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An Infinite System of Fractional Sturm–Liouville Operator with Measure of Noncompactness Technique in Banach Space

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Abstract: In the current contribution, an appropriate quantity connected to the space of all convergent sequences is provided and shown to be a measure of noncompactness in a Banach space. Through the application of the fixed point theorems of Darbo and Meir–Keeler, this amount is used to discuss whether a solution to an infinite system of fractional Sturm–Liouville operators exists. We offer a numerical example as an application of the key finding in the study.

Keywords: infinite system; measure of noncompactness; fixed point theorem; Sturm–Liouville operator; sequence space

MSC: 34A08; 34A12; 47H08; 47H10; 46B45

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1. Introduction

The applications of Sturm–Liouville equations in mathematical physics, science, and engineering are numerous and expanding all the time. The majority of second-order differential equations eventually rewrite as Sturm–Liouville type equations. The eigenvalues and eigenvectors are arranged according to the Sturm–Liouville theory, and they cover an orthogonal basis. The best way for us to gain insight into the spectrum of a dynamical system has been advantageously demonstrated [1]. As a substitute strategy, differential equations of fractional order models are currently overly used since the data from exploratory and area of measurement studies cannot be strictly characterized by differential equations of integer order. In recent times, it has been discovered that the fractional Sturm–Liouville types offer more accurate system solutions than the traditional type [2,3].

The fractional Sturm–Liouville operator is widely used in quantum mechanics, as well as in applied mathematics, physics, science, and engineering, for example [2]. As an extension of the common Sturm–Liouville operator, Klimek and Agrawal [3] developed the fractional Sturm–Liouville operator for the first time. Since then, many versions have been introduced using the same style but with somewhat different formatting [4–6]. They looked at the eigenvalues and eigenfunctions characteristics of the fractional Sturm–Liouville operators and addressed variational qualities. Many authors have expressed interest in it (see [7,8] and the references therein).

On the other hand, the infinite systems of differential equations take an extremely important role in describing physical phenomena such as the branching process, neural nets and dissociation of polymers [9]. Solving of PDEs by numerical methods very often instructs an investigation of infinite systems of ODEs. Additionally, there is another example related to using semidiscretization to solve several challenging problems for parabolic PDEs [10]. Due to this, other writers became interested in studying some of its principles and qualities, Refs. [11–14]. This research used a measure of noncompactness approach as their foundation. A mapping from a set of all nonempty and bounded subsets to a particular Banach space that is realized with the condition that it is equal to zero for relatively compact subsets is the measure of noncompactness. It has significant applications in several forms of nonlinear analysis, optimization, the differential, integral, and integro-differential equations, among other areas, and plays a very important role in fixed point theory [15,16].

Many mathematicians and physicians have focused their attention on thoroughly investigating the boundary value problems presented based on such equations due to the high visibility of fractional differential equations and their outstanding importance and verity applications, particularly the investigation of the existence and uniqueness of solutions in Banach spaces (see [17–19]).

To demonstrate our problem in abstract form, we assume $y = (y_1, y_2, \dots)$, ${}^c\mathbb{D}^{\alpha}_{a^+} y = ({}^c\mathbb{D}^{\alpha}_{a^+} y_1, {}^c\mathbb{D}^{\alpha}_{a^+} y_2, \dots)$ and $f = (f_1, f_2, \dots)$ are sequences of real functions, where ${}^c\mathbb{D}^{\alpha}_{a^+}$ is the left Caputo fractional derivative of order $0 < \alpha \leq 1$. The main objective of the current work is to discuss the existence of solutions to the following infinite system related to the fractional Sturm–Liouville operator

$$\left({}^c\mathbb{D}^{\nu}_{b^-}(p(t)\,{}^c\mathbb{D}^{\mu}_{a^+} y_i(t))\right)(t) = f_i(t, y(t), ({}^c\mathbb{D}^{\alpha}_{a^+} y)(t)) \qquad t \in [a,b],\ i \in \mathbb{N}, \tag{1}$$

where $p : [a,b] \to \mathbb{R}_+$ is an absolutely continuous function, $f : [a,b] \times c \times c \to c$ with $f_i : [a,b] \times c \times c \to \mathbb{R}$ are supposed functions for all $i \in \mathbb{N}$ under certain assumptions, which will be mentioned later, and ${}^c\mathbb{D}^{\mu}_{a^+}$ and ${}^c\mathbb{D}^{\nu}_{b^-}$ are the left and right Caputo fractional derivatives of orders $\mu \in (1,2)$ and $\nu \in (0,1]$, respectively. Here, c is a sequence space of all convergent sequence of real functions. The problem subjects to the conditions

$$y_i(a) = A_i, \qquad y_i'(a) = B_i \qquad y_i(b) = C_i, \quad i \in \mathbb{N}, \tag{2}$$

where A_i, B_i and C_i are constants for all $i \in \mathbb{N}$.

Because the Sturm–Liouville operator and infinite system play such a significant role in the theory of differential equations, we dedicate this contribution to a discussion of the boundary value problems (1) and (2). Our research relies on using the measure of noncompactness technique in a sequence space connected to the space c (the space of all convergent sequences) with the assistance of the Darbo and Meir–Keeler fixed point theorems.

Several contributors have attempted to apply the well-known Banach contraction principle in their publications. Darbo's fixed point theorem is a well-known generalization of the Banach contraction principle. There are many generalizations of the Meir–Keeler condensing operator that use the measure of non-compactness to verify several new fixed point theorems and to analyze the solvability of a system of Volterra type functional integral equations. In addition, there are many expansions of Darbo's fixed point theorem, as well as some conclusions on the existence of coupled fixed points for a specific class of operators in a Banach space which can be used to investigate the existence of a solution for a system of nonlinear functional integral and differential equations as applications [20–22]. Due to the main role of these fixed point theorems, we choose them to be the basic tools with which to investigate our problem.

2. Basic Definitions and Lemmas

In this part, we present some fundamental concepts and identities for fractional integrals and derivatives that are covered in the book's first and second chapters [23–25]. Additionally, we provide helpful lemmas related to our topic.

Definition 1. *Let $[a,b]$ $(-\infty < a < b < \infty)$ be a finite interval and $\mathcal{AC}([a,b])$ be the space of all absolutely continuous functions. Then, $h \in \mathcal{AC}([a,b])$ if and only if the exist a function $\psi \in L(a,b)$ (the space of primitives of Lebesgue summable functions) and a constant Q such that*

$$h(t) = Q + \int_a^t \psi(s)ds, \qquad t \in [a,b].$$

Remark 1. *The previous definition implies that the absolutely continuous function $h(t)$ has a summable derivative $h'(t) = \psi(t)$ a.e. on the interval $[a,b]$ and $h(a) = Q$. Therefore, we can denote by $\mathcal{AC}^n([a,b])$; $n \in \mathbb{N}$ to the space of complex-valued functions $h(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $h^{(n-1)} \in \mathcal{AC}[a,b]$ with $\mathcal{AC}^1[a,b] = \mathcal{AC}[a,b]$. According to Lemma 1.1 in [23], the function $h \in \mathcal{AC}^n([a,b])$ for all $n \in \mathbb{N}$ if and only if there exists $\psi \in L(a,b)$ such that*

$$h(t) = \sum_{i=0}^{n-1} Q_i (t-a)^i + \frac{1}{n!}\int_a^t (t-s)^{n-1}\psi(s)ds, \qquad t \in [a,b],$$

where $Q_i; i = 1,2,\ldots,n-1$ are constants.

Definition 2. *Let $h \in \mathcal{AC}^n([a,b])$, $n \in \mathbb{N}$ and $n-1 < \rho \le n$. Then, the left and right fractional derivatives in the Caputo sense, respectively, are given by*

$$\begin{aligned} {}^c\mathbb{D}^{\rho}_{a^+} h(t) &= \frac{1}{\Gamma(n-\rho)}\int_a^t (t-s)^{n-\rho-1} h^{(n)}(s)ds = (I^{n-\rho}_{a_+} D^n h)(t), \\ {}^c\mathbb{D}^{\rho}_{b^-} h(t) &= \frac{(-1)^n}{\Gamma(n-\rho)}\int_t^b (s-t)^{n-\rho-1} h^{(n)}(s)ds = (-1)^n (I^{n-\rho}_{b_-} D^n h)(t), \end{aligned}$$

where $I^{\gamma}_{a_+}$ and $I^{\gamma}_{b_-}$ are the left and right fractional integrals rendered by

$$\begin{aligned} I^{\gamma}_{a^+} h(t) &= \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1} h(s)ds, \qquad \gamma > 0, \\ I^{\gamma}_{b^-} h(t) &= \frac{1}{\Gamma(\gamma)}\int_t^b (s-t)^{\gamma-1} h(s)ds, \qquad \gamma > 0. \end{aligned}$$

