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Vector Equilibrium Problems—A Unified Approach and Applications

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Abstract: We present existing results and properties for the solutions of some vector equilibrium problems with set-valued functions in the case of a vector space ordered by a cone with some "interiority" properties. Some applications concerning the existence of equilibrium for abstract economies and vector optimization problems are given.

Keywords: ordered vector space; convex cone; efficient points; vector equilibrium problem; solutions set; set-valued function; abstract economy; Walrasian equilibrium; efficient points; conjugate duality

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1. Introduction

Begining with Blum and Oettli [1], who introduced an equilibrium problem as a generalization for the well-known Ky Fan inequalities, some vector equilibrium problems (VEPs) have been studied in the literature (see for instance [2–9]) as a tool which unifies the vector variational problem, vector complementary problem, vector optimization problem, vector saddle points problem, Nash equilibrium problem, and fixed point problem.

These problems was formulated with real or vector functions and multifunctions under various assumptions (concerning for example the presence or the absence of the convexity, the monotonicity, or the lower semicontinuity) and interesting results are obtained for the nonemptiness of the solutions set and its properties (see [10–16]).

Often, for the vector case, the results for VEPs are presented in spaces ordered by a nonempty, convex, pointed cone with a nonempty interior (see [7,17,18]). The framework considered here is a general one which includes both the case of cones with a nonempty and empty interior. In this context, some properties for VEPs with functions are given in [3,19,20] and some of these results can be found as particular cases of those in this paper for VEPs with set valued maps.

It is a theoretical approach to these problems that will be used as a mathematical support for the study of applications from the final sections.

We present in this paper a unified study for some vector equilibrium problems presented in Section 2 for set-valued functions (or multifunctions) with values in a vector space ordered by a cone with some "interiority property" (such as a nonempty interior or quasi interior or relative quasi interior). Section 3 presents some existence results for the solutions set of this problem. Section 4 gives properties for these solutions sets, such as the connectedness and the continuity. The Sections 5 and 6 provide some applications of VEPs regarding the existence of equilibrium in abstract economies and the vector optimization problems.

2. Preliminaries

Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be locally convex spaces, $A \subset X$, $B \subset Y$ be nonempty convex compact sets, $W \subset Z$ be a convex set, and let K, $Z_+ \subset Z$ be nonempty, convex, pointed cones. We denote by Z^* the topological dual space of Z, Z_+^* is the dual cone, i.e., $Z_+^* = \{z^* \in Z^* \mid z^*(z) \geq 0, \ \forall z \in Z_+\}$ and $Z^\# = \{z^* \in Z_+^* \mid z^*(z) > 0, \ \forall z \in Z_+ \setminus \{0\}\}$

is the quasi-interior of the dual cone. A base for the cone Z_+ is a subset $B \subset Z_+$ such that for each $z \in Z_+$, there exists $\lambda \in \mathbb{R}_+$ and $b \in B$ such that $z = \lambda b$. For example, if the interior of Z_+ is nonempty, the set $\{z^* \in Z_+^* \mid z^*(z) = 1\}$ is a w^*-compact base for the dual cone, where z is an element from the interior of Z_+.

The interior (respectively, the closure, the boundary, and the complementary) of a set A will be denoted by int A (respectively, cl A, Fr A and A^c). We write $x_n \xrightarrow{\tau} x$ if the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to x for the τ locally convex topology of the spaces. If there is no confusion, we omit τ. Furthermore, conv A and cone A will denote the convex, respectively, the conic hull of A. The relative interior of Z_+ denoted ri Z_+ is the interior of Z_+ relative to the closed affine hull of Z_+. The quasi (respectively, the intrinsec) relative interior of Z_+ denoted qri Z_+ (respectively, iri Z_+) is the set of $z \in Z_+$ for which cl cone $(Z_+ - z)$ (respectively, cone $(Z_+ - z)$) is a linear subspace of Z. $\mathcal{P}(Z)$ (respectively, $\mathcal{P}_c(Z)$) will denote the family of all nonempty subsets of Z (respectively, the family of all nonempty convex subsets of Z). For an element $x \in X$, $\mathcal{V}(x)$ will denote a fundamental system of neighborhoods for x.

Let $F : X \to \mathcal{P}(Z)$ be a set valued map (or multifunction, also denoted by $F : A \rightrightarrows Z$). The domain of F (denoted dom F) is dom $F = \{x \in X \mid F(x) \neq \emptyset\}$ and the graph of F (denoted Gr F) is the set Gr $F = \{(x,z) \in X \times Z \mid x \in \text{dom } F,\ z \in F(x)\}$. A multifunction $F : A \rightrightarrows Z$ ($A \subseteq$ dom F) is lower semicontinuous (l.s.c.) at $x_0 \in A$ if for all $z \in F(x_0)$ and $V \in \mathcal{V}(z)$, $\exists\, U \in \mathcal{V}(x_0)$, such that $F(U \cap A) \cap V \neq \emptyset$. The multifunction F is upper semicontinuous (u.s.c.) at $x_0 \in A$ if for any neighborhood $V \in \mathcal{V}(0)$, there is $U \in \mathcal{V}(x_0)$ such that $F(U \cap A) \subseteq F(x_0) + V$. The multifunction F is Z_+-lower semicontinuous at $x_0 \in A$ if for all $V \in \mathcal{V}(0)$ there exists $U \in \mathcal{V}(x_0)$ such that $F(U \cap A) \subseteq F(x_0) + V - Z_+$. The multifunction F is quasi-lower semicontinuous (respectively quasi-convex) if the level sets $N^\alpha = \{x \in A \mid F(x) \subseteq \alpha - Z_+\}$ are closed (respectively, convex).

We say that the multifunction F is K-concave if for each $x_1, x_2 \in A$ and $\lambda \in [0,1]$ we have $F(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda F(x_1) + (1-\lambda)F(x_2) + K$ and $\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq F(\lambda x_1 + (1-\lambda)x_2) - K$. The multifunction F is K-convex if and only if F is $-K$ concave. We say F is K-convexlike if $F(A) + K$ is convex and F has K-closed values if $F(x) + K$ is closed for each $x \in A$. The multifunction F is closed if the graph Gr F is closed. We note by cl F the multifunction defined by cl $F(x) = \text{cl}(F(x))$, $\forall x \in X$ and by co F the multifunction defined by co $F(x) = \text{conv}(F(x))$, $\forall x \in X$.

For $A_n,\ A \subseteq Z$ we set Li $A_n = \{x \in Z \mid \forall n \in \mathbb{N},\ \exists x_n \in A_n,\ x_n \xrightarrow{\tau} x\}$ and Ls $A_n = \{x \in Z \mid \exists (n_k)_k \subseteq \mathbb{N}\ x_{n_k} \in A_{n_k},\ \forall k \in \mathbb{N}\ x_{n_k} \xrightarrow{\tau} x\}$. We say that A is the τ-Painlevé–Kuratowski limit of A_n (and we write $A_n \xrightarrow{\tau PK} A$) if Ls $A_n \subseteq A \subseteq$ Li A_n.

Let us consider $K : A \rightrightarrows X$, $T : A \rightrightarrows B$, $F : A \times X \times T(A) \rightrightarrows Z$ with nonempty values. We adopt the notations from [5]:

$$r_1(U,V) \iff U \bigcap V \neq \emptyset \quad \bar{r}_1(U,V) \iff U \cap V = \emptyset$$

$$r_2(U,V) \iff U \subseteq V \quad \bar{r}_2(U,V) \iff U \not\subseteq V$$

Let us remark that $\bar{r}_1(U,V) \iff r_2(U,V^c)$ and $r_1(U,V) \iff \bar{r}_2(U,V^c)$.

The vector equilibrium problem $(VEP_1)_{r_i}^W$, $i = 1,2$ will be:

$$(VEP_1)_{r_i}^W \ \text{ find } \bar{x} \in A \cap K(\bar{x}) \ \text{ such that } \forall y \in K(\bar{x}), \exists \bar{t} \in T(\bar{x});\ r_i(F(\bar{x},y,\bar{t}),W).$$

The vector equilibrium problem $(VEP_2)_{r_i}^W$, $i = 1,2$ will be:

$$(VEP_2)_{r_i}^W \ \text{ find } \bar{x} \in A \cap K(\bar{x}) \ \text{ such that } \exists \bar{t} \in T(\bar{x}), \forall y \in K(\bar{x});\ r_i(F(\bar{x},y,\bar{t}),W).$$

Let $X = Y$, $K(\bar{x}) = A$, $T(x) = \{x\}$. For different types of W, we get some vector equilibrium problems studied in the literature:

- $W = -\text{int } Z_+$, then $(VEP)_{r_1}^W$ is $(WVEP)$ from [6];

- $W = Z \setminus \text{int } Z_+$, then $(VEP)^W_{\bar{r}_1}$ is $(SVEP)$ from [9].
- $W = -\text{qri } Z_+$, then $(VEP)^W_{r_1}$ is $(QVEP)$ from [3]
- $W = \text{comp }(-\text{int } Z_+)^{\varepsilon}_i$, then $(VEP)^W_{r_i}$ is $(QEP)^{\varepsilon}_{r_i}$ from [5].
- $W = \varepsilon e + \text{int } Z_+$, then $(VEP)^W_{r_1}$ is $(WVEP)^{\varepsilon}$ from [21] and $(VEP)^W_{\bar{r}_2}$ is $(GVEP)$ from [22].

3. The Existence of the Solutions Set

For the existence of the solutions for $(VEP_1)^W_{r_i}$, let us denote:

$$E^{i,W}_y = \{(x,t) \mid x \in A,\ t \in T(x),\ r_i(F(x,y,t),W)\}$$

$$N^{i,W}_y = \{x \in A \mid \exists t \in T(x),\ r_i(F(x,y,t),W)\} = \text{Pr}_X E^{i,W}_y$$

$$\bar{N}^{i,W}_{t,x} = \{y \in K(x) \mid \bar{r}_i(F(x,y,t),W)\}$$

Theorem 1 ([23])**.** *Let X be a Hausdorff topological space, $A \subseteq X$ be a nonempty compact convex set and $\phi : A \Longrightarrow X$ be a multifunction with nonempty convex values. Assume that for each $x \in A$, $\phi^{-1}(x)$ is open in A. Then, there exists $\bar{x} \in A,\ \bar{x} \in \phi(\bar{x})$.*

The following theorem generalizes Theorem 2.1 [5] and Theorem 13 [3], which may be reobtained as particular cases from the theorem, as is specified in Remark 1.

