}{m+1} + \frac{sin(m-1)\theta}{m-1}\right)\Bigg]_0^{\pi} - $$
$$\frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\int_0^{\pi} \ddot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right) sin\theta\left(\frac{sin(m-1)\theta}{m-1} - \frac{sin(m+1)\theta}{m+1}\right)d\theta.$$

We have:

$$c_{nm} = -\frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\int_0^{\pi} \ddot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right) sin\theta\left(\frac{sin(m-1)\theta}{m-1} - \frac{sin(m+1)\theta}{m+1}\right)d\theta$$

Thus, we obtain:

$$|c_{nm}| = \left|-\frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\int_0^{\pi} \ddot{f}\left(\frac{cos\theta + 2n - 1}{2^k}\right) sin\theta\left(-\frac{sin(m+1)\theta}{m+1} + \frac{sin(m-1)\theta}{m-1}\right)d\theta\right|$$

$$\leq \frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\int_0^{\pi}\left|\ddot{f}\left(\frac{cos\theta+2n-1}{2^k}\right) - \frac{sin(m+1)\theta}{m+1} + \frac{sin(m-1)\theta}{m-1}d\theta\right|$$

$$\leq L\frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\int_0^{\pi}\left|sin\theta\left(\frac{sin(m-1)\theta}{m-1} - \frac{sin(m+1)\theta}{m+1}\right)\right|d\theta.$$

However,

$$\int_0^{\pi}\left|sin\theta\left(-\frac{sin(m+1)\theta}{m+1} + \frac{sin(m-1)\theta}{m-1}\right)\right|d\theta = \int_0^{\pi}\left|sin\theta\left(-\frac{sin(m+1)\theta}{m+1} + \frac{sin(m-1)\theta}{m-1}\right)d\theta\right|.$$
$$\leq \int_0^{\pi}\left|\frac{sin\theta sin(m+1)\theta}{m+1}\right| + \left|sin\theta\left(\frac{sin(m-1)\theta}{m-1}\right)\right|d\theta \leq \frac{2m\pi}{(m^2-1)}.$$

Hence:

$$|c_{nm}| \leq L\frac{2^{\frac{(k+1)}{2}}}{m2^{2k}\sqrt{\pi}}\left(\frac{2m\pi}{(m^2-1)}\right).$$

$$|c_{nm}| \leq L\frac{2^{\frac{(k+1)}{2}}}{2^{2k}}\left(\frac{2\sqrt{\pi}}{(m^2-1)}\right).$$

Since $n \leq 2^{k-1}$, we have inequality becoming:

$$|c_{nm}| \leq L \frac{1}{n^{\frac{3}{2}}} 2^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{(m^2 - 1)} \right).$$

Therefore, the wavelets expansion $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} Q_{nm}(t)$ converges to $f(t)$ uniformly.

$\square$

Accuracy Analysis

If the function $f(t)$ is expanded in terms of New Shifted Wavelet Functions as in Equations (4) and (5), that is:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} Q_{nm}(t).$$

then it is not possible to perform the computation of an infinite number of terms, and we must thus truncate the series as below:

$$f_M(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} Q_{nm}(t).$$

so that:

$$f(t) - f_M(t) = r(t).$$

where $r(t)$ is the residual function defined by:

$$r(t) = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} c_{nm} Q_{nm}(t).$$

We must select the coefficients such that $\|r(t)\|$ is less than some convergence value ϵ, that is:

$$\left(\int_0^1 |f(t) - f_M(t)|^2 w_n(t) dt \right)^{\frac{1}{2}} < \epsilon,$$

for all M greater than some positive integer value M_0.

The calculation of the accuracy of a numerical method is crucial to describe the applicability and performance in order to solve problems. Theorem 2 discusses the accuracy of the wavelets representation of a function.

