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BV Solutions to Evolution Inclusion with a Time and Space Dependent Maximal Monotone Operator

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Abstract: This paper deals with the research of solutions of bounded variation (BV) to evolution inclusion coupled with a time and state dependent maximal monotone operator. Different problems are studied: existence of solutions, unicity of the solution, existence of periodic and bounded variation right continuous (BVRC) solutions. Second-order evolution inclusions and fractional (Caputo and Riemann–Liouville) differential inclusions are also considered. A result of the Skorohod problem driven by a time- and space-dependent operator under rough signal and a Volterra integral perturbation in the BRC setting is given. The paper finishes with some results for fractional differential inclusions under rough signals and Young integrals. Many of the given results are novel.

Keywords: bounded variation; differential inclusion; maximal monotone operator; pseudo-distance; right continuous; second order; fractional derivative; fixed point

MSC: 34A60; 26A33; 34H05; 34A08; 34G25; 47H10; 49J52; 49J53

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1. Introduction and Preliminaries

The main objective of this paper is to present the existence theory of a class of fractional equation coupled with a time- and state-dependent maximal monotone operator with domain $D(A_{(t,x)})$ in a separable Hilbert space in the BV setting. Taking account of the complexity of the study, we present in Sections 1 and 2 various new properties of the perturbed dynamic

$$\begin{cases} u(0) = a \in D(A_0) \\ u(t) \in D(A_t), t \in [0,T] \\ -Du(t) \in A_t u(t) + F(t, x(t)),\ t \in [0,T] \end{cases}$$

where A_t is a time-dependent maximal monotone operator with domain $D(A_t)$ in the Hilbert space H and $F : I \times H \to H$ is a multivalued mapping. This dynamic has enjoyed intense activity, with applications in economics, mechanics, medicine, biology, etc. As a direct application, we establish in Section 3 several variants concerning the existence of periodic and bounded variation right continuous (BVRC) solution for the aforementioned differential inclusion. The perturbation of the second-order differential inclusion by a time-dependent maximal monotone operator is studied in Section 4. We continue in Section 5 with fractional equations coupled with time and state dependent maximal monotone operators $A_{t,x}$ in the BVRC setting. In Section 6, we present a new version of the Skorohod problem for differential inclusion driven by time and state dependent maximal monotone operator $A_{t,x}$ in the vein of Castaing et al. [1,2], Rascanu [3], and L.Maticiuc, A. Rascanu,

L. Slominski, and M. Topolewski [4]. Let $a \in D(A_{(0,0)})$. Our aim is to find a continuous, bounded variation (BVC) function $x : [0,T] \to H$ and a continuous, bounded variation function (BVC) $u : [0,T] \to H$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \ \forall t \in [0,T] \\ h(t) = \int_0^t b(\tau, x(\tau)) dz_\tau, \ \forall t \in [0,T] \\ k(t) = \int_0^t g(t,\tau,x(\tau)) d\tau, \ \forall t \in [0,T] \\ u(t) \in D(A_{(t,h(t))}), \ \forall t \in [0,T] \\ -du \in A_{(t,h(t))} u(t) + \int_0^t g(t,\tau,x(\tau)) d\tau \end{cases}$$

where the functions $b(\tau, x)$ and $g(t,\tau,x)$ are continuous and uniformly bounded, and $\int_0^t b(\tau, x(\tau)) dz_\tau$ denotes the Riemann–Stieltjes integral of continuous function $b(., x(.))$ with respect to $z \in C^{1-var}([0,T], \mathbb{R}^e)$, the space of continuous functions of bounded variation defined on $[0,T]$ with values in $\mathbb{R}^e$. The principal novelties are that $A_{t,x}$ is a time- and state-dependent maximal monotone operator, and the integral Volterra perturbation and the Young integral perturbation are considered. Section 7 is devoted to evolution problems driven by time and state dependent operators under rough signal (Young integral) with applications in optimization. We refer to Brogliato et al. [5] for a large synthesis of applications in the study of dynamic systems coupled with time and state dependent maximal monotone operators. In particular, the second-order evolution inclusions governed by time- and state-dependent maximal monotone operators arise from unilateral mechanic problems with dry friction; see [6–9]. Currently, this work is a continuation of the pioneering ones [10,11] dealing with absolute continuous solutions to the fractional differential inclusion coupled with a time and state dependent maximal monotone operator and particularly the second-order evolution inclusion. It is known that this study is a difficult one and contains as a particular case the convex sweeping process [12–14], namely $A_{(t,x)} = N_{C(t,x)}$, the normal cone of a closed convex moving set $C(t,x)$ in H. In recent years, there has been intense activity around the second-order sweeping process [15–25]. In addition, there has been a significant development in fractional differential theory and applications; see [26–44].

Withing the BV setting, the study of differential inclusions driven by fractional equations and a time- and state-dependent maximal monotone operator under rough signal is a great novelty. We provide the existence of a BVC solution to an evolution problem driven by a time and state dependent maximal monotone operator perturbed by a rough signal with application in optimization problems. Likewise, the existence of BVRC periodic solutions in this framework is stated for the first time in the literature.

Throughout the paper, $I := [0,T]$ $(0 < T < +\infty)$ is an interval of $\mathbb{R}$ and H is a real *separable* Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$.

We use the following definitions and notations. We denote by $\overline{B}_H$ the closed unit ball of H. On the space $\mathcal{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence on I. By $L^p_H(I)$ for $p \in [1, +\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|x(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\| \ \|$). By $W^{1,2}_H(I)$ and $W^{1,1}_H(I)$, we denote the space of absolutely continuous functions from I to H with derivatives in $L^2_H(I)$ and $L^1_H(I)$, respectively. If $H = \mathbb{R}$, we note $W^{1,2}(I)$ for simplicity. By $W^{2,1}_H(I)$, we denote the set of all continuous functions in $\mathcal{C}_H(I)$ such that their first derivatives are continuous and their second derivatives belong to $L^1_H(I)$.

We introduce in the following the definition and some properties of maximal monotone operators needed in the proofs of our results, and we refer the reader to [45,46] for their basic theory and more details.

