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Quantum Temporal Winds: Turbulence in Financial Markets

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Abstract: This paper leverages turbulence theory from physics to examine the similarities and differences between financial market volatility and turbulent phenomena on a statistical physics level. By drawing analogies between the dynamics of financial markets and fluid turbulence, an innovative analytical framework has been developed to enhance our understanding of the complexity inherent in financial markets. The research methodology involves a comparative analysis of several national stock market indices and simulated turbulent velocity time series, with a particular focus on key statistical properties such as probability distributions, correlation structures, and power spectral densities. Furthermore, a financial market capital flow model has been established, and corresponding solutions have been proposed. Through computational simulations and data analysis, it was discovered that financial market volatility shares some statistical characteristics with turbulence, yet there are significant differences in the shape of probability distributions and the timescales of correlations. This indicates that although financial markets exhibit patterns similar to turbulence, as a multivariate-driven complex system, their behavioral patterns do not completely correspond to natural turbulence phenomena, highlighting the limitations of directly applying turbulence theory to financial market analysis. Additionally, the study explores the use of Bézier curves to simulate market volatility and, based on these analyses, formulates trading strategies that demonstrate practical applications in risk management. This research provides fresh perspectives for the fields of financial market theory and econophysics, offering new insights into the complexity of financial markets and the prevention and management of financial risks.

Keywords: quantum finance; financial markets; turbulence; time series analysis

MSC: 81P99; 91G15

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1. Introduction

Econophysics, as an interdisciplinary field, predominantly employs theories and methods from physics to decipher and forecast the behavior and dynamics of financial markets. The evolution of this field signifies the convergence and intersection between traditional finance and natural sciences, particularly the application of statistical physics, nonlinear dynamics, and complex systems theory from physics. In recent years, econophysics has made notable strides in understanding market volatility, the distributional characteristics of asset prices, and the structure and stability of financial complex networks.

As a complex system, the financial market exhibits behaviors that are replete with nonlinearity, uncertainty, and intricate dynamic correlations. These patterns are particularly evident in the fluctuations of financial time series such as stock prices, exchange rates, and interest rates. In the natural sciences, turbulence is a prevalent nonlinear phenomenon that has long posed a challenge for physicists and engineers. The complexity and unpredictability displayed by turbulence share many similarities with the fluctuations observed in financial markets. Figure 1 illustrates the pressure distribution of a fluid in motion, while Figure 2 displays the fluctuation pattern of financial market prices. This paper proposes a framework based on the inflection points of price fluctuations, constructing an upper and lower Bézier curve track. When prices move to these extremities, they exhibit a turbulent reversion phenomenon akin to that observed in fluid turbulence.

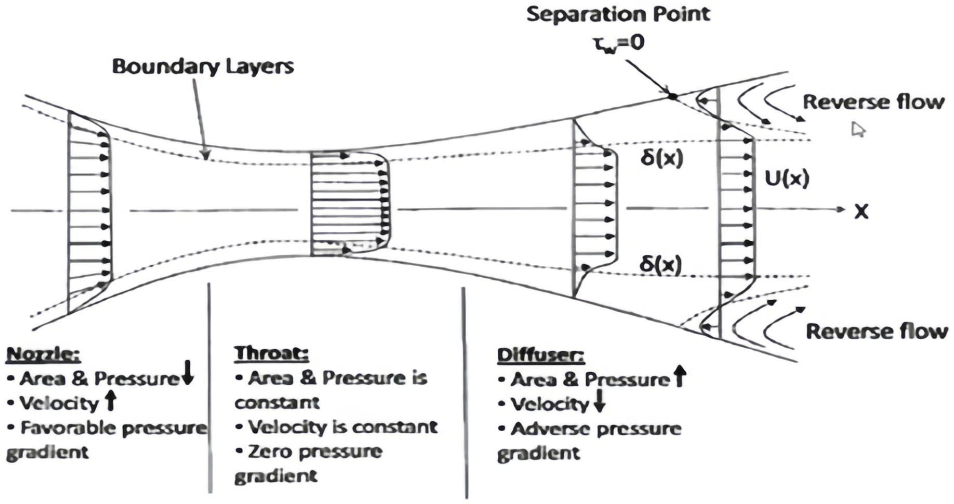

Figure 1. Fluid force diagram (https://doi.org/10.1115/1.3448767 (accessed on 1 December 2023)).

Figure 2. Financial market price chart.

Advancements in the study of turbulence have provided new perspectives and tools for this phenomenon, offering theories and methods that can be applied to analyze the dynamic behavior of financial markets. In particular, turbulence theory offers a novel framework for time series analysis, enhancing our understanding of the complex fluctuation structures within market time series data. The specific contributions of this paper are as follows:

We first introduce the concept of temporal turbulence, positing that time-space can be quantized. The paper then details the construction process of the temporal turbulence model, including theoretical assumptions, mathematical formulations, and statistical simulation methods. By comparing the statistical properties of financial markets and turbulence, we investigate the similarities between the two, analyzing the complex dynamics of financial markets from the perspective of turbulence theory. The Standard & Poor's 500 Index is chosen as a representative case, with its time series compared to the turbulent velocity time series.

Subsequently, the paper presents the empirical analysis results of the model, discovering certain degrees of similarity in probability distributions, correlations, and power spectra between the two, both exhibiting highly complex nonlinear dynamical behavior. However, differences also exist, such as in the shapes of probability distributions and the temporal scales of correlations. Moreover, the paper constructs a financial market trading strategy based on Bézier curves—a mathematical tool for describing turbulence—demonstrating that the application of Bézier curves in financial markets, combined with time series analysis, observation of price fluctuations, and changes in curve curvature, can form a dynamic model for analyzing market volatility. This model aids investors in assessing market trends and risks, thereby informing their trading strategies.

Through an in-depth study of market dynamics, this paper aims to provide novel insights into understanding the complexity of financial markets and to offer new strategies for the prevention and management of financial risk. This research clarifies the operating mechanisms of another complex system from an already understood system, serving as a reference for the application of econophysics methods in the analysis of financial markets. It shows that similar comparative analysis methods can be adopted in the field of econophysics. However, the quantitative correspondence between the two is limited, and further research is required to ascertain the dynamical mechanisms linking them.

This paper aims to explore the application of physical turbulence theory to the analysis of financial market dynamics, with a particular focus on deepening our understanding of financial market complexities through a statistical physics perspective. Initially, the paper introduces the interdisciplinary field of financial physics, explaining how this field has helped us understand market volatility, asset price distributions, and the structure and stability of financial networks. Subsequently, through a literature review, the paper establishes a theoretical and methodological foundation, thoroughly discussing the key concepts of quantum finance and financial market turbulence theory, especially the quantization of time and space. In the following sections, the paper constructs a financial market turbulence model and explores key mathematical equations used to describe the dynamics of financial markets, including continuity equations and momentum conservation equations modeled after the Navier–Stokes equations from fluid dynamics. The paper also applies these models and theories to real financial market data and turbulence data simulated through computational fluid dynamics (CFD), comparing the statistical physical properties of financial markets with fully developed turbulence. The final section summarizes the research findings and discusses the implications of applying turbulence theory to financial markets, debating how this approach can provide new insights for the prevention and management of financial risks, and reflecting on the broader impact of this study on the fields of economic physics and financial theory.

2. Literature Review

Quantum finance, an innovative field that merges quantum mechanics with financial theory, offers novel insights into the complexities of financial markets that challenge traditional economic models. This integration was systematically introduced by Baaquie (2007) [1] in his seminal work, "Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates," establishing a foundational framework for applying quantum mechanical principles to financial instruments. Subsequent theoretical advancements by Nakayama (2009) [2] and Schaden (2010) [3] expanded this foundation by introducing concepts such as gravity duals and exploring their implications for interest rates and bonds. Baaquie (2009) [4] further explored additional theoretical frameworks, thereby enhancing the understanding of interest rates in quantum finance.

These theoretical explorations were supported by empirical investigations, such as those conducted by Baaquie and Yang (2009) [5], which validated the efficacy of quantum models in capturing the dynamics of financial markets, particularly for interest rate predictions. Additionally, Agrawal and Sharda (2010) [6] extended the quantum approach to human decision-making processes, offering a fresh perspective on behavioral finance by modeling decision-making under uncertainty with quantum probabilistic frameworks. Baaquie (2018) [7] also explored the implications of bonds with index-linked stochastic coupons in quantum finance.

Lee (2021) [8] introduced the Quantum Finance Forecast System, utilizing a quantum anharmonic oscillator model for modeling quantum price levels. The application of quantum mechanics to finance has been greatly enhanced by developments in quantum computing, which provides the computational power necessary to handle the complex calculations required by quantum financial models. The work of Zoufal, Lucchi, and Woerner (2019) [9], along with the theoretical exploration by Bouland et al. (2020) [10], highlights the role of quantum generative adversarial networks and the broader prospects and challenges of quantum finance.

Further theoretical insights have been provided by Arraut (2023) [11], discussing gauge symmetries and the Higgs mechanism in quantum finance, which opens up new avenues for financial theories. Additionally, Choi et al. (2023) [12] from Hanyang University have emphasized the implications of quantum finance in their extensive review, broadening the academic discussion on this topic.

Technological advancements in quantum computing, such as those by Braine, Egger, Glick, and Woerner (2021) [13], further discussed in a survey by Herman et al. (2022) [14,15], emphasize the development of quantum algorithms for mixed binary optimization and their broader applications in finance. These technologies facilitate more efficient market analyses and risk assessments, as also explored by Lee and Constantinides (2023) [16] in real-life finance applications.

Practical implementations of quantum technologies in finance, as explored by Canabarro et al. (2022) [17], Wang and Lee (2022) [18], and further discussed by Naik et al. (2023) [19] and Chang et al. (2023) [20], underscore the ongoing integration of quantum computing into financial practices, promising to enhance the robustness and efficiency of financial markets.

The study of financial market turbulence through the lens of quantum mechanics and fluid dynamics offers profound insights into market behaviors during periods of high volatility. Gao, Cai, and Wang (2012) [21] utilized multifractal analysis to model the hierarchical structure of stock price fluctuations. This approach is complemented by the work of Gustafson (2011, 2012) [22,23] and Seyfert (2016) [24], who explore the statistical and epistemic challenges within financial markets.

Recent contributions by Li and Liang (2020) [25] on extending option pricing models, Widdows and Bhattacharyya (2024) [26] on quantum financial modeling, and insights by Evanoff (2009) [27] on globalization and systemic risk further reinforce the transformative potential of quantum finance. Additional perspectives by Gkillas and Longin (2018) [28] on Bitcoin, and by Zhou et al. (2019) [29] on marketing agility under market turbulence, alongside a review of evidence for financial contagion (2015) [30], contribute to an understanding of complex economic environments.

3. Theoretical Basis

In this section, we delve into the fundamental concepts of quantum finance and financial market turbulence theory, with a particular focus on the quantization of time and space, which provides a precise analytical framework for financial market analysis. The process of quantization involves dividing time and space into discrete minimum units, a concept borrowed from physics' treatment of atoms and fundamental particles. By simulating the dynamics of these minimal units, we can model and analyze the micro-fluctuations and macro trends of the market, better capturing the nonlinear characteristics and complexities of market behavior. In financial markets, particle-based quantization allows us to observe and record market changes in a discrete manner, revealing important micro-dynamics that might be overlooked in traditional continuous models. Moreover, applying these theories to actual financial data analysis supports our use of advanced mathematical tools and physical models to predict and understand potential market behaviors, offering a new perspective on mastering market volatility and unpredictability. Therefore, the theoretical discussion in this section not only deepens our theoretical understanding but also provides a solid scientific foundation for subsequent empirical research and model applications, ensuring the rigor of the research methodology and the reliability of the results.