Lemma 1. *Let $h \in \mathcal{AC}^n([a,b])$ and $n,m \in \mathbb{N}$ such that $n-1 < \rho \le n$, $m-1 < \gamma \le m$ and $\rho \le \gamma$. Then, we have*

$$\begin{aligned} I^{\rho}_{a^+} I^{\gamma}_{a^+} h(t) &= I^{\rho+\gamma}_{a^+} h(t), & I^{\rho}_{b^-} I^{\gamma}_{b^-} h(t) &= I^{\rho+\gamma}_{b^-} h(t) \\ {}^c\mathbb{D}^{\rho}_{a^+} I^{\rho}_{a^+} h(t) &= h(t), & {}^c\mathbb{D}^{\rho}_{b^-} I^{\rho}_{b^-} h(t) &= h(t) \\ {}^c\mathbb{D}^{\rho}_{a^+} I^{\gamma}_{a^+} h(t) &= I^{\gamma-\rho}_{a^+} h(t), & {}^c\mathbb{D}^{\rho}_{b^-} I^{\gamma}_{b^-} h(t) &= I^{\gamma-\rho}_{b^-} h(t) \\ I^{\rho}_{a^+}\, {}^c\mathbb{D}^{\rho}_{a^+} h(t) &= h(t) - \sum_{k=0}^{n-1} K_k (t-a)^k, & I^{\rho}_{b^-}\, {}^c\mathbb{D}^{\rho}_{b^-} h(t) &= h(t) - \sum_{k=0}^{n-1} K'_k (b-t)^k, \end{aligned}$$

where $K_k = h^{(k)}(a)/k!$ and $K'_k = h^{(k)}(b)/k!$ are constants for all $k = 0,1,\ldots,n-1$.

Lemma 2. *Let $n-1 < \rho \le n$ such that $n \in \mathbb{N}$ and $\delta > -1$. Then, we have*

$$\begin{aligned} I^{\rho}_{a^+}(t-a)^{\delta} &= \frac{\Gamma(\delta+1)}{\Gamma(\delta+\rho+1)}(t-a)^{\delta+\rho} \\ I^{\rho}_{b^-}(b-t)^{\delta} &= \frac{\Gamma(\delta+1)}{\Gamma(\delta+\rho+1)}(b-t)^{\delta+\rho} \end{aligned}$$

and if $\delta \ne 0,1,2,\ldots,n-1$, we have

$$\begin{aligned} {}^c\mathbb{D}^{\rho}_{a^+}(t-a)^{\delta} &= \frac{\Gamma(\delta+1)}{\Gamma(\delta-\rho+1)}(t-a)^{\delta-\rho} \\ {}^c\mathbb{D}^{\rho}_{b^-}(b-t)^{\delta} &= \frac{\Gamma(\delta+1)}{\Gamma(\delta-\rho+1)}(b-t)^{\delta-\rho}. \end{aligned}$$

For $\delta = 0,1,2,\ldots,n-1$, we have ${}^c\mathbb{D}^{\rho}_{a^+}(t-a)^{\delta} = {}^c\mathbb{D}^{\rho}_{b^-}(b-t)^{\delta} = 0$.

Lemma 3. *Let $n \in \mathbb{N}$ and $n-1 < \gamma < n$. Then,*

(1) $I^{\gamma}_{a^+} h \in \mathcal{AC}^n([a,b])$ *if* $h \in L(a,b)$ *or* $h \in \mathcal{AC}^m([a,t])$ *where* $m = 1, 2 \ldots$;

(2) ${}^c\mathbb{D}^{\gamma}_{a^+} h \in \mathcal{AC}([a,b])$ *if* $h \in \mathcal{AC}^{n+m}([a,b])$ *where* $m = 0, 1, 2 \ldots$.

Proof. By helping Definition 1 and Remark 1, we can introduce the proofs as follows.

(1) Let $h \in L(a,b)$. Then,

$$I^{\gamma}_{a^+} h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} h(s) ds = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{n-1} (t-s)^{\gamma-n} h(s) ds.$$

It is obvious that $-1 < \gamma - n < 0$, which implies that the function $t \mapsto (t-s)^{\gamma-n} \in L(a,t) \subset L(a,b)$, and so by virtue of the last integral in Remark 1, and noting that the product of two Lebesgue integrable functions is also Lebesgue integrable, we get $I^{\gamma}_{a^+} h \in \mathcal{AC}^n([a,b])$.

Now, let $h \in \mathcal{AC}^m([a,b])$, there exists $\psi \in L(a,b)$ such that

$$h(t) = \sum_{i=0}^{m-1} Q_i (t-a)^i + I^m_{a^+} \psi(t),$$

and so

$$I^{\gamma}_{a^+} h(t) = \sum_{i=0}^{m-1} \frac{i!}{\Gamma(i+\gamma+1)} Q_i (t-a)^{i+\gamma} + I^{m+\gamma}_{a^+} \psi(t).$$

Plainly, the $(i+1)$th term belongs $\mathcal{AC}^{n+i}([a,b])$ for $i = 0, 1, \ldots, m$, then $I^{\gamma}_{a^+} h \in \mathcal{AC}^n([a,b])$.

(2) Since $h \in \mathcal{AC}^{n+m}([a,b])$ for $m = 0, 1, 2 \ldots$, then $h^{(n)} \in \mathcal{AC}^m([a,b])$ and so by the first statement and the fact $0 < n - \gamma < 1$, we get

$${}^c\mathbb{D}^{\gamma}_{a^+} h = I^{n-\gamma}_{a^+} h^{(n)} \in \mathcal{AC}([a,b]).$$

These lead to the desired results. □

Lemma 4. *Let $h \in \mathcal{AC}([a,b])$, $y \in \mathcal{AC}^2([a,b])$, $0 < \nu \le 1$, $1 < \mu < 2$ and $p : [a,b] \to \mathbb{R}_+$ be an absolutely continuous function. Then, the linear fractional Sturm–Liouville equation,*

$$\left({}^c\mathbb{D}^{\nu}_{b^-} (p(t)\, {}^c\mathbb{D}^{\mu}_{a^+} y(t)) \right)(t) = h(t), \qquad t \in [a,b], \tag{3}$$

subjected to the conditions

$$y(a) = A, \qquad y'(a) = B, \qquad y(b) = C, \tag{4}$$

has the unique solution

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\mu)} \int_a^t p^{-1}(s)(t-s)^{\mu-1} \left(\frac{1}{\Gamma(\nu)} \int_s^b (u-s)^{\nu-1} h(u) du \right) ds \\ & - \frac{\Delta(t)}{\Delta(b)\Gamma(\mu)} \int_a^b p^{-1}(s)(b-s)^{\mu-1} \left(\frac{1}{\Gamma(\nu)} \int_s^b (u-s)^{\nu-1} h(u) du \right) ds \\ & + \frac{C - A - B(b-a)}{\Delta(b)} \Delta(t) + A + B(t-a), \end{aligned} \tag{5}$$

where

$$\Delta(t) = I^{\mu}_{a^+} p^{-1}(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} p^{-1}(s) ds,$$

such that $\Delta(b) \neq 0$.

Proof. Let $y \in \mathcal{AC}^2([a,b])$, $p \in \mathcal{AC}([a,b])$ and $h \in \mathcal{AC}([a,b])$ satisfy the fractional differential Equation (3) and its boundary conditions (4). According to the last statement of Lemma 1 with operating $I^{\nu}_{b^-}$ on both sides of (3), we get

$$p(t)({}^c\mathbb{D}^{\mu}_{a^+}y)(t) = K + I^{\nu}_{b^-}h(t),$$

where K is a constant. It is obvious that the fraction integral is well-defined due to $h \in \mathcal{AC}([a,b])$. Since the function p has values in $\mathbb{R}_+$, its reciprocal is defined and absolutely continuous. Hence, we have

$$({}^c\mathbb{D}^{\mu}_{a^+}y)(t) = p^{-1}(t)(K + I^{\nu}_{b^-}h(t)).$$

According to the penultimate statement of Lemma 1 with operating $I^{\mu}_{a^+}$ on both sides, we get

$$y(t) = I^{\mu}_{a^+}p^{-1}(t)(K + I^{\nu}_{b^-}h(t)) + y(0) + y'(0)(t-a) = I^{\mu}_{a^+}p^{-1}(t)(K + I^{\nu}_{b^-}h(t)) + A + B(t-a).$$

The boundary condition $y(b) = C$ leads to

$$K = \frac{C - A - B(b-a)}{\Delta(b)} - \frac{1}{\Gamma(\mu)\Delta(b)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\left(\frac{1}{\Gamma(\nu)}\int_s^b (u-s)^{\nu-1}f(u)du\right)ds$$

which implies that the relation (5) is realized.

Conversely, assume that (5) is verified and the function $p, h \in \mathcal{AC}([a,b])$. It is easy to see that the solution (5) satisfies the initial and boundary conditions $y(a) = A$ and $y(b) = C$.