Let $A : D(A) \subset H \rightrightarrows H$ be a set-valued operator. We use classical definitions of the domain $D(A)$, the range $R(A)$, and the graph $gph(A)$ of A. We say that $A : D(A) \subset H \rightrightarrows H$ is monotone, if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ whenever $(x_i, y_i) \in gph(A)$, $i = 1, 2$. It is maximal monotone if its graph could not be contained strictly in the graph of any other monotone operator, in this case, for all $\lambda > 0$, $R(I_H + \lambda A) = H$, where I_H stands for the identity mapping of H.

If A is a maximal monotone operator, then for every $x \in D(A)$, Ax is non-empty, closed, and convex, such that the projection of the origin into Ax, $A^0(x)$, exists and is unique.

If the maximal monotone operator is time-dependent, it will be noted A_t. If it is time and space dependent, it will be noted $A_{(t,x)}$.

Let $A : D(A) \subset H \rightrightarrows H$ and $B : D(B) \subset H \rightrightarrows H$ be two maximal monotone operators; then, we denote by $dis(A, B)$ (see [47]) the pseudo-distance between A and B defined by

$$dis(A, B) = \sup\left\{ \frac{\langle y - y', x' - x \rangle}{1 + ||y|| + ||y'||} : \ (x, y) \in gph(A), \ (x', y') \in gph(B) \right\}. \tag{1}$$

Clearly, $dis(A, B) \in [0, +\infty]$, $dis(A, B) = dis(B, A)$ and $dis(A, B) = 0$ iff $A = B$. But dis is not a distance, since in a general case, the triangle inequality is not fulfilled.

An interesting property of dis is the following. If $C(t)$ is a family of closed, convex sets for $t \in [0, T]$, and $A(t) = N_{C(t)}$ their normal cones, $dis(A(t), A(s)) = d_H(C(t), C(s))$ for $t, s \in [0, T]$, where d_H denotes the Hausdorff distance.

To prove our main results, we need the following lemmas (see [48]).

Lemma 1. *Let A be a maximal monotone operator of H. If $x \in \overline{D(A)}$ and $y \in H$ are such that*

$$\langle A^0 z - y, z - x \rangle \geq 0 \ \ \forall z \in D(A),$$

then $x \in D(A)$ and $y \in Ax$.

Lemma 2. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that $dis(A_n, A) \to 0$. Suppose also that $x_n \in D(A_n)$ with $x_n \to x$ and $y_n \in A_n(x_n)$ with $y_n \to y$ weakly for some $x, y \in H$. Then $x \in D(A)$ and $y \in Ax$.*

Lemma 3. *Let A_n $(n \in \mathbb{N})$, A be maximal monotone operators of H such that $dis(A_n, A) \to 0$ and $\|A_n^0 x\| \leq c(1 + \|x\|)$ for some $c > 0$, all $n \in \mathbb{N}$ and $x \in D(A_n)$. Then for every $z \in D(A)$, there exists a sequence (z_n) such that*

$$z_n \in D(A_n), \quad z_n \to z \ \text{ and } \ A_n^0 z_n \to A^0 z. \tag{2}$$

We finish this section with some types of Gronwall's lemma, which are crucial for our purposes.

Lemma 4. *Let μ be a positive Radon measure on I. Let $g \in L^1(I, \mathbb{R}_+; \mu)$ and $\beta \geq 0$ be such that $\forall t \in I$, $0 \leq \mu(\{t\}) g(t) \leq \beta < 1$. Let $\varphi \in L^\infty(I, \mathbb{R}_+; \mu)$ satisfying*

$$\varphi(t) \leq \alpha + \int_{]0,t]} g(s)\varphi(s)\mu(ds) \quad \forall t \in I,$$

where α is a nonnegative constant. Then

$$\varphi(t) \leq \alpha \exp\left(\frac{1}{1 - \beta} \int_{]0,t]} g(s)\mu(ds) \right) \quad \forall t \in I.$$

Proof. This lemma is due to M.M. Marques. For a proof, see, e.g., ([49], Lemma 2.1). □

Lemma 5. *Let μ be a non-atomic positive Radon measure on the interval I. Let c, p be nonnegative real functions such that $c \in L^1(I, \mathbb{R}; \mu), p \in L^\infty(I, \mathbb{R}; \mu)$, and let $\alpha \geq 0$. Assume that for $\mu-a.e.\ t \in I$*

$$p(t) \leq \alpha + \int_0^t c(s)p(s)\mu(ds).$$

Then, for $\mu - a.e.\ t \in I$

$$p(t) \leq \alpha \exp\Big(\int_0^t c(s)\mu(ds)\Big).$$

The proof (see [50], Lemma 2.7, or [51], Lemma 4, taking $\eta = 0$) is not a consequence of the classical Gronwall lemma dealing with Lebesgue measure dt on I. It relies on a deep result of Moreau–Valadier on the derivation of (vector) functions of bounded variation [52]. Let us recall Schauder's fixed point theorem [53].

Theorem 1. *Let C be a non-empty closed bounded convex subset of a Banach space E and let $f : C \to C$ be a continuous mapping. If $f(C)$ is relatively compact, then f has a fixed point.*

For the sake of completeness, we give a result about the existence of BVRC solutions for an evolution inclusion with time-dependent m-accretive operator. Its proof is given in [54]. Let E be a separable Banach space and let $ccwl(E)$ denote the closed convex weakly locally compact class which contains no line ([55]).

Theorem 2. *Let $I = [0, T]$. Let $t \mapsto: D(A_t) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}_1^A)$; there exists a nonnegative real number c such that*

$$\|A_t^0 x\| = \|A^0(t,x)\| \leq c(1 + \|x\|) \ \text{for}\ \ t \in I,\ x \in D(A_t)$$

$(\mathcal{H}_2^A)$ $\Gamma : t \mapsto D(A_t)$ has right closed graph, $gph(\Gamma)$, and for each $t \in I$, for each $k > 0$, the set $\{x \in D(A_t) : ||x||| \leq k\}$ is relatively compact, and in short, $D(A_t)$ is ball-compact.