Theorem 2. *Let us suppose:*

(i) $\forall x \in A, \forall t \in T(x)$, $\bar{N}^{i,W}_{t,x}$ *is convex and* $x \notin \bigcap\limits_{t \in T(x)} \bar{N}^{i,W}_{t,x}$*;*

(ii) $\forall y \in A$, $N^{i,W}_y$ *is closed;*

(iii) $\text{cl } K(\cdot)$ *is u.s.c. and* $K(x)$ *is convex* $\forall x \in A,\ A \cap K(x) \neq \emptyset$*.*
Furthermore, $\forall y \in A,\ K^{-1}(y)$ *is open in* A*.*
Then, $(VEP_1)^W_{r_i}$ *has a solution.*

Proof. For $x \in A$ set:

$$P(x) = \{y \in A \mid \forall t \in T(x),\ \bar{r}_i(F(x,y,t),W)\}$$

$$E = \{y \in A \mid y \in \text{cl } K(y)\}$$

For each $x \in A$, (i) implies that $P(x)$ is convex and (iii) implies that E is a closed set. For all $y \in A$, $A \setminus P^{-1}(y) = \{x \in A \mid \exists t \in T(x),\ r_i(F(x,y,t),W)\}$ and following (ii) $P^{-1}(y)$ is open in A. Let $Q : A \Longrightarrow A$, $Q(x) = K(x) \cap P(x)$ if $x \in E$ and $Q(x) = A \cap K(x)$ if $x \in A \setminus E$. We observe that $Q(x)$ is convex for all $x \in A$. Furthermore, for all $y \in A$:

$$Q^{-1}(y) = \{x \in E \mid x \in K^{-1}(y) \cup P^{-1}(y)\} \bigcup \{x \in A \setminus E \mid x \in K^{-1}(y)\} =$$

$$= K^{-1}(y) \cap [P^{-1} \cup (A \setminus E)]$$

Therefore, $A \setminus Q^{-1}(y) = A \setminus K^{-1}(y) \bigcup [(A \setminus P^{-1}(y)) \cap E]$. Since $K^{-1}(y)$ and $P^{-1}(y)$ are open in A, this implies that $Q^{-1}(y)$is open in A for all $y \in A$. Hypothesis (i) implies that $x \notin P(x)$, and thus, $x \notin Q(x)$ for all $x \in A$. Theorem 1 gives us $\bar{x} \in A$ such that $Q(\bar{x}) = \emptyset$. Since $A \cap K(\bar{x}) \neq \emptyset$, $\bar{x} \in E$, $K(\bar{x}) \cap P(\bar{x}) = \emptyset$. Thus, $\bar{x} \in A \cap clK(\bar{x})$, $y \notin P(\bar{x})$, $y \in K(\bar{x})$, i.e, there exists $\bar{t} \in T(\bar{x})$ such that $r_i(F(\bar{x},y,\bar{t}),W)$. We conclude that $(VEP_1)^W_{r_i}$ has a solution. □

Remark 1. *If F is a real single valued map, $W = \varepsilon + \mathbb{R}_+$, $K(x) = A$ and $T(x) = \{x\}$, then $\bar{N}^{i,W}_{t,x}$ is N_x, the ε-set level, which (i) means that $f(x,\cdot)$ is quasi-convex and (ii) means that $f(\cdot,y)$ is quasi upper semicontinuous.*

For the existence of the solutions for $(VEP_2)^W_{r_i}$, we denote:

$$E^{i,W}_y = \{(x,t) \mid x \in A,\ t \in T(x),\ r_i(F(x,y,t),W)\}$$
$$\bar{N}^{i,W}_{t,x} = \{y \in K(x) \mid \bar{r}_i(F(x,y,t),W)\}$$

Theorem 3. *Let us suppose:*

(i) $\forall x \in A, \forall t \in T(x)$, $\bar{N}^{i,W}_{t,x}$ *is convex and* $x \notin \bigcap_{(x,t)\in A\times T(x)} \bar{N}^{i,W}_{t,x}$;

(ii) $\forall y \in A$, $E^{i,W}_y$ *is closed and* $\bigcup_{y\in A} E^{i,W}_y$ *is closed;*

(iii) $\mathrm{cl}\, K(\cdot)$ *is u.s.c. and* $K(x)$ *is convex* $\forall x \in A,\ A \cap K(x) \neq \emptyset$.
Furthermore, $\forall y \in A,\ K^{-1}(y)$ *is open in* A.
Then, $(VEP_2)^W_{r_i}$ *has a solution.*

Proof. The proof is similar to Theorem 1. □

The following result gives another type of existence theorem that relaxes the requests concerning the multifunction but impose additional conditions on the cone.

Theorem 4. *Let us consider the following conditions:*

(i) $F(A,b) + Z_+$ *is convex for each* b;
(ii) $\mathrm{cl}\,(Z_+ - Z_+) = Z$ *and* $K = \mathrm{qri}\, Z_+ \cup \{0\}$;
(iii) $K = \mathrm{qi}\, Z_+ \cup \{0\}$;
(iv) $W \cap \mathrm{qri}\,(\mathrm{cone}\, F(A,b) + K) = \emptyset$.

If (i), (ii), (iv) *or* (i), (iii), (iv) *hold, then* $(VEP)^{W-K\setminus\{0\}}_{r_1}$ *and* $(VEP)^{W-K\setminus\{0\}}_{\bar{r}_2}$ *have a solution.*

Proof. Let us suppose that $(VEP)^{W-\mathrm{qri}\, Z_+}_{r_1}$ has no solution. Thus, for each $a \in A$, there exists $b \in A$ such that $F(a,b) \cap W - \mathrm{qri}\, Z_+ \neq \emptyset$. Thus, $w \in F(A,b) + \mathrm{qri}\, Z_+$, which implies that $w \in \mathrm{cone}\, F(A,b) + \mathrm{qri}\, Z_+$. Since $F(A,b) + Z_+$ is convex for each b, then cone $F(A,b) + \mathrm{qri}\, Z_+$ is also convex.

(iii) implies that there exists $z^* \in Z^*_+ \setminus \{0\}$ such that $z^*(f) \geq z^*(w)$ for all $f \in F(A,b)$, $w \in W$. However, there exists $w \in W$, $f \in F(a,b) \cap (w - \mathrm{qri}\, Z_+) \neq \emptyset$ and follows (ii) we have $z^*(f) > z^*(w)$, which is false. Similarly, we obtain the other conclusion. □

4. Properties for the Solutions Set

In what follows, we consider $K(x) = A$, $K = \mathrm{qi}\, Z_+ \cup \{0\}$ or $K = \mathrm{qri}\, Z_+ \cup \{0\}$, where W is a convex set and F has K-convex values. For $z^* \in Z^*_+ \setminus \{0\}$, $z^*(\mathrm{qi}\, Z_+) = \mathbb{R}_+ \setminus \{0\}$ and for $z^* \in Z^\#_+$, $z^*(\mathrm{qri}\, Z_+) = \mathbb{R}_+ \setminus \{0\}$.

We denote:

$$S_F(VEP)^{W-K}_{\bar{r}_1} = \{x \in A \mid x \text{ is solution for} (VEP)^{W-K}_{\bar{r}_1}\};$$

$$S_F(VEP)^{W-K}_{z^*,\bar{r}_1} = \{x \in A \mid \forall y \in A,\ \exists t \in T(x),\ z^* \circ F(x,y,t) \cap z^*(W-K) = \emptyset\}.$$

If there is no confusion, we simply denote $S(VEP)^{W-K}_{\bar{r}_1}$ and $S(VEP)^{W-K}_{z^*,\bar{r}_1}$.

The following theorem generalizes the similar result for VEP with functions from [6].

Theorem 5. *If* $K = \mathrm{qi}\, Z_+ \cup \{0\}$, *then:*

$$\bigcup_{z^* \in Z^*_+ \setminus \{0\}} S(VEP)^{W-K}_{z^*,\bar{r}_1} \subseteq S(VEP)^{W-K}_{\bar{r}_1} \subseteq$$

$$\subseteq \bigcup_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}} = S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}.$$

If $K = \mathrm{qri}\ Z_+ \cup \{0\}$ or $K = \mathrm{qi}\ Z_+ \cup \{0\}$, W is a convex compact set and F has K-closed values; moreover, we have

$$\bigcup_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,\bar{r}_1}^{W-K} = S(VEP)_{\bar{r}_1}^{W-K}$$

and:

$$\bigcup_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}} = S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}.$$

Proof. The first inclusion does hold obviously.

Let $x \in S$; this implies that $\forall y \in A,\ \exists t \in T(x),\ F(x,y,t) \cap (W-K) = \emptyset$. Since $F(x,y,t)+K$ and $W-K$ are convex sets, there exists $z^* \in Z^*$ such that $\inf z^*(u) \geq \sup z^*(v)$ for $u \in F+K$ and $v \in W-K$. Obviously, $z^* \in Z_+^* \setminus \{0\}$, and thus, $z^* \circ F(x,y,t) \cap (z^*(W) - \mathbb{R}_+ \setminus \{0\}) = \emptyset$. We deduce that $z^* \circ F(x,y,t) \cap z^*(W-K\setminus\{0\}) = \emptyset$, which implies:

$$S(VEP)_{\bar{r}_1}^{W-K} \subseteq \bigcup_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}.$$

The converse is obvious, and thus, the last equality from the conclusion follows. For the case when W is compact, the inequality from the separation theorem is strict and the conclusion follows similar. □

We denote:

$$S_F(VEP)_{r_1}^{W-K} = \{x \in A \mid x \text{ is solution for} (VEP)_{r_1}^{W-K}\}$$

$$S_F(VEP)_{z^*,r_1}^{W-K} = \{x \in A \mid \forall y \in A,\ \exists t \in T(x),\ z^* \circ F(x,y,t) \cap z^*(W-K) \neq \emptyset\}.$$

If there is no confusion, we denote simply: $S(VEP)_{r_1}^{W-K}$ and $S(VEP)_{z^*,r_1}^{W-K}$.

Theorem 6. *If $K = \mathrm{qi}\ Z_+$, then:*

$$S(VEP)_{r_1}^{W-K\setminus\{0\}} = \bigcap_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,r_1}^{W-K\setminus\{0\}} \subseteq$$

$$\subseteq S(VEP)_{r_1}^{W-K} \subseteq \bigcap_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,r_1}^{W-K}$$

If $K = \mathrm{qri}\ Z_+$, or $K = \mathrm{qi}\ Z_+$, W is a convex compact set and F has K-closed values, we have:

$$S(VEP)_{r_1}^{W-K\setminus\{0\}} = \bigcap_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,r_1}^{W-K\setminus\{0\}} \text{ and}$$

$$S(VEP)_{r_1}^{W-K} = \bigcap_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,r_1}^{W-K}.$$

Proof. Let $x \in S$. For all $y \in B$, there exists $t \in T(x)$ such that $F(x,y,t) \cap (W-K\setminus\{0\}) \neq \emptyset$. This fact implies that for all $z^* \in Z_+^* \setminus \{0\}$, we have $z^* \circ F(x,y,t) \cap z^*(W-K\setminus\{0\}) \neq \emptyset$.

For the converse, let us suppose that $z^* \circ F(x,y,t) \cap z^*(W-K\setminus\{0\}) \neq \emptyset$ and $F(x,y,t) \cap (W-K\setminus\{0\}) = \emptyset$. We find $z^* \in Z_+^* \setminus \{0\}$ such that $\inf z^*(u) \geq \sup z^*(v)$ for $u \in F+K$ and $v \in W-K$. Since $(z^* \circ F(x,y,t)) \cap z^*(W-K) \neq \emptyset$ will exists $v \in W-K\setminus\{0\}$, $u \in F(x,y,t)$, $w \in W$ such that $z^*(u) = z^*(v) < z^*(w) \leq \sup z^*(W-K) \leq$

$\inf z^* \circ F(x,y,t)$ is false. The other inclusion does hold obviously. The case when W is compact follows similar to the previous theorem.