3.1. Quantization of Time

Figure 3 exhibits a candlestick chart detailing the volatility of market prices within a financial context. The horizontal axis denotes the chronological sequence, while the vertical axis quantifies the price movements of the asset, reflecting the market's volatility and the vigor of trade activities over the observed period. The fluctuating curves depicted in the chart symbolize the market's depth and liquidity, serving to concretize the price variations across time and in relation to another dimension such as trading volume or volatility rates. The lines positioned at the apex of the chart may represent the relative strength index (RSI) or alternative types of oscillators, which gauge the overbought or oversold status of the asset. This graphical representation is employed to elucidate the intricacies of market dynamics and resonates with the notion of "temporal turbulence" discussed in the article, intimating the potential statistical and physical congruences between the financial market behaviors and the turbulent phenomena observed in fluid dynamics.

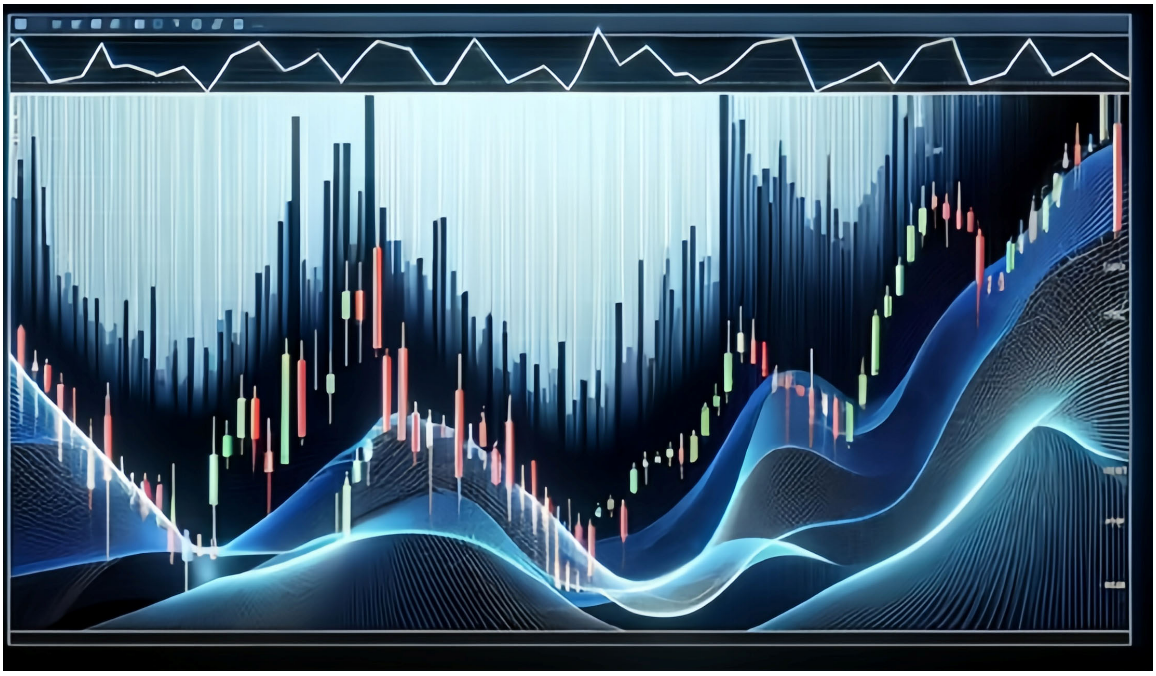

Figure 3. Quantum time wind concept diagram.

In the financial markets, candlestick charts, known as "K lines", serve as a graphical representation of price action, with each candlestick representing the price behavior of an asset over a specific time frame. The analogy of comparing K lines to "time particles" vividly emphasizes the observation of price behavior at discrete points in time. Each candlestick carries crucial information about the price within that period, including the opening price, the highest price, the lowest price, and the closing price. Expressed in mathematical language, if we consider a series of time intervals $\Delta t_1, \Delta t_2, \ldots, \Delta t_n$ with each interval corresponding to a candlestick, then a series of candlesticks can be represented as follows: $K_{\Delta t_1}, K_{\Delta t_2}, \ldots, K_{\Delta t_n}$ Each candlestick $K_{\Delta t_i}$ can be represented by a tuple: In the analysis of financial time series, the opening price $S_{\text{open},i}$ the highest price $S_{\text{high},i}$, and the closing price $S_{\text{close},i}$ are considered key indicators of the price movement of an asset within the i-th time interval. These indicators collectively form the candlestick chart, providing investors and analysts with a powerful tool for interpreting market trends, identifying levels of support and resistance, and recognizing price patterns. Each candlestick represents the price volatility within a time period, and the aggregated information allows for a complete presentation of the price history, enabling predictions about future price behavior based on historical data.

The log-normal distribution is a probability distribution of a continuous variable X, where the natural logarithm of X, $\ln(X)$, follows a normal distribution. If a random variable X is log-normally distributed, its probability density function (PDF) is given by Equation (1):

$$f(x \mid \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln\ x-\mu)^2}{2\sigma^2}\right), \text{for } x > 0 \quad (1)$$

where μ and σ are the mean and standard deviations of the variable's logarithm, respectively. This formula characterizes the distribution for variables whose log-transform follows a normal distribution, commonly used to model the behavior of asset prices in finance due to its ability to represent prices that are always positive and potentially skewed. For an asset price s, if the logarithm of s, denoted as $\ln\ (S)$, follows a normal distribution, then s itself follows a log-normal distribution. In discrete-time simulations, a time step Δt is typically defined, and price changes are considered at each step. If S_t is the asset price at time t, then $s_{t+\Delta t}$ is the price at $t+\Delta t$. The logarithmic return of the asset price can be expressed as follows:

$$\log\left(\frac{s_{t+\Delta t}}{s_t}\right) \sim \mathcal{N}\left(\left(\mu-\frac{\sigma^2}{2}\right)\Delta t, \sigma^2\Delta t\right) \quad (2)$$

In financial modeling, the simulation of asset price paths is an iterative process. Initially, the asset price s_0, expected return rate μ, volatility σ, time step Δt, and the total number of steps N are established. For each time step $\mathrm{n} = 0, 1, 2, \ldots, N-1$, a standard normally distributed variable z_n is generated to simulate the randomness of price movements. The asset price is updated according to Equation (3):

$$s_{(n+1)\Delta t} = s_{n\Delta t}\exp\left(\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}Z_n\right) \quad (3)$$

This reflects the expected return and random volatility at each time step. The process is repeated until the N iterations are completed. Subsequently, the generated paths are subject to statistical analysis to estimate key financial metrics such as the average price path, volatility, and the probability of extreme events. These simulation techniques have extensive applications in options pricing, risk management, portfolio optimization, and scenario analysis.

3.2. Space Quantization

3.2.1. Einstein Thought Experiment

Einstein's pollen experiments typically refer to his studies concerning Brownian motion. In 1905, Einstein published a series of revolutionary scientific papers, one of which elaborated in detail on Brownian motion—the random movement of pollen particles suspended in water. This discovery had a profound impact on the physics community of the time because it provided direct evidence for the existence of atoms and molecules. Prior to this, many scientists still harbored doubts about the actual existence of microscopic particles.

As shown in Figure 4, the central premise is that macroscopically observable random motion is caused by the incessant collisions with numerous microscopic particles within the fluid. Einstein further developed a mathematical model to describe this phenomenon, which is based on the assumption that the movement of pollen particles is induced by their collisions with the water molecules in the fluid. He derived an equation for the mean displacement of pollen particles, $\langle D(t)\rangle = \sqrt{2At}$, where $\langle D(t)\rangle$ represents the mean displacement within time t, and A is a constant that involves the temperature and the viscosity of the fluid. This equation not only clarified the possibility of revealing the properties of the microscopic world through macroscopic phenomena but also provided a quantitative method for measuring the existence of microscopic particles. Einstein's theory was subsequently confirmed by experimental physicists such as Jean Perrin, whose experimental results not only agreed with Einstein's predictions but also provided direct evidence for the existence of atoms, thereby deepening our understanding of the fundamental structure of matter.

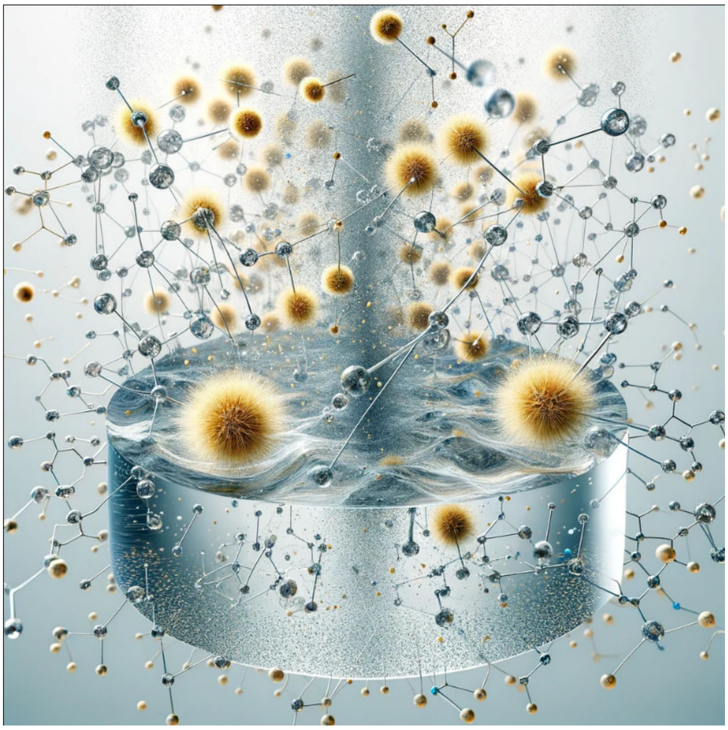

Figure 4. Einstein's Pollen thought experiment.

3.2.2. Space Quantization Analysis

Minimum Unit under Uncertainty Perspective

The financial market is a real-time dynamic interaction system involving tens of thousands of participants from around the world. The price fluctuation within an extremely short time frame is inherently uncertain and ambiguous, making it exceedingly difficult to accurately predict every price movement. This is illustrated by the underlying transaction mechanism of financial markets, as depicted in Figure 5, where the number of buy and sell orders for each price can change at any moment.

Order	Volume	Buy	Sell	Volume	Order
2	1452	2,842	2,850	647	2
4	2635	2,840	2,852	1179	1
2	2756	2,838	2,854	2656	4
3	3621	2,836	2,856	3009	2
2	979	2,834	2,858	829	3
3	8566	2,832	2,860	5000	1
4	6382	2,830	2,862	13587	3
2	958	2,828	2,864	804	2
2	3596	2,826	2,866	5000	1
5	22030	2,824	2,868	2714	2
29	52975			35425	21

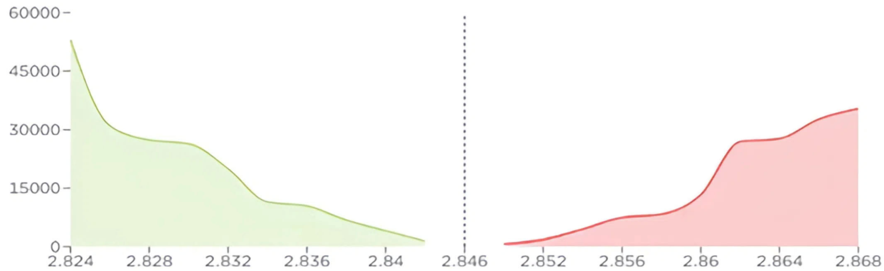

Figure 5. Transaction mechanism behind financial markets (www.highcharts.com (accessed on 1 December 2023)).

The principle of uncertainty in financial markets can be interpreted as follows: a specific transaction establishes the price of a particular asset, while the decision by investors to buy or sell the asset at a higher or lower price determines the asset's price trend. At a given moment, it is impossible to know both the precise price of a trading asset and the direction of its price change in the next moment. The uncertainty principle is frequently observable in the market. For example, at a certain point in time, if an individual does not know the exact price of a stock, they certainly cannot anticipate the speed and direction of the next price change. In other words, the uncertainty of trends seems infinite.