$(\mathcal{H}_3^A)$ $(t,x) \mapsto A_t(x) : gph(\Gamma) \to ccwl(E)$ is scalar upper semicontinuous: for $t_n \downarrow t$, for $x_n \to x$ with $x_n \in D(A_{t_n})$ and $x \in D(A_t)$,

$$\forall x^* \in E^*,\ \ \limsup_n \delta^*(x^*, A_{t_n}x_n) \leq \delta^*(x^*, A_t x)$$

$(\mathcal{H}_4^A)$ There exists a non-decreasing and right continuous function $r : I \to [0, \infty[$ such that $r(T) < \infty$ with the Stieltjes measure dr such that, for $t < \tau \subset I$, for $\lambda > 0$ and $x \in D(A_t)$

$$\|x - J_\lambda^{A(\tau)}(x)\| \leq (r(\tau) - r(t))(1 + \lambda\|A_t^0 x\|)$$

$(\mathcal{H}^F)$ Let $F : I \times E \to cwk(E)$ be a convex weakly compact valued mapping such that:

- *(i) F is scalarly $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable, that is, for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(.,.))$ is $\mathcal{L}(I) \otimes \mathcal{B}(E)$-measurable;*
- *(ii) For each $t \in I$, $F(t,.)$ is scalarly upper semicontinuous, that is, for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(t,.))$ is upper semicontinuous on E;*
- *(iii) $F(t,x) \subset M(1 + ||x||)\overline{B}_E$ for all $(t,x) \in I \times E$ for some positive constant M.*

Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}$ be the density of λ relative to the measure ν. Then for all $u_0 \in D(A_0)$, the evolution problem

$$-Du(t) \in A_t u(t) + F(t, u(t))$$

admits a BVRC solution u with $u(0) = u_0$, that is, there exists a BVRC mapping $u : I \to E$ and a Lebesgue-integrable mapping $z : I \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in \ I \\ \frac{du}{dv}(t) \in L^\infty_E(I, \nu) \\ z(t) \in F(t, u(t)), \ \lambda \ \ a.e \\ -\frac{du}{dv}(t) \in A_t u(t) + z(t)\frac{d\lambda}{dv}(t), \nu \ a.e, \ t \in I \end{cases}$$

We now provide two corollaries of Theorem 2 which will be useful in the following.

Corollary 1. *Let $t \mapsto A_t : D(A_t) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}^A_1)$, $(\mathcal{H}^A_2)$, $(\mathcal{H}^A_3)$, $(\mathcal{H}^A_4)$. Let $f : I \times E \to E$ such that:*

(i) $f(.,x)$ *is* $\mathcal{L}(I)$*-measurable on I for all* $x \in E$*;*
(ii) $f(t,.)$ *is continuous on E for all* $t \in I$*;*
(iii) $||f(t,x)|| \le M(1 + ||x||)$ *for all* $(t,x) \in I \times E$*.*

Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ with respect to the measure ν. Then for all $u_0 \in D(A_0)$, the evolution problem

$$-Du(t) \in A_t u(t) + f(t, u(t))$$

admits at least a BVRC solution u with $u(0) = u_0$, that is, there exists a BVRC function $u : I \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in \ I \\ \frac{du}{dv}(t) \in L^\infty_E(I, \nu) \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t))\frac{d\lambda}{dv}(t), \ \nu \ a.e. \end{cases}$$

Corollary 2. *Let $t \mapsto A_t : D(A_t) \to ccwl(E)$ be a time-dependent maximal monotone operator satisfying $(\mathcal{H}^A_1)$, $(\mathcal{H}^A_2)$, $(\mathcal{H}^A_3)$, $(\mathcal{H}^A_4)$. Let $f : I \times E \to E$ such that*

(i) $f(.,x)$ *is* $\mathcal{L}(I)$*-measurable on I for all* $x \in E$*;*
(ii) $||f(t,x) - f(t,y)|| \le M||x - y||$ *for all* $t, x, y \in I \times E \times E$*;*
(iii) $||f(t,x)|| \le M(1 + ||x||)$ *for all* $(t,x) \in I \times E$*,*

For some constant $M > 0$, let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ relative to the measure ν. Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I, 0 \le 2M\frac{d\lambda}{d\nu}(t)d\nu(\{t\}) \le \beta < 1$. Then for all $u_0 \in D(A_0)$, the evolution problem

$$-Du(t) \in A_t u(t) + f(t, u(t))$$

*admits a **unique** BVRC solution u with $u(0) = u_0$, that is, there exists a unique BVRC function $u : I \to E$ such that*

$$\begin{cases} u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in \ I \\ \frac{du}{dv}(t) \in L^\infty_E(I, d\nu) \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t))\frac{d\lambda}{dv}(t), \ \nu \ a.e. \end{cases}$$

Proof. We need only to prove the uniqueness. Suppose that there are two BVRC solutions u and v:

$$-\frac{du}{d\nu}(t) - f(t, u(t))\frac{d\lambda}{d\nu}(t) \in A_t u(t)$$

$$-\frac{dv}{d\nu}(t) - f(t, v(t))\frac{d\lambda}{d\nu}(t) \in A_t v(t)$$

By the monotonicity of A_t, we obtain

$$\left\langle \frac{dv}{d\nu}(t) - \frac{du}{d\nu}(t) + \frac{d\lambda}{d\nu}(t) f(t, v(t)) - \frac{d\lambda}{d\nu}(t) f(t, u(t)), v(t) - u(t) \right\rangle \leq 0$$

and by the Lipschitz condition on $f(t, \cdot)$,

$$\left\langle \frac{dv}{d\nu}(t) - \frac{du}{d\nu}(t), v(t) - u(t) \right\rangle \leq \left\langle \frac{d\lambda}{d\nu}(t) f(t, u(t)) - \frac{d\lambda}{d\nu}(t) f(t, v(t)), v(t) - u(t) \right\rangle$$

$$\leq M \frac{d\lambda}{d\nu}(t) \|v(t) - u(t)\|^2.$$

Then, u and v are bounded variation and right continuous and have the density $\frac{du}{d\nu}$ and $\frac{dv}{d\nu}$ relative to $d\nu$, by a result of Moreau concerning the differential measure [56], $||v - u||^2$ is BVRC and we have