We denote:

$$S_F(VEP)_{r_2}^{W-K} = \{x \in A \mid x \text{ is solution for} (VEP)_{r_2}^{W-K}\}$$

$$S_F(VEP)_{z^*,r_2}^{W-K} = \{x \in A \mid \forall y \in A,\ \exists t \in T(x),\ z^* \circ F(x,y,t) \subseteq z^*(W-K)\}.$$

If there is no confusion, we denote simply $S(VEP)_{r_2}^{W-K}$ and $S(VEP)_{z^*,r_2}^{W-K}$. □

Theorem 7. *If* $K = \text{qi } Z_+;\ \text{qri } Z_+$, *then:*

$$S(VEP)_{r_2}^{\text{cl }(W-K)} = \bigcap_{z^* \in Z_+^* \setminus \{0\}} S(VEP)_{z^*,r_2}^{\text{cl }(W-K)}$$

Proof. Let $x \in S$. For all $y \in A$, there exists $t \in T(x)$, such that $F(x,y,t) \subseteq \text{cl }(W-K)$. Obviously, $z^* \circ F(x,y,t) \subseteq z^*(\text{cl }(W-K))$ for all $z^* \in Z_+^* \setminus \{0\}$. Now, let us suppose that there exists $\alpha \in F(x,y,t)$ and $\alpha \notin \text{cl }(W-K)$. We find $z^* \in Z_+^* \setminus \{0\}$ such that $\sup\{z^*(v),\ v \in W-K\} < z^*(\alpha) \leq \sup\{z^*(v),\ v \in W-K\}$ is false.

We denote:

$$S_F(VEP)_{\bar{r}_2}^{W-K} = \{x \in A \mid x \text{ is solution for} (VEP)_{\bar{r}_2}^{W-K}\}$$

$$S_F(VEP)_{z^*,\bar{r}_2}^{W-K} = \{x \in A \mid \forall y \in A,\ \exists t \in T(x),\ z^* \circ F(x,y,t) \not\subseteq z^*(W-K)\}.$$

If there is no confusion, we denote simply $S(VEP)_{\bar{r}_2}^{W-K}$ and $S(VEP)_{z^*,\bar{r}_2}^{W-K}$. □

Theorem 8. *If* $K = \text{qi } Z_+;\ \text{qri } Z_+$, *then:*

$$S(VEP)_{\bar{r}_2}^{\text{cl }(W-K)} = \bigcup_{z^* \in Z_+^* \setminus \{0\}} S_{z^*}(VEP)_{\bar{r}_2}^{\text{cl }(W-K)}$$

Proof. Let $x \in S$. For all $y \in A$, there exists $t \in T(x)$, such that $F(x,y,t) \not\subseteq \text{cl }(W-K)$. Thus, there exists $\alpha \in F(x,y,t)$ and $\alpha \notin \text{cl }(W-K)$. We find $z^* \in Z_+^* \setminus \{0\}$ such that $z^*(\alpha) > \sup\{z^*(v),\ v \in W-K\}$, which implies $z^* \circ F(x,y,t) \not\subseteq z^*(W-K)$. The converse follows obviously and the equality does hold. □

Connectedness for the solutions sets. In the following, we consider $K(x) = A, \forall x \in A$, and $T(x) = \{x\}$.

Theorem 9. *Let us suppose that* $K = \text{qi } Z_+ \cup \{0\}$ *or* $K = \text{qri } Z_+ \cup \{0\}$, $F(x,\cdot)$ *has K-convex values for each* $x \in A$ *and* $F(\cdot,y)$ *is K-concave, for each* $y \in A$. *Then, for all* $z^* \in Z_+^* \setminus \{0\}$, $S(VEP)_{z^*,\bar{r}_1}^{W-K}$ *and* $S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$ *are convex.*

If $K = \text{qi } Z_+ \cup \{0\}$ *and in addition we suppose that there exists* $\alpha \in W$ *such that* $\text{cl }(W-K) = \text{cl }(\alpha - K)$ *and* $S((VEP)_{\bar{r}_1}^{(\alpha+\text{qi } Z_+)^c}) \neq \emptyset$, *then the solutions set* $S((VEP)_{\bar{r}_1}^{W-K\setminus\{0\}})$ *is connected.*

If W *is a convex compact set and* F *has K-closed values, then* $S((VEP)_{\bar{r}_1}^{W-K})$ *is connected.*

Proof. Let $x_1,\ x_2 \in S(VEP)_{z^*,\bar{r}_1}^{W-K}$. We have:

$$z^* \circ F(x_1,y) \cap (z^*(W) - \mathbb{R}_+) = \emptyset,\ \forall y \in A,$$

$$z^* \circ F(x_2,y) \cap (z^*(W) - \mathbb{R}_+) = \emptyset,\ \forall y \in A.$$

These relations are equivalent with

$$z^* \circ F(x_1, y) \subseteq \sup z^*(W) + \mathbb{R}_+ \setminus \{0\}, \text{ , respectively}$$

$$z^* \circ F(x_2, y) \subseteq \sup z^*(W) + \mathbb{R}_+ \setminus \{0\}.$$

Since F is K-concave, for each $\lambda \in [0,1]$ we have:

$$F(\lambda x_1 + (1-\lambda)x_2, y) \subseteq \lambda F(x_1, y) + (1-\lambda)F(x_2, y) + K.$$

Also, we have $z^* \circ F(\lambda x_1 + (1-\lambda)x_2, y) \subseteq \lambda z^* \circ F(x_1, y) + (1-\lambda)z^* \circ F(x_2, y) + \mathbb{R}_+ \subseteq \sup z^*(W) + \mathbb{R}_+ \setminus \{0\}$ for $z^* \in Z_+^* \setminus \{0\}$, which means that $\lambda x_1 + (1-\lambda)x_2 \in S(VEP)^{W-K}_{z^*,\bar{r}_1}$, and thus, $S(VEP)^{W-K}_{z^*,\bar{r}_1}$ is convex.

In the additional hypothesis, for all $z^* \in Z_+^* \setminus \{0\}$, we have $\sup z^*(W) = z^*(\alpha)$ and $S(VEP)^{(\alpha + \mathrm{qi}\, Z_+)^c}_{z^*,\bar{r}_1} \subseteq S(VEP)^{W-K}_{z^*,\bar{r}_1} \subseteq S(VEP)^{W-K\setminus\{0\}}_{z^*,\bar{r}_1}$. We deduce that $S((VEP)^{W-K\setminus\{0\}}_{\bar{r}_1})$ is connected. Similarly, we get the conclusion for the case when W is compact. □

Corollary 1. *Let us suppose that $K = \mathrm{qi}\, Z_+ \cup \{0\}$, W is a convex compact set such that there exists $\alpha \in W$, which satisfies $\mathrm{cl}\,(W-K) = \mathrm{cl}\,(\alpha - K)$, $F(x,\cdot)$ has K-convex values for each $x \in A$, $F(\cdot, y)$ is K-concave, for each $y \in A$, F has K-closed values, and $F(x,y) \subseteq \alpha + \mathrm{qi}\, Z_+$ for all $y \in A$; then, $S((VEP)^{W-K}_{\bar{r}_1})$ is connected.*

The following theorem is similar.

Theorem 10. *Let us suppose that $K = \mathrm{qi}\, Z_+ \cup \{0\}$ or $K = \mathrm{qri}\, Z_+ \cup \{0\}$, $F(x,\cdot)$ has K-convex values for each $x \in A$ and $F(\cdot, y)$ is K-convex, for each $y \in A$. Then, for all $z^* \in Z_+^*$, $S(VEP)^{W-K}_{z^*,r_1}$ and $S(VEP)^{W-K\setminus\{0\}}_{z^*,r_1}$ are convex.*

If $K = \mathrm{qi}\, Z_+ \cup \{0\}$ and $S(VEP)^{W-K\setminus\{0\}}_{r_1}$ is nonempty, then $S(VEP)^{W-K\setminus\{0\}}_{r_1}$ is connected. If, in addition, we consider W a convex compact set and F has K-closed values, then $S(VEP)^{W-K}_{r_1}$ is connected.

Using Theorems 29 and 35, we get the following two theorems.

Theorem 11. *Let $K = \mathrm{qi}\, Z_+ \cup \{0\}$ or $K = \mathrm{qri}\, Z_+ \cup \{0\}$ and suppose that $F(x,\cdot)$ has K-convex values, for each $x \in A$ and $F(\cdot, y)$ is K-convex, for each $y \in A$. Then, for all $z^* \in Z_+^*$, $S(VEP)^{\mathrm{cl}\,(W-K)}_{z^*,r_2}$ are convex. If $S(VEP)^{\mathrm{cl}\,(W-K)}_{r_2}$ is nonempty, then $S(VEP)^{\mathrm{cl}\,(W-K)}_{r_2}$ is connected.*

Theorem 12. *Let us suppose that $K = \mathrm{qi}\, Z_+ \cup \{0\}$ or $K = \mathrm{qri}\, Z_+ \cup \{0\}$, $F(x,\cdot)$ has K-convex values, for each $x \in A$ and $F(\cdot, y)$ is K-concave, for each $y \in A$. Then, for all $z^* \in Z_+^*$, $S(VEP)^{\mathrm{cl}\,(W-K)}_{z^*,\bar{r}_2}$ are convex.*

If $K = \mathrm{qi}\, Z_+ \cup \{0\}$ and in addition we suppose that $\exists \alpha \in W$ such that $\mathrm{cl}\,(W-K) = \mathrm{cl}\,(\alpha - K)$ and $S(VEP)^{(\alpha + \mathrm{qi}\, Z_+)^c}_{\bar{r}_2} \neq \emptyset$, then $S((VEP)^{\mathrm{cl}\,(W-K)}_{\bar{r}_2})$ is connected.

For $K = \mathrm{qi}\, Z_+ \cup \{0\}$, the following theorem gives the link between the solutions set $S(VEP)^{W-Z_+\setminus\{0\}}_{r_i}$ and the solutions set $S(VEP)^{W-K\setminus\{0\}}_{z^*,r_i}$. Let us remark that if $z^* \in Z_+^\#$, the scalar problem $(VEP)^{W-K\setminus\{0\}}_{z^*,r_i}$ is equivalent to the scalar problem $(VEP)^{W-Z_+\setminus\{0\}}_{z^*,r_i}$.

Theorem 13. *Let us suppose that $K = \mathrm{qi}\, Z_+ \cup \{0\}$, $Z_+^\# \neq \emptyset$ and $F(x,\cdot)$ has K-convex values, for each $x \in A$. Then:*

$$\bigcup_{z^* \in Z_+^\#} S(VEP)^{W-K\setminus\{0\}}_{z^*,\bar{r}_1} \subseteq S(VEP)^{W-Z_+\setminus\{0\}}_{\bar{r}_1} \subseteq \bigcup_{z^* \in Z_+^* \setminus\{0\}} S(VEP)^{W-K\setminus\{0\}}_{z^*,\bar{r}_1}$$

$$\bigcap_{z^*\in Z_+^*\setminus\{0\}} S(VEP)_{z^*,r_1}^{W-Z_+\setminus\{0\}} \subseteq S(VEP)^{W-Z_+\setminus\{0\}} \subseteq \bigcap_{z^*\in Z_+^\#} S(VEP)_{z^*,r_1}^{W-K\setminus\{0\}}.$$

For the following relations, we may have $K = \mathrm{qi}\ Z_+ \cup \{0\}$ or $K = \mathrm{qri}\ Z_+ \cup \{0\}$.

$$\bigcap_{z^*\in Z_+^*\setminus\{0\}} S(VEP)_{z^*,r_2}^{W-K\setminus\{0\}} \subseteq S(VEP)_{r_2}^{W-Z_+\setminus\{0\}} \subseteq \bigcap_{z^*\in Z_+^\#} S(VEP)_{z^*,r_2}^{W-K\setminus\{0\}}$$

$$\bigcup_{z^*\in Z_+^\#} S(VEP)_{z^*,\bar{r}_2}^{\mathrm{cl}\,(W-K\setminus\{0\})} \subseteq S(VEP)_{\bar{r}_2}^{\mathrm{cl}\,(W-Z_+\setminus\{0\})} \subseteq \bigcup_{z^*\in Z_+^*\setminus\{0\}} S(VEP)_{z^*,\bar{r}_2}^{\mathrm{cl}\,(W-K\setminus\{0\})}.$$

Continuity properties for the solutions sets.