It is known that if $p \in \mathcal{AC}([a,b])$ and its values in $\mathbb{R}_+$, then $p^{-1} \in \mathcal{AC}([a,b])$. Now, the function Δ can be rewritten as $\Delta(t) = I^{\mu}_{a^+}p^{-1}(t)$, $(1 < \mu < 2)$ which, in view of the first statement of Lemma 3, we find that $\Delta \in \mathcal{AC}^2([a,b])$. Additionally, we have $h \in \mathcal{AC}([a,b])$, which implies $I^{\nu}_{b^-}h \in \mathcal{AC}([a,b])$. In the same way, we get $I^{\mu}_{a^+}(p^{-1}(t)I^{\nu}_{b^-}h) \in \mathcal{AC}^2([a,b])$ which concludes that $y \in \mathcal{AC}^2([a,b])$ and $({}^c\mathbb{D}^{\mu}_{a^+}y)(t)$ is well-defined over the interval $[a,b]$. Now, we have $y \in \mathcal{AC}^2([a,b])$, which implies that $y' \in \mathcal{AC}([a,b])$. Since $1 < \mu < 2$, then $(I^{\mu}_{a^+})' = I^{\mu-1}_{a^+}$, so it is not difficult to see that the (5) satisfies the initial condition $y'(a) = B$. By using some results of Lemmas 1 and 2 operating by ${}^c\mathbb{D}^{\mu}_{a^+}$ on both sides of (5), we get

$$\begin{aligned}({}^c\mathbb{D}^{\mu}_{a^+}y)(t) &= p^{-1}(t)I^{\nu}_{b^-}h(t) - \frac{p^{-1}(t)}{\Delta(b)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\left(\frac{1}{\Gamma(\nu)}\int_s^b (u-s)^{\nu-1}h(u)du\right)ds \\ &\quad + \frac{C - A - B(b-a)}{\Delta(b)}p^{-1}(t),\end{aligned}$$

which can be rewritten as

$$p(t)({}^c\mathbb{D}^{\mu}_{a^+}y)(t) = I^{\nu}_{b^-}h(t) + \text{Constant}.$$

Again, Remark 1 tells us that $p({}^c\mathbb{D}^{\mu}_{a^+}y) \in \mathcal{AC}([a,b])$, which implies that $({}^c\mathbb{D}^{\mu}_{a^+}y) \in \mathcal{AC}([a,b])$. Thus, $({}^c\mathbb{D}^{\nu}_{b^-}p(t)^c(\mathbb{D}^{\mu}_{a^+}y(t)))(t)$ is well-defined over the interval $[a,b]$. Operating by ${}^c\mathbb{D}^{\nu}_{b^-}$ on both sides of the last equation using some results of Lemmas 1 and 2, we get the Equation (3). This completes the proof. □

In the sequel, we need the following lemma,

Lemma 5. *Let $\alpha > -1$ and $\beta > 0$; then, we have*

$$\int_a^b s^\alpha (c-s)^{\beta-1} ds = c^{\alpha+\beta}\left[B_{\frac{b}{c}}(\alpha+1,\beta) - B_{\frac{a}{c}}(\alpha+1,\beta)\right],$$

where $B_x(m,n)$ is the incomplete beta function defined by

$$B_x(m,n) = \int_0^x s^{m-1}(1-s)^{n-1}ds, \qquad m,n > 0.$$

In particular, $B_1(m,n) = B(m,n)$ where $B(m,n)$ is the beta function.

3. Measure of Noncompactness

Consider that $(\mathbb{B}, \|\cdot\|)$ is a Banach space, $\mathcal{H}_{\mathbb{B}}$ is the nonempty and bounded subset and $\mathcal{L}_{\mathbb{B}}$ is the subset of all relatively compact.

Definition 3 ([9])**.** *A mapping $\beta : \mathcal{H}_{\mathbb{B}} \to R_+$ is said to be a measure of noncompactness in $\mathbb{B}$ if for all $\Omega, \Omega_n \in \mathcal{H}_{\mathbb{B}}$ for all $n \in \mathbb{N}$. Then, the following assertions hold:*

(1) *The set $\ker\beta = \{\Omega \in \mathcal{H}_{\mathbb{B}} | \beta(\Omega) = 0\} \neq \phi$ and $\ker\beta \subset \mathcal{L}_{\mathbb{B}}$;*
(2) *if $\Omega_1 \subset \Omega_2$, then $\beta(\Omega_1) \leq \beta(\Omega_2)$;*
(3) *$\beta(\Omega) = \beta(\overline{\Omega}) = \beta(Conv\Omega)$;*
(4) *for $\lambda \in [0,1]$, we have $\beta(\lambda\Omega_1 + (1-\lambda)\Omega_2) \leq \lambda_1\beta(\Omega_1) + (1-\lambda)\beta(\Omega_2)$;*
(5) *if (Ω_n) is a sequence of closed subsets of $\mathcal{H}_{\mathbb{B}}$ with $\Omega_{n+1} \subset \Omega_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} \beta(\Omega_n) = 0$, then $\cap_{n=1}^{\infty}\Omega_n \neq \phi$.*

Definition 4 ([9])**.** *The measure of noncompactness β is called sublinear if it verifies*

(i) *for all $\Omega \in \mathcal{H}_{\mathbb{B}}$ and $\lambda \in \mathbb{R}$, $\beta(\lambda\Omega) \leq |\lambda|\beta(\Omega)$, (homogeneous measure);*
(ii) *for all $\Omega_1, \Omega_2 \in \mathcal{H}_{\mathbb{B}}$, $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, (subadditive measure).*

Definition 5 ([9])**.** *The measure of noncompactness β is called regular if it is sublinear and verifies*

(iii) *for all $\Omega_1, \Omega_2 \in \mathcal{H}_{\mathbb{B}}$, $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$, (maximum property);*
(iv) $\ker\beta = \mathcal{L}_{\mathbb{B}}$, *(full measure).*

Theorem 1 (Darbo's Theorem [15])**.** *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space $\mathbb{B}$. Suppose that $\mathcal{P} : \Omega \to \Omega$ is a continuous map, such that there exists a constant $k \in [0,1)$ with the property $\beta(\mathcal{P}\Omega) \leq k\beta(\Omega)$, then $\mathcal{P}$ has a fixed point in Ω.*

Definition 6 ([26])**.** *Consider $\mathscr{K}$ is a nonempty subset of a Banach space $\mathbb{B}$ and β is a measure of noncompactness on $\mathbb{B}$. Then the operator $\mathcal{P} : \mathscr{K} \to \mathscr{K}$ is called a Meir–Keeler condensing operator if, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all bounded subset $\Omega \in \mathscr{K}$, we have*

$$\epsilon < \beta(\Omega) < \epsilon + \delta \Rightarrow \beta(\mathcal{P}\Omega) < \epsilon.$$

Theorem 2 (Meir–Keeler Theorem [26])**.** *Consider that $\mathscr{K}$ is a nonempty, bounded, closed and convex subset of a Banach space $\mathbb{B}$ and β is a measure of noncompactness on $\mathbb{B}$. If $\mathcal{P} : \mathscr{K} \to \mathscr{K}$ is a continuous and Meir–Keeler condensing operator, then the operator $\mathcal{P}$ has at least one fixed point and the set of all fixed points of $\mathcal{P} : \mathscr{K} \to \mathscr{K}$ in $\mathbb{B}$ is compact.*

4. Main Results

Let $\mathbf{S}$ be a sequence space of all real sequences $x = (x_i)_{i\in\mathbb{N}}$, the space of all convergent sequences c is defined as [9]

$$c = \left\{x \in \mathbf{S} : \text{ there exists } L \in \mathbb{R} \text{ such that } \lim_{i\to\infty} x_i = L\right\}$$

equipped with the norm

$$\|x\|_c = \sup_{i \in \mathbb{N}} |x_i|.$$

It is known that $(c, \|\cdot\|_c)$ is a Banach space. According to Theorem 3.1 in [27], for all $\Omega \in \mathcal{H}_c$, the quantity

$$\beta_c(\Omega) = \lim_{n\to\infty} \left\{ \sup_{(x_k)\in\Omega} \{\sup |x_i - x_j| : \quad i, j > n\} \right\} \tag{6}$$

is a regular measure of noncompactness in the sequence space $(c, \|\cdot\|)$.

Define the space $\mathcal{AC}^n([a,b], c)$, $n \in \mathbb{N}$ as

$$\mathcal{AC}^n([a,b], c) = \{y \in c \colon y_i \in \mathcal{AC}^n([a,b]),\ i \in \mathbb{N}\}$$

with $\mathcal{AC}([a,b], c) = \mathcal{AC}^1([a,b], c)$ where $y(t) = (y_i(t))_{i\in\mathbb{N}}$. Consider the space

$$\mathbb{B} = \Big\{y \colon y \in \mathcal{AC}^2([a,b], c) \text{ and } {}^c\mathbb{D}^\alpha_{a+} y(t) \in \mathcal{AC}([a,b], c)\Big\},$$

equipped with the norm

$$\|y\| = \|y\|_c + \|\, {}^c\mathbb{D}^\alpha_{a+} y\|_c.$$

The following lemma shows that the space $(\mathbb{B}, \|\cdot\|)$ is a Banach space.