$$d||v - u||^2 \leq 2\langle v(.) - u(.), \frac{dv}{d\nu}(.) - \frac{du}{d\nu}(.)\rangle d\nu$$

such that, by integrating on $]0, t]$ and using the above estimate, we obtain

$$||v(t) - u(t)||^2 = \int_{]0,t]} d||u - v||^2 \leq \int_{]0,t]} 2\langle v(.) - u(.), \frac{dv}{d\nu}(.) - \frac{du}{d\nu}(.)\rangle d\nu(t)$$

$$\leq \int_{]0,t]} 2M \frac{d\lambda}{d\nu}(t) \|v(t) - u(t)\|^2 d\nu(t).$$

According to the assumption $0 \leq 2M\frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$ and using Grownwall's Lemma 4, we deduce from the last inequality that $u = v$ in I. This completes the proof. □

2. Existence of BVRC Solution to Differential Inclusion with Time-Dependent Maximal Monotone Operator and Perturbation

We present a specific study on the existence of bounded variation right continuous (BVRC) solutions in a separable Hilbert space H to the inclusion of the form

$$-Du \in A_t u(t)) + F(t, u(t))$$

where $t \in I = [0, T]$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone time-dependent operator satisfying some conditions and the perturbation F is a convex weakly compact-valued $\mathcal{B}(I) \otimes \mathcal{B}(H)$-measurable such that $F(t, .)$ is upper semicontinuous and satisfying some growth condition.

First, we fix some notations and preliminary facts. Let μ a positive Borel regular measure (alias Radon measure) on $I = [0, T]$ and let us denote by $L^1_H(I, \mathcal{B}(I)), \mu)$ the space of $(\mathcal{B}(I), B(H))$-measurable and μ-integrable mappings $f : I \to H$. If g is a positive $(\mathcal{B}(I), B(\mathbb{R}))$-measurable and μ-integrable, then the set

$$\{f \in L^1_H(I, \mathcal{B}(I), \mu) : ||f(t)|| \leq g(t),\ \mu\ a.e.\}$$

is convex and weakly compact; in particular, the set

$$\mathcal{S}^{\infty}_{M\overline{B}_H}(\mu) := \{f \in L^1_H(I, \mathcal{B}(I), \mu) : ||f(t)|| \leq M,\ \mu\ a.e.\}$$

where M is a positive constant, is convex and weakly compact. In most usual applications, μ is the Lebesgue measure λ on I and

$$\mathcal{S}^{\infty}_{M\overline{B}_H}(\lambda) := \{f \in L^1_H(I, \mathcal{L}(I), \lambda) : ||f(t)|| \leq M,\ \lambda\ a.e.\}$$

where $\mathcal{L}(I)$ is the σ-algebra of Lebesgue sets in I.

Our results are proved using the following assumptions for the operators A_t:

($\mathcal{H}_1^*$) There exists a nonnegative real number c such that $\|A_t^0 x)\| \leq c(1 + \|x\|)$ for all $(t, x) \in I \times D(A_t)$.

($\mathcal{H}_2^*$) $dis(A_t, A_\tau) \leq dr(]\tau, t])$, for all $0 \leq \tau \leq t \leq T$ where $r : I \to \mathbb{R}^+$ is non-decreasing right continuous on I with $r(0) = 0, r(T) < +\infty$.

($\mathcal{H}_2^{*c}$) $dis(A_t, A_\tau) \leq dr(]\tau, t])$, for all $0 \leq \tau \leq t \leq T$ where $r : I \to \mathbb{R}^+$ is non-decreasing continuous on I with $r(0) = 0, r(T) < +\infty$.

($\mathcal{H}_3^*$) $D(A_t)$ is boundedly compactly measurable, in the sense that there is a convex compact-valued Borel -measurable mapping $X : [0,1] \to H$ such that $D(A_t) \subset X(t) \subset \kappa(t)\overline{B}_H$ for all $t \in [0,1]$ where κ is a positive $L^1(I, \lambda)$-integrable function.

($\mathcal{H}_4^*$) $D(A_t)$ is ball-compact.

($\mathcal{H}^*$) $D(A_T)$ is convex compact and $D(A_T) \subset D(A_0)$.

($\mathcal{H}_g^*$) $\Gamma : t \mapsto D(A_t)$ has right closed graph, $gph(\Gamma)$.

($\mathcal{H}_g^{**}$) $\Gamma : t \mapsto D(A_t)$ has closed graph, $gph(\Gamma)$.

Lemma 6. *Assume that for every $t \in I = I$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying ($\mathcal{H}_1^*$) and ($\mathcal{H}_2^*$). Let $f \in \mathcal{S}^\infty_{M\overline{B}_H}(\mu)$ and $x_0 \in D(A_0)$.*
Then the evolution inclusion

$$\begin{cases} -Du \in A_t u(t) + f(t), t \in I, \\ u(0) = u_0 \in D(A_0), \end{cases}$$

admits a unique BVRC solution, in the sense that there is a positive Radon measure ν on I, a BVRC mapping $u : I \to H$ satisfying

$$\begin{cases} u(t) = u_0 + \int_{]0,t]} \frac{du}{d\nu}(s) d\nu(s), t \in \ I \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t)\frac{d\mu}{d\nu}(t) \end{cases}$$

where $\frac{d\mu}{d\nu}$ is the density of the measure μ with respect to the measure ν, and $\frac{du}{d\nu}$ is the density of the differential measure du with respect to the measure ν.

Moreover, one has the estimate

$$||u(t) - u(\tau)|| \leq L\nu(]\tau, t]), 0 \leq \tau \leq t \leq T$$

where L is a positive constant depending on I, x_0, dr, μ, and M.