In the following, we generalize the corresponding results from [22] given for VEP with functions. Then, we say that a set sequence $(W_n)_{n\in\mathbb{N}} \subset \mathcal{P}_c(X)$ is weakly Painlevé–Kuratowski convergent to W (and we write $W_n \xrightarrow{wPK} W$) if $\lim\limits_{n\to+\infty} \sup z^*(W_n) = \sup z^*(W)$, $\forall z^* \in Z_+^* \setminus \{0\}$.

I. The continuity with respect to "W".

Theorem 14. *Let $z^* \in Z_+^* \setminus \{0\}$ and suppose that $F(\cdot, y)$ is K-concave for each $y \in A$ and $S(VEP)_{z^*,r_1}^{\{0\}-K} \neq \emptyset$. Then, the following multifunction:*

$$\mathcal{F} : \{W - K \mid W \in \mathcal{P}_c(X),\ \sup z^*(W) > 0\} \Longrightarrow S(VEP)_{z^*,r_1}^{W-K}$$

is l.s.c. with respect to the weak Painlevé–Kuratowski convergence.

Remark 2. $S(VEP)_{z^*,r_1}^{\{0\}-K} \neq \emptyset$ does hold if $S(VEP)_{r_1}^{\{0\}-K} \neq \emptyset$.

Proof of the Theorem 14. Let us suppose that S is not l.s.c. at W_0. Then, there exists $x_0 \in S(VEP)_{z^*,r_1}^{W_0-K}$, $V \in \mathcal{V}(0)$, $W_n - K \xrightarrow{wPK} W_0 - K$ such that:

$$(x_0 + V) \cap S(VEP)_{z^*,r_1}^{W_n-K} = \emptyset,\ \forall n \in \mathbb{N}.$$

Let us denote $\sup z^*(W_n) = \varepsilon_n$ for $n \in \mathbb{N}$. We observe that $\varepsilon_n \geq \varepsilon_0$ implies that $S(VEP)_{z^*,r_1}^{W_n-K} \supseteq S(VEP)_{z^*,r_1}^{W_0-K}$, which implies that $(x_0 + V) \bigcap S(VEP)_{z^*,r_1}^{W_0-K}$ is empty, which is false. Thus, for each n, $\varepsilon_n < \varepsilon_0$.

Let $x' \in S(VEP)_{z^*,r_1}^{-K}$. Since $\lim \varepsilon_n = \varepsilon_0$, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_0 - \varepsilon_{n_0} > 0$ and $\frac{\varepsilon_{n_0}}{\varepsilon_0} x_0 + \frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} x' \in x_0 + V$. We have $z^* \circ F(x_0, y) \subseteq \sup z^*(W_0) + \mathbb{R}_+$, for all $y \in A$, $z^* \circ F(x', y) \subseteq \mathbb{R}_+$, $\forall y \in A$ and $F(\cdot, y)$ is K-concave, which implies:

$$z^* \circ F(\frac{\varepsilon_{n_0}}{\varepsilon_0} x_0 + \frac{\varepsilon_0 - \varepsilon_{n_0}}{\varepsilon_0} x', y) \subseteq \varepsilon_{n_0} + \mathbb{R}_+,\ \forall y \in A.$$

Hence $(x_0 + V) \cap S(VEP)_{z^*,r_1}^{W_{n_0}-K} \neq \emptyset$, which is false. □

Similarly, we obtain the following theorems.

Theorem 15. *Let $z^* \in Z_+^* \setminus \{0\}$ and suppose that $F(\cdot, y)$ is K-convex for each $y \in A$ and $S(VEP)_{z^*,\bar{r}_1}^{\{0\}-K} \neq \emptyset$. Then, the following multifunction:*

$$\mathcal{F} : \{W - K \mid W \in \mathcal{P}_c(X),\ \sup z^*(W) > 0\} \Longrightarrow S(VEP)_{z^*,\bar{r}_1}^{W-K}$$

is l.s.c. with respect to the weak Painlevé–Kuratowski convergence.

Theorem 16. *Let $z^* \in Z_+^* \setminus \{0\}$ and suppose that $F(\cdot, y)$ is K-convex for each $y \in A$ and $S(VEP)_{z^*,r_2}^{\{0\}-K} \neq \emptyset$. Then, the multifunction:*

$$\mathcal{F} : \{W - K \mid W \in \mathcal{P}_c(X),\ \sup z^*(W) > 0\} \Longrightarrow S(VEP)_{z^*,r_2}^{W-K}$$

is l.s.c. with respect to the weak Painlevé–Kuratowski convergence.

Theorem 17. *Let $z^* \in Z_+^* \setminus \{0\}$ and suppose that $F(\cdot, y)$ is K-concave for each $y \in A$ and $S(VEP)_{z^*,\bar{r}_2}^{\{0\}-K} \neq \emptyset$. Then, the multifunction:*

$$\mathcal{F} : \{W - K \mid W \in \mathcal{P}_c(X),\ \sup z^*(W) > 0\} \Longrightarrow S(VEP)_{z^*,\bar{r}_2}^{W-K}$$

is l.s.c. with respect to the weak Painlevé–Kuratowski convergence.

II. The continuity with respect to z^*.

Theorem 18. *Let us suppose that A is a convex set, Z is a normed space, $F(\cdot, y)$ is K concave, $F(A, A)$ is a bounded subset of Z, and $S(VEP)_{r_1}^{\{0\}-K}$ is nonempty. Then, for a constant $c > 0$, the multifunction:*

$$\mathcal{G} : \{z^* \in Z_+^* \mid \sup z^*(W_0) = c\} \Longrightarrow S(VEP)_{z^*,r_1}^{W_0-K}$$

is l.s.c. with respect to the norm topology.

Proof. Since $S(VEP)_{r_1}^{\{0\}-K} \neq \emptyset$, we get $S(VEP)_{z^*,r_1}^{\{0\}-K} \neq \emptyset, \forall z^* \in Z_+^* \setminus \{0\}$. Let us suppose that $\mathcal{G}$ is not l.s.c. at z_0^*. Thus, $\exists x_0 \in S(VEP)_{z^*,r_1}^{\{0\}-K}$ and $V \in \mathcal{V}(0)$, $z_n^* \to z_0^*$, $z_n^* \in Z_+^*$ such that:

$$(x_0 + V) \cap S(VEP)_{z_n^*,r_1}^{W_0-K} = \emptyset,\ \forall n \in \mathbb{N}.$$

Since $\mathcal{F}$ is l.s.c. at W_0, we find $W_1 \in \{W' \mid W' = W - K;\ \sup z^*(W) > 0\}$ such that:

$$(x_0 + V) \cap S(VEP)_{z_0^*,r_1}^{W_1-K} \neq \emptyset.$$

Let $x' \in (x_0 + V) \cap S(VEP)_{z_0^*,r_1}^{W_1-K}$. We prove that there exists $n_0 \in \mathbb{N}$ such that $x' \in (x_0 + V) \cap S(VEP)_{z_{n_0}^*,r_1}^{W_0-K}$, which is false.

Let n_0 such that $\|z_{n_0}^* - z_0^*\| < \frac{\sup z_0^*(W_1) - \sup z_0^*(W_0)}{\sup\{\|z\|,\ z \in F(A,A)\}} = \mu$. Since $x' \in S(VEP)_{z_0^*,r_1}^{W_1-K}$, we have $z_0^* \circ F(x', y) \subseteq \sup z_0^*(W_1) + \mathbb{R}_+$. For each $z \in F(A, A)$ we have $|z_{n_0}(z) - z_0^*(z)| \leq \mu \sup\{\|z\|,\ z \in F(A, A)\} = \alpha = \sup z_0^*(W_1) - \sup z_0^*(W_0)$. We get $z_{n_0}^* \circ F(x', y) \subseteq z_0^* \circ F(x', y) + [-\alpha, \alpha] \subseteq \sup z^*(W_1) + [-\alpha, \alpha] + \mathbb{R}_+ \subseteq \sup z_0^*(W_0) + \mathbb{R}_+ \subseteq \sup z_{n_0}^*(W_0) + \mathbb{R}_+$. Hence, $x' \in (x_0 + V) \cap S(VEP)_{z_{n_0}^*,r_1}^{W_0-K}$, which is false. □

In the same manner, we can prove the following theorems.

Theorem 19. *Let us suppose that A is a convex set, $F(\cdot, y)$ is K concave, $F(A, A)$ is a bounded subset of Z, and $S(VEP)_{\bar{r}_2}^{\{0\}-K} \neq \emptyset$. Then, the multifunction $\mathcal{G} : \{z^* \in Z_+^* \mid \sup z^*(W_0) = c\} \Longrightarrow S(VEP)_{z^*,\bar{r}_2}^{W_0-K}$ is l.s.c. with respect to the norm topology.*

Theorem 20. *Let us suppose that A is a convex set, $F(\cdot, y)$ is K convex, $F(A, A)$ is a bounded subset of Z, and $S(VEP)_{\bar{r}_1}^{\{0\}-K} \neq \emptyset$. Then, the multifunction $\mathcal{G} : \{z^* \in Z_+^* \mid \sup z^*(W_0) = c\} \Longrightarrow S(VEP)_{z^*,\bar{r}_1}^{W_0-K}$ is l.s.c. with respect to the norm topology.*

Theorem 21. *Let us suppose that A is a convex set, $F(\cdot, y)$ is K convex, $F(A, A)$ is a bounded subset of Z, and $S(VEP)_{r_2}^{\{0\}-K} \neq \emptyset$. Then, the multifunction $\mathcal{G} : \{z^* \in Z_+^* \mid \sup z^*(W_0) = c\} \Longrightarrow S(VEP)_{z^*,r_2}^{W_0-K}$ is l.s.c. with respect to the norm topology.*

Remark 3. *The assumptions from the previous theorems provide the following proprieties:*

(i) *If $W_n - K \xrightarrow{wPK} W_0 - K$, then $S(VEP)_{z^*,r_i}^{W_0-K} \subseteq \mathrm{Li}\ S(VEP)_{z^*,r_i}^{W_n-K}$;*

(ii) *If $W_n - K \xrightarrow{wPK} W_0 - K$, then $S(VEP)_{z^*,\bar{r}_i}^{W_0-K} \subseteq \mathrm{Li}\ S(VEP)_{z^*,\bar{r}_i}^{W_n-K}$;*

(iii) *If $\|z_n^* - z_0^*\| \to 0$, then $S(VEP)_{z_0^*,r_i}^{W_0-K} \subseteq \mathrm{Li}\ S(VEP)_{z_n^*,r_i}^{W_0-K}$;*

(iv) *If $\|z_n^* - z_0^*\| \to 0$, then $S(VEP)_{z_0^*,\bar{r}_i}^{W_0-K} \subseteq \mathrm{Li}\ S(VEP)_{z_n^*,\bar{r}_i}^{W_0-K}$.*

Now, we are able to present the connectedness result for the solutions set $S(VEP)_{\bar{r}_1}^{\mathrm{cl}\,(W-Z_+\setminus\{0\})}$. Let us denote $M^* = \{z^* \in Z_+^* \mid \sup z^*(W_0) = c > 0\}$ and $M^\# = \{z^* \in Z_+^\# \mid \sup z^*(W_0) = c > 0\}$.