However, in the real financial market, one always has information beyond the price of the stock itself at any given time, thus allowing for the prediction of localized price fluctuations within a certain range.

Just as in quantum mechanics, the position of a particle is indeterminate. The uncertainty principle describes the relationship between two non-commuting variables. For instance, the product of the uncertainties in position and momentum is greater than or equal to a specific constant, which means it is impossible to simultaneously obtain precise values for both position and momentum. According to Heisenberg's uncertainty principle, there exists a relationship between energy ΔE and time Δt in Equation (4):

$$\Delta E \Delta t \ \geq \ \frac{\hbar}{2} \tag{4}$$

If time is quantized, then there exists a minimum time unit Δt, which defines the lower bound of the time interval that can be measured in this context. Within this minimum time unit, any measurement of energy will have an uncertainty ΔE. Based on the uncertainty principle, it can be deduced that during the time Δt, the uncertainty in energy is at least

$$\Delta E \ \geq \ \frac{\hbar}{2\Delta t} \tag{5}$$

In Equations (4) and (5), Planck's constant h serves as a fundamental constant in quantum mechanics that describes the quantization of energy, with its reduced form $\hbar$ (Planck's constant divided by 2π) embodying the fundamental limit on the relationship between energy and time as delineated by the Heisenberg uncertainty principle. The introduction of this constant reveals the inherent uncertainty in measurements of physical quantities at the microscopic scale. In its interdisciplinary application to financial markets, $\hbar$ is utilized to simulate the volatility of financial asset prices, endeavoring to explain and predict the complex dynamics within financial markets through the lens of quantum mechanics. By drawing an analogy to the Heisenberg uncertainty principle in physics, Planck's constant symbolizes the limitations on certainty between price and time in financial models, suggesting that the volatility observed in financial market data may parallel the behavior of fundamental particles to some extent. Here, Planck's constant serves as a scaling factor, aiding in the quantification and understanding of this uncertainty.

This implies that if time is quantized, then during the duration of the minimum time particle Δt, there is a fixed lower bound to the uncertainty in energy, which could have significant implications for the energy states of quantum systems. Within a quantized time framework, physical processes (such as the transition of electrons, emission of photons, etc.) would be restricted to occur at discrete points in time. This means that the occurrence of these processes is no longer possible within arbitrarily small time intervals but must occur in multiples of Δt. Therefore, the duration T of any process can be expressed as in Equation (6):

$$T = n\Delta t \tag{6}$$

where n is an integer. This notion of quantized time could lead to new characteristics of physical processes, such as the rate of energy level transitions potentially exhibiting new quantized patterns.

3.3. The Smallest Spatial Unit in the Financial Market

The method of analysis based on Einstein's thought experiments can likewise be applied to financial markets, treating market price fluctuations as a form of financial "Brownian motion". In this model, each transaction is akin to an impact on the asset price, causing random fluctuations and, thus, aiding our understanding of the complex dynamics of financial markets. Applying Einstein's approach to financial markets allows us to view the price movements of financial assets as phenomena analogous to Brownian motion. In this metaphor, the current price of financial markets is comparable to pollen particles suspended in a fluid, while the orders of global investors are akin to the collisions of water molecules.

Each order, regardless of size, impacts the asset price similarly to how water molecules strike pollen particles. Although the effect of a single order might be negligible, the collective effect of all orders causes price fluctuations on a macro level, resulting in the stock market's "Brownian motion".

In the field of financial market analysis, the mathematical expression of Einstein's Brownian motion, $\langle D(t)\rangle = \sqrt{2At}$, can be reinterpreted as a model for asset price volatility, where $\langle D(t)\rangle$ represents the average magnitude of change in asset price over time t, and the constant A in this context is associated with market liquidity and trading activity. This model provides a new perspective for understanding the fundamental nature of financial market price movements, revealing how external factors and investor behavior jointly impact asset prices. This theoretical framework not only offers insights into the deeper analysis of financial market fluctuations but also provides solid theoretical support for the formulation of investment strategies and the practice of risk management.

3.3.1. Argument Basic Assumptions

As shown in Figure 6,the minimal unit of stock price movement: Let us assume the existence of a minimal price movement unit, ΔP, which can be regarded as the "quantum of space" for the stock market. The randomness of transactions: The price change caused by each transaction can either increase by one ΔP or decrease by one ΔP, akin to a random walk model.

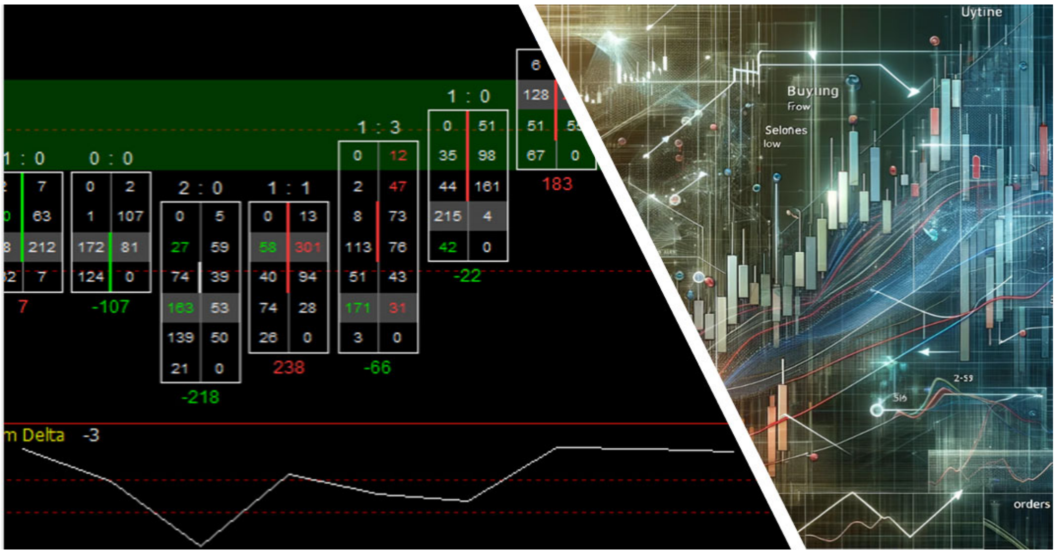

Figure 6. Brownian motion of orders behind the K−line.

In stock markets, Brownian motion is commonly applied to the mathematical model known as geometric Brownian motion (GBM), which is utilized to simulate the fluctuations of asset prices, such as stocks or indices. This model is based on key assumptions where the price movements are considered continuous and exhibit both a determinable trend and stochastic volatility. The mathematical expression for geometric Brownian motion can be described by the following stochastic differential Equation (7):

$$dS = \mu S dt + \sigma S dW \tag{7}$$

Here, S represents the asset price, μ denotes the expected rate of return of the asset, also known as the drift rate; σ represents the volatility of the asset price; dW is a Wiener process term representing random shocks, and dt is the time increment. The model assumes that the logarithmic returns of asset prices are normally distributed, which simplifies the calculation of probabilities at different price levels. The geometric Brownian motion model is an essential tool in financial engineering for risk management and derivative pricing, with the Black–Scholes option pricing model being one notable application based on this framework. Nevertheless, the actual behavior of stock markets can be more complex than this model describes, as it may also be influenced by a variety of factors including market psychology, informational asymmetry, and changes in supply and demand. Analogous to Einstein's analysis of Brownian motion, this paper attempts to construct a simplified model from a statistical physics perspective to describe this phenomenon.

3.3.2. Mathematical Description of Argumentation

Consider a simplified model where stock price movements follow the rules of a one-dimensional random walk: within a time interval, Δt, the price change ΔS can be either $+\Delta P$ or $-\Delta P$. Assuming that the probabilities for upward or downward movements are equal; that is, $P(\Delta S = +\Delta P) = P(\Delta S = -\Delta P) = 0.5$.

The derivation process is as follows: We define the expected value $E[\Delta S]$ and variance Var $[\Delta S]$ of stock price movements. Since each movement is independent and the probabilities of rising or falling are equal, as shown in Equation (8):

$$\begin{aligned} &E[\Delta S] = 0 \\ &\text{Var}\,[\Delta S] = E[\Delta S^2] - (E[\Delta S])^2 = (\Delta P)^2 \end{aligned} \tag{8}$$

Considering the stock price movements over a long period t, one can invoke the Central Limit Theorem to predict that the sum of stock price movements approximately follows a normal distribution with a mean of 0 and a variance of $n(\Delta P)^2$, where $n = t/\Delta t$ represents the number of trades that occur over time t. Therefore, the model for stock price movements can be represented as follows:

$$\Delta S(t) \sim \mathcal{N}(0, n\Big(\Delta P)^2\,) \tag{9}$$

In this model, the stock price changes are modeled as a stochastic process that, over a long time scale, tends to exhibit Gaussian fluctuations around the initial price, which is consistent with the empirical distributions observed in the financial markets under certain conditions.

In Equation (9), the normal distribution $\mathcal{N}(0, n(\Delta P)^2\,)$ indicates that, over time t, the summation of stock price movements is distributed as a normal distribution with a mean of 0 and a variance of $n(\Delta P)^2$. Here, n independent changes in ΔS (each with a mean of 0 and a variance of $(\Delta P)^2$) are summed to produce the total change $\Delta S(t)$. This is just a theoretical model, and such a simplified model usually requires further adjustments to accommodate the complexities of real markets. For instance, in actual markets, the probabilities of price increases and decreases may not always be equal, and the magnitude of price change ΔP may vary with changing market conditions. Additionally, factors such as transaction costs, market impact, and the speed of information dissemination must be considered for their effects on stock prices. Despite these limitations, the random walk model remains one of the very important theoretical tools in the field of finance and has had a significant impact on the development of modern financial mathematics.

In this section, we have explored in detail the fundamental principles of quantum finance and financial market turbulence theory, particularly the quantization of time and space and its application in simulating financial market dynamics. By establishing these theoretical foundations, we have provided new perspectives and tools for an in-depth analysis of financial markets. These tools not only reveal the micro-dynamics within the markets but also help us understand and predict macro trends.

4. Turbulence Models in Financial Markets

This section discusses several key mathematical equations that describe the dynamics of financial markets. First, the continuity equation provides a framework for the conservation of capital flow in financial markets, indicating that the change in capital density within a closed system is equal to the negative divergence of capital flow.

Next, the conservation of momentum equation, inspired by the Navier–Stokes equations from fluid dynamics, describes the changes in market behavior momentum, taking into account market pressure, market friction, and external influences. Following this, a quantized version of the Navier–Stokes equation is introduced as a theoretical model, attempting to describe the behavior of fluids at the microscale using concepts from quantum fluid dynamics. However, this model remains largely in the theoretical research phase.

In the realm of quantum mechanics, the Schrödinger equation is the fundamental equation describing the temporal evolution of quantum systems. By employing the method of separation of variables, the wave function is decomposed into a product of time and space functions, thereby splitting the original equation into two independent equations that handle the temporal and spatial dependencies of the wave function separately. The solution for the time-dependent part reveals the phase factor with which the wave function varies over time, while the spatial-dependent part provides the wave function solution for the state of a free particle.

Finally, the information diffusion equation is explored, which simulates the spread of information through financial markets, analogous to the heat diffusion equation in physics. Through these equations, a better understanding and prediction of the behavior and information flow in financial markets can be achieved.

4.1. Continuity Equation

The continuity equation is a mathematical representation of conservation laws, utilized in financial market models to characterize the conservation of capital flow. Let $\rho(x,t)$ represent the market density at time t and location x, where the density could be the amount of capital or trading volume. Concurrently, $v(x,t)$ denotes the velocity of capital flow, meaning the rate at which funds enter or exit per unit time. The continuity equation can be expressed as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \tag{10}$$

Equation (10) indicates that in a system without external capital injections or withdrawals, the market's capital is conserved. The rate of change in capital density $\frac{\partial\rho}{\partial t}$ plus the divergence of capital flow $\nabla\cdot(\rho v)$ sums to zero.