Proof. Consider, for every $t \in I$, the operator $B_t : D(B_t) \subset H \rightrightarrows H$ defined by

$$D(B_t) = D(A_t) + \int_{]0,t]} f(s) d\mu(s)$$

and

$$B_t x = A_t\Big(x - \int_{]0,t]} f(s) d\mu(s)\Big), \quad \forall x \in D(B_t).$$

It is clear that for each $t \in I$, B_t is a maximal monotone operator. Let us show that $t \mapsto B_t$ is of BVRC in variation. Let $s, t \in I$ $(s < t)$, $x_1 \in D(B_t)$, $x_2 \in D(B_s)$, $y_1 \in B_t x_1 = A_t\big(x_1 - \int_{]0,t]} f(\tau) d\mu(\tau)\big)$, and $y_2 \in B_s x_2 = A_s\big(x_2 - \int_{]0,s]} f(\tau) d\mu(\tau)\big)$.

We have

$$\begin{aligned}
\langle y_1 - y_2, x_2 - x_1\rangle &= \Big\langle y_1 - y_2, \Big(x_2 - \int_{]0,s]} f(\tau)d\mu(\tau)\Big) - \Big(x_1 - \int_{]0,t]} f(\tau)d\mu(\tau)\Big) - \int_{]s,t]} f(\tau)d\mu(\tau)\Big\rangle \\
&= \Big\langle y_1 - y_2, \Big(x_2 - \int_{]0,s]} f(\tau)d\mu(\tau)\Big) - \Big(x_1 - \int_{]0,t]} f(\tau)d\mu(\tau)\Big)\Big\rangle \\
&- \Big\langle y_1 - y_2, \int_{]s,t]} f(\tau)d\mu(\tau)\Big\rangle, \\
&\leq \Big\langle y_1 - y_2, \Big(x_2 - \int_{]0,s]} f(\tau)d\mu(\tau)\Big) - \Big(x_1 - \int_{]0,t]} f(\tau)d\mu(\tau)\Big)\Big\rangle \\
&+ \Big(\|y_1\| + \|y_2\|\Big) \int_{]s,t]} ||f(\tau)||d\mu(\tau), \\
&\leq \Big\langle y_1 - y_2, \Big(x_2 - \int_{]0,s]} f(\tau)d\mu(\tau)) - (x_1 - \int_{]0,t]} f(\tau)d\mu(\tau))\Big\rangle \\
&+ \Big(\|y_1\| + \|y_2\| + 1\Big) \int_{]s,t]} ||f(\tau)||d\mu(\tau),
\end{aligned}$$

then

$$\begin{aligned}
\frac{\Big\langle y_1 - y_2, x_2 - x_1\Big\rangle}{\|y_1\| + \|y_2\| + 1} &\leq \frac{\Big\langle y_1 - y_2, \Big(x_2 - \int_{]0,s]} f(\tau)d\mu(\tau)\Big) - \Big(x_1 - \int_{]0,t]} f(\tau)d\mu(\tau)\Big)\Big\rangle}{\|y_1\| + \|y_2\| + 1} \\
&+ \int_{]s,t]} ||f(\tau)\|d\mu(\tau),
\end{aligned}$$

and

$$\begin{aligned}
dis\big(B_t, B_s\big) &\leq dis\big(A_t, A_s\big) + \int_{]s,t]} ||f(\tau)||d\mu(\tau) \\
&\leq dr(]s,t]) + \int_{]s,t]} Md\mu(\tau) \\
&= dr(]s,t]) + M\mu(]s,t]) := \nu(]s,t]),
\end{aligned}$$

where $\nu = dr + M\mu$. On the other hand, for every $t \in I$ and $x \in D(B_t)$, we have

$$\begin{aligned}
||B_t^0 x\| &= \|A_t^0\big(x - \int_{]0,t]} f(\tau)d\mu(\tau)\big)\| \leq c\big(1 + \|x - \int_{]0,t]} f(\tau)d\mu(\tau)\|\big) \\
&\leq c\big(1 + \|x\| + M\mu(I)\big) \\
&\leq c_1\big(1 + \|x\|\big)
\end{aligned}$$

Consequently, all the hypotheses of Theorem 3.1 in [48] are verified with the operator B_t, where the measure dr is replaced by the Radon measure ν; then we conclude the existence of a unique BVRC solution $v(.)$ to the problem

$$\begin{cases} -\frac{dv}{d\nu}(t) \in B_t v(t) = A_t\Big(v(t) - \int_{]0,t]} f(s)d\mu(s)\Big) \quad d\nu \quad a.e. \\ v(0) = u_0 \in D(B_0) = D(A_0) \end{cases}$$

with $v(t) = u_0 + \int_{]0,t]} \frac{dv}{d\nu}(s)d\nu(s), t \in I$, $||v(t) - v(s)|| \leq K\nu(]s,t]), 0 \leq s \leq t \leq T$ and the density $\frac{dv}{d\nu}$ of the differential measure dv with respect to the measure ν satisfies $\frac{dv}{d\nu}(t) \in K\overline{B}_H$, ν *a.e.* where K is positive constant depending on I, x_0, dr, μ, M. Set $u(t) = v(t) -$

$\int_{]0,t]} f(s)d\mu(s), t \in I$. Then u is BVRC with $u(0) = v(0) = u_0$ and the density $\frac{du}{d\nu}$ of du with respect to the measure ν satisfies

$$\frac{du}{d\nu}(t) = \frac{dv}{d\nu}(t) - f(t)\frac{d\mu}{d\nu}(t), t \in I$$

where $\frac{d\mu}{d\nu}$ is the density of the measure μ with respect to the measure ν so that

$$-\frac{du}{d\nu}(t) - f(t)\frac{d\mu}{d\nu}(t) \in B_t v(t) = A_t\Big(v(t) - \int_{]0,t]} f(s)d\mu(s)\Big) = A_t u(t),\ \nu\ a.e.$$

with the estimate

$$||u(t) - u(s)|| \leq L\nu(]s,t]) = (K+M)\nu(]s,t])$$

This completes the proof. □

Remark 1. *The proof of Lemma 6 uses a technique due to Azzam-Boutana ([57], Theorem 4) dealing with A_t absolutely continuous in variation. Actually, the tool is constructive and allows us to give a precise sense of BVRC solution to the inclusion*

$$(\mathcal{P}_f) \qquad \begin{cases} -Du \in A_t u(t) + f(t), t \in I, \\ u(0) = u_0 \in D(A_0), \end{cases}$$