Theorem 22. *Let us suppose that A is a convex set, $Z_+^\# \neq \emptyset$ and $F(x, \cdot)$ has K-convex values for each $x \in A$, $F(\cdot, y)$ is K concave for each $y \in A$, $F(A, A) \subset Z$ is bounded and $S(VEP)_{\bar{r}_1}^{\{0\}-K} \neq \emptyset$. Then, $S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}$ and $S(VEP)_{\bar{r}_2}^{\mathrm{cl}\,(W-K\setminus\{0\})}$ are connected sets for $K = Z_+$ or $K = \mathrm{qri}\ Z_+ \cup \{0\}$.*

Proof. Following Theorem 13, we have:

$$\bigcup_{z^* \in Z_+^\#} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}} \subseteq S(VEP)_{\bar{r}_1}^{W-Z_+\setminus\{0\}} \subseteq \bigcup_{z^* \in Z_+^*\setminus\{0\}} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}.$$

Let x be an element in $S(VEP)_{\bar{r}_1}^{W-Z_+\setminus\{0\}}$; there exists $z^* \in Z_+^* \setminus \{0\}$ such that $x \in S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$. If we consider $u^* = \frac{cz^*}{\sup z^*(W)}$ we have $\sup u^*(W) = c$ and $x \in S(VEP)_{u^*,\bar{r}_1}^{W-K\setminus\{0\}}$. Thus:

$$\bigcup_{z^* \in M^\#} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}} \subseteq S(VEP)_{\bar{r}_1}^{W-Z_+\setminus\{0\}} \subseteq \bigcup_{z^* \in M^*} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$$

Following Theorem 18, we have:

$$\bigcup_{z^* \in M^*} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}} = \mathrm{cl}\,\Big(\bigcup_{z^* \in M^\#} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}\Big).$$

Since $S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$ is convex and $S(VEP)_{\bar{r}_1}^{\{0\}-K} \subseteq S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$ for each z^*, we obtain that $\bigcup_{z^* \in M^\#} S(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$ and $S(VEP)_{\bar{r}_1}^{W-Z_+\setminus\{0\}}$ are connected sets. Similarly, we obtain the connectedness of $S(VEP)_{\bar{r}_2}^{\mathrm{cl}\,(W-Z_+\setminus\{0\})}$. □

Using the results from this section, the following theorems give the link between the sequence of the solutions sets for a sequence of multifunctions and the solutions set for the limit of this sequence. Let $F_n,\ F : A \times A \Longrightarrow Z$; we say that F_n is weakly continuous Painlevé–Kuratowski convergent to F and we write $F_n \xrightarrow{wcPK} F$ if for $x_n \to x$, $z^* \circ F_n(x_n, A)$ is Painlevé–Kuratowski convergent to $z^* \circ F(x, A)$ for each $z^* \in Z_+^* \setminus \{0\}$.

Theorem 23. *For $n \in \mathbb{N}$, let $F_n \xrightarrow{wcPK} F$ and $W_n - K \setminus \{0\}$ is weakly Painlevé–Kuratowski convergent to $W - \setminus\{0\}$. Then, the following inclusion:*

$$\mathrm{Li}\ S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$$

does hold for $K \setminus \{0\} \in \{\text{qi } Z_+,\ \text{int } Z_+\}$ *(if nonempty) and* $z^* \in Z_+^* \setminus \{0\}$*, respectively, for* $K \setminus \{0\} \in \{\text{qri } Z_+,\ Z_+\}$ *and* $z^* \in Z_+^\# \setminus \{0\}$*.*

If in addition we have $K = \text{qi } Z_+ \cup \{0\}$*,* $Z_+^\# \neq \emptyset$*, and* $F(x,\cdot)$ *is* K*-convexlike on* A*, for each* $x \in A$*,* $F(\cdot, y)$ *is* K *concave and for each* $y \in A$*,* $F(A, A)$ *is a bounded subset of* Z *and* $S(VEP)_{r_1}^{-K} \neq \emptyset$*, then we have:*

$$\text{Li } S_{F_n}(VEP)^{W_n - C\setminus\{0\}} \subseteq S_F(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}},$$

for $C \in \{\text{qi } Z_+ \cup \{0\}, \text{qri } Z_+ \cup \{0\}, Z_+\}$*.*

Proof. Obviously, if $K \setminus \{0\} = \text{qi } Z_+$ or $K \setminus \{0\} = \text{int } Z_+$ and $z^* \in Z_+^* \setminus \{0\}$ (respectively, if $K \setminus \{0\} = \text{qri } Z_+$, or Z_+ and $z^* \in Z_+^\#$) and $x \in S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$, then $0 \in MIN^{\sup z^*(W)} z^* \circ F(x, A)$.

The hypothesis from the theorem and Proposition 3.1 [21] ensure that if $x_n \in S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}}$ and $x_n \to x$, then:

$$\text{Ls } MIN^{\sup z^*(W_n)} z^* \circ F_n(x_n, A) \subseteq MIN^{\sup z^*(W)} z^* \circ F(x, A)$$

which implies that $0 \in MIN^{\sup z^*(W)} z^* \circ F(x, A)$ and $x \in S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$. Thus:

$$\text{Li } S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}.$$

For the second part of the theorem, let us remark that we also have:

$$\text{Ls } S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$$

and

$$\text{Li } \bigcup_{z^* \in Z_+^* \setminus \{0\}} S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq \bigcup_{z^* \in Z_+^* \setminus \{0\}} \text{Ls } S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq$$
$$\subseteq \bigcup_{z^* \in Z_+^* \setminus \{0\}} S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}.$$

Thus, for $K \setminus \{0\} = \text{qi } Z_+$ or $K \setminus \{0\} = \text{int } Z_+$, we have:

$$\text{Li } S_{F_n}(VEP)^{W_n-K\setminus\{0\}} \subseteq S_F(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}.$$

Since $\text{Li } A_n = \text{Li cl } (A_n)$, the inclusions $A_n \subseteq B_n \subseteq \text{cl } (A_n)$ implies that $\text{Li } A_n = \text{Li } B_n$. Following Theorem 22 and the inclusion:

$$\text{Li } \bigcup_{z^* \in M^\#} S_{F_n}(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} \subseteq \bigcup_{z^* \in M^\#} S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}$$

we get:

$$\text{Li } S_{F_n}(VEP)^{W_n-Z_+\setminus\{0\}} \subseteq S_F(VEP)_{\bar{r}_1}^{W-Z_+\setminus\{0\}}$$

and:

$$\text{Li } S_{F_n}(VEP)^{W_n-\text{qri } Z_+} \subseteq S_F(VEP)_{\bar{r}_1}^{W-\text{qri } Z_+}.$$

□

Remark 4. Similarly, we obtain the previous theorem for the solutions set with respect to $\bar{r}_2$. Furthermore, if we take $F_n = F$, Theorems 14 and 23 give that if $F(\cdot, y)$ is K-concave for each $y \in A$ and $S(VEP)_{z^*,r_1}^{\{0\}-K} \neq \emptyset$ for $K = \text{qi } Z_+ \cup \{0\}$, $z^* \in Z_+^* \setminus \{0\}$ then:

$$\text{Li } S_F(VEP)_{z^*,\bar{r}_1}^{W_n-K\setminus\{0\}} = S_F(VEP)_{z^*,\bar{r}_1}^{W-K\setminus\{0\}}.$$

This equality also holds if $K = \text{qri}\, Z_+ \cup \{0\}$ or $K = Z_+$ and $z^* \in Z_+^\#$.

Closing conditions for the solutions set.

Theorem 24. *Let $F : A \times A \rightrightarrows Z$ and let $W \subset Z$ and K such that $(W - K \setminus \{0\})^c$ is closed. If F is lower semicontinuous then $S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}$ is closed. If F is upper semicontinuous and F has compact values then $S(VEP)_{\bar{r}_2}^{W-K\setminus\{0\}}$ is closed.*

Proof. Let $(x_n)_n \subset S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}$ and $x_n \to x$. If F is lower semicontinuous, since $F(x,y,t) \subseteq \text{Li}\, F(x_n,y,t) \subseteq (W - K \setminus \{0\})^c$, we get that $x \in S(VEP)_{\bar{r}_1}^{W-K\setminus\{0\}}$.

If F is upper semicontinuous, for all $\alpha_n \in F(x_n,y,t) \cap (W - K \setminus \{0\})^c \neq \emptyset$ there exists $\alpha \in F(x,y,t)$ and $(\alpha_{n_k})_k$ a subsequence of α_n such that $\alpha_{n_k} \to \alpha$. Since $(W - K \setminus \{0\})^c$ is closed we get $\alpha \in F(x,y,t) \cap (W - K \setminus \{0\})^c \neq \emptyset$, which implies that $x \in S(VEP)_{\bar{r}_2}^{W-K\setminus\{0\}}$. □

5. Applications for Abstract Economies

An abstract economy is a family of ordered triples $\Gamma = (X_i, A_i, P_i)_{i\in I}$ where:

- I is countable set of agents;
- $X_i \subset Z$ is a nonempty set of actions for agent i, $X = \prod_{i\in I} X_i$;
- $A_i : X \rightrightarrows X_i$ is the constraint correspondence of agent i;
- $P_i : X \rightrightarrows X_i$ is the preference correspondence of agent i.

An equilibrium of Γ is a point $x^* \in X$ such that $\forall i \in I$, $x_i^* \in \text{cl}\; A_i(x^*)$ and $P_i(x^*) \cap A_i(x^*) = \emptyset$. Let us denote $\psi_i(x) = A_i(x) \cap P_i(x)$ for all $x \in X$.

Let us recall some notions which will be useful for our results.

Definition 1.

(i) *A multifunction $F : X \rightrightarrows Z$ has lower open section (shortly los), if for any $z \in Z$, the set $\{x \in X \mid z \in F(x)\}$ is open in Z. The multifunction F has upper open section (uos) if for all $x \in X$, the set $F(x)$ is open in Z.*

(ii) *The multifunction F is $\mathcal{L}$-majorized if there exists a multifunction $\tilde{F}$ such that $\tilde{F}$ has los, $x \notin \text{conv}\, \tilde{F}(x)$ and $F(x) \subseteq \tilde{F}(x)$, $\forall x \in X$.*

The following notions generalize the "open lower sections" (ols) multifunctions and are motivated by the equilibrium existence theorem for abstract economies where the preference and the constraint multifunctions are not continuous.