4.2. Market Turbulence Model

The conservation of momentum in financial markets can be analogized to the Navier–Stokes equations from fluid dynamics, which describe the changes in the momentum of market participants' behavior and its relation to market pressure and external forces, as Equation (10) shows:

$$\rho\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) = -\nabla p + \eta\nabla^2 v + F_{\text{ext}} \tag{11}$$

Here, $\rho(x,t)$ denotes market pressure, which could be a measure of information density or signal strength; η is the viscosity coefficient, representing the magnitude of market friction; F_{ext} represents external forces, such as policy changes, macroeconomic indicators, and other factors. This equation accounts for the inertia of market participants' behavior (the first term), local acceleration (the second term), the gradient of market pressure, market friction, and external influences.

Below is a potential form of the quantized Navier–Stokes equation, introducing a construct akin to a wave function, which we can refer to as the "quantum fluid wave function" ψ. This is not a wave function in the strict quantum mechanical sense but rather a mathematical tool attempting to analogize quantum mechanics to describe the probability distribution of fluid flow:

$$\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2\psi + V(r,t)\psi + U(\psi,\psi^*)\right] \tag{12}$$

In the theoretical nature of this quantized Navier–Stokes equation, we introduce a quantum constant $\hbar$, analogous to Planck's constant, as well as an effective mass m for the "fluid particle", and consider an external potential term $V(r,t)$, which may relate to the fluid pressure, along with a nonlinear term $U(\psi,\psi^*)$, to describe the fluid's self-interaction, similar to the interaction potential in quantum many-body problems. Although this form of equation is highly theoretical and does not directly correspond to physical processes in reality, it has been used in certain research domains to simulate fluid behavior, especially where fluid dynamics and quantum effects might intersect at microscopic scales. However, this model remains primarily at the stage of theoretical inquiry and has not yet been widely applied in practical applications.

4.3. Specific Solution Methods for Market Turbulence

4.3.1. Separate Variables

In the mathematical formalism of quantum mechanics, the original Equation (13):

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi \tag{13}$$

i as the imaginary unit, defined by $i^2 = -1$, and $\hbar$ to represent the reduced Planck constant, a fundamental physical constant with an approximate value of $1.0545718 \times 10^{-34}$ m^2 kg/s at the quantum scale. The wave function ψ, as a fundamental physical entity, encapsulates the quantum system's temporal evolution through its partial derivative with respect to time $\partial\psi/\partial t$, and its spatial variation via the second-order partial derivative with respect to position $\partial^2\psi/\partial x^2$. The mass of the particle m and the function $V(x)$, which defines the system's potential energy, are critical for understanding how particles are influenced by forces in space. The wave function ψ itself encodes the comprehensive information of the particle state, being an indispensable element in the quantum description. Assuming the solution takes the form:

$$\psi(x,t) = \phi(x)T(t) \tag{14}$$

The wave function $\psi(x,t)$ is a function of both time t and position x, representing the state of the quantum system, where $\phi(x)$ denotes the spatial component of the wave function, dependent solely on position x, and $T(t)$ represents the temporal component of the wave function, dependent solely on time t. When the assumed solution $\psi(x,t)$ as the product of these two functions is substituted into the original wave equation, we can derive two independent equations that describe the temporal evolution and spatial variation of the wave function, respectively. This method is known as the separation of variables.

$$\frac{i\hbar}{T}\frac{dT}{dt} = \frac{1}{\phi}\left(-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi\right) = E \tag{15}$$

In Equation (15), the temporal derivative of the temporal part of the wave function $T(t)$ is represented by $\frac{dT}{dt}$, while the second-order spatial derivative of the spatial part of the wave function $\phi(x)$ is denoted by $\frac{d^2\phi}{dx^2}$. The energy E, as a conserved quantity, appears on both sides of the equation, signifying its constancy and reflecting the energy conservation characteristic of the system. These symbols play a crucial role in analyzing quantum systems, allowing for the separation of the wave function's dependence on time and space while ensuring energy conservation throughout the system.

4.3.2. The Solution to the Time-Dependent Part

In mathematics and physics, the letter e denotes the base of the natural logarithm, approximately equal to 2.71828, which is a significant mathematical constant. The term $-iEt/\hbar$ serves as the exponent in a complex exponential function, where i is the imaginary unit, E signifies energy, t represents time, and $\hbar$ stands for the reduced Planck constant. This exponentiation plays a pivotal role in quantum mechanics, as it constitutes the phase factor that characterizes the temporal evolution of a wave function. The time-dependent equation and its solution are given by:

$$i\hbar\frac{dT}{dt} = ET \tag{16}$$

$$T(t) = e^{-\frac{iEt}{\hbar}} \tag{17}$$

This solution delineates the temporal component of the wave function, illustrating the phase evolution of the wave function in time. The complex exponential $e^{-iEt/\hbar}$ is a unitary operation, crucial in maintaining the normalization of the wave function consistent with the probabilistic interpretation of quantum states.

4.3.3. The Solution of the Spatial Dependency Part

The time-independent Schrödinger Equation (18) is as follows:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi = E\phi \tag{18}$$

For a free particle, where the potential $V(x) = 0$, the solution is as follows:

$$\phi(x) = Ae^{ikx} + Be^{-ikx} \tag{19}$$

In the context of quantum mechanics or wave physics, constants A and B are determined by the boundary conditions or initial conditions of the system. The exponential functions e^{ikx} and e^{-ikx} represent the forward and backward propagation of the wave in the spatial x-direction, respectively. The wave number k is related to the wavelength and frequency of the wave and mathematically defines the spatial structure and propagation characteristics of the wave.

4.4. Information Diffusion Equation

The propagation of information in financial markets can indeed be modeled using a diffusion equation, analogous to the heat diffusion equation or Fick's second law in physics. The equation can be expressed as follows:

$$\frac{\partial I}{\partial t} = D\nabla^2 I + S \tag{20}$$

In Equation (20), $I(x,t)$ represents the density of information at location x and time t; D is the diffusion coefficient for information, which measures the speed at which information spreads through the market; S is the source term for information, representing the rate of generation of new information. Information sources can include corporate news, earnings releases, policy changes, and other market-relevant events.

This diffusion equation suggests that information spreads out over the market, affecting the prices and trading behavior as it goes. The rate at which this information spreads is crucial for traders, as it can influence their ability to capitalize on new information before it becomes widely known and is reflected in the asset prices. The efficiency of a market is often gauged by how quickly and accurately prices reflect all available information—a concept known as market efficiency.

4.5. Differences from Real Fluid Mechanics

Although both financial market models and fluid dynamics models utilize mathematical equations to describe system dynamics, they exhibit significant differences in their application contexts and core characteristics. Fluid dynamics equations primarily describe the physical motion and interactions of fluid particles, which are generally simple and linear, involving fluid inertia, internal pressure gradients, viscosity, and external forces. In contrast, financial market models must address complex nonlinear interactions and sensitivity to market risks, phenomena unique to financial markets.

In financial markets, the nonlinear interaction term $N(\psi) = \alpha|\psi|^2\psi$ characterizes complex interactions among market participants, such as herd behavior and the propagation of market sentiments. These interactions often lead to extreme market behaviors including financial bubbles, market crashes, and long-tail distributions. Additionally, the risk-adjusted momentum term $R(x,t) = \beta\sigma(t)$ captures the impact of market risk changes on financial market dynamics, reflecting the market's response to various risk factors (such as changes in interest rates, economic policy adjustments, and global events). These model features highlight the high sensitivity of financial markets to the concept of "risk", which is absent in fluid dynamics.

Therefore, while both fields employ mathematical models to predict and analyze system behavior, financial market models must capture more complexity and nonlinear factors, whereas fluid dynamics models focus on describing the physical behavior of fluids. This fundamental difference suggests that although some analytical techniques from fluid dynamics can be borrowed, direct application of these physical theories to financial market analysis requires caution and a deep understanding of the linkages and distinctions between the two domains.

The nonlinear interaction term $N(\psi)$ represents complex nonlinear interactions among participants in financial markets, such as herding behavior and the propagation of market sentiments. These nonlinear interactions are phenomena unique to financial markets and can lead to extreme market behaviors, such as financial bubbles and crashes, as well as long-tail distributions and other complex dynamics. In mathematical terms, $N(\psi) = \alpha|\psi|^2\psi$ (where α is a constant) is a typical form of nonlinearity that can model the intricate interplay among market participants. Such nonlinear characteristics are not present in the simple particle interactions of fluid dynamics.

In financial models, R(x,t) represents a risk-adjusted momentum term, expressed as R(x,t) = βσ(t), where β is an adjustment coefficient, and σ(t) denotes the level of market risk at a specific point in time. This term crucially describes how changes in market risk impact the dynamics of financial markets, allowing for a more precise reflection of changes in asset prices and market behavior. R(x,t) plays a significant role in risk management, market analysis, and model optimization, ensuring that other parameters and variables within the model adapt to its effects to maintain consistency and practicality. Additionally, the dynamics of financial markets are influenced not only by the behavior of internal participants but also significantly by external environments and market risks. By modeling the market's response to various risk factors such as changes in interest rates, economic policy adjustments, and global events, R(x,t) demonstrates its sensitivity to market risk.

Fluid dynamics models describe the motion and interaction of fluid particles, and their core is encapsulated in the Navier–Stokes equations:

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = -\nabla p + \eta\nabla^2 v + F_{\text{ext}} + R(x,t) \tag{21}$$

Equation (21) governs the dynamics of fluid flow, where ρ is the fluid density, $\mathbf{v}$ is the flow velocity field, p is the pressure field, η is the dynamic viscosity, and $\mathbf{F}_{\text{ext}}$ represents external forces acting on the fluid. The term R(x,t) = βσ(t) reflects the market's response to changes in economic policies, interest rates, and other global events, serving as a mathematical expression of risk sensitivity in financial models.

This equation encompasses the inertia of the fluid, internal pressure gradients, viscous forces, and externally applied forces. The interactions in fluid dynamics are relatively simple and mostly linear, lacking the complex nonlinear behaviors and sensitivity to the concept of "risk" found in financial markets.

By contrasting the two models, the financial market model, with its inclusion of nonlinear interaction terms and risk-adjusted momentum terms, is capable of capturing the market's complex dynamics and sensitivity to external risk factors attributes unique to financial markets. In contrast, the fluid dynamics model focuses on describing the physical behavior of fluids, with interactions that are simpler and do not involve the complexity and risk sensitivity akin to those in financial markets.

The differences between financial market models and fluid dynamics models ultimately stem from the nature of the systems they describe, i.e., financial markets are complex systems driven by human behavior, replete with uncertainties and risk perception, whereas fluid dynamics describes natural phenomena that are comparatively simpler and governed by physical laws.

5. Demonstration

In this study, to investigate the similarities and differences between financial markets and turbulent dynamics, we have modeled the dynamics of financial markets and physical systems respectively. Specifically, time series data for the Standard & Poor's 500 index was sourced from Bloomberg terminals, while the time series data of the velocity field for a fully developed three-dimensional turbulent fluid was obtained through computational fluid dynamics (CFD) simulations. Utilizing these datasets, our aim is to reveal the commonalities and disparities in the statistical physical properties of these two systems and to perform relevant analyses within the realm of econophysics.