Indeed, given dr, μ and $\mathcal{S}^\infty_{M\overline{B}_H}(\mu)$, let us consider the Radon measure $\nu = dr + M\mu$. Then μ is absolutely continuous with respect to the measure ν and let $\frac{d\mu}{d\nu}$ be the density of the measure μ with respect to the measure ν. Then by the proof of Lemma 6, there is a unique BVRC solution to the inclusion

$$-\frac{du}{d\nu}(t) \in A_t u(t) + f(t)\frac{d\mu}{d\nu}(t),\ \nu\ a.e.$$

with initial condition $u(0) = u_0 \in D(A_0)$ and required estimation. So it amounts to saying that a mapping u is a solution to the above inclusion $(\mathcal{P}_f)$ with perturbation f meaning that u is BVRC and the couple (u, f) satisfies the above inclusion. And so this allows us to give the definition of the solution to the inclusion with A_t and perturbation $F(t, x)$.

$$(\mathcal{P}_F) \qquad \begin{cases} -Du \in A_t u(t) + F(t, u(t)), t \in\ I \\ u(0) = u_0 \in D(A_0) \end{cases}$$

By the solution of $(\mathcal{P}_F)$, it amounts to finding a pair (u_g, g) where u_g is BVRC and $g \in L^1_H(I, \mathcal{B}(I), \mu)$ such that $g(t) \in F(t, u_g(t))$, ν *a.e.* and such that

$$-\frac{du_g}{d\nu}(t) \in A_t u_g(t) + g(t)\frac{d\mu}{d\nu}(t),\ \ \nu\ \ a.e.$$

Lemma 7. *Assume that for every $t \in I = [0, T]$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying $(\mathcal{H}^*_1)$, $(\mathcal{H}^*_2)$ and $(\mathcal{H}^*_4)$.*

Let $X : I \to cwk(H)$ be a $\mathcal{B}(I)$-measurable convex weakly compact-valued mapping with $X(t) \subset M\overline{B}_H$ for all $t \in I$, where M is a positive constant and let

$$\mathcal{S}^\infty_X(\mu) := \{g \in L^1_H(I, \mathcal{B}(I), \mu) : g(t) \in X(t),\ \mu\ a.e.\}$$

be the set of all $(\mathcal{B}(I), \mathcal{B}(H))$-measurable and μ-integrable selections of X. Then the BVRC solution set $\mathcal{T}_X := \{u_g : g \in \mathcal{S}^\infty_X(\mu)\}$ to the inclusion

$$\begin{cases} -Du_g \in A_t g(t) + g(t), g \in \mathcal{S}^\infty_X(\mu), t \in I, \\ u_g(0) = u_0 \in D(A_0), \end{cases}$$

is sequentially compact with respect to the pointwise convergence.

Proof. Carefully apply the results and notations of Lemma 6. First we note that $\mathcal{S}^\infty_X(\mu)$ is convex weakly compact in $L^1_H(I, \mathcal{B}(I), \mu)$. For each $g \in \mathcal{S}^\infty_X(\mu)$, consider for every $t \in I$, the operator $G_t : D(G_t)) \subset H \rightrightarrows H$ defined by

$$D(G_t) = D(A_t) + \int_{]0,t]} g(s)d\mu(s)$$

and

$$G_t x = A_t\Big(x - \int_{]0,t]} g(s)d\mu(s)\Big), \quad \forall x \in D(G_t).$$

It is already seen that for each $t \in I$, G_t is a maximal monotone operator satisfying the conditions $\|G^0_t x\| \leq d(1 + \|x\|)$ for all $(t, x) \in I \times D(G_t)$, for some positive constant d, and the operators G are **equi-BVRC in variation**:

$$dis(G_t, G_\tau) \leq dr(]s, t]) + M\mu(]s, t]) = \nu(]s, t])$$

where $\nu = \mu + M\nu$. Then by ([48], Theorem 3), we assert the existence of a unique BVRC solution v_g to the problem

$$\begin{cases} -\frac{dv_g}{d\nu}(t) \in G_t v_g(t) = A_t\Big(v_g(t) - \int_{]0,t]} g(s)d\mu(s)\Big), \ \nu \ \ a.e. \\ v_g(0) = x_0 \in D(G_0) = D(A_0) \end{cases}$$

with $v_g(t) = x_0 + \int_{]0,t]} \frac{dv_g}{d\nu}(s)d\nu(s), t \in I$, $||v_g(t) - v_g(s)|| \leq K\nu(]s, t]), 0 \leq s \leq t \leq T$ and the density $\frac{dv_g}{d\nu}$ of the differential measure dv_g with respect to the measure ν satisfies $\frac{dv_g}{d\nu}(t) \in K\overline{B}_H$, ν a.e. where K is positive constant depending on I, x_0, dr, μ, M. Set $u_g(t) = v_g(t) - \int_{]0,t]} g(s)d\mu(s), t \in I$. Then u_g is BVRC with $u_g(0) = v_g(0) = x_0$ and the density of $\frac{du_g}{d\nu}$ of the differential measure du_g with respect to the measure ν satisfies

$$\frac{du_g}{d\nu}(t) = \frac{dv_g}{d\nu}(t) - g(t)\frac{d\mu}{d\nu}(t), t \in I$$

such that

$$-\frac{du_g}{d\nu}(t) - g(t)\frac{d\nu}{d\nu}(t) \in G_t v_g(t) = A_t\Big(v_g(t) - \int_{]0,t]} g(s)d\mu(s)\Big) = A_t u_g(t), \ \nu \ a.e. \quad (3)$$

with the estimate

$$||u_g(t) - u_g(s)|| \leq L\nu(]s, t]) := (K + M)\nu(]s, t]). \quad (4)$$

This shows that the BVRC solution set $\mathcal{T}_X := \{u_g : g \in \mathcal{S}^\infty_X(\mu)\}$ of the inclusion

$$\begin{cases} -Du_g \in A_t u_g(t) + g(t), g \in \mathcal{S}^\infty_X(\mu), t \in I, \\ u_g(0) = u_0 \in D(A_0), \end{cases}$$