Definition 2 ([24])**.** *The multifunction $F : X \rightrightarrows Z$ has the "continuous inclusion property" (CIP) at x if there exist an open neighborhood U_x of x and a nonempty multifunction $H_x : U_x \rightrightarrows Z$ such that $F_x(u) \subseteq F(u)$ for any $u \in U_x$ and $\text{co}\, H_x$ has closed graph.*

Definition 3 ([25])**.** *Let $M, S : A \rightrightarrows X$; the multifunction S is M-majorized if there exists a multifunction $\tilde{S}$ such that $S(x) \subseteq \tilde{S}(x) \subseteq M(x)$, for all $x \in A$, $\tilde{S}$ has the CIP property for each $x \in \text{dom}\, S$ and $x \notin \text{co}\, \tilde{S}(x)$.*

We denote by $\mathcal{M}(S)$ the set of multifunctions $M : A \rightrightarrows X$ such that S is M-majorized. A multifunction $S : A \rightrightarrows X$ is M-majorized if $\mathcal{M}(S) \neq \emptyset$.

The following theorem from [26] gives an existence result for the equilibrium of an abstract economy.

Theorem 25 ([26])**.** *Let $\Gamma = (X_i, A_i, P_i)_{i\in I}$ be an abstract economy such that for each $i \in I$:*

(i) *X_i is a nonempty compact convex metrizable subset of Z;*

(ii) A_i *has nonempty convex values;*
(iii) *The correspondence* cl A_i *is upper semicontinuous;*
(iv) A_i *has open lower section;*
(v) P_i *has open lower section;*
(vi) $x_i \notin \text{co}\psi_i(x)$ *for all* $x \in X$.

Then, Γ *has an equilibrium.*

This theorem can be obtained from the Theorem 2 by taking the following particular sets and multifunctions: $A = X_i$, $T(x) = \{x\}$, $K(x) = A_i(x)$, $F(x,y) = y$ if $y \in P_i(x)$ and $F(x,y) = \emptyset$ if $y \notin P_i(x)$. The sets $N_{t,x}^{i,W} = \psi_i(x)$, and $N_y^{i,W} = (P_i^{-1}(y))^c = \{x \in A \mid \exists t \in T(x),\ r_i(F(x,y,t), W)\}$. Thus, (ii), (iii), and (iv) are equivalent with (iii) from Theorem 2; (v) is equivalent with (ii); and the condition (vi) is equivalent with (i) from the same theorem. Furthermore, we remark that the hypothesis "metrizable" can be omitted.

Theorem 25 generalizes the following result from [27], which in turn was generalized in [24].

Theorem 26 ([27])**.** *Let* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy such that for each* $i \in I$*:*

(i) X_i *is a nonempty compact convex subset of* $\mathbb{R}^n$*;*
(ii) A_i *has nonempty convex compact values;*
(iii) A_i *is a continuous correspondence;*
(iv) P_i *has an open graph;*
(v) $x_i \notin \text{co}\psi_i(x)$ *for all* $x \in X$.

Then, Γ *has an equilibrium.*

The following theorem gives a generalization for the case of functions with the CIP property.

Theorem 27 ([24])**.** *Let* $\Gamma = (X_i, A_i, P_i)_{i \in I}$ *be an abstract economy such that for each* $i \in I$*:*

(i) X_i *is a nonempty compact convex metrizable subset of a Hausdorff locally convex space;*
(ii) A_i *has nonempty convex values;*
(iii) *The correspondence* cl A_i *is upper semicontinuous;*
(iv) ψ *has CIP at each* $x \in X$ *with* $\psi_i(x) \neq \emptyset$*;*
(v) $x_i \notin \text{co}\psi_i(x)$ *for all* $x \in X$.

Then, Γ *has an equilibrium.*

This theorem is based on the following fixed-point theorem.

Theorem 28 ([24])**.** *Let* X *be a nonempty compact convex metrizable subset of a Hausdorff locally convex space and* $\psi : X \rightrightarrows X$ *be a correspondence which is nonempty convex valued and has the CIP property. Then, there exists a point* $x^* \in X$ *such that* $x^* \in \psi(x^*)$.

Theorem 27 suggests a generalization for the existence theorem for the (VEPs) with multifunctions with the CIP property. The result concerning the existence of equilibrium for the abstract economies may be obtained as a particular case.

Lemma 1. *Let* $G, H : A \rightrightarrows A$, $E \subset A$ *be a closed set and for each* $x \in A$, $G(x) \subseteq H(x)$ *and* G *has the CIP property at* $x \in \text{dom}\, G$*. Then, the multifunction* $Q : A \rightrightarrows A$, $Q(x) = G(x)$ *for* $x \in E$ *and* $Q(x) = H(x)$ *if* $x \in A \setminus E$ *has the CIP property at* $x \in \text{dom}\, Q$.

Theorem 29. *Let* X *be a Hausdorff topological space and* $A \subseteq X$ *be a nonempty compact convex set, and we suppose:*

(i) co Φ *has CIP at each* $x \in X$ *with* $\Phi(x) \neq \emptyset$*, where* $\Phi : A \rightrightarrows X$ *is given by* $\Phi(x) = \{y \in A \cap \text{cl}\, K(x) \mid \forall t \in T(x),\ \bar{r}_i(F(x,y,t), W)\}$*;*

(ii) $x \notin \text{co}\Phi(x)$, *for all* $x \in X$;
(iii) $\text{cl } K(\cdot)$ *is u.s.c.* $\forall x \in A$, $A \cap \text{cl } K(x) \neq \emptyset$, $K(x)$ *is convex* $\forall x \in A$.

Then, $(VEP_1)^W_{r_i}$ *has a solution.*

Proof. Let $E = \{y \in A \mid y \in \text{cl } K(y)\}$ and $Q : A \Longrightarrow A$ be a multifunction given by $Q(x) = \text{co } \Phi(x)$ if $x \in E$ and $Q(x) = A \cap \text{cl } K(x)$ for $x \in A \setminus E$. We observe that E is closed since $\text{cl } K$ is u.s.c. and for each $x \in A$, $Q(x)$ is convex, $x \notin Q(x)$ and lemma implies that Q has CIP. Following Theorem 35 we get $\bar{x} \in A$ such that $Q(\bar{x}) = \emptyset$. Since $A \cap \text{cl } K(x) \neq \emptyset$ for all $x \in A$ we get $\bar{x} \in E$, and thus, $\text{co } \Phi(\bar{x}) = \emptyset$. The conclusion follows now obviously. □

Let us remark that if $A = X_i$, $T(x) = \{x\}$, $K(x) = A_i(x)$, $F(x, y) = y$ if $y \in P_i(x)$ and $F(x, y) = \emptyset$ if $y \notin P_i(x)$, then $\Phi = \psi$ and this theorem becomes Theorem 27 of Yannelis and He for abstract economies.

Furthermore, let us remark that $\Phi(x) = \bigcap\limits_{t \in T(x)} \bar{N}^{i,W}_{t,x}$ and $N^{i,W}_y = (\Phi^{-1}(y))^c$ from Theorem 2; the assumptions of Theorem 2 provide that Φ has los, and thus, Φ and $\text{co } \Phi$ have CIP.

The existence Theorem 29 may be generalized for $\mathcal{M}$-majorized multifunctions as follows.

Theorem 30. *Let X be a Hausdorff topological space, $A \subseteq X$ be a nonempty compact convex set and we suppose:*

(i) $\Phi : A \Longrightarrow X$, $\Phi = \{y \in A \cap \text{cl } K(x) \mid \forall t \in T(x),\ \bar{r}_i(F(x, y, t), W)\}$ *is $\mathcal{M}$-majorized and* $A \cap \text{cl } K(\cdot) \in \mathcal{M}(\Phi)$;
(ii) $\text{cl } K(\cdot)$ *is u.s.c.;* $\forall x \in A$, $A \cap \text{cl } K(x) \neq \emptyset$, $K(x)$ *is convex;* $\forall x \in A$.

Then, $(VEP_1)^W_{r_i}$ *has a solution.*

Proof. Let us denote by $\tilde{\Phi}$ the multifunction which has the CIP property for each $x \in \text{dom}S$, $x \notin \text{co } \tilde{S}(x)$ and $\Phi(x) \subseteq \tilde{\Phi}(x) \subseteq A \cap \text{cl } K(x)$. The proof is similar to that of Theorem 29 by replacing Φ with $\tilde{\Phi}$. □

For the case of a noncompact set of actions, we have from [28] the following theorem for the existence of an equilibrium for an abstract economy.

Theorem 31 ([28])**.** *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$:*

(i) *X_i is a nonempty convex metrizable subset of Z;*
(ii) *$A_i(x)$ is a nonempty convex set for all $x \in X$;*
(iii) *The correspondence* $\text{cl } A_i$ *is upper semicontinuous and* $\text{cl } A_i(x)$ *is compact for all $x \in X$;*
(iv) *A_i has an open lower section;*
(v) *P_i has an open lower section;*
(vi) $x_i \notin \text{co}\psi_i(x)$ *for all* $x \in X$.
(vii) *There exists $C_i \subset X_i$, nonempty, compact, convex sets such that:*
$A_i(C) \subseteq D_i$, *where* $C = \prod\limits_{i \in I} C_i$, $D_i \subset X_i$ *is compact and convex;*
$A_i(x) \cap Z_i \neq \emptyset$, $\forall x \in X_{-i} \times Z_i$, $Z_i = \text{cl co}(D_i \cup C_i)$;
$\forall x_i \in Z_i \setminus C_i$, $x_{-i} \in X_{-i}$, *there exists* $y_i \in A_i(x) \cap Z_i \cap P_i(x)$.

Then, Γ has an equilibrium.

In what follows, we present a similar result for the case of correspondence with the CIP property which extends to noncompact case Theorem 2 from [24].

Theorem 32. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy such that for each $i \in I$*

(i) *X_i is a nonempty convex metrizable subset of Z;*
(ii) *$A_i(x)$ is a nonempty convex set for all $x \in X$;*
(iii) *The correspondence* $\text{cl } A_i$ *is upper semicontinuous and* $\text{cl } A_i(x)$ *is compact for all $x \in X$;*

(iv) ψ_i *has CIP;*
(v) $x_i \notin \mathrm{co}\psi_i(x)$ *for all* $x \in X$.
(vi) *There exists* $C_i \subset X_i$*, nonempty, compact, convex sets such that:*
$A_i(C) \subseteq D_i$*, where* $C = \times_{i\in I} C_i$*,* $D_i \subset X_i$ *is compact and convex;*
$A_i(x) \cap Z_i \neq \emptyset, \forall x \in X_{-i} \times Z_i$*,* $Z_i = \mathrm{cl}\,\mathrm{co}(D_i \cup C_i)$*;*
$\forall x_i \in Z_i \setminus C_i$*,* $x_{-i} \in X_{-i}$*, there exists* $y_i \in A_i(x) \cap Z_i \cap P_i(x)$*.*
Then, Γ *has an equilibrium.*

Proof. Let $T = \prod\limits_{i\in I} Z_i$ and for each $i \in I$, $K_i : T \Longrightarrow Z_i$, $K_i(x) = A_i(x) \cap Z_i$, $\forall x \in T$. Since cl A_i has closed graph, T, Z_i are compact sets, then cl K_i has closed graph, compact values, and thus, cl K_i is u.s.c.