Computational fluid dynamics (CFD) utilizes physical equations, primarily the Navier–Stokes equations, to simulate the dynamic behavior of fluids. The Navier–Stokes equations are described by the equation $\rho(\partial \mathrm{u}/\partial \mathrm{t} + \mathrm{u} \cdot \nabla \mathrm{u}) = -\nabla \mathrm{p} + \mu\nabla^\wedge 2\mathrm{u} + \mathrm{f}$, which represents the continuity, momentum conservation, and energy conservation of fluids, where ρ is the fluid density, u is the velocity vector, p is the pressure, μ is the dynamic viscosity, and f represents external forces. To numerically solve these equations, the commonly used finite volume method (FVM) in CFD divides the computational domain into control volumes and applies the laws of conservation to each volume. Additionally, to simulate complex turbulent flows, this paper employs the $\mathrm{k} - \varepsilon$ turbulence model, which simplifies the computation of turbulence through two additional equations describing the turbulent kinetic energy k and the dissipation rate ϵ. In terms of financial data simulation, this paper simulates the time series of stock prices by generating normally distributed random numbers and calculating their cumulative sum, thus mimicking the random walk characteristic of markets. The paper also calculates the standard deviation of these time series and performs power spectral analysis to assess the volatility and frequency characteristics of prices. These additional details aim to enhance the transparency and reproducibility of the research, enabling other researchers to understand and apply the methods of this paper for similar analyses.

In the core part of the program, a series of random numbers is generated, and their cumulative sum is calculated to simulate time series data, formulated as $S(t) = \sum_{i=1}^{t} \epsilon_i$, where $S(t)$ represents the value of the time series at time point t, and ϵ_i is a randomly generated value representing the change at time point i. Subsequently, the difference of the time series is calculated as $D(t) = S(t) - S(t-1)$, where $D(t)$ denotes the difference value at time point t, serving as a foundation for analyzing the changes in the series. To assess the volatility or degree of change of the series, the standard deviation of the difference series is computed using $\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N} (D(i) - \mu)^2}$, here N represents the length of the difference series, and μ is the mean of the difference series. Lastly, by calculating the power spectrum $P(f) = |FFT(S(t))|^2$, where $P(f)$ indicates the power spectral density of the time series at frequency f, utilizing fast Fourier transform (FFT) and taking the square of its modulus, this step aids in revealing the energy distribution of the time series across different frequencies, identifying its periodic characteristics or frequency response. These mathematical formulas and steps constitute the foundation for time series analysis, applicable from financial market data to other domains of time series data processing and analysis.

Simulation framework as shown in Table 1:

Step 1: Data preparation.

We collect time series data of the S&P 500 index to serve as the financial data and generate three-dimensional turbulent velocity field time series data using computational fluid dynamics (CFD). This establishes the foundational datasets necessary for our subsequent analysis.

Step 2: Data processing.

Initially, we perform first-order differencing on both the financial and turbulent time series to extract sequences of incremental changes. We then calculate the standard deviations of these differential sequences to analyze their volatility. Finally, the fast Fourier transform (FFT) is applied to the time series to conduct power spectrum analysis, which allows us to assess the response across various frequencies.

Step 3: Results analysis.

By comparing the volatility of the financial market and physical turbulence data at different time points and analyzing as well as contrasting their energy distribution features within the frequency domain, we investigate the similarities and differences in volatility and frequency responses between these two systems.

This article selects the daily data from January 2020 to April 2024 as the sample period to analyze the behavior of the S&P 500 index. This timeframe is particularly representative as it encompasses the severe market fluctuations triggered by the COVID-19 pandemic followed by a robust rebound, as well as the oscillating market conditions under the influence of inflationary pressures and interest rate cycles. The selection of this period not only reflects the market's extreme behaviors but also demonstrates its dynamic changes in a volatile economic environment, making it crucial for understanding current and future market trends. Financial market data are typically assumed to follow a heavy-tailed distribution, such as the t-distribution, while turbulence data are assumed to follow a normal distribution. A substantial number of iterations (e.g., over 1000) were performed during the simulation process to ensure the robustness of the results. Furthermore, by altering distribution parameters such as shape and scale, we evaluated the sensitivity of the simulation outcomes to changes in the DGP assumptions, thus verifying the robustness and adaptability of the model.

Table 1. Empirical procedure steps.

	Step	Program Execution Content
1.	Data definitions	Y(t) = S&P 500 index at time t V(t) = Velocity of turbulent fluid at time t historical_data = {Y(t), V(t)}
2.	Calculate differences	for i in range(1, len(series)): diffs[i] = series[i] − series[i − 1] return diffs
3.	Standard deviation	Function calculate_standard_deviation(diffs): mean_diff = mean(diffs) sum_squared_diffs = sum((diff − mean_diff)ˆ2 for diff in diffs) standard_deviation = sqrt(sum_squared_diffs/len(diffs)) return standard_deviation
4.	Calculate Power spectrum	Function calculate_power_spectrum(series): power_spectrum = FFT(series)ˆ2 return power_spectrum
5.	Main program	diffs_Y = calculate_differences(Y) diffs_V = calculate_differences(V) σ_Y = calculate_standard_deviation(diffs_Y) σ_V = calculate_standard_deviation(diffs_V) power_spectrum_Y = calculate_power_spectrum(Y) power_spectrum_V = calculate_power_spectrum(V)
6.	Plotting	plot_standard_deviations(σ_Y, σ_V) compare_power_spectra(power_spectrum_Y, power_spectrum_V)

The fast Fourier transform (FFT) is an efficacious algorithm for converting time series data into the frequency domain. This transformation allows for the analysis and understanding of frequency components within the data, such as periodic signals or noise. FFT stands as a pivotal tool in achieving this objective, as it swiftly extracts frequency information from the data, aiding researchers in identifying the primary characteristics and potential patterns within the signal. By applying FFT, we are able to decompose complex time series data into a combination of various frequencies, thereby facilitating a deeper exploration and analysis of the data's structure and dynamic changes.

Power spectral analysis constitutes a process that reveals the intrinsic frequency components of a signal by calculating the power density corresponding to each frequency. This analysis method relies on the outcomes of FFT, converting the time series data into a series of values that represent the energy distribution across different frequencies. Through the illustration of the power spectrum, researchers can visually perceive the intensity of energy at various frequencies, thereby identifying the major frequency components and potential periodic signals within the data. This approach is crucial for comprehending the essential properties of the signal as well as for further signal processing and analysis, particularly in scientific research and technical applications where extracting valuable information from complex data is requisite.

5.1. Data Definitions

Firstly, we define $Y(t)$ as the value of the S&P 500 index at time t, a time series data designed to capture the volatility and unpredictability of the financial markets, thus reflecting the actual dynamic changes of the market. Secondly, $V(t)$ is defined as the velocity of a three-dimensional fully developed turbulent fluid at the same time point t. This simulation aims to intricately depict the complex behaviors of turbulence within physical systems. Through this methodology, the paper has successfully generated a set of time series data that embodies the characteristics of fluid dynamics, providing robust data support for understanding the dynamic interactions between financial markets and physical turbulence.

5.2. Differential Sequence Calculation

To further analyze the dynamic characteristics of these two systems, this study computed the differenced series for both sets of time series data. Specifically, $Z(t) = \Delta Y(t)$ calculates the change in the Standard & Poor's 500 index between consecutive time points, obtained by taking the first difference of the $Y(t)$ series. Similarly, $U(t) = \Delta V(t)$ represents the change in turbulent velocity at consecutive time points, achieved through the first differencing of the $V(t)$ series. These differenced series, $Z(t)$ and $U(t)$, provide the foundation for subsequent volatility analysis, enabling the paper to assess the dynamics of the systems from the perspective of changes in quantities.

5.3. Volatility Analysis

Building upon the differenced series, this study further investigates the relationship of the standard deviation of $Z(t)$ and $U(t)$ as a function of time t. This analysis aims to assess the dynamic behavior of volatility in the financial market and the variability in turbulent fluid velocity changes over time. By calculating the standard deviation, the paper is able to quantify the extent of fluctuations in the differenced series, thus providing a metric for understanding the similarities and differences between financial markets and turbulent dynamics. Moreover, the volatility analysis reveals the inherent instability and complexity within the time series data, laying the groundwork for a deeper understanding of the statistical physical characteristics of both systems.

5.4. Power Spectrum Analysis

In this study, we calculated and compared the power spectrum $S(f)$, where f represents frequency. Power spectral analysis is a key technique for exploring the energy distribution of time series in the frequency domain. By analyzing the power spectra of the financial time series $Y(t)$ and the physical turbulent time series $V(t)$, we can identify similarities and differences in frequency responses between market volatility and physical turbulent behavior. This method not only reveals the periodicity and randomness of the time series but also aids in a deeper understanding of the universality and specificity of dynamic behaviors across different systems.

5.5. Result Analysis

In this study, we compute and compare the power spectrum $S(f)$, where f denotes frequency. Power spectral analysis is a pivotal technique for exploring the distribution of energy across different frequencies within a time series.

5.5.1. Time Series Dynamic Analysis

As shown in Figure 7, the time series data $Y(t)$ and $V(t)$, respectively, represent the Standard & Poor's 500 index of the financial market and the velocity of a fully developed three-dimensional turbulent fluid, reflecting complex dynamic behaviors triggered by cumulative random changes. This approach ensures that the simulated data not only captures the characteristics of volatility in financial markets but also reproduces the nonlinear dynamics present in turbulent fluids within physical systems. In this way, the paper is able to portray the similarities in time series dynamics between financial markets and turbulence phenomena at the model level, providing a solid foundation for further understanding the interactions and behavioral patterns between the two systems. The detailed code can be found in Bezier Curve A. The detailed code can be found in Appendices A and B.

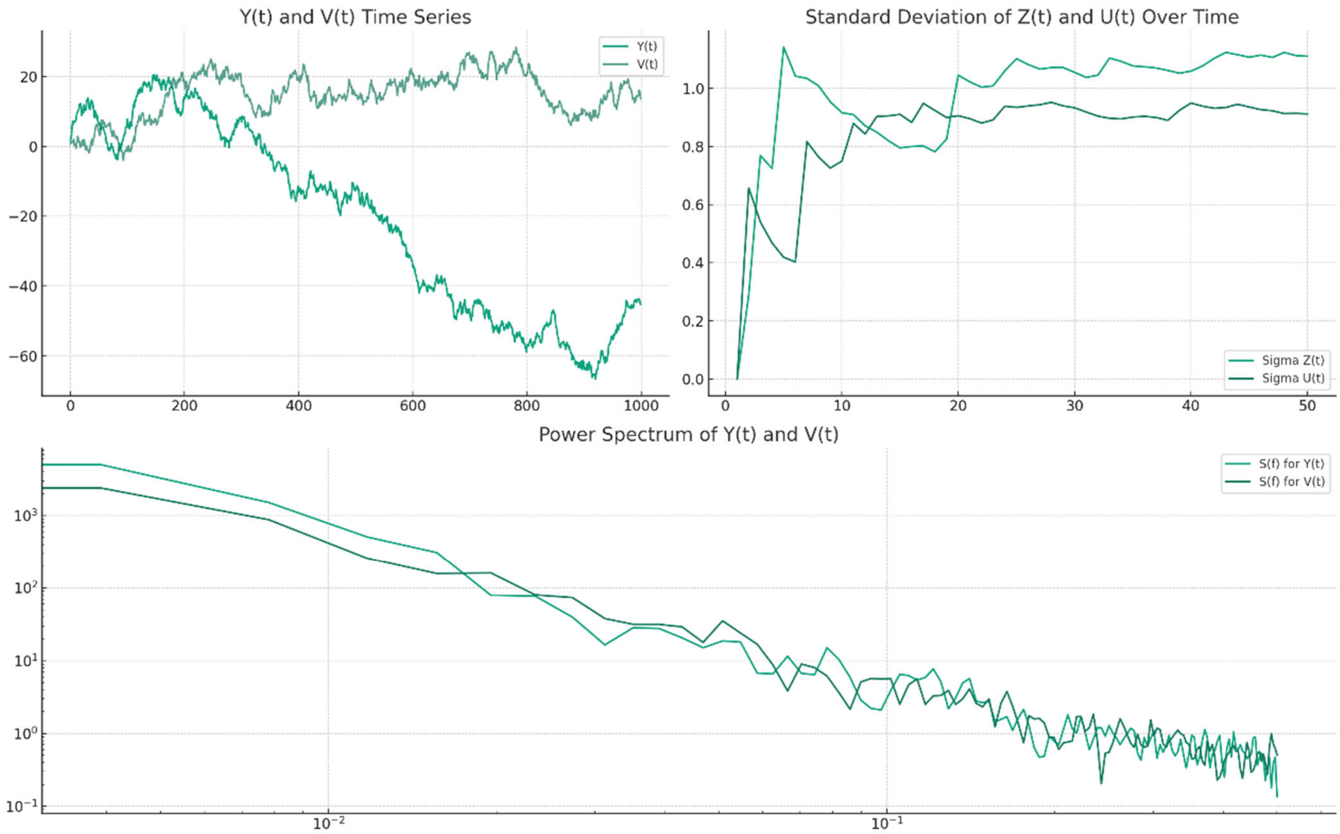

Figure 7. Analysis of financial market time series and fluid time series.