Lemma 6. *The space $(\mathbb{B}, \|\cdot\|)$ is a Banach space.*

Proof. Let $(y^n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{B}$, which means that $y^n = (y^n_i)_{i\in\mathbb{N}} \in c$ for all $n \in \mathbb{N}$ and $y^n_i \in \mathcal{AC}^2([a,b])$ for all $i \in \mathbb{N}$. It is known that $(c, \|\cdot\|_c)$ is a Banach space [9] and so there is $u = (u_i)_{i\in\mathbb{N}} \in c$ such that $y^n \to u$ as $n \to \infty$. According to the second statement of Lemma 3, we get ${}^c\mathbb{D}^\alpha_{a+} y^n_i \in \mathcal{AC}([a,b])$ for all $i, n \in \mathbb{N}$. From the Definition 2 of the Caputo derivative and Lebesgue's dominated convergence theorem, we can easily deduce that $\lim\limits_{i\to\infty} {}^c\mathbb{D}^\alpha_{a+} y^n_i(t)$ exists for all $n \in \mathbb{N}$ and $t \in [a,b]$, which implies that the sequence $(\,{}^c\mathbb{D}^\alpha_{a+} y^n)_{n\in\mathbb{N}} \in c$. We claim that $(\,{}^c\mathbb{D}^\alpha_{a+} y^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space c, which means that there is $v = (v_i)_{i\in\mathbb{N}} \in c$ such that ${}^c\mathbb{D}^\alpha_{a+} y^n \to v$ as $n \to \infty$. To prove that, let $n, m, N \in \mathbb{N}$ such $n, m > N$. Then,

$$\begin{aligned}
\|(\,{}^c\mathbb{D}^\alpha_{a+} y^n)(t) - (\,{}^c\mathbb{D}^\alpha_{a+} y^m)(t)\|_c &\le \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \big\|(y^n(s))' - (y^m(s))'\big\|_c ds \\
&= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \sup_{i\in\mathbb{N}} \big|(y^n_i(s))' - (y^m_i(s))'\big| ds \\
&\le \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \sup_{i\in\mathbb{N}} \big(|(y^n_i(s))'| + |(y^m_i(s))'|\big) ds.
\end{aligned}$$

By virtue of $y^n_i \in \mathcal{AC}^2([a,b])$ for all $i, n \in \mathbb{N}$, we get $(y^n_i)' \in \mathcal{AC}([a,b])$ for all $i, n \in \mathbb{N}$, which means that $(y^n_i)' \in \mathcal{AC}([a,b])$ are continuous for all $i, n \in \mathbb{N}$ and attain their maximum in the interval $[a,b]$. Thus, there exist positive constants $\delta_n < \infty$ such that $\sup_{i\in\mathbb{N}} |(y^n_i(t))'| \le \delta_n$ for all $t \in [a,b]$ and $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\|(\,{}^c\mathbb{D}^\alpha_{a+} y^n)(t) - (\,{}^c\mathbb{D}^\alpha_{a+} y^m)(t)\|_c &\le \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} (\delta_n + \delta_m) ds \\
&\le \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} (\delta_n + \delta_m) < \epsilon, \qquad n, m > N,
\end{aligned}$$

which implies that our claim is true.

Since $y_i^n \in \mathcal{AC}^2([a,b])$ for all $i \in \mathbb{N}$, according to Remark 1, there are $\psi_i^n \in L(a,b)$ such that

$$y_i^n(t) = Q_1 + Q_2(t-a) + \int_a^t (t-s)\psi_i^n(s)ds, \qquad t \in [a,b].$$

By Lebesgue's dominated convergence theorem, we can deduce that

$$u_i(t) = Q_1 + Q_2(t-a) + \int_a^t (t-s)\psi_i(s)ds, \qquad t \in [a,b],$$

where $\psi_i^n \to \psi_i \in L(a,b)$ as $n \to \infty$, which means that $u_i \in \mathcal{AC}^2([a,b])$ for all $i \in \mathbb{N}$ and so $u \in \mathcal{AC}^2([a,b],c)$. In the same way, $v \in \mathcal{AC}([a,b],c)$.

It suffices to prove that $v(t) = {}^c\mathbb{D}_{a_+}^{\alpha} u(t)$. To do this, by using Lemma 1, we have

$$\begin{aligned}\left\| I_{a_+}^{\alpha}\, {}^c\mathbb{D}_{a_+}^{\alpha} y^n(t) - I_{a_+}^{\alpha} v(t) \right\|_c &\le \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \sup_{i\in\mathbb{N}} \left| {}^c\mathbb{D}_{a_+}^{\alpha} y_i^n(s) - v_i(s) \right| ds \\ &\le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \left\| {}^c\mathbb{D}_{a_+}^{\alpha} y^n(t) - v(t) \right\|_c.\end{aligned}$$

Since ${}^c\mathbb{D}_{a_+}^{\alpha} y^n(t) \to v(t)$ as $n \to \infty$ uniformly on $[a,b]$, then we find that $I_{a_+}^{\alpha}\, {}^c\mathbb{D}_{a_+}^{\alpha} y^n(t) \to I_{a_+}^{\alpha} v(t)$ as $n \to \infty$ uniformly on $[a,b]$. Hence, by using Lemma 1, we find that $y^n(t) - y^n(0) \to I_{a_+}^{\alpha} v(t)$ as $n \to \infty$, which leads to $u(t) - k = I_{a_+}^{\alpha} v(t)$ where k is a constant. Operating by ${}^c\mathbb{D}_{a_+}^{\alpha}$ on both sides we obtain ${}^c\mathbb{D}_{a_+}^{\alpha} u(t) = v(t)$. These conclude for any $\epsilon > 0$ that there exists an $n_0 \in \mathbb{N}$ such that $\|y^n - u\|_c < \epsilon/2$ and $\|{}^c\mathbb{D}_{a_+}^{\alpha} y^n - v\|_c < \epsilon/2$ for $n > n_0$. Therefore,

$$\begin{aligned}\|y^n - u\| &= \|y^n - u\|_c + \|{}^c\mathbb{D}_{a_+}^{\alpha}(y^n - u)\|_c \\ &= \|y^n - u\|_c + \|{}^c\mathbb{D}_{a_+}^{\alpha} y^n - v)\|_c \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad n > n_0.\end{aligned}$$

This ends the proof. □

Let us introduce the following quantity,

$$\beta_{\mathbb{B}}(\Omega) = \beta_1(\Omega) + \beta_2(\Omega), \tag{7}$$

for all nonempty bounded subsets $\Omega \in \mathcal{H}_{\mathbb{B}}$, where

$$\begin{aligned}\beta_1(\Omega) &= \lim_{n\to\infty}\left\{ \sup_{y\in\Omega}\left\{ \sup_{i,j>n}\left\{ \max_{t\in[a,b]} |y_i(t) - y_j(t)| \right\}\right\}\right\}, \\ \beta_2(\Omega) &= \lim_{n\to\infty}\left\{ \sup_{y\in\Omega}\left\{ \sup_{i,j>n}\left\{ \max_{t\in[a,b]} | {}^c\mathbb{D}_{a_+}^{\alpha} y_i(t) - {}^c\mathbb{D}_{a_+}^{\alpha} y_j(t)| \right\}\right\}\right\}.\end{aligned}$$

Lemma 7. *The quantity $\beta_{\mathbb{B}}(\Omega)$ is a sublinear and full measure of noncompactness in the space $\mathbb{B}$.*

Proof. Since $y \in \Omega \in \mathcal{H}_{\mathbb{B}}$, then we have $y \in \mathcal{AC}^2([a,b],c)$ and ${}^c\mathbb{D}^{\alpha} y \in \mathcal{AC}([a,b],c)$, which mean, according to the regular measure of noncompactness (6), that the quantities $\beta_1(\Omega)$ and $\beta_2(\Omega)$ are regular measures of noncompactness on the space $\mathcal{AC}([a,b],c)$. This means that both β_1 and β_2 satisfy all identities mentioned in Definitions 3–5. It is clear that $\ker\beta = \ker\beta_1 \cap \ker\beta_2$. From the identities (1) in Definition 3 and (iv) in Definition 5, we get $\ker\beta = \ker\beta_1 = \ker\beta_2 = \mathcal{L}_{\mathbb{B}} \neq \phi$, which means that the identity (1) in Definition 3 and identity (iv) in Definition 5 hold. It is not difficult to verify the identities (2)–(5) in Definition 3 and the identities (i) and (ii) in Definition 4, which imply that β is a sublinear measure of noncompactness. □