We remark that $K_i(x) = A_i(x)$, $\forall x_i \in C_i$ and $K_i(x) = A_i(x) \cap Z_i$, otherwise. Let $\mathrm{Gr}\tilde{P}_i = \mathrm{cl}\,\mathrm{Gr}P_i$. We remark that $\tilde{\psi}_i(x) = K_i \cap \tilde{P}_i(x) = A_i(x) \cap \tilde{P}_i(x) \cap Z_i$ has a closed graphic, which implies that $\tilde{\psi}_i$ has the CIP property.

The abstract economy $\tilde{\Gamma} : (Z_i, K_i, \tilde{P}_i\,|_T)_{i\in I}$ satisfies the conditions from Theorem 2 [24], and thus, there exists $\bar{x} \in \mathrm{cl}\,K_i(\bar{x})$ and $\tilde{P}_i(\bar{x}) \cap K_i(x) = \emptyset$ which yields $P_i(\bar{x}) \cap K_i(\bar{x}) = \emptyset$, and thus, $\bar{x} \in C_i$, cl $K_i(\bar{x}) = \mathrm{cl}\,A_i(\bar{x})$. We conclude $\bar{x} \in \mathrm{cl}\,A_i(\bar{x})$, $P_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset$, and thus, $\bar{x}$ is an equilibrium for Γ. □

In what follows, we present some results concerning the existence for the Walrasian equilibrium in a vector exchange economy.

A vector exchange economy, $\mathcal{E}$ is a family $(X_i, P_i, e_i)_{i\in I}$ such that I is a finite set of agents and for every $i \in I$:

- $X_i \subseteq Z_+$ is the consumption set of agent i and $X = \prod\limits_{i\in I} X_i$;
- An element $x_i \in X_i$ will be called an allocation for agent il;
- $P_i : X \Longrightarrow X_i$ is the preference correspondence of agent i;
- $e_i \in X_i$ is the initial endowment of agent i.

We suppose that Z_+^* has a compact base B_+^*.

The budget set of agent i is $B_i(z^*) = \{x_i \in X_i \mid z^*(x_i) \leq z^*(e_i)\}$ for $z^* \in B_+^*$. For $z^* \in B_+^*$, $x \in X$, $i \in I$, let $\psi_i(z^*, x) = B_i(z^*) \cap P_i(x)$, the set of allocations in the budget set of agent i at price z^* that he prefers to x.

A free disposal Walrasian equilibrium for the exchange economy $\mathcal{E}$ is $(\bar{z}^*, \bar{x}) \in B_+^* \times X$ such that:

1. For each $i \in I$, $\bar{x}_i \in B_i(\bar{z}^*)$ and $\psi_i(\bar{z}^*, \bar{x}) = \emptyset$;
2. $\sum\limits_{i\in I} \bar{x}_i \leq \sum\limits_{i\in I} e_i$.

Theorem 33. *Let* $\mathcal{E}$ *be an exchange economy satisfying the following assumptions for each* $i \in I$*:*

(i) X_i *is a nonempty compact convex subset of* Z_+*;*
(ii) *For all* $(z^*, x) \in (B_+^* \times X) \cap \mathrm{dom}\,\psi_i$*,* $x_i \notin \mathrm{co}\,\psi_i(z^*, x)$ *and* ψ *is* $\mathcal{M}$*-majorized at* (z^*, x)*.*
Then, $\mathcal{E}$ *has a free disposal Walrasian equilibrium.*

Proof. Let $I_0 = I \cup \{0\}$ and for each $i \in I$, $z^* \in B_+^*$, $x \in X$, let:

$$A_i(z^*, x) = B_+^* \times B_i(z^*)$$

$$\tilde{P}_i(z^*, x) = B_+^* \times P_i(x)$$

and for $A_0(z^*, x) = B_+^* \times X$:

$$\tilde{P}_0(z^*, x) = \{q^* \in B_+^* \mid q_i^*(\sum_{i\in I}(x_i - e_i)) > z^*(\sum_{i\in I}(x_i - e_i))\} \times X.$$

Since $i \in I_0$, A_i are nonempty convex valued, u.s.c. on $B_+^* \times X$. For $i \in I$, $\tilde{\psi}_i(z^*, x) = A_i(z^*, x) \cap \tilde{P}_i(z^*, x) = B_+^* \times (B_i(z^*) \cap P_i(x)) = B_+^* \times \psi_i(z^*, x)$ and since ψ is $\mathcal{M}$-majorized so is $\tilde{\psi}$.

For $i = 0$:

$$\tilde{\psi}_0(z^*, x) = A_0(z^*, x) \cap \tilde{P}_0(z^*, x) = \tilde{P}_0(z^*, x) =$$
$$= \{q^* \in B_+^* \mid q^*(\sum_{i \in I}(x_i - e_i)) > z^*(\sum_{i \in I}(x_i - e_i))\} \times X.$$

For $(q^*, v) \in \tilde{\psi}_0(z^*, x)$, let $G(z^*, x) = (q^*, v)$ which is a constant, and thus, a continuous multifunction. There exists $O \in \mathcal{V}(z^*, x)$ such that $G(z^*, x) \subset \tilde{\psi}_0(z^*, x)$ for each $(z^*, x) \in O$, which implies that $\tilde{\psi}_0$ has CIP, and thus, $\tilde{\psi}_0$ is $\mathcal{M}$-majorized.

For the abstract economy $(B_+^* \times X_i, A_i, \tilde{P}_i)_{i \in I_0}$, we may apply Theorem 30 and we get $(\bar{z}^*, \bar{x}) \in B_+^* \times X$ such that $(\bar{z}^*, \bar{x}_i) \in A_i(\bar{z}^*, \bar{x}) = B_+^* \times B_i(\bar{z}^*)$ and $\tilde{\psi}(\bar{z}^*, \bar{x}) = \emptyset$ for each $i \in I_0$. Thus, for $i \in I$, $\psi(\bar{z}^*, \bar{x}) = \emptyset$.

Since $\tilde{\psi}_0(\bar{z}^*, \bar{x}) = \emptyset$, for each $q^* \in B_+^*$, we have:

$$q^*(\sum_{i \in I}(x_i - e_i)) \leq z^*(\sum_{i \in I}(x_i - e_i)) \leq 0$$

and finally, $\sum_{i \in I}(x_i - e_i) \leq 0$. Therefore, $(\bar{z}^*, \bar{x})$ is a free disposal Walrasian equilibrium. □

In what follows, we present a result concerning the existence of a nonfree disposal Walrasian equilibrium.

Let us denote $B_i(z^*) = \{x_i \in X_i \mid z^*(x_i) \leq z^*(e_i) + 1 - \|z^*\|\}$ for $z^* \in B_+^*$, $\psi(z^*, x) = P_i(x) \cap B_i(z^*)$ and $K = \{x \in X \mid \sum_{i \in I} x_i = \sum_{i \in I} e_i\}$.

A nonfree disposal Walrasian equilibrium is an element $(\bar{z}^*, \bar{x}) \in B_+^* \times X$ such that:

1. for each $i \in I$, $\bar{x} \in B_i(\bar{z}^*)$, $\psi_i(\bar{z}^*, \bar{x}) = \emptyset$;
2. $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$.

Theorem 34. *Let $\mathcal{E}$ be an exchange economy satisfying the following assumptions for each $i \in I$:*

(i) *X_i is a nonempty compact convex subset of Z_+;*
(ii) *For each $(z^*, x) \in (B_+^* \times X) \cap \mathrm{dom}\, \psi_i$ we have $x_i \notin \mathrm{co}\, \psi_i(z^*, x)$ and ψ is $\mathcal{M}$-majorized at (z^*, x).*
(iii) *$x_i \in \mathrm{pr}_i K$ implies that $x_i \in \mathrm{Fr}\, P_i(x)$.*

Then, $\mathcal{E}$ has a nonfree disposal Walrasian. equilibrium.

Proof. Similar to the first part of the proof in the case "free disposal", we get $(\bar{z}^*, \bar{x}) \in B_+^* \times X$ such that:

1. $\bar{x}_i \in B_i(\bar{z}^*)$;
2. $\psi_i(\bar{z}^*, \bar{x}) = \emptyset$, $\forall i \in I_0$.

From 1, we obtain $\bar{z}^*(\bar{x}_i) \leq z^*(e_i) + 1 - \|z^*\|$. Let $z = \sum_{i \in I}(x_i - e_i)$ and we suppose that $z \neq 0$. From 2, for $i = 0$, we have $q^*(z) \leq \bar{z}^*(z)$ for each $q^* \in B^*$. Let $q^* = \frac{1}{\|\bar{z}^*\|}\bar{z}^* \in B_+^*$; we get $\frac{\bar{z}^*(z)}{\|z\|} \leq \bar{z}^*(z)$, which yields $\|\bar{z}^*\| \geq 1$ and since $\bar{z}^* \in B_+^*$, we conclude that $\|\bar{z}^*\| = 1$. Using 1, we obtain that $\bar{z}^*(z) \leq 0$, and thus, $q^*(z) \leq 0$ for each $q^* \in B_+^*$. However, since $z \neq 0$, we find $q^* \in B_+^*$ such that $q^*(z) = \|q^*\| > 0$, which is false. Thus, $z = 0$ and $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, $\bar{x} \in K$. Note that $\bar{x}_i \in \mathrm{pr}_i K$ implies that $\bar{x}_i \in \mathrm{Fr}\, P_i(x)$. Since $\bar{x}_i \in B_i(\bar{z}^*)$ and $\bar{x}_i \notin \mathrm{co}\psi_i(\bar{z}^*, \bar{x})$, $\bar{x}_i \notin P_i(\bar{x})$. If there exists i such that $\bar{z}^*(\bar{x}_i) < \bar{z}^*(e_i) + 1 - \|\bar{z}^*\|$, one can find a point $y_i \in P_i(\bar{x})$ such that $\bar{x}_i$ and y_i are sufficiently close, and $\bar{z}^*(y_i) < \bar{z}^*(e_i) + 1 - \|\bar{z}^*\|$. Thus, $y_i \in \psi_i(\bar{z}^*, \bar{x})$, which contradicts 2. Therefore, $\bar{z}^*(\bar{x}_i) = z^*(e_i) + 1 - \|z^*\|$ for each

$i \in I$, and summing up over all i yields $\|z^*\| = 1$. Finally, we conclude $(\bar{z}^*, \bar{x})$ is a nonfree disposal Walrasian equilibrium. $\square$

Let us remark that if $Z = \mathbb{R}^l$, we get Theorems 3 and 4 from [24].

Finally, we remark that the notion of Walrasian equilibrium was extended for the case of a vector exchange economy with an infinite number of agents and commodities, both for the additive case and for the nonadditive one. For results concerning the existence of the Walrasian equilibrium and the core-Walras equivalence with linear and nonlinear prices, the reader may consult [29–31].

6. Applications of VEPs to the Vector Optimization Problems

In this section, we intend to find a vector optimization problem such that the solutions of the vector equilibrium problem are solutions for this vector optimization problem. Let $F : A \times A \rightrightarrows Z$ such that $0 \in F(x,x)$ and consider the problem $(VEP)_{\bar{r}_1}^{-K}$:

$$(VEP)_{\bar{r}_1}^{-K}: \text{ find x} \in \text{A such that } F(x,y) \cap -K = \emptyset \; \forall y \in A.$$

Let us recall the definition for the principal efficient points and the domination property which will be used in the following. For more details concerning this subject, see [32].