In our research, by conducting a detailed comparison between the S&P 500 index and the velocity series of a fully developed three-dimensional turbulent flow simulated through (CFD), we identified several key statistical similarities. Firstly, the probability distributions of both systems exhibit long-tailed characteristics, indicating that extreme events occur more frequently than would be expected under a normal distribution; this manifests as significant fluctuations in the financial market and extreme velocity variations in turbulent flows. Secondly, both systems display scale invariance across different time scales, suggesting an intrinsic connection between large-scale market trends and small-scale fluctuations, similar to the energy cascade process in turbulence. Additionally, the analysis of the autocorrelation functions reveals the persistence and inertia of information effects in both series, with autocorrelation declining rapidly in the short term but remaining significant over longer time scales. Further spectral analysis shows that despite differences in the frequency compositions of the S&P 500 and turbulent velocity series, they exhibit similar energy distribution patterns in certain key frequency bands, suggesting that the two systems may share similar dynamics in energy distribution and transfer mechanisms. These findings not only deepen our understanding of the dynamics of financial markets but also provide a theoretical and empirical foundation for interdisciplinary research methods, aiding in the development of new risk management strategies.

5.5.2. Analysis of Changes in Volatility over Time

By examining the standard deviations of the differenced series $Z(t)$ and $U(t)$, this study meticulously explores the evolution of volatility over time in both the financial market and the physical system. The analysis reveals that both the change in the Standard & Poor's 500 index, $Z(t)$, and the change in turbulent velocity, $U(t)$, exhibit significant time-dependency, and both display similar volatility characteristics across different time scales. Through this quantitative assessment of the standard deviation of the differenced series, the research not only reveals patterns of volatility in financial markets but also gains a deeper understanding of the dynamic changes in turbulent fluids, particularly emphasizing the similarity in statistical characteristics and fluctuating behavior between the two systems.

5.5.3. Exploration of Frequency Domain Characteristics

By calculating and comparing the power spectra $S(f)$ of the time series $Y(t)$ and $V(t)$, this study explores the characteristics of energy distribution in the frequency domain. Power spectral analysis, as an indispensable tool, allows us to understand the periodicity and randomness of time series from the perspective of frequency response. This analysis reveals similarities and differences in the contributions of frequency components between the financial market and turbulent phenomena, providing insights into the frequency characteristics of complex system dynamics in both domains. Moreover, the comparative results of the power spectra further underscore the importance of frequency domain properties in the analysis of econophysics phenomena, offering an effective analytical framework for understanding and interpreting the behavioral patterns of these systems.

5.6. Difference Analysis

By simulating time series for financial markets and fluid dynamics and comparing their power spectra, this paper is able to observe differences in their frequency responses. In the Figure 8 presented, the solid line represents the power spectrum of the financial market time series, while the dashed line corresponds to the power spectrum of the simulated fluid dynamics time series.

Power spectral analysis is a method for quantifying the distribution of energy across various frequencies in a signal or time series. It can be observed from the Figure 8 that the power spectra of the two-time series are similar in the low-frequency region, suggesting that both may exhibit similar dynamic behaviors over long-term trends. However, as the frequency increases, the power spectra begin to display noticeable differences, which may reflect dissimilarities in behavior between financial market data and fluid dynamics data on shorter time scales.

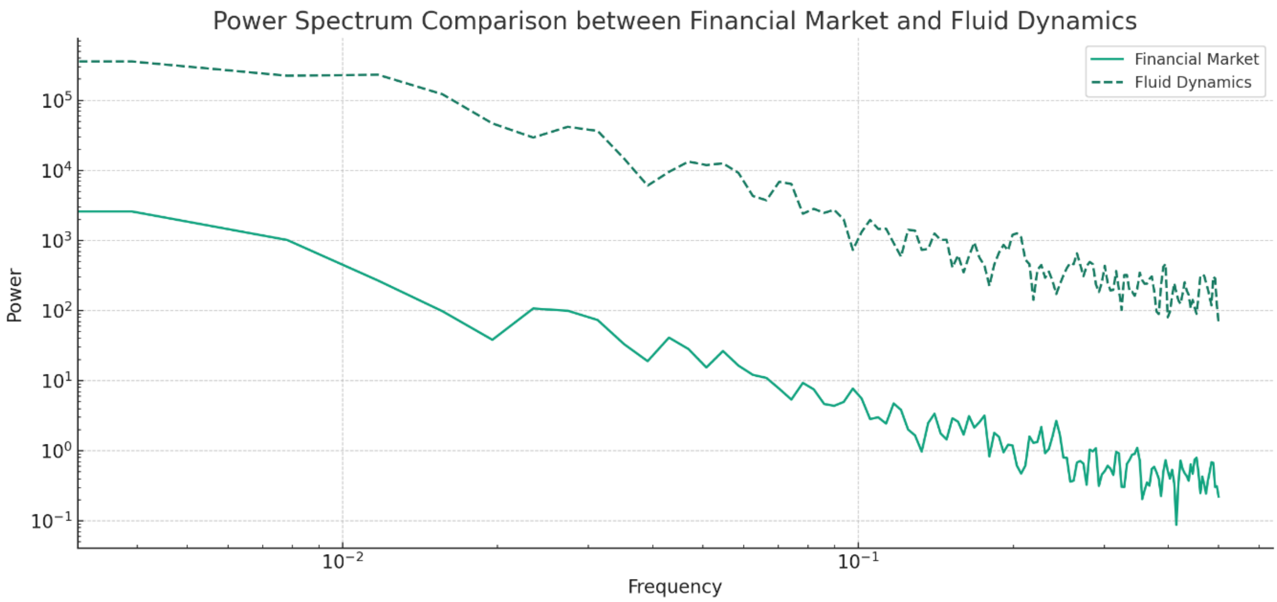

Figure 8. Power spectrum difference diagram.

Specifically, the power spectrum of the financial market time series may show peaks at certain frequencies, reflecting periodic behaviors in the market data or the influence of specific events. In contrast, the simulated fluid dynamics time series exhibits a more uniform distribution of energy across a wider range of frequencies, consistent with the turbulent behavior of fluids where flows at various scales contribute to the overall behavior.

This analysis reveals differences in the statistical physical properties between financial market models and real fluid dynamics models, particularly in their time series dynamic behaviors and frequency responses. Although both financial markets and fluid dynamics can be described using similar mathematical frameworks, they exhibit fundamental differences in their specific behavioral characteristics. This underscores the complexity and nuance that must be considered when applying physical theories to explain economic phenomena.

In physics, Bézier curves are utilized to simulate and analyze the path lines of turbulent flow fields. These curves are recognized for their mathematical smoothness and the flexibility of their control points adjustment, offering an efficient means of approximating complex turbulent structures. As such, they provide valuable support in numerical simulations and graphical visualizations for understanding and predicting turbulent behavior. In the analysis conducted here, this paper employs a strategy based on constructed financial Bézier curves to assess the changes in market volatility over time. Specifically, we will focus on the relationship between the time series (X-axis), price fluctuations (Y-axis), and the curvature of the Bézier curve (Z-axis).

Table 2 is designed to demonstrate how we utilize Bézier curves to monitor and predict fluctuations in financial market prices. Bézier curves serve not merely as graphical tools but also as dynamic analytical methods for identifying crucial turning points and persistent trends in market prices. During the construction process, we select key price points from historical market data, such as highs, lows, opening, and closing prices. These points are used as control points to generate a smooth curve that serves as a baseline reference for market price fluctuations. In the application of the model, the upper and lower bounds of the curve represent potential resistance and support levels in the market, respectively. Contact with the upper bound is generally viewed as a signal for selling or short-selling, indicating that the price may be excessively high and poised to fall.

Table 2. Empirical Celler of Bezier Curve.

	Step	Program Execution Content
1.	Generate Bezier curve	Function generate_bezier_curve(data): inflections = detect_inflection_points(data) bezier_curve = construct_bezier_curve(inflections) return bezier_curve

2.	Monitor price fluctuations	Function monitor_price_fluctuations(price, bezier_curve):
		if price > bezier_curve.upper_bound:
		check_for_turbulence(price)
		elif price < bezier_curve.lower_bound:
		check_for_turbulence(price)

3.	Check turbulence	Function check_for_turbulence(price):
		MA_short = calculate_moving_average(price, 20)
		MA_long = calculate_moving_average(price, 50)
		if abs(MA_short − MA_long) > threshold:
		return True
		else:
		return False

4.	Execute transactions	Functionexecute_transactions(price, bezier_curve, is_turbulent): if price > bezier_curve.upper_bound and is_turbulent: execute_sell_order(price) elif price < bezier_curve.lower_bound and is_turbulent:execute_buy_order(price)
5.	Main programs	Main Programs data = get_financial_price_data() bezier_curve= generate_bezier_curve(data) for each price inis_turbulent=monitor_price_fluctuations(price, bezier_curve)execute_transactions(price, bezier_curve, is_turbulent)
6.	Record Results	Function log_trades(trade): log(trade)

Check market turbulence: Calculate the short-term moving average (MA_short) over a 20-day period and the long-term moving average (MA_long) over a 50-day period. Significant deviations between MA_short and MA_long suggest potential market turbulence, indicative of rapid changes in price trends. To facilitate practical implementation, we use the crossover of MA_short through MA_long as an indicator of turbulence. In this paper, by applying Bézier curves to simulate fluctuations in financial market prices and constructing trading strategies based on these simulations (as shown in Table 2), our empirical analysis demonstrates stable profitability across multiple market cycles. This success not only illustrates the practical utility of Bézier curves in capturing market extremes but also validates the theoretical effectiveness of applying fluid dynamics' turbulence models to financial market analysis. Therefore, this combination of theoretical framework and empirical evidence effectively supports the theories presented in this paper, reinforcing the scientific foundation of employing physical models in the field of economic finance.

In this study, we conduct an in-depth analysis of the temporal series data, representing the temporal dimension of price datasets. As illustrated in Figure 9, we observe the evolution of price volatility and Bezier curve curvature over time. Typically, time series analysis reveals long-term trends, cyclical patterns, and potential seasonal factors in price behaviors. Throughout the analysis, the trends in time series reflect shifts in market sentiment as well as influences from economic cycles and macroeconomic events.

Price volatility (Y-axis) analysis: The price dimension reflects the market value at specific time points. The magnitude and direction of price fluctuations provide critical insights into market heat, investor sentiment, and stock liquidity. High price volatility might indicate significant discrepancies in market participants' perceptions of stock value, or the market's reaction to sudden events. Conversely, stable price movements generally suggest uniformity in market information or a lack of significant news events.