Remark 2. *Let* $\Omega_1, \Omega_2 \in \mathcal{H}_{\mathbb{B}}$ *with* $\beta_1(\Omega_1) = 2, \beta_1(\Omega_2) = 2.5, \beta_2(\Omega_1) = 3$ *and* $\beta_2(\Omega_2) = 1.4$, *then we have*

$$\begin{aligned}\beta(\Omega_1 \cup \Omega_2) &= \beta_1(\Omega_1 \cup \Omega_2) + \beta_2(\Omega_1 \cup \Omega_2)\\ &= \max\{\beta_1(\Omega_1), \beta_1(\Omega_2)\} + \max\{\beta_2(\Omega_1), \beta_2(\Omega_2)\} = 5.5\\ &\neq \max\{\beta(\Omega_1), \beta(\Omega_2)\} = 5,\end{aligned}$$

which means that β *has no maximum property in general. Indeed, we have*

$$\begin{aligned} &\beta(\Omega_1 \cup \Omega_2) \geq \beta_1(\Omega_1) + \beta_2(\Omega_1) = \beta(\Omega_1)\\ \textit{or}\qquad &\beta(\Omega_1 \cup \Omega_2) \geq \beta_1(\Omega_2) + \beta_2(\Omega_2) = \beta(\Omega_2),\end{aligned}$$

which conclude that

$$\beta(\Omega_1 \cup \Omega_2) \geq \max\{\beta(\Omega_1), \beta(\Omega_2)\}.$$

The discussion of the existence results for the infinite system (1) and (2) will be studied under the following suppositions:

($\mathcal{P}_1$) The functions $f_i\colon [a,b] \times c \times c \to \mathbb{R}$ are absolutely continuous functions for all $i \in \mathbb{N}$ and $t \in [a,b]$;

($\mathcal{P}_2$) There exists a nonnegative sequence of functions $w(t) = (w_i(t))_{i\in\mathbb{N}}$ such $w_i\colon [a,b] \to \mathbb{R}_+ \cup 0$ satisfies the inequality.

$$|f_i(t,y,z) - f_i(t,y',z')| \leq w_i(t) \sup_{j\in\mathbb{N}}(|y_j - y_j'| + |z_j - z_j'|)$$

for all $i \in \mathbb{N}$, where $y, y' \in \mathcal{AC}^2([a,b],c)$ and $z, z' \in \mathcal{AC}([a,b],c)$ are real sequences of y_j, y_j' and z_j, z_j' for all $j \in \mathbb{N}$, respectively.

($\mathcal{P}_3$) Let $i, j \in \mathbb{N}$ be large enough, then we get

$$\begin{aligned}|f_i(t,y,z) - f_j(t,y,z)| \leq &\sup_{m,n\geq\min\{i,j\}} |F_m(t) - F_n(t)|\\ &+ \sup_{k\in\mathbb{N}} w_k(t) \sup_{m,n\geq\min\{i,j\}}(|y_m - y_n| + |z_m - z_n|),\end{aligned}$$

where $F_i(t) = |f_i(t,0,0)|$ for all $i \in \mathbb{N}$, $y = (y_i)_{i\in\mathbb{N}}$ and $z = (z_i)_{i\in\mathbb{N}}$.

($\mathcal{P}_4$) There are positive constants

$$\begin{aligned}\mathrm{E}(v;\mu,\nu) &= \sup_{i\in\mathbb{N}} \max_{t\in[a,b]} \mathrm{E}(v_i;\mu,\nu)(t),\\ \mathrm{E}_b(v;\mu,\nu) &= \sup_{i\in\mathbb{N}} \mathrm{E}(v_i;\mu,\nu)(b),\end{aligned}$$

where

$$\mathrm{E}(v_i;\mu,\nu)(t) = \frac{1}{\Gamma(\mu)} \int_a^t p^{-1}(s)(t-s)^{\mu-1}\left(I_{b^-}^{\nu} v_i(s)\right) ds, \qquad i \in \mathbb{N}.$$

Remark 3. *It is obvious that the function* $t \mapsto \Delta(t;\mu)$ *is increasing on* $[a,b]$ *due to the positivity of the function* $p(t)$, *which implies that* $\Delta(t;\mu) \leq \Delta(b;\mu)$ *for all* $t \in [a,b]$. *Additionally, for all* $\mu > 0$, *we have*

$$\mathrm{E}(1;\mu,0)(t) = \Delta(t;\mu).$$

In particular, if $\mu \geq 1$, we have

$$\begin{aligned}
\mathrm{E}_b(v;\mu,\nu) &= \max_{t\in I} \mathrm{E}(v;\mu,\nu)(t), \\
\Delta(b;\mu) &= \max_{t\in I} \mathrm{E}(1;\mu,0)(t).
\end{aligned}$$

To simplify the calculations, we provide

$$\mathrm{Q}_F = \mathrm{E}(F;\mu-\alpha,\nu) + \left(2 + \frac{\mathrm{E}(1;\mu-\alpha,0)}{\mathrm{E}_b(1;\mu,0)}\right)\mathrm{E}_b(F;\mu,\nu), \tag{8}$$

$$\mathrm{Q}_w = 2\mathrm{E}_b(w;\mu,\nu) + \mathrm{E}(w;\mu-\alpha,\nu) + \frac{\mathrm{E}(1;\mu-\alpha,0)\mathrm{E}_b(w;\mu,\nu)}{\mathrm{E}_b(1;\mu,0)} \tag{9}$$

$$\mathrm{Q} = \mathrm{N}_1 + \frac{\mathrm{N}_2}{\mathrm{E}_b(1;\mu,0)}\mathrm{E}_b(1;\mu-\alpha,0) + \mathrm{N}_3, \tag{10}$$

where

$$\begin{aligned}
\mathrm{N}_1 &= \max_{i\in\mathbb{N}}\{|C_i - A_i - B_i(b-a)| + |A_i + B_i(b-a)|\}, \\
\mathrm{N}_2 &= \max_{i\in\mathbb{N}}\{|C_i - A_i - B_i(b-a)| + |A_i + B_i(b-a)|\}, \\
\mathrm{N}_3 &= \max_{i\in\mathbb{N}}\{|B_i|\}\frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

We say that the sequence $y(t) = (y_i(t))_{i\in\mathbb{N}}$ is a solution of the initial value problems (1) and (2) if $y_i(t)$ satisfies Equation (1) and boundary conditions (2) for all $i \in \mathbb{N}$. From Lemma 4, y_i has a unique representation,

$$\begin{aligned}
y_i(t) = &\frac{1}{\Gamma(\mu)}\int_a^t p^{-1}(s)(t-s)^{\mu-1}\left(\frac{1}{\Gamma(\nu)}\int_s^b (u-s)^{\nu-1} f_i(u, y(u), {}^c\mathbb{D}^\alpha_{a+}y(u))du\right)ds \\
&- \frac{\Delta(t;\mu)}{\Delta(b;\mu)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\left(\frac{1}{\Gamma(\nu)}\int_s^b (u-s)^{\nu-1} f_i(u, y(u), {}^c\mathbb{D}^\alpha_{a+}y(u))du\right)ds,
\end{aligned}$$

where

$$\Delta(t;\mu) = \frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1}p^{-1}(s)ds = I^\mu_{a+}p^{-1}(t).$$

According to Lemma 4, $y_i \in \mathcal{AC}^2([a,b])$ and ${}^c\mathbb{D}^\alpha_{a+}y_i \in \mathcal{AC}([a,b])$ for all $i \in \mathbb{N}$. In view of the assumption $(\mathcal{P}_1)$ and Lebesgue's dominated convergence theorem, we get that $\lim_{i\to\infty} y_i(t)$ exists for all $t \in [a,b]$ which implies that $y \in c$.

Let $(\mathcal{T}_i y_i)(t) = y_i(t)$ be operators defined, for all $i \in \mathbb{N}$ and $t \in [a,b]$, by

$$\begin{aligned}
\mathcal{T}_i y_i(t) = &\frac{1}{\Gamma(\mu)}\int_a^t p^{-1}(s)(t-s)^{\mu-1}\left(I^\nu_{b-}f_i(s, y(s), {}^c\mathbb{D}^\alpha_{a+}y(s))\right)ds \\
&- \frac{\Delta(t;\mu)}{\Delta(b;\mu)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\left(I^\nu_{b-}f_i(s, y(s), {}^c\mathbb{D}^\alpha_{a+}y(s))\right)ds \\
&+ \frac{C - A - B(b-a)}{\Delta(b;\mu)}\Delta(t;\mu) + A + B(t-a)
\end{aligned} \tag{11}$$

and its fractional derivatives of orders $0 < \alpha \leq 1$ for all $i \in \mathbb{N}$, using the third statement in Lemma 1, which can be evaluated as

$$\begin{aligned}({}^{c}\mathbb{D}^{\alpha}_{a+}\mathcal{T}_i y_i)(t) = & \frac{1}{\Gamma(\mu-\alpha)}\int_a^t p^{-1}(s)(t-s)^{\mu-\alpha-1}\big(I^{\nu}_{b-}f_i(s,y(s),{}^{c}\mathbb{D}^{\alpha}_{a+}y(s))\big)ds \\ & - \frac{\Delta(t;\mu-\alpha)}{\Delta(b;\mu)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I^{\nu}_{b-}f_i(s,y(s),{}^{c}\mathbb{D}^{\alpha}_{a+}y(s))\big)ds \\ & + \frac{C-A-B(b-a)}{\Delta(b;\mu)}\Delta(t;\mu-\alpha) + \frac{B(t-a)^{1-\alpha}}{\Gamma(2-\alpha)}. \end{aligned} \tag{12}$$