is non-empty and satisfies the conditions (3) and (4). Let (u_{g_n}) be a sequence in $\mathcal{T}_X$. We have to prove that there is a (not relabeled) subsequence (u_{g_n}) that converge pointwise to a u_g with $g \in \mathcal{S}^\infty_X(\mu)$. First by weak compactness, we may assume that (g_n) weakly converges in $L^1_H(I, \mathcal{B}(I), \mu)$ to g with $g(t) \in X(t)$ for all $t \in I$ such that $g_n\frac{d\mu}{d\nu}$ weakly converges to $g\frac{d\mu}{d\nu}$ in $L^1_H(I, \mathcal{B}(I), \nu)$. Furthermore, since (u_{g_n}) is bounded in norm and in variation, and $D(A_t)$ is ball-compact (cf $(\mathcal{H}^*_4)$), by the Helly principle [58], we may ensure that (u_{g_n}) converges pointwise to a BVRC function u. So we may ensure that $u_{g_n}(t)) = x_0 + \int_{]0,t]} \frac{du_{g_n}}{d\nu}(s)d\nu(s) \to u(t) = x_0 + \int_{]0,t]} \frac{du}{d\nu}(s)d\nu(s)$ with $\frac{du_{g_n}}{d\nu} \to \frac{du}{d\nu}$ weakly in $L^1_H(I,\mathcal{B}(I),\nu)$. As $\frac{du_{g_n}}{d\nu} + g_n\frac{d\mu}{d\nu} \to \frac{du_g}{d\nu} + g\frac{d\mu}{d\nu}$ weakly in $L^1_H(I,\mathcal{B}(I),\nu)$. We may assume that $\frac{du_{g_n}}{d\nu} + g_n\frac{d\mu}{d\nu}$ Komlos converges to $\frac{du_g}{d\nu} + g\frac{d\mu}{d\nu}$. Further, we note that $u(t) \in D(A_t)$ for all $t \in I$. It is clear that $(y_n = A^0_t u_{g_n}(t))$ is bounded, and hence relatively weakly compact. By applying Lemma 2 to $u_{g_n}(t) \to u(t)$ and to a weakly convergent subsequence of (y_n) to show that $u(t) \in D(A_t)$, it remains to establish the **main fact**,

$$-\frac{du}{d\nu}(t) \in A_t u(t) + g(t)\frac{d\mu}{d\nu}(t), \nu a.e.$$

There is a ν-negligible set N such that

$$-\frac{du_{h_n}}{d\nu}(t) - g_n(t)\frac{d\mu}{d\nu}(t) \in A_t u_{g_n}(t),\ t \in I \setminus N \tag{5}$$

$$\lim_n \frac{1}{n}\sum_{j=1}^n [\frac{du_{g_j}}{d\nu}(t) + g_j(t)\frac{d\mu}{d\nu}(t)] = \frac{du}{d\nu}(t) + g(t)\frac{d\mu}{d\nu}(t),\ t \in I \setminus N.$$

Let $t \in I \setminus N$. Let $\eta \in D(A_t)$. From (5)

$$-\frac{du_{g_n}}{d\nu}(t) - g_n(t)\frac{d\mu}{d\nu}(t) \in A_t u_{g_n}(t)$$

and by monotonicity

$$\langle \frac{du_{g_n}}{d\nu}(t) + g_n(t)\frac{d\mu}{d\nu}(t), u_{g_n}(t) - \eta\rangle \le \langle A^0_t\eta, \eta - u_{g_n}(t)\rangle. \tag{6}$$

From (6), we deduce that

$$\frac{1}{n}\sum_{j=1}^n \Big\langle \frac{du_{g_j}}{d\nu}(t) + g_j(t)\frac{d\mu}{d\nu}(t), u_{g_j}(t) - \eta\Big\rangle \le \frac{1}{n}\sum_{j=1}^n \Big\langle A^0_t\eta, \eta - u_{g_j}(t)\Big\rangle \tag{7}$$

Passing to the limit when $n \to \infty$, this last inequality (7) immediately gives

$$\Big\langle \frac{du}{d\nu}(t) + g(t)\frac{d\mu}{d\nu}(t), u(t) - \eta\Big\rangle \le \Big\langle A^0_t\eta, \eta - u(t)\Big\rangle \quad a.e.$$

As a consequence, by Lemma 1, we obtain $-\frac{du}{d\nu}(t) - g(t)\frac{d\mu}{d\nu}(t) \in A_t u(t)$, ν *a.e.* with $u(t) \in D(A_t)$ for all $t \in I$. The proof is complete. □

Lemmas 6 and 7 are important for our purposes.

Theorem 3. *Assume that for every $t \in I = I$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying $(\mathcal{H}^*_1)$, $(\mathcal{H}^*_2)$ and $(\mathcal{H}^*_4)$. Let $X : I \to cwk(H)$ be a $\mathcal{B}(I)$-measurable convex weakly compact-valued mapping with $X(t) \subset M\overline{B}_H$ for all $t \in I$, where M is a positive constant, and let*

$$\mathcal{S}^\infty_X(\mu) := \{g \in L^1_H(I,\mathcal{B}(I),\mu) : g(t) \in X(t),\ \mu\ a.e.\}$$

be the set of all $(\mathcal{B}(I),\mathcal{B}(H))$-measurable and μ-integrable selections of X.

Let $F : I \times H \to H$ be a convex weakly compact-valued mapping satisfying:

(i) $F(t,x) \subset X(t)$ for all $(t,x) \in I \times H$, where M is a positive constant;

(ii) For every $e \in H$, the mapping $(t,x) \to \delta^*(e, F(t,x))$ is $\mathcal{B}(I) \otimes \mathcal{B}(H)$-measurable;

(iii) For every $e \in H$, for every $t \in I$, the mapping $x \to \delta^*(e, F(t,x))$ is upper semicontinuous.