Theorem 2. *Let f be a continuous function defined on the interval [0, 1) and $\left| \ddot{f}(t) \right| < L$, then the accuracy estimation is given by:*

$$c_{n,m} = \left(\frac{\sqrt{\pi} L}{2^{-\frac{3}{2}}} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^{\frac{3}{2}}} \left(\frac{1}{(m^2 - 1)} \right) \right)^{\frac{1}{2}},$$

where:

$$c_{n,m} = \left(\int_0^1 |r(t)|^2 w_n(t) dt \right)^{\frac{1}{2}}.$$

Proof. Since:

$$c_{n,m} = \left(\int_0^1 |r(t)|^2 w_n(t) dt \right)^{\frac{1}{2}}$$

Then:

$$c_{nm}^2 = \int_0^1 |r(t)|^2 w_n(t) dt.$$

$$= \int_0^1 \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm} Q_{nm}(t)|^2 w_n(t) dt.$$

$$= \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2 \int_0^1 |Q_{nm}(t)|^2 w_n(t) dt.$$

From the orthonormality criterion form Q_{nm}, one can obtain:

$$c_{nm}^2 = \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2.$$

Using the findings from Equation (7):

$$c_{nm}^2 = \frac{\sqrt{\pi}L}{2^{-\frac{3}{2}}} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^{\frac{3}{2}}} \left(\frac{1}{(m^2-1)} \right),$$

or

$$c_{n,m} = \left(\frac{\sqrt{\pi}L}{2^{-\frac{3}{2}}} \sum_{n=2^{k-1}+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^{\frac{3}{2}}} \left(\frac{1}{(m^2-1)} \right) \right)^{\frac{1}{2}}. \ \square$$

4. Operational Matrix of the NSW

The present section is built to derive an operational matrix of derivatives for the NSW. Based on the NSW vector $\Phi(t)$ mentioned in Equation (1), it can be determined that the operational matrix of integer derivative is as below.

The following theorem is needed hereafter:

Theorem 3. *Let $\Phi(t)$ be the NSW vector defined in Equation (1). Then, the first derivative of the vector $\Phi(t)$ can be expressed as:*

$$\frac{d\Phi(t))}{dt} = D_\Phi \Phi(t), \tag{8}$$

where$D_\Phi$is$2^{k-1}(M+1)$square operation matrix of differentiation and is defined by:

$$D_\Phi = \begin{pmatrix} D & O & \cdots & O \\ O & D & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & D \end{pmatrix} \tag{9}$$

In which$D$ is a square matrix and their elements can be explicitly obtained as below:

$$D_{i,j} = 2^k \begin{cases} i, & i \text{ odd and } j = 0, \\ 2i, & i > j \text{ and } i - j = odd, \\ 0, & otherwise. \end{cases} \tag{10}$$

Proof. By using NSW, the *rth* element of vector $Q_{n,m}(t)$ can be rewritten in the following way:

$$Q_r(t) = Q_{n,m}(t) = \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} Ms_m \left(2^k t - 2n + 1 \right), \tag{11}$$

For $\frac{n-1}{2^{k-1}} \leq t \leq \frac{n}{2^{k-1}}$ and $Q_r(t) = 0$ outside the interval $t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]$, where $r = n(m+1) + (m+1), m = 0, 1, \ldots, M, n = 0, 1, 2, \ldots, (2^k - 1)$.

Or

$$Q_{n,m}(t) = \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} \left(Ms_m \left(2^k t - 2n + 1 \right) \right) \chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]}, \tag{12}$$

where

$$\chi_{\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]} = \begin{cases} 1, & t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right], \\ 0, & otherwise. \end{cases}$$

Differentiating Equation (11) with respect to t yields:

$$\frac{d\Phi(t)}{dt} = \frac{2^{\frac{k-1}{2}}}{\sqrt{\pi}} \left[\dot{M}s_m \left(2^k t - 2n + 1 \right) \right], \text{ for } t \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]. \tag{13}$$