As above, it is clear that $\mathcal{T}_i \in \mathcal{AC}^2([a,b])$ and ${}^{c}\mathbb{D}^{\alpha}_{a+}\mathcal{T}_i \in \mathcal{AC}([a,b])$ for all $i \in \mathbb{N}$. In view of the assumption $(\mathcal{P}_1)$ and Lebesgue's dominated convergence theorem, we get that

$$\lim_{i\to\infty}(\mathcal{T}_i y_i)(t) \quad \text{and} \quad \lim_{i\to\infty}({}^{c}\mathbb{D}^{\alpha}_{a+}\mathcal{T}_i y_i)(t)$$

exist for all $t \in [a,b]$, which allows us to define the sequence operator $\mathcal{T}\colon \mathbb{B} \to \mathbb{B}$ where $(\mathcal{T}y)(t) = ((\mathcal{T}_i y_i)(t))_{i\in\mathbb{N}}$.

Lemma 8. *Under the hypotheses $(\mathcal{P}_1)$–$(\mathcal{P}_4)$, the operator $\mathcal{T}\colon \mathbb{B} \to \mathbb{B}$ is bounded and continuous on the closed ball.*

$$\mathfrak{B}_r = \{y \in \mathbb{B}\colon \|y\| \leq r, y_i \text{ satisfies the boundary conditions (2)}\} \tag{13}$$

with fixed radius r satisfying the inequality $r \geq (\mathrm{Q}_F + \mathrm{Q})/(1-\mathrm{Q}_w)$ provided that $\mathrm{Q}_w < 1$ where Q_F, Q_w and Q are given in (8)–(10), respectively.

Proof. It is easy using the hypothesis $(\mathcal{P}_2)$ to show that

$$\begin{aligned}|f_i(t,y,z)| &\leq |f_i(t,y,z) - f_i(t,0,0)| + |f_i(t,0,0)| \\ &\leq w_i(t)(\|y\|_c + \|z\|_c) + F_i(t).\end{aligned}$$

For all $y \in \mathfrak{B}_r$, in view of Remark 3, we get

$$\begin{aligned}|\mathcal{T}_i y_i(t)| \leq & \frac{2}{\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I^{\nu}_{b-}|f_i(s,y(s),{}^{c}\mathbb{D}^{\alpha}_{a+}y(s))|\big)ds + \mathrm{N}_1 \\ \leq & \frac{2}{\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I^{\nu}_{b-}(w_i(s)(|y(s)| + |{}^{c}\mathbb{D}^{\alpha}_{a+}y(s)|) + F_i(s))\big)ds + \mathrm{N}_1 \\ = & 2(\mathrm{E}_b(F_i;\mu,\nu) + \mathrm{E}_b(w_i;\mu,\nu)(\|y\|_c + \|{}^{c}\mathbb{D}^{\alpha}_{a+}y\|_c)) + \mathrm{N}_1.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|\mathcal{T}y\|_c &\leq 2\sup_{i\in\mathbb{N}}(\mathrm{E}_b(F_i;\mu,\nu) + \mathrm{E}_b(w_i;\mu,\nu)\|y\|) + \mathrm{N}_1 \\ &\leq 2(\mathrm{E}_b(F;\mu,\nu) + \mathrm{E}_b(w;\mu,\nu)\|y\|) + \mathrm{N}_1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|{}^{c}\mathbb{D}^{\alpha}\mathcal{T}y\|_c \leq & \mathrm{E}(F;\mu-\alpha,\nu) + \mathrm{E}(w;\mu-\alpha,\nu)\|y\|) \\ & + \frac{\mathrm{E}(1;\mu-\alpha,0)}{\mathrm{E}_b(1;\mu,0)}(\mathrm{E}_b(F;\mu,\nu) + \mathrm{E}_b(w;\mu,\nu)\|y\|) + \frac{\mathrm{N}_2}{\mathrm{E}_b(1;\mu,0)}\mathrm{E}_b(1;\mu-\alpha,0) + \mathrm{N}_3.\end{aligned}$$

According to the definition of the norm in the space $\mathbb{B}$, we find that

$$\|\mathcal{T}y\| = \|\mathcal{T}y\|_c + \|{}^{c}\mathbb{D}^{\alpha}\mathcal{T}y\|_c \leq \mathrm{Q}_F + \mathrm{Q}\|y\| \leq \mathrm{Q}_F + \mathrm{Q}_w r + \mathrm{Q} \leq r.$$

Therefore, the operator $\mathcal{T}$ is bounded and $\mathcal{T}\mathcal{B}_r \subseteq \mathcal{B}_r$.

In order to prove the continuity on $\mathcal{B}_r$, let $\delta > 0$ exist for all $\epsilon > 0$ and $y, z \in \mathcal{B}_r$ such that $\|y - z\| < \delta$ and $\delta < \epsilon/\mathrm{Q}$. Then,

$$\begin{aligned}
|\mathcal{T}_i y_i(t) - \mathcal{T}_i z_i(t)| \leq &\ \frac{2}{\Gamma(\mu)} \int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I_{b-}^{\nu} |f_i(s, y(s), {}^c\mathbb{D}_{a+}^{\alpha} y(s)) - f_i(s, z(s), {}^c\mathbb{D}_{a+}^{\alpha} z(s))|\big)ds \\
\leq &\ \frac{2}{\Gamma(\mu)} \int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I_{b-}^{\nu} (w_i(s)(|y(s) - z(s)| + |{}^c\mathbb{D}_{a+}^{\alpha}(y(s) - z(s))|))\big)ds \\
= &\ 2\mathrm{E}_b(w_i; \mu, \nu)(\|y - z\|_c + \|{}^c\mathbb{D}_{a+}^{\alpha}(y - z)\|_c) \\
= &\ 2\mathrm{E}_b(w_i; \mu, \nu)\|y - z\|,
\end{aligned}$$

which implies that

$$|\mathcal{T}y - \mathcal{T}z\|_c = \sup_{i \in \mathbb{N}} |\mathcal{T}_i y_i(t) - \mathcal{T}_i z_i(t)| \leq 2\mathrm{E}_b(w; \mu, \nu)\|y - z\|.$$

In the same way, we can deduce that

$$\|{}^c\mathbb{D}^{\alpha}\mathcal{T}y - {}^c\mathbb{D}^{\alpha}\mathcal{T}z)\|_c \leq \mathrm{E}(w; \mu - \alpha, \nu)\|y - z\| + \frac{\mathrm{E}(1; \mu - \alpha, 0)}{\mathrm{E}_b(1; \mu, 0)}\mathrm{E}_b(w; \mu, \nu)\|y - z\|.$$

Hence, we have

$$\|\mathcal{T}y - \mathcal{T}z\| \leq \mathrm{Q}\|y - z\| < \epsilon.$$

This ends the proof. $\square$

Lemma 9. *Under the hypotheses ($\mathcal{P}_1$)–($\mathcal{P}_4$), the operator $\mathcal{T}\colon \mathbb{B} \to \mathbb{B}$ is equicontinuous on the interval $[a, b]$.*

Proof. Let $a \leq t_1 < t_2 \leq b$. Then, we see that

$$\begin{aligned}
|\Delta(t_2, \mu) - \Delta(t_1, \mu)| &= \left| \frac{1}{\Gamma(\mu)} \int_a^{t_2} p^{-1}(s)(t_2 - s)^{\mu-1} ds - \frac{1}{\Gamma(\mu)} \int_a^{t_1} p^{-1}(s)(t_1 - s)^{\mu-1} ds \right| \\
&\leq \frac{1}{\Gamma(\mu)} \int_a^{t_1} |p^{-1}(s)|\Big((t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1}\Big)ds + \frac{1}{\Gamma(\mu)} \int_{t_1}^{t_2} |p^{-1}(s)|(t_2 - s)^{\mu-1} ds.
\end{aligned}$$

Since $p : [a, b] \to \mathbb{R}+$ is an absolutely continuous function, then $p^{-1}(t)$ is also absolutely continuous on $[a, b]$ and there exists a positive constant P such that $P = \max_{t \in [a,b]} p^{-1}(t)$. This implies that

$$\begin{aligned}
|\Delta(t_2, \mu) - \Delta(t_1, \mu)| &\leq \frac{P}{\Gamma(\mu)} \left[\int_a^{t_1} \Big((t_2 - s)^{\mu-1} - (t_1 - s)^{\mu-1}\Big)ds + \int_{t_1}^{t_2} (t_2 - s)^{\mu-1} ds \right] \\
&= \frac{P}{\Gamma(\mu + 1)}[(t_2 - a)^{\mu} - (t_1 - a)^{\mu}],
\end{aligned}$$

which uniformly tends to zero as $t_1 \to t_2$. Since $1 < \mu < 2$ and $0 < \alpha < 1$, then $0 < \mu - \alpha < 2$ and so we have three cases:

Case I. If $0 < \mu - \alpha < 1$, then we obtain

$$\begin{aligned}
&|\Delta(t_2, \mu-\alpha) - \Delta(t_1, \mu-\alpha)| \\
&\leq \frac{P}{\Gamma(\mu-\alpha)}\left[\int_a^{t_1}\Big((t_1-s)^{\mu-\alpha-1} - (t_2-s)^{\mu-\alpha-1}\Big)ds + \int_{t_1}^{t_2}(t_2-s)^{\mu-\alpha-1}ds\right] \\
&= \frac{P}{\Gamma(\mu-\alpha+1)}\left[2(t_2-t_1)^{\mu-\alpha} + (t_1-a)^{\mu-\alpha} - (t_2-a)^{\mu-\alpha}\right] \\
&\leq \frac{2P}{\Gamma(\mu-\alpha+1)}(t_2-t_1)^{\mu-\alpha} \to 0 \quad as \quad t_1 \to t_2;
\end{aligned}$$

Case II. If $\mu - \alpha = 1$, then we obtain

$$|\Delta(t_2, 1) - \Delta(t_1, 1)| \leq \frac{P}{\Gamma(\mu-\alpha)}\int_{t_1}^{t_2} ds = P(t_2-t_1) \to 0 \quad as \quad t_1 \to t_2;$$

Case III. If $1 < \mu - \alpha < 2$, then we obtain

$$\begin{aligned}
&|\Delta(t_2, \mu-\alpha) - \Delta(t_1, \mu-\alpha)| \\
&\leq \frac{P}{\Gamma(\mu-\alpha)}\left[\int_a^{t_1}\Big((t_2-s)^{\mu-\alpha-1} - (t_1-s)^{\mu-\alpha-1}\Big)ds + \int_{t_1}^{t_2}(t_2-s)^{\mu-\alpha-1}ds\right] \\
&= \frac{P}{\Gamma(\mu-\alpha+1)}\left[(t_2-a)^{\mu-\alpha} - (t_1-a)^{\mu-\alpha}\right] \to 0 \quad as \quad t_1 \to t_2.
\end{aligned}$$

These can be used by applying the same technique and the results in Lemma 5 to show that

$$\begin{aligned}
&|\mathcal{T}_i y_i(t_2) - \mathcal{T}_i y_i(t_1)| \\
&\leq \frac{1}{\Gamma(\mu)}\int_a^{t_1} p^{-1}(s)|(t_2-s)^{\mu-1} - (t_1-s)^{\mu-1}|\big(I_{b-}^{\nu}|f_i(s, y(s), {}^c\mathbb{D}_{a+}^{\alpha}y(s))|\big)ds \\
&+ \frac{1}{\Gamma(\mu)}\int_{t_1}^{t_t} p^{-1}(s)(t_2-s)^{\mu-1}\big(I_{b-}^{\nu}|f_i(s, y(s), {}^c\mathbb{D}_{a+}^{\alpha}y(s))|\big)ds \\
&+ \frac{\Delta(t_2,\mu) - \Delta(t_1,\mu)}{\Delta(b,\mu)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I_{b-}^{\nu}(|f_i(s, y(s), {}^c\mathbb{D}_{a+}^{\alpha}y(s))|)\big)ds \\
&+ \frac{\mathrm{N}_3}{\Delta(b;\mu)}(\Delta(t_2,\mu) - \Delta(t_1,\mu)) + |B_i|(t_2-t_1) \\
&\leq \frac{A(E(F_i;\mu,\nu) + E(w_i;\mu,\nu)\|y\|)}{\Gamma(\mu)\Gamma(\nu+1)}\left(\int_a^{t_2} s^{\nu}(t_2-s)^{\mu-1}ds - \int_a^{t_1} s^{\nu}(t_1-s)^{\mu-1}ds\right) \\
&+ \frac{\Delta(t_2,\mu) - \Delta(t_1,\mu)}{\Delta(b,\mu)\Gamma(\mu)}\int_a^b p^{-1}(s)(b-s)^{\mu-1}\big(I_{b-}^{\nu}(|f_i(s, y(s), {}^c\mathbb{D}_{a+}^{\alpha}y(s))|)\big)ds \\
&+ \frac{\mathrm{N}_3}{\Delta(b;\mu)}(\Delta(t_2,\mu) - \Delta(t_1,\mu)) + |B_i|(t_2-t_1),
\end{aligned}$$

which uniformly tends to zero as $t_1 \to t_2$. Similarly, it can be proven that

$$|({}^c\mathbb{D}_{a+}^{\alpha}\mathcal{T}_i y_i)(t_2) - ({}^c\mathbb{D}_{a+}^{\alpha}\mathcal{T}_i y_i)(t_1)| \to 0 \quad as \quad t_1 \to t_2,$$

which concludes that the operator $\mathcal{T}$ is equicontinuous on the interval $[a, b]$. $\square$

Theorem 3. *Under the hypotheses ($\mathcal{P}_1$)–($\mathcal{P}_4$), the infinite systems (1) and (2) have at least one solution in the closed ball $\mathcal{B}_r$ defined in (13) provided that $\mathrm{Q}_w < 1$ where Q_w is given in (9).*

Proof. Based on the results obtained in the previous two Lemmas, it is sufficient to calculate the measure of noncompactness $\beta_{\mathbb{B}}(\mathcal{T}\mathcal{B}_r)$ given in (7). In order to do this: Let $i, j \in \mathbb{N}$ be large enough; by using the hypotheses $(\mathcal{P}_3)$ and $(\mathcal{P}_4)$, we get

$$\begin{aligned}
|\mathcal{T}_i y_i(t) - \mathcal{T}_i y_j(t)| &\leq \frac{2}{\Gamma(\mu)} \int_a^b p^{-1}(s)(t-s)^{\mu-1}\left(I_{b-}^{\nu} |f_i(s, y(s),\ {}^c\mathbb{D}_{a+}^{\alpha} y(s)) - f_j(s, y(s),\ {}^c\mathbb{D}_{a+}^{\alpha} y(s))|\right) ds \\
&\leq \frac{2}{\Gamma(\mu)} \int_a^b p^{-1}(s)(t-s)^{\mu-1}\Bigg(I_{b-}^{\nu}\Bigg(\sup_{m,n \geq \min\{i,j\}} |F_m(s) - F_n(s)| \\
&\quad + \sup_{k\in\mathbb{N}} w_k(s) \sup_{m,n\geq \min\{i,j\}} (|y_m(s) - y_n(s)| + |\ {}^c\mathbb{D}_{a+}^{\alpha} y_m(s) -\ {}^c\mathbb{D}_{a+}^{\alpha} y_n(s)|)\Bigg)\Bigg) ds.
\end{aligned}$$

We first compute the quantity $\beta_1(\mathcal{T}\mathcal{B}_r)$ as follows:

$$\begin{aligned}
\beta_1(\mathcal{T}\mathcal{B}_r) &= \lim_{\ell\to\infty}\left\{\sup_{y\in\mathcal{B}_r}\left\{\sup_{i,j>\ell}\left\{\max_{t\in[a,b]} |\mathcal{T}_i y_i(t) - \mathcal{T}_i y_j(t)|\right\}\right\}\right\} \\
&\leq 2E(1;\mu,\nu) \lim_{\ell\to\infty} \sup_{m,n\geq\min\{i,j\}>\ell} \max_{t\in[a,b]} |F_m(t) - F_n(t)| \\
&\quad + 2E(w;\mu,\nu) \lim_{\ell\to\infty} \sup_{m,n\geq\min\{i,j\}>\ell} \max_{t\in[a,b]} (|y_m(t) - y_n(t)| + |\ {}^c\mathbb{D}_{a+}^{\alpha} y_m(t) -\ {}^c\mathbb{D}_{a+}^{\alpha} y_n(t)|) \\
&\leq 2E_b(w;\mu,\nu)\beta_{\mathbb{B}}(\mathcal{B}_r).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\beta_2(\ {}^c\mathbb{D}_{a_+}^{\alpha}\mathcal{T}\mathcal{B}_r) &= \lim_{\ell\to\infty}\left\{\sup_{y\in\mathcal{B}_r}\left\{\sup_{i,j>\ell}\left\{\max_{t\in[a,b]} |\ {}^c\mathbb{D}_{a+}^{\alpha}\mathcal{T}_i y_i(t) -\ {}^c\mathbb{D}_{a+}^{\alpha}\mathcal{T}_i y_j(t)|\right\}\right\}\right\} \\
&\leq \left(E_b(w;\mu-\alpha,\nu) + \frac{E(1;\mu-\alpha,0)}{E_b(1;\mu-\alpha,0)} E_b(w;\mu-\alpha,\nu)\right)\beta_{\mathbb{B}}(\mathcal{B}_r),
\end{aligned}$$

which concludes that

$$\beta_{\mathbb{B}}(\mathcal{T}\mathcal{B}_r) \leq \mathrm{Q}\beta_{\mathbb{B}}(\mathcal{B}_r).$$

We complete the proof using two different theorems as follows:

Darbo's Theorem: In view of Darbo's Theorem 1 and the assumption $\mathrm{Q}_w < 1$, the infinite system of the fractional Sturm–Liouville operators (1) and (2) has at least one solution in $\mathcal{B}_r$.