Then the BVRC solution set $\mathcal{T}_F := \{u\}$ to the inclusion

$$\begin{cases} -Du \in A_t u(t) + F(t, u(t)), t \in I, \\ u(0) = u_0 \in D(A_0), \end{cases}$$

is sequentially compact with respect to the pointwise convergence.

Proof. We make explicit the notion of BVRC solutions and prove first the existence according to the above results.

Step 1. For each $g \in \mathcal{S}^{\infty}_X(\mu)$, let us define

$$\Phi(g) = \left\{ f \in L^1_H(I, \mathcal{B}(I), \mu) : \ f(t) \in F(t, u_g(t)), \ \mu \ a.e. \ t \in I \right\},$$

where u_g is the unique BVRC solution (see Lemma 6) to the inclusion

$$\begin{cases} -\frac{du_g}{dv}(t) \in A_t u_g(t) + g(t)\frac{d\mu}{dv}(t), \ \nu \ \ a.e. \\ u_g(0) = x_0 \in D(A_0) \end{cases}$$

By (i)–(iii), it is clear that $\Phi(g)$ is nonempty with $\Phi(g) \subset \mathcal{S}^{\infty}_X(\mu)$ because of condition (i). In fact, $\Phi(g)$ is the set of $L^1_H(I, \mathcal{B}(I), \mu)$-selections of the convex weakly compact-valued scalarly $\mathcal{B}(I)$-measurable mapping $t \mapsto F(t, u_g(t))$ by noting that u_g is BVRC right continuous, then u_g is Borel, i.e., $(\mathcal{B}(I), B(H))$-measurable; hence, by (ii), $t \to \delta^*(e, F(t, u_g(t)))$ is $\mathcal{B}(I)$-measurable. Clearly, if g is a fixed point of Φ ($g \in \Phi(g)$), then u_g is a BVRC solution to the inclusion under consideration:

$$\begin{cases} -\frac{du_g}{dv}(t) \in A_t u_g(t) + F(t, u_g(t))\frac{d\mu}{dv}(t), \ \nu \ \ a.e. \\ u_g(0) = x_0 \in D(A_0) \end{cases}$$

Now we show that $\Phi : \mathcal{S}^{\infty}_X(\mu) \hookrightarrow \mathcal{S}^{\infty}_X(\mu)$ is a convex $\sigma(L^1_H(I, \mathcal{B}(I), \mu), L^{\infty}_H(I, \mathcal{B}(I), \mu))$-compact-valued upper semicontinuous mapping. By weak compactness, it is enough to show that the graph of Φ is sequentially $\sigma(L^1_H(I, \mathcal{B}(I), \mu), L^{\infty}_H(I, \mathcal{B}(I), \mu))$-compact. Let $(h_n) \subset \Phi(g_n)$ such that

(g_n) $\sigma(L^1_H(I, \mathcal{B}(I), \mu), L^{\infty}_H(I, \mathcal{B}(I), \mu))$-converges to $g \in \mathcal{S}^{\infty}_X(\mu)$,

(h_n) $\sigma(L^1_H(I, \mathcal{B}(I), \mu), L^{\infty}_H(I, \mathcal{B}(I), \mu))$-converges to $h \in \mathcal{S}^{\infty}_X(\mu)$.

We need to show that $h \in \Phi(g)$. By virtue of Lemma 7, it is already known that the set $\mathcal{T}_X := \{u_g : g \in \mathcal{S}^{\infty}_X(\mu)\}$ of solutions to

$$\begin{cases} -\frac{du_g}{dv}(t) \in A_t u_g(t)) + g(t)\frac{d\mu}{dv}(t), \ t \in I, \ g \in S^{\infty}_X(\mu) \\ u_g(0) = x_0 \in D(A_0) \end{cases}$$

is sequentially compact with respect to the pointwise convergence. Hence, we may assume that (u_{g_n}) converges pointwise to $u_g \in \mathcal{T}_X$. Since $h_n(t) \in F(t, u_{g_n}(t))$,

$$\langle 1_E(t)x, h_n(t) \rangle \leq \delta^*\big(1_E(t)x, F(t, u_{g_n}(t))\big),$$

holds in I, for every $\mathcal{B}(I)$-measurable $E \subset I$ and for every $x \in H$. Thus, by integrating

$$\int_E \langle x, h_n(t) \rangle d\mu \leq \int_E \delta^*\big(x, F(t, u_{g_n}(t))\big) d\mu,$$

it follows that

$$\lim_n \int_E \langle x, h_n(t) \rangle d\mu = \int_E \langle x, h(t) \rangle d\mu \leq \limsup_n \int_E \delta^*\big(x, F(t, u_{g_n}(t))\big) d\mu$$

$$\leq \int_E \limsup_n \delta^*\big(x, F(t, u_{g_n}(t))\big) d\mu \leq \int_E \delta^*\big(x, F(t, u_g(t))\big) d\mu.$$

Whence we obtain

$$\int_E \langle x, h(t) \rangle d\mu \leq \int_E \delta^*\big(x, F(t, u_g(t))\big) d\mu$$

for every $\mathcal{B}(I)$-measurable $E \subset I$. Consequently, $\langle x, h(t) \rangle \leq \delta^*(x, F(t, u_g(t)))$, μ *a.e.*

By the separability of H and by ([55], Prop. III.35), we obtain $h(t) \in F(t, u_g(t))$, μ *a.e.*

Applying the Kakutani–Ky Fan fixed point theorem to the convex weakly compact-valued upper semicontinuous mapping Φ shows that Φ admits a fixed point, $g \in \Phi(g)$, thus proving the existence of at least one BVRC solution to our inclusion.

Step 2. Compactness follows easily from the above arguments and the pointwise compactness of $\mathcal{T}_X$ given Lemma 7. $\square$

The following result has some importance in further applications

Corollary 3. *Assume that for every $t \in I = [0, T]$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_2^*)$ and $(\mathcal{H}_4^*)$.*
Let $f : I \times H \to H$ satisfying:

(i) $f(., x) \in L^1_H(I, \mathcal{B}(I), \mu)$ *for all* $x \in H$*;*
(ii) $\|f(t, x) - f(t, y)\| \leq M\|x - y\|$ *for all* $(t, x, y) \in