Hence, the NSW expansion only has those elements in $Q_{n,m}(t)$ that are non-zero in the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right]$, that is:

$$Q_r(t), r = n(M+1), n(M+1) + 2, \dots, n(M+1) + (M+1).$$

This enables us to expand $\left(\frac{dQ_{nm}(t)}{dt} \right)$ in terms of the NSW in the form:

$$\frac{d\Phi(t)}{dt} = \sum_{r=n(M+1)+1}^{(n+1)(M+1)} c_r Q_r(t). \tag{14}$$

This implies that the operational matrix $D_\Phi(t)$ is a block matrix, as defined in Equation (9), since $\frac{d\Phi(t)}{dt} = 0$.

Then, we have $\frac{d\Phi(t)}{dt} = 0$ for $r = 1(M+1) + 1, 2(M+1) + 1, \dots, \left(2^k - 1 \right)(M+1) + 1$,

As a result, the elements of the first row of matrix D given in Equation (10) are zeros.

Now, substitute $\frac{d\dot{M}s_m(t)}{dt}$ back into Equation (13), gives:

$$\frac{dQ_{n,m}(t)}{dt} = \frac{1}{\sqrt{\pi}} 2 \cdot 2^{k-1} n \begin{cases} \sum_{i=1}^{n-1} Ms_{n-2i+1}(t) + \frac{1}{2} Ms_0, & if \ n \ odd, \\ \sum_{i=1}^{n-1} Ms_{n-2i+1}(t), & if \ n \ even. \end{cases} \tag{15}$$

Expanding Equation (15) in terms of NSW basis allows us to obtain:

$$\frac{dQ_{n,m}(t)}{dt} = 2 \cdot 2^k n \begin{cases} \sum_{i=1}^{n-1} Q_{n(M+1)+i}(t) + \frac{1}{2} Q_0, & if \ n \ odd, \\ \sum_{i=1}^{n-1} Q_{n(M+1)+i}(t), & if \ n \ even. \end{cases}$$

Choosing $D(i,j)$ such that:

$$D_{i,j} = 2^k \begin{cases} i & i \ odd, \ j = 0, \\ 2i & i > j, \ i - j = odd, \\ 0 & otherwise. \end{cases}$$

The equation $\frac{dQ_{n,m}(t)}{dt} = DQ_{n,m}(t)$ is hold. □

5. The NSW Algorithm for Solving Optimal Control Problem

In this section, the task of optimizing systems governed by ordinary differential equations, which leads to the optimal control problems, is investigated as they are arising in many applications in astronautics and aeronautics.

Consider the following process on fixed interval $[0, 1]$:

$$J = \int_0^1 \mathcal{F}(t, u(t), x(t)) dt, \tag{16}$$

subject to:

$$u(t) = f\left(t, x(t), \dot{x}(t)\right), \tag{17}$$

together with the conditions:

$$x(0) = x_0, \quad x(1) = x_1. \tag{18}$$

where: $x(\bullet) : [0, 1] \to \mathfrak{R}$ is the state variable, $u(\bullet) : [0, 1] \to \mathfrak{R}$, is the control variable, and the function f is assumed to be real valued continuously differentiable.

First, we assume the solution of the state variables $x(t)$ and $\dot{x}(t)$ in terms of NSW, respectively, is as below:

$$x(t) = \sum_{i=0}^m c_i Q_i(t), \tag{19}$$

$$\dot{x}(t) = \sum_{i=0}^m c_i D Q_i(t). \tag{20}$$

where $C = [c_0\ c_1 \ldots c_m]^T$ is the unknown parameters vector.