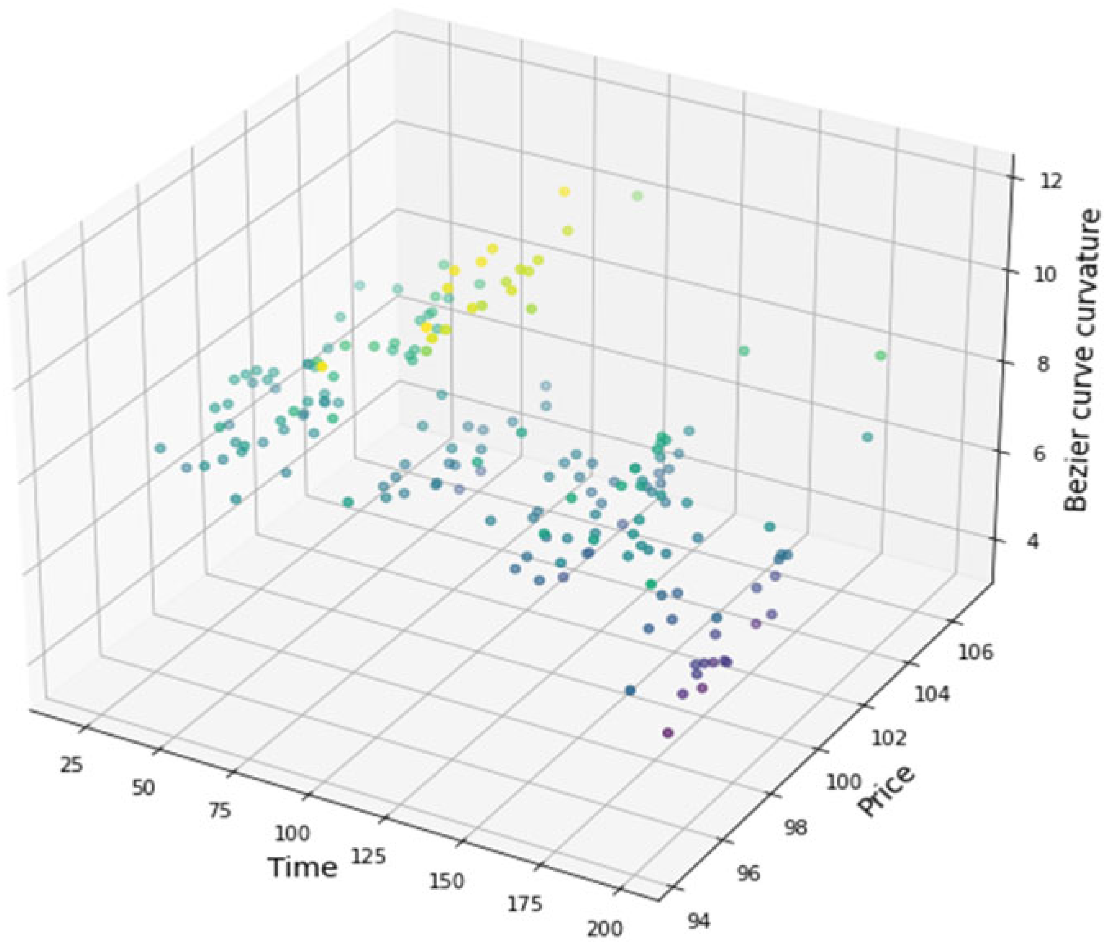

Figure 9. Fluctuation chart of price and Bezier curvature.

Bezier Curve curvature (Z-axis) Analysis: In this analytical framework, Bezier curve curvature, characterized by Bollinger bandwidth, quantifies the range of price volatility. This attribute of the Bezier curve serves as an indicator of the intensity of price fluctuations. A higher curvature typically corresponds with increased market volatility, which might be due to inconsistent interpretations of market information by investors or responses to changes in macroeconomic indicators. On the other hand, a lower curvature, with reduced Bollinger bandwidth, reflects decreased market volatility, possibly indicating more stable market information and consistent investor expectations, leading to limited price movements.

By integrating these three dimensions, we obtain a dynamic view of market volatility over time. Observing these variations allows investors and analysts to make more informed judgments about market trends and risk levels. For instance, during periods of high Bezier curve curvature, the market may exhibit instability, prompting investors to adopt more cautious investment strategies to mitigate potential risks. Conversely, during periods with lower curvature, the market stability may be higher, encouraging investors to consider increasing their investments to capture potential gains. Thus, Bezier curve curvature becomes a powerful tool in understanding and predicting market dynamics.

Specifically, by examining the time series (X-axis), we can track trends and patterns in prices (Y-axis) as they evolve over time. These patterns may arise due to the market's digestion of new information or reactions to macroeconomic events. For example, anticipation of significant policy changes might manifest as notable increases or decreases in stock prices within the time series.

Detailed analysis of price fluctuations on the Y-axis reveals immediate changes in the market's supply-demand relationship. During periods of high price volatility, there may be strong divergences between buyers and sellers, or the market may be rapidly responding to sudden news. For investors, understanding these price fluctuations is crucial, as they provide immediate signals about market sentiment and directions.

On the Z-axis, Bezier curve curvature provides a quantifiable measure of market volatility. An increase in curvature typically accompanies an increase in market volatility. During these phases, prices may exhibit sharp fluctuations, reflecting potentially rapid changes in market information or diverse future expectations and interpretations among market participants. For risk-averse investors, this could signal a time to reduce holdings or seek hedging strategies.

Conversely, a decrease in curvature usually indicates reduced market volatility. During these periods, price movements may be more stable, and investor expectations more aligned. This could signify a period of higher market confidence, where investors might consider expanding investments to reap potential rewards. By combining time, price, and Bezier curve curvature into a three-dimensional view, we gain a deeper understanding of market dynamics. This multidimensional analytical approach can provide a complex but comprehensive perspective for investment decisions, helping investors recognize cyclical patterns, anticipate changes in volatility, and thus make wiser investment choices.

Figure 10 presents a data analysis of the results of strategies applied to various market instruments as outlined in Table 2, where Bézier curves are utilized to display the results of market price trend analyses, demonstrating how these curves effectively capture key turning points and trends within financial market dynamics. By integrating historical market data points—such as highs, lows, opening, and closing prices—as control points, Bézier curves provide a smooth, continuous representation to trace the primary movements of the market. Different colors are used to distinguish between various market behavior patterns, assisting investors in identifying potential buy or sell signals. For instance, if market prices rise to the upper boundary of a Bézier curve, it may indicate an overvalued market, presenting a potential selling or short-selling opportunity; conversely, if prices fall to the lower boundary, it may suggest that the market is undervalued, indicating a buying opportunity. The dynamic adjustment of the curve's control points in response to new market information highlights the practicality of Bézier curves in tracking and predicting market trends. Thus, Figure 10 is a key visual representation of the application of Bézier curves in financial analysis, showcasing their capability to provide actionable insights for investment decision-making and risk management.

In the theoretical frameworks of quantum finance and market turbulence, the analysis of market data from indices such as GER40, HK50, and US500 reveals profound dynamics and complexities in market behavior. The cumulative profit graphs exemplify the evolution of quantum states in trading strategies, where each transaction point serves as a real-time observation of the market's quantum state, unveiling the non-linear dynamics characteristic of fluid-like turbulence in financial markets. The profit distribution histograms, viewed through the lens of quantum probability, display the probability distributions of returns, highlighting potential quantum effects such as entanglement and superposition that are often overlooked in traditional financial models. These distributions, within the context of market turbulence, reveal the collective adaptive behaviors of market participants under high volatility, leading to skewed or heavy-tailed characteristics in profit distributions. The box plots of returns provide clear visuals of market volatility and extreme events, interpretable as manifestations of "superposition" states in the market, where outliers may represent extreme market reactions caused by quantum fluctuations. This comprehensive analysis of the graphs offers unique and profound insights into capturing and preventing systemic risks from extreme events in financial markets, demonstrating a holistic perspective from micro-level quantum effects to macro-level economic behaviors.

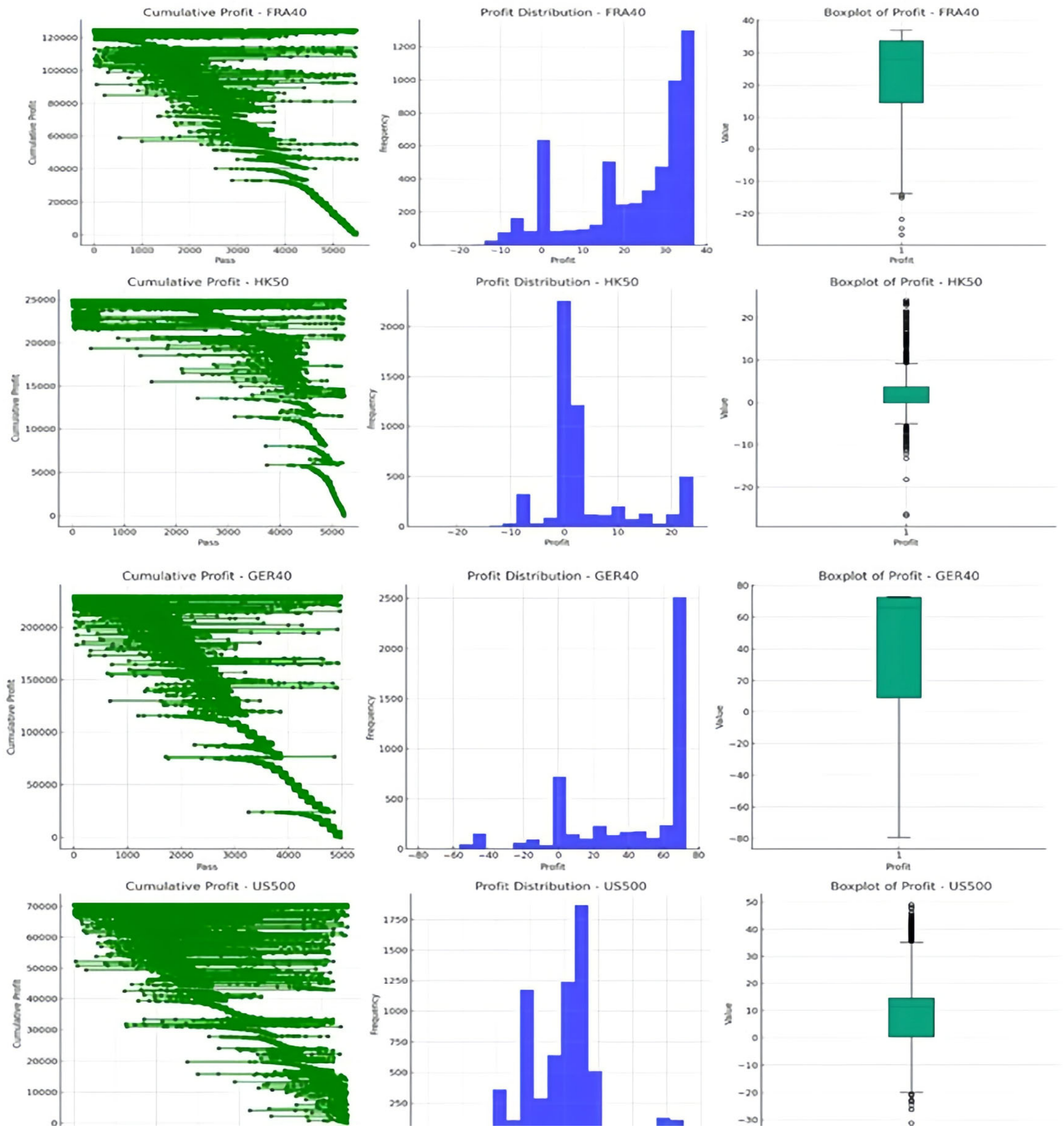

Figure 10. Table results: analysis charts.

6. Conclusions

This study pioneers a new direction in the field of econophysics by constructing a theoretical framework that draws an analogy between financial markets and fluid turbulence, utilizing time series analysis to simulate and reveal the statistical physical similarities between the two. Specifically, financial markets and turbulence dynamics exhibit statistically similar patterns in probability distributions, correlations, and spectral analysis. This offers valuable insights into interpreting the complex dynamics of financial markets using theories and techniques from fluid dynamics.