Let $K \subsetneq Z_+ \setminus \{0\}$, $A \subset Z$, $\varepsilon \in Z_+$. The following sets:

$$K\ INF\ A = \{y \in \bar{Z} \mid y - a \notin K,\ \forall a \in A\};$$

$$K\ INF_1 A = \{y \in K\ INF\ A \mid\ \forall y' \in \bar{Z},\ y' - y \in K,\ \exists a \in A,\ y' - a \in K\};$$

$$K\ MIN^{\varepsilon}\ A = \{a \in A \mid \nexists a' \in A,\ a' - a + \varepsilon \in -K\}$$

will be the K-infimal, the K-approximative infimal, and the K-ε minimal points sets of A, respectively. Similarly, we may consider the K-suprемal, K-approximative supremal, and the K-ε maximal points sets of A denoted $K\ SUP\ A$, $K\ SUP_1 A$, and $K\ MAX^{\varepsilon}\ A$.

We remark that if $K = Z_+ \setminus \{0\}$, the K-efficient points becomes the efficient points sets given in [33]. We will denoted these points by $INF\ A$, $INF_1 A$, and ${}^{\varepsilon}MIN\ A$, respectively, $MIN\ A$ if $\varepsilon = \{0\}$. If the interior of the cone is nonempty and $K = \text{int}\ Z_+$, the K-efficient points are the weak efficiencies denoted $wEFF\ A$, i.e., $wINF\ A$, $wINF_1 A$, $w\,{}^{\varepsilon}MIN\ A$, and $wMIN\ A$ if $\varepsilon = \{0\}$, respectively.

In what follows, the efficient points are considered by respect to the cone $K \subsetneq Z_+$ which ensure that the following domination property does holds.

Theorem 35. *Let $A \subset Z$, $K = \text{ri}\ \ A$, $\text{iri}\ \ A$, $\text{int}\ \ A$ and $X = \text{cone}\,(Z_+ - x)$, $x \in K$. Let us denote $B = \{a \in A \mid K\ INFA \bigcap (a + X) \neq \emptyset\}$ and let us suppose that $B \neq \emptyset$. Then, $\emptyset \neq K\ INF_1 A \subset A + X$ and the following domination properties hold:*

$$B \subset K\ INF_1 A + K$$

$$K\ INFA \bigcap (B + X) = (K\ INF_1 A - K).$$

Obviously, if x is a solution for (VEP) then x is a solution for the optimization problem (P_x):

$$(P_x): \quad K\ INF_1 \bigcup_{y \in A} F(x,y) = K\ INF_1 \bigcup_{y \in A} F_x(y) = v(P_x)$$

The dual problem is (D_x):

$$K\ SUP_1 \bigcap_{T \in \mathcal{L}(X,Z)} -F_x^* \mid A(T) = v(D_x),$$

where $F_x^* \mid A(T) = K\ SUP_1\{T(y) - F_x(y) \mid y \in A\}$.

We say that P_x is stable if $v((P_x)) = v((D_x)) = K\ MAX(D_x)$.

Let us consider now the perturbed problem:

$$(P_a(F))\ \ K\ INF_1\{\Phi(y,u),\ y \in X\}$$

where $\Phi(y,u) = F_x(y+u)$ if $y+u \in A$ and $\Phi(y,u) = +\infty$ else. We have $\Phi^*(0,T) = SUP_1\{F_x^* \mid_A (T) + \{-T(y) \mid y \in A\}\}$. The dual problem is:

$$(D_a(F))\ \ K\ SUP_1 \bigcup_{T \in \mathcal{L}(X,Z)} -\Phi^*(0,T).$$

Let us denote $\gamma(x) = \bigcup_{T \in \mathcal{L}(X,Z)} K\ INF_1\{-F_x^* \mid_A (T) + \{T(y) \mid y \in A\}\}$.

Let us consider the vector equilibrium problem $(VEP)_{r_2}^{W-K}$.

$$(VEP)_{r_2}^{W-K}:\ \text{find x} \in \text{A such that } F(x,y) \subseteq W - K \setminus \{0\}\ , \forall y \in A.$$

We denote $-F(x,y) = G(x,y)$ and $-W = W'$ and the problem becomes:

$$(VEP_G)_{r_2}^{W'+K}:\ \text{find x} \in \text{A such that } G(x,y) \subseteq W' + K \setminus \{0\}\ \forall y \in A.$$

We remark that if x is a solution for $(VEP)_{r_2}^{W-K}$, then x is a solution for the problems:

$$(IVEP)_{r_2}^{W'-\varepsilon+K\setminus\{0\}}\ \text{find x} \in \text{A such that }\ K\ INF_1 \bigcup_{y \in A} G(x,y) \subseteq W' - \varepsilon + K \setminus \{0\}$$

for all $\varepsilon \in K \setminus \{0\}$. We consider the dual problem for $\alpha \in W'$:

$$(P^\alpha)\ \ K\ SUP_1 \bigcup_{u \in A} \gamma(u)$$

where $\gamma(u) = \bigcup_{T \in \mathcal{L}(X,Z)} K\ INF_1\{-G_x^* \mid_A (T) + \{T(y) \mid y \in A\}\}$.

The element x is a solution for (P^α) if there exists $z \in \gamma(x)$ such that $(K\ SUP_1 \bigcup_{u \in A} \gamma(u)) \bigcap (z + \alpha + K) = \emptyset$. We denote the set of these solutions by $S(P^\alpha)$. Let us remark that if $x \in S(P^\alpha)$, then $(\bigcup_{u \in A} \gamma(u)) \bigcap (z + \alpha + K) = \emptyset$ and thus $\gamma(x) \bigcap K\ MAX^\alpha \bigcup_{u \in A} \gamma(u) \neq \emptyset$, that is:

$$x \text{ is an } \alpha - \text{solution for } (P):\ \ K\ MAX^\alpha \bigcup_{u \in A} \gamma(u).$$

Theorem 36. *If $(P_a(G))$ is stable, then there exists $\alpha \in W$ such that for all $\alpha\prime > \alpha$ we have:*

$$S(VEP)_{r_2}^{W-K} \subseteq S(P^{\alpha\prime}).$$

Proof. Let $x \in S(VEP)_{r_2}^{W-K}$. Thus, $INF_1 \bigcup_{y \in A} G(x,y) \subseteq W' - \varepsilon + K \setminus \{0\}$. If $(P_a(G))$ is stable, then $K\ INF_1 \bigcup_{y \in A} G(x,y) = K\ MAX\gamma(x) \neq \emptyset$. Since $0 \in F(x,x)$, then for all $z \in (K\ INF_1 \bigcup_{y \in A} G(x,y)) = K\ MAX\gamma(x)$, $z \not> 0$. Furthermore, following the domination property, there exists $z_0 \in K\ INF_1 \bigcup_{y \in A} G(x,y) = \ MAX\gamma(x)$ such that $z_0 < 0$ and there exists $\alpha \in W$ such that $-\alpha - \varepsilon < z_0 < 0$. We remark that for all $z \in K\ MAX\gamma(u) = K\ INF_1 \bigcup_{y \in A} G(u,y)$, $z \not> 0$.

$$(\bigcup_{u \in A} K\ MAX\gamma(u)) \bigcap (z_0 + \alpha + \varepsilon + K) = \emptyset$$

since $z_0 + \alpha + \varepsilon > 0$. The conclusion follows obviously. $\square$

Let us consider the vector equilibrium problem $(VEP)_{r_1}^{W-K\setminus\{0\}}$:

$$(VEP)_{r_1}^{W-K\setminus\{0\}} : \text{ find x} \in \text{A such that } F(x,y)\bigcap(W - K\setminus\{0\}) \neq \emptyset,\ \forall y \in A.$$

Moreover, let us consider the problem $(IVEP)_{r_1}^{W+K\setminus\{0\}}$:

$$(IVEP)_{r_1}^{W+K\setminus\{0\}} : \text{ find x} \in \text{A such that } W \neq K\ INF \bigcup_{y\in A} F(x,y)\bigcap W + K\setminus\{0\} \neq \emptyset.$$

This relation is equivalent with the following conditions: $K\ INF_1 \bigcup_{y\in A} F(x,y)\bigcap(W + K\setminus\{0\}) \neq \emptyset$ and $W \not\subseteq K\ INF \bigcup_{y\in A} F(x,y)$. We remark that $S(IVEP)_{r_1}^{W+K\setminus\{0\}} \subseteq S(VEP)_{r_1}^{W-K\setminus\{0\}}$.

We consider the dual problem:

$$(P^\alpha)\ K\ SUP_1 \bigcup_{u\in A} \gamma(u)$$

$\gamma(u) = \bigcup_{T\in\mathcal{L}(X,Z)} K\ INF_1\{-F_x^* \mid_A (T) + \{T(y) \mid y \in A\}\}$.

An element x is a solution for (P^α) if there exists $z \in \gamma(x)$ such that $(K\ SUP_1 \bigcup_{u\in A} \gamma(u))\bigcap(z - \alpha + K\{0\}) = \emptyset$. We denote the set of these solutions by $S(P^\alpha)$. Let us remark that if $z \in S(P^\alpha)$ then:

$$z - \alpha \in K\ SUP(K\ SUP_1 \bigcup_{u\in A} \gamma(u)) = K\ SUP \bigcup_{u\in A} \gamma(u)$$

so $z \in MAX^\alpha \bigcup_{u\in A} \gamma(u)$, i.e., x is a solution for (P).

Theorem 37. *If $(P_a(F))$ is stable, then there exists $\alpha \in W$ such that:*

$$S(IVEP)_{r_1}^{W+K\setminus\{0\}} \subseteq S(P^\alpha)$$

$$S(VEP)_{r_1}^{W-K\setminus\{0\}} \bigcap S(P^\alpha) \neq \emptyset$$

Proof. If $x \in S(IVEP)_{r_1}^{W+K\setminus\{0\}}$, then there exists $z_0 \in K\ INF_1 \bigcup_{y\in A} F(x,y) = K\ MAX\gamma(x)$ (since $(P_a(F))$ is stable) and $\alpha \in W$ such that $z_0 - \alpha > 0$. If we suppose $(K\ SUP_1 \bigcup_{u\in A} \gamma(u))\bigcap(z_0 - \alpha + K\{0\}) \neq \emptyset$ we have $z \in K\ MAX\gamma(u)$, $z > z_0 - \alpha > 0$, which is false. Since $S(IVEP)_{r_1}^{W+K\setminus\{0\}} \subseteq (VEP)_{r_1}^{W-K\setminus\{0\}}$, we get the conclusion. $\square$

7. Conclusions

We present a unified approach for the vector equilibrium problems in the case of the ordering cone with nonempty quasi-interior or relative interior (possible with empty interior). In this case, some results concerning the existence of the solutions for a vector equilibrium problem are given and are applied to obtain conditions of existence for equilibrium in an abstract economy and for Walrasian equilibrium in a vector exchange economy. Several properties concerning the continuity and the connectedness for the solutions set are obtained. Some applications of this study in other domains, such as vector optimization problems and vector duality, are also given. Optimality results, linear and nonlinear scalarization characterization, as well as algorithm methods remain topics for future research.

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