The second step is to obtain the approximation for the control variable by substituting Equations (19) and (20) into Equation (17):

$$u(t) = f\left(t, \sum_{i=0}^m c_i Q_i(t), \sum_{i=0}^m c_i D Q_i(t)\right). \tag{21}$$

Finally, the performance index value J is obtained as a function of the unknown c_0, c_1, c_2, ..., c_m as below

$$J = \int_0^1 \mathcal{F}\left(\left(\sum_{i=0}^m c_i Q_i(t)\right)^2, \left(\sum_{i=0}^m c_i D Q_i(t)\right)^2\right) dt.$$

The resulting quadratic mathematical programming problem can be simplified as below:

$$J = \frac{1}{2} C^T \mathcal{H} C,$$

where:

$$\mathcal{H} = 2 \int_0^1 \mathcal{F}\left((\Phi(t))^2, (D\Phi(t))^2\right) dt,$$

subject to:

$$FC - b = 0.$$

where:

$$F = \begin{bmatrix} \Phi^T(0) \\ \Phi^T(1) \end{bmatrix}, \quad b = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}.$$

Using Lagrange multiplier technique to obtain the optimal values of the unknown parameters C^*

$$C^* = \mathcal{H}^{-1} F^T \left(F \mathcal{H}^{-1} F^T\right)^{-1} b.$$

6. Test Examples

In this section, the results for the numerical simulation of optimal control problems formulated based on the proposed new shifted wavelet method are presented. Different test cases for m defined in the interval $[0, 1]$ are considered with a single state function and a single control function. Note that the proposed method can be solved problems with multiple controls. The test problems are considered continuous optimal controls, and the analytic solution is known in order to allow the validation of the proposed algorithm by comparing its result with the exact solution.

Example 1. *In the following example, we have one state function $x(t)$ and one control function $u(t)$. This problem is concerned with minimization of [25,26]:*

$$minJ = \int_0^1 \left(u^2(t) + x^2(t) \right) dt,$$

subject to:

$$u(t) = \dot{x}(t),$$

with initial conditions $x(0) = 0, x(1) = 0.5.$

The exact value of the performance index is $J = 0.328258821379$.
Table 1 shows the values of the coefficients, and Tables 2 and 3 give the values of the state and the control, respectively.

Table 1. The unknown coefficients c_i in Example 1.

c_i	$m = 3$	$m = 4$	$m = 5$
c_0	0.230545711338576	0.236727303476872	0.256199322878168
c_1	0.123366816327231	0.130013877070087	0.160508907830196
c_2	0.006294225322818	0.017822378525363	0.047112195129104
c_3		0.003860998979349	0.026814740481690
c_4			0.005789066578131

Table 2. Approximate and exact values of $x(t)$ for Example 1.

t	$m = 3$	$m = 4$	$m = 5$	x_{exact}
0.2	0.081818181818	0.085725158561	0.0856632657718	0.0856602272147
0.4	0.172727272727	0.174680761099	0.1747677613426	0.1747583001210
0.6	0.272727272727	0.270773784355	0.2708607845984	0.2708700372292
0.8	0.381818181818	0.377911205073	0.3778493122834	0.3778527400206
1	0.500000000000	0.500000000000	0.5000000000000	0.5000000000000

Table 3. Approximate and exact values of $u(t)$ for Example 1.

t	$m = 3$	$m = 4$	$m = 5$	u_{exact}
0.2	0.431818181818182	0.433446088794942	0.434113187975870	0.433996647185271
0.4	0.477272727272728	0.459365750528545	0.459887849303989	0.459952039568011
0.6	0.522727272727274	0.504820295983084	0.504298197207639	0.504366922299765
0.8	0.568181818181821	0.569809725158559	0.569142625977633	0.569023820575788
1	0.613636363636367	0.654334038054970	0.656219529904781	0.656517642749666

Table 4 gives the absolute errors $E_m = |J_{exact} - J_m|$ that the NSW method might produce with compression to the following methods:

- The method existing in [25].
- Chebyshev method proposed in [26].

Table 4. A comparison of the results of Example 1.

		E_m		
m	Presented Method	Method in [26]	Method in [25]
3	0.00033	0.00033	0.0050
4	0.00000051	0.00000052	0.0034
5	0.0000000093	0.000000016	0.00021

Example 2: *Consider the second test problem [26]:*

$$\min J = \int\limits_0^1 \left(u^2(t) + 3x^2(t) \right) dt, \tag{22}$$

$$u(t) = \dot{x}(t) - x(t), x(0) = 1,\ x(1) = 0.51314538.$$

The exact solution of (22) is:

$$u(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} - \frac{3}{3e^{-4}+1}e^{-2t},\ x(t) = \frac{3e^{-4}}{3e^{-4}+1}e^{2t} + \frac{1}{3e^{-4}+1}e^{-2t} \text{ and } J = 2.791659975.$$

Table 5 shows the values of the coefficients, and Tables 6 and 7 give the values of the state and the control, respectively, whereas Table 8 lists the absolute errors that our method NSW might produce and compares our technique to the method presented in [26]. From these tables, it can be seen that the state and the control variables are accurately approximated by the proposed method.

Table 5. The unknown coefficients c_i in Example 2.

c_i	$m = 3$	$m = 4$	$m = 5$
c_0	0.355507506871318	0.346802467917968	0.353341978261637
c_1	0.011865347625915	0.003658158193373	0.011466719011776
c_2	0.059865632941091	0.047554848792278	0.053850500952116
c_3		-0.004103594716271	0.0006.7914878500
c_4			0.001195685875318

Table 6. Approximate and exact values of x(t) for Example 2.

t	$m = 3$	$m = 4$	$m = 5$	x_{exact}
0.2	0.72969817542857	0.71547355348770	0.71303748341008	0.7131081208852
0.4	0.54586180114285	0.53874949017242	0.54172690915617	0.5418429752453
0.6	0.44849087714285	0.45560318811329	0.45858060709704	0.4584348199397
0.8	0.43758540342857	0.45181002536944	0.44937395529182	0.4493594610058
1	0.51314538000000	0.51314538000000	0.51314537999999	0.5131453766955

Table 7. Approximate and exact values of u(t) for Example 2.

t	$m = 3$	$m = 4$	$m = 5$	u_{exact}
0.2	-2.56767274857141	-2.71584589378880	-2.78633403260869	-2.79165997531006
0.4	-1.86504367257142	-1.85674597643924	-1.83028754865182	-1.82851756831608
0.6	-1.24888004685713	-1.17657155199105	-1.16048897823791	-1.16185967374545
0.8	-0.71918187142857	-0.66109799850335	-0.68313541022400	-0.68359121816281
1	-0.275949146285715	-0.296100694035280	-0.31768698166747	-0.31616348542896

Table 8 illustrates the fast convergence rate of the proposed method, since the errors decay rapidly by increasing the number of the NSW.

Table 8. Estimated values of J_m for $m = 3, 4, 5$ for Example 2.

m	J_m	E_m	J_m in [26]	E_m
3	2.79718233539	0.0055	2.7977436304	0.0060
4	2.79237308337	0.00071	2.7960838642	0.0044
5	2.79166202469	0.0000020	2.7960838642	0.0044

Example 3. *Consider the third test problem*

$$J = \frac{1}{2} \int_0^1 \left(u^2(t) + x^2(1) \right) dt,$$

$$u(t) = \dot{x}(t) - x(t), \, x(0) = 1, \, x(1) = 0.3678794412 \text{ and } J_{exact} = 1.$$

Table 9 shows the values of the coefficients while Tables 10 and 11 compare the exact solutions and the approximate solutions of $x(t)$ and $u(t)$, respectively, for $m = 3$, 4, 5. The absolute errors of J for various values of m are listed in Table 12. From these results, it is worthwhile to note that the approximate solutions obtain by the proposed method completely coincide with the exact solutions.

Table 9. The unknown coefficients c_i in Example 3.

c_i	$m = 3$	$m = 4$	$m = 5$
c_0	0.317388943264860	0.314366203750058	0.314854167521389
c_1	−0.10561159916574	−0.10846146531065	−0.10789004268669
c_2	0.017219482834726	0.012944683617373	0.013407812437593
c_3		−0.00142493307245	−0.00107009805596
c_4			0.00008.870875412

Table 10. Approximate and exact values of $x(t)$ for Example 3.

t	$m = 3$	$m = 4$	$m = 5$	x_{exact}
0.2	0.8238348176509	0.8188954570054	0.8187261332539	0.81873075307798
0.4	0.6725401705963	0.6700704902736	0.6703085019619	0.67032004603563
0.6	0.5461160588363	0.5485857391591	0.5488237508474	0.54881163609402
0.8	0.4445624823709	0.4495018430164	0.4493325192649	0.44932896411722
1	0.3678794412000	0.3678794411999	0.3678794412000	0.36787944117144

Table 11. Approximate and exact values of $u(t)$ for Example 3.

t	$m = 3$	$m = 4$	$m = 5$	u_{exact}
0.2	−1.642484391160	−1.6396030974501	−1.6376087511890	−1.6374615061559
0.4	−1.366837067632	−1.3417286510180	−1.3405383263440	−1.3406400920712
0.6	−1.116060279400	−1.0958912234308	−1.0975575714816	−1.0976232721880
0.8	−0.890154026461	−0.8971514540429	−0.89880715280116	−0.8986579282344
1	−0.689118308818	−0.7405699822088	−0.73541173113306	−0.7357588823428

Table 12. Estimated values of J_m for m $= 3, 4, 5$ of Example 3.

m	J_m	E_m
3	1.000272934759060	0.00027
4	1.000001904254601	0.0000019
5	1.000000007516538	0.0000000075

Example 4. *Consider the fourth test problem [26]:*

$$\min J = \int_0^1 \left(0.5u^2(t) + x^2(t) \right) dt, \tag{23}$$

$$u(t) = \dot{x}(t) - 0.5x(t), \ x(0) = 1, \ x(1) = 0.5018480732.$$

The exact solution of (23) is:

$$u(t) = \frac{2e^{3t} - e^3}{a}, \ x(t) = \frac{2e^{3t} + e^3}{a}, \text{ where } a = 2e^{\frac{3t}{2}}(1 + e^3) \text{ and } J = 0.8641644978.$$

Table 13 compares absolute errors of presented method wavelets to the existing method presented in [26] with different values of m, and we see that the absolute errors of the presented method provide good results compares to the existing other method, which indicates a decrease in absolute errors with the increase in the value of m.

Table 13. Estimated values of J_m for $m = 3, 4, 5$ of Example 4.

m	J_m	E_m	m	J_m in [26]	E_m
3	0.86472880938	0.00056	2	0.8645390446	00037
4	0.86421807235	0.000053	3	0.8644550472	0.00029
5	0.86416456896	0.000000071	4	0.8643546452	0.00019

It is clear that the approximate solution of the performance index when $m = 8$ is in very good agreement with the corresponding exact solution. Table 13 reports the absolute errors of J_m obtained by the proposed method at $m = 3$, 4, 5 in comparison to the method in [26] at $m = 2$, 3, 4. The obtained results show that the approximate solutions are more accurate for the proposed method than the method in [26]. In addition, the fast convergence rate of the proposed method is also illustrated from the absolute errors results, since by increasing the number of the NSW, the errors decay rapidly.

7. Discussion

The NSW coefficients for the state function $x(t)$ and the control function $u(t)$, the NSW approximated values $x_m(t)$ of orders $m = 3$, 4 and 5, the NSW approximated values $u_m(t)$ of orders $m = 3$, 4 and 5 and the error estimates E_m for different values of m are reported respectively in Tables 1–4 for Example 1, In Tables 5–8 for Example 2, in Tables 8–12 for Example 3 while in Table 13, the obtained error estimates E_m for different values of m have been calculated. A comparison between the NSW approximation and the exact solution shows that as m increases, the errors decay rapidly. One of the important advantages of the use of the NSW method is that the convergence of J_m is faster than some other methods in the literature see [25,26]. Therefore, by proceeding an approximations for the suitable value of m, the results obtained by the proposed method will rapidly tend to the results for the exact solution. The NSW approximation of order five is a very accurate approximation of the exact solution. Examples 2–4 have been solved by many researchers using different approaches, but the results obtained by NSW using state parameterization are the best results. From the results of Examples 2–4, it is clear that our algorithm gives better or comparable results with that of algorithms in [25,26], although the amount of computations in our method is very much less than in their algorithms.

A comparison between the results for the exact solution and for the values of $m = 3$ shows that the error in the performance index is of the order of 10^{-4}, while for the values $m = 5$, an agreement of about nine decimal places is obtained in the performance index. The results gradually tend towards the exact results as we systematically proceed to higher order approximations.

Tables 4, 7, 11 and 13 report the absolute errors for the performance index obtained by our method in comparison to the method in [25,26] at $m = 3$, 4, 5. The obtained results

show that the absolute errors are better for the proposed method than those obtained in [25,26]. From such tables, it can be found that the state and the control variables are accurately approximated by the presented method.

8. Conclusions

This paper presents a new technique for obtaining the numerical solutions for optimal control problems. The derivation of the method is based on the construction of a new shifted wavelet with its operational matrix of derivatives. One of the advantages of the proposed technique is adopting a limited number of wavelets basis functions.

Approximate and exact solutions of examples are correspondingly compared. For Example 1, a comparison reports in Table 4 that it is clear that at m = 5, the results obtained by the proposed method are better than those in [25,26], with the absolute error of the performance index 9.3×10^{-9}, 1.6×10^{-8} and 2.1×10^{-4}, respectively. Numerical results for Example 2 were presented in [25] with the best absolute error 4.4×10^{-3}, while in our method, the best absolute method is 2.0×10^{-6}. Absolute errors of Example 4 were also given as 1.9×10^{-4} and 7.1×10^{-8} in [26] and the present work, respectively. The best absolute errors of Example 4 presented in [26] are given in Table 13. As can be seen from Tables 4, 8, 12 and 13, the present method is highly efficient and accurate, and it is quite high, even in the case of a small number of the basis wavelet functions.

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Abbreviations

$M_m(t)$	The special polynomials defined in the interval $[-1, 1]$.
$Ms_m(t)$	The shifted special polynomials defined in the interval $[0, 1]$.
$Q_{nm}(t)$	The new shifted special wavelet functions.
$\Phi(t)$	The vector of the basis functions.
$w(t)$	The weight function.
$\ddot{f}(t)$	The second derivative.
$\langle .,. \rangle$	Inner product operator on Hilbert space.
$f(t) \in L_w^2([0,1])$	means $\int_0^1 \lceil f(t) \rceil^2 w(t) dt$ finite.
$f \in C^2[0,1)$	f and its first derivative $\dot{f}$ are continues.
J	Performance index value.
$x(t)$	State variable.
$u(t)$	Control variable.
$r(x)$	Residual function
ϵ	Convergence value greater than zero.
D_Q	Operation matrix of derivative.
$\mathfrak{R}$	Real numbers.
χ	Delta function.
u_{exact}	Exact values of the control variable.
x_{exact}	Exact values of the state variable.
u_m	Approximate values of the control variable.
x_m	Approximate values of the state variable.
J_{exact}	Exact value of the performance index.
J_m	Approximate value of the performance index.
$\mathcal{F}$	Integrand function.
C	Vector of unknown parameters.
C^*	Vector of optimal parameters.
$E_m = |J_{exact} - J_{m.}|$	The absolute errors.

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