However, the research also identifies significant differences between financial markets and turbulence, particularly in the specific shapes of probability distributions and the time scales of correlations. This finding highlights the complexity of financial markets as systems influenced by various economic variables, whose behavioral patterns are not entirely akin to natural turbulence phenomena, pointing out the potential limitations of directly applying turbulence theory to financial market analysis.

Although the current findings provide a new perspective for studying the nonlinear dynamics of financial markets, it remains a preliminary exploration. Future research could build on this model for further quantitative analysis to establish a more precise quantitative correspondence between financial market dynamics and turbulence, thereby enhancing the predictive accuracy of the model. Additionally, integrating modern big data technologies to collect and analyze high-frequency financial market data is crucial for a deeper understanding of the intrinsic connections between financial market dynamics and fluid turbulence dynamics. Such profound understanding has potentially significant value for practical applications in financial risk management.

The interdisciplinary nature of this research provides an innovative framework that contributes to the advancement of financial market theory. While the current study is still in its initial stages and requires further empirical data and model refinement, it has already charted a course for deeper exploration into financial markets and envisaged the possibility of establishing a robust quantitative analysis platform. Future work will further quantify the relationship between financial markets and turbulence, extract valuable signals for market prediction, and delve deeper into analyzing high-frequency financial market data using modern data analysis techniques to enhance understanding and predictive capabilities of market dynamics.

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Appendix A. Fluid Model

```
# Import necessary libraries
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import welch
# Step 1: Generate a simulated time series
def generate_time_series(length):
# Generate a cumulative sum of a random walk to simulate financial asset prices and fluid velocity
return np.cumsum(np.random.randn(length))
# Step 2: Calculate differences to simulate financial asset returns and changes in fluid velocity
def calculate_differences(series):
# Calculate the first-order difference of the given time series
return np.diff(series)
# Step 3: Calculate the relationship of standard deviation over time
def calculate_std_over_time(differences):
# Calculate the standard deviation of the difference series
return np.std(differences)
# Step 4: Calculate and compare power spectra
def calculate_power_spectrum(series, nperseg = 256):
# Use Welch's method to calculate the power spectrum of the given time series
frequencies, power = welch(series, nperseg = nperseg)
return frequencies, power
# Execute simulation and analysis
length = 1000
```

Y_t = generate_time_series(length) # Simulate financial asset prices
V_t = generate_time_series(length) # Simulate fluid velocity
Z_t = calculate_differences(Y_t) # Calculate financial asset returns
U_t = calculate_differences(V_t) # Calculate changes in fluid velocity
std_Z_t = calculate_std_over_time(Z_t) # Calculate the standard deviation of financial asset returns
std_U_t = calculate_std_over_time(U_t) # Calculate the standard deviation of changes in fluid velocity
f_Y, P_Y = calculate_power_spectrum(Y_t) # Calculate the power spectrum of financial asset prices
f_V, P_V = calculate_power_spectrum(V_t) # Calculate the power spectrum of fluid velocity
# Plotting the results
plt.figure(figsize=(12, 8))
# Plot the standard deviation of financial asset returns and changes in fluid velocity
plt.subplot(2, 1, 1)
plt.bar([‘Financial Asset Returns’, ‘Fluid Velocity Changes’], [std_Z_t, std_U_t])
plt.title(“Comparison of Standard Deviation of Financial Asset Returns and Fluid Velocity Changes”)
# Plot the power spectrum comparison
plt.subplot(2, 1, 2)
plt.loglog(f_Y, P_Y, label=‘Financial Asset Prices’)
plt.loglog(f_V, P_V, label=‘Fluid Velocity’)
plt.title(“Power Spectrum Comparison of Financial Asset Prices and Fluid Velocity”)
plt.legend()
plt.tight_layout()
plt.show()

Appendix B. Bezier Curve Approximation of Flow Field

Assuming that the values of $u(x,y)$ and $v(x,y)$ are known at specific control points, we will use cubic Bézier curves to approximate these velocities. To simplify the problem, we can first consider only the approximation of $u(x,y)$; the approximation of $v(x,y)$ will follow the same method. For $u(x,y)$, suppose its values at the four control points of the Bézier curve are u_0, u_1, u_2, u_3, then we have the following:

$$u(x,y) = \sum_{i=0}^{3} B_i^3(t) u_i$$

where $B_i^3(t)$ is the basis function of the cubic Bézier curve. For a cubic curve, we have the following:

$$B_0^3(t) = \Big(1-t\Big)^3, B_1^3(t) = 3\Big(1-t\Big)^2 t, B_2^3(t) = 3(1-t)t^2, B_3^3(t) = t^3$$

According to the continuity equation $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, we need to ensure that the derivatives of $u(x,y)$ and $v(x,y)$ satisfy this condition. Since we are approximating these functions, we actually need to satisfy this condition in some average sense. To this end, we can consider minimizing the sum of squares of the degree of violation of the continuity equation throughout the entire region of interest. Specifically, we need to solve the following optimization problem:

$$\min_{\{u_i\},\{v_i\}} \int \Omega \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 dA$$

Here, Ω represents the region of fluid flow, and dA is the infinitesimal area element within the region.

References

1. Baaquie, B.E. *Quantum Finance: Path Integrals and Hamiltonians for Options and Interest Rates*; Cambridge University Press: Cambridge, UK, 2007.
2. Nakayama, Y. Gravity Dual for Reggeon Field Theory and Non-linear Quantum Finance. *Int. J. Mod. Phys. A* **2009**, *24*, 6197–6222. [CrossRef]
3. Schaden, M. Interest Rates and Coupon Bonds in Quantum Finance. *Am. J. Phys.* **2010**, *78*, 654–656. [CrossRef]
4. Baaquie, B.E. Interest rates in quantum finance: The Wilson expansion and Hamiltonian. *Phys. Rev. E* **2009**, *80*, 046119. [CrossRef] [PubMed]
5. Baaquie, B.E.; Yang, C. Empirical analysis of quantum finance interest rates models. *Phys. A Stat. Mech. Its Appl.* **2009**, *388*, 2666–2681. [CrossRef]
6. Agrawal, P.M.; Sharda, R. Quantum Mechanics and Human Decision Making. *SSRN J.* **2010**, 1–49. [CrossRef]
7. Baaquie, B.E. Bonds with index-linked stochastic coupons in quantum finance. *Phys. A Stat. Mech. Its Appl.* **2018**, *499*, 148–169. [CrossRef]
8. Lee, R.S.T. Quantum Finance Forecast System with Quantum Anharmonic Oscillator Model for Quantum Price Level Modeling. *Int. Adv. J. Eng. Res.* **2021**, *4*, 1–21.
9. Zoufal, C.; Lucchi, A.; Woerner, S. Quantum Generative Adversarial Networks for learning and loading random distributions. *npj Quantum Inf.* **2019**, *5*, 103. [CrossRef]
10. Bouland, A.; van Dam, W.; Joorati, H.; Kerenidis, I.; Prakash, A. Prospects and challenges of quantum finance. *arXiv* **2020**, arXiv:2011.06492. [CrossRef]
11. Arraut, I. Gauge symmetries and the Higgs mechanism in Quantum Finance. *Europhys. Lett.* **2023**, *143*, 42001. [CrossRef]
12. Choi, M.; Eom, C.; Kang, H.-G. Quantum Finance and Its Implications. *Asian Rev. Financ. Res.* **2023**, *36*, 103–129. [CrossRef]
13. Braine, L.; Egger, D.J.; Glick, J.; Woerner, S. Quantum Algorithms for Mixed Binary Optimization Applied to Transaction Settlement. *IEEE Trans. Quantum Eng.* **2021**, *2*, 3101208. [CrossRef]
14. Herman, D.; Googin, C.; Liu, X.; Galda, A.; Safro, I.; Sun, Y.; Pistoia, M.; Alexeev, Y. A Survey of Quantum Computing for Finance. *arXiv* **2022**, arXiv:2201.02773. [CrossRef]
15. Herman, D.; Googin, C.; Liu, X.; Sun, Y.; Galda, A.; Safro, I.; Pistoia, M.; Alexeev, Y. Quantum computing for finance. *Nat. Rev. Phys.* **2023**, *5*, 450–465. [CrossRef]
16. Lee, W.B.; Constantinides, A.G. Computational Results for a Quantum Computing Application in Real-Life Finance. In Proceedings of the 2023 IEEE International Conference on Quantum Computing and Engineering (QCE), Bellevue, WA, USA, 18–23 September 2023; IEEE: Piscataway, NJ, USA, 2023; pp. 414–423. [CrossRef]
17. Canabarro, A.; Mendonça, T.M.; Nery, R.; Moreno, G.; Albino, A.S.; de Jesus, G.F.; Chaves, R. Quantum Finance: A tutorial on quantum computing applied to the financial market. *arXiv* **2022**, arXiv:2208.04382. [CrossRef]
18. Wang, L.; Lee, R.S.T. The Design and Implementation of Quantum Finance Software Development Kit (QFSDK) for AI Edu-cation. In Proceedings of the 2022 20th International Conference on Information Technology Based Higher Education and Training (ITHET), Antalya, Turkey, 7–9 November 2022; IEEE: Piscataway, NJ, USA, 2022; pp. 1–7. [CrossRef]
19. Naik, A.; Yeniaras, E.; Hellstern, G.; Prasad, G.; Vishwakarma, S.K.L.P. From Portfolio Optimization to Quantum Blockchain and Security: A Systematic Review of Quantum Computing in Finance. *arXiv* **2023**, arXiv:2307.01155. [CrossRef]
20. Chang, Y.-J.; Sie, M.-F.; Liao, S.-W.; Chang, C.-R. The Prospects of Quantum Computing for Quantitative Finance and beyond. *IEEE Nanotechnol. Mag.* **2023**, *17*, 31–37. [CrossRef]
21. Gao, Y.-C.; Cai, S.-M.; Wang, B.-H. Hierarchical structure of stock price fluctuations in financial markets. *J. Stat. Mech.* **2012**, *2012*, P12016. [CrossRef]
22. Gustafson, K.; Gustafson, K.E. *Antieigenvalue Analysis: With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance and Optimization*; World Scientific: Hackensack, NJ, USA, 2011.
23. Puntanen, S. Antieigenvalue Analysis: With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance, and Optimization by Karl Gustafson. *Int. Stat. Rev.* **2012**, *80*, 325–326. [CrossRef]
24. Seyfert, R. Bugs, predations or manipulations? Incompatible epistemic regimes of high-frequency trading. *Econ. Soc.* **2016**, *45*, 251–277. [CrossRef]
25. Li, P.; Liang, S. Option Pricing beyond Black-Scholes Model: Quantum Mechanics Approach. *J. Econ. Sci. Res.* **2020**, *3*. [CrossRef]
26. Widdows, D.; Bhattacharyya, A. Quantum Financial Modeling on Noisy Intermediate-Scale Quantum Hardware: Random Walks Using Approximate Quantum Counting. *Quantum Econ. Financ.* **2024**, *0*, 29767032231217444. [CrossRef]
27. Evanoff, D.D. (Ed.) *Globalization and Systemic Risk*; World Scientific Studies in International Economics; World Scientific: Hackensack, NJ, USA, 2009.
28. Gkillas, K.; Longin, F. Is Bitcoin the New Digital Gold? Evidence From Extreme Price Movements in Financial Markets. *SSRN J.* **2018**, *45*, 251–277. [CrossRef]

29. Zhou, J.; Mavondo, F.T.; Saunders, S.G. The relationship between marketing agility and financial performance under dif-ferent levels of market turbulence. *Ind. Mark. Manag.* **2019**, *83*, 31–41. [CrossRef]
30. Evidence for Financial Contagion in Endogenous Volatile Periods. Review of Development Economics. Wiley Online Li-brary. 2015. Available online: https://onlinelibrary.wiley.com/doi/abs/10.1111/rode.12126 (accessed on 21 April 2024).