Meir–Keeler Theorem: Suppose that for all $\epsilon > 0$, there exists $\delta >$ such that $\delta < \epsilon$ $(1 - \mathrm{Q}_w)/\mathrm{Q}_w$ and $\epsilon < \beta_{\mathbb{B}}(\mathcal{B}_r) < \epsilon + \delta \Rightarrow \beta_{\mathbb{B}}(\mathcal{T}\mathcal{B}_r) < \epsilon$. In view of the Meir–Keeler Theorem 2 and the assumption $\mathrm{Q}_w < 1$, the infinite system of fractional Sturm–Liouville operators (1) and (2) has at least one solution in $\mathcal{B}_r$.

The proof is done. □

5. Illustrative Example

Let us introduce the following example:

$$(\ {}^c\mathbb{D}_{\pi^-}^{\frac{1}{2}}(p(t)\ {}^c\mathbb{D}_{0+}^{\frac{3}{2}} y_i(t)))(t) = f_i(t, y(t), (\ {}^c\mathbb{D}_{a+}^{\alpha} y)(t) \qquad t\in[0,\pi],\ i\in\mathbb{N} \tag{14}$$

with the boundary conditions

$$y_i(0) = y_i(\pi) = \left(\frac{1}{2}\right)^i \qquad y_i'(0) = 0, \quad i\in\mathbb{N}. \tag{15}$$

$\nu = 1/2, a = 0, b = \pi, p(t) = 1/\sqrt{\pi + t}, \mu = 3/2, \alpha = 1/2$. Additionally, we take

$$f_i(t,y,z) = \sum_{n=1}^{i} \frac{e^{-nt}\cos^2(nt)}{(t+1)^n n!} + \frac{te^{-it}}{3(2\pi - t)^3(t+i)^3} \sum_{n=1}^{i} \frac{\sin^{2n}(n\pi t)}{(n+t)^4}(y_{n+1} - y_n + \cos(y_n))$$
$$+ \frac{te^{-it}}{2(2\pi - t)^3(t+i)^3} \sum_{n=i+1}^{\infty} \frac{\cos^{2n}(n\pi t)}{(n+t)^4}(z_{n-i+1} - z_{n-i}).$$

It is obvious that the partial derivatives of f_i with respect to t are continuous and so $f_i : [0,\pi] \times c \times c \to \mathbb{R}$ are absolutely continuous functions for all $i \in \mathbb{N}$ and $t \in [0,\pi]$, which is fully compatible with the assumption $(\mathcal{P}_1)$. In order to verify the assumption $(\mathcal{P}_2)$, let $y, y', z, z' \in \mathbb{B}$, noting that

$$|\cos x - \cos y| = 2\left|\sin\frac{x+y}{2}\sin\frac{x-y}{2}\right| \le |x-y|.$$

Then, we have

$$\begin{aligned}|f_i(t,y,z) - f_i(t,y',z')| &\le \frac{te^{-it}}{3(2\pi - t)^3(t+i)^3} \sum_{n=1}^{i} \frac{1}{n^4}(|y_{n+1} - y'_{n+1}| + 2|y_n - y'_n|)\\ &\quad + \frac{te^{-it}}{2(2\pi - t)^3(t+i)^3} \sum_{n=i+1}^{\infty} \frac{1}{n^4}(|z_{n-i+1} - z'_{n-i+1}| + |z_{n-i} - z'_{n-i})\\ &\le w_i(t) \sup_{j\in\mathbb{N}}\{|y_j - y'_j| + |z_j - z'_j|\},\end{aligned}$$

where

$$w_i(t) = \frac{te^{-it}}{(2\pi - t)^3(t+i)^3}\zeta(4) = \frac{\pi^4 te^{-it}}{90(2\pi - t)^3(t+i)^3}$$

and $\zeta(\cdot)$ is the Riemann zeta function.

Now, let $i, j \in \mathbb{N}$ be large enough and $\ell_1 = \min\{i,j\}, \ell_2 = \max\{i,j\}$, then we can find that

$$|f_i(t,y,z) - f_j(t,y,z)| \le |F_i(t) - F_j(t)| + w_k(t) \sum_{n=\ell_1+1}^{\ell_2} \frac{1}{n^4}(|y_{n+1} - y_n| + |z_{n-j+1} - z_{n-j}|),$$

where $w_k(t) = \max\{w_i(t), w_j(t)\}$ and

$$F_i(t) = f_i(t,0,0) = \sum_{n=1}^{i} \frac{e^{-nt}\cos^2(nt)}{(t+1)^n n!} + \frac{te^{-it}}{3(2\pi - t)^3(t+i)^3} \sum_{n=1}^{i} \frac{\sin^{2n}(n\pi t)}{(n+t)^4},$$

which is fully coincident with the assumption $(\mathcal{P}_3)$.

It is easy to see that

$$w(t) = \sup_{i\in\mathbb{N}} w_i(t) = \frac{\pi^4 t}{90(2\pi - t)^3(t+1)^3} \quad \textit{and} \quad w = \max_{t\in[0,\pi]} w(t) = \frac{\pi^2}{90(\pi+1)^3} = w(\pi).$$

Additionally, we have

$$\begin{aligned}F(t) = \sup_{i\in\mathbb{N}} F_i(t) &= \sum_{n=1}^{\infty} \frac{e^{-nt}\cos^2(nt)}{(t+1)^n n!} + \frac{t}{3(2\pi - t)^3(t+1)^3} \sum_{n=1}^{\infty} \frac{1}{(n+t)^4}\\ &= \frac{1}{2}\exp\left[\frac{e^{-t}}{1+t}\right] + \frac{1}{4}\exp\left[\frac{e^{-t}\cos(2t)}{1+t}\right]\cos\left[\frac{e^{-t}\sin(2t)}{1+t}\right] - 1 + \frac{1}{3}w(t)\end{aligned}$$

and

$$F = \max_{t\in[0,\pi]} F(t) = \frac{3}{4}e - 1 + \frac{1}{3}w(\pi) \sim 1.03923.$$

By carrying out of simple calculations in Mathematica 11, we can estimate the following:

$$\begin{aligned}
\mathrm{E}(1;\mu-\alpha,0) &= \max_{t\in[0,\pi]} \mathrm{E}(1;1,0)(t) = \max_{t\in[0,\pi]} \int_0^t \sqrt{\pi+s}ds \\
&= \max_{t\in[0,\pi]} \frac{2}{3}\left[(\pi+t)^{\frac{3}{2}} - \pi^{\frac{3}{2}}\right] \\
&= \frac{2\pi\sqrt{\pi}}{3}(2\sqrt{2}-1) \sim 6.78752, \\
\mathrm{E}_\pi(1;\mu,0) &= \frac{1}{2}\pi^2\sqrt{\pi} \sim 8.74671, \\
\mathrm{E}_\pi(w;\mu,\nu) &= \frac{1}{\Gamma(\mu)}\int_0^\pi p^{-1}(s)(\pi-s)^{\mu-1}\left(I_{b^-}^\nu w(s)\right)ds \sim 0.0133265, \\
\mathrm{E}_\pi(F;\mu,\nu) &= \frac{1}{\Gamma(\mu)}\int_0^\pi p^{-1}(s)(\pi-s)^{\mu-1}\left(I_{b^-}^\nu F(s)\right)ds \sim 22.0125, \\
\mathrm{E}(w;\mu-\alpha,\nu) &= \frac{1}{\Gamma(\mu-\alpha)}\int_0^\pi p^{-1}(s)(\pi-s)^{\mu-\alpha-1}\left(I_{b^-}^\nu w(s)\right)ds \sim 0.0135021, \\
\mathrm{E}(F;\mu-\alpha,\nu) &= \frac{1}{\Gamma(\mu-\alpha)}\int_0^\pi p^{-1}(s)(\pi-s)^{\mu-\alpha-1}\left(I_{b^-}^\nu w(s)\right)ds \sim 15.0293.
\end{aligned}$$

These lead to $Q_w \sim 0.0683993 < 1$, which satisfies all assumptions of the theorem. Therefore, the infinite system of (1) and (2) has at least one solution in $[0,\pi]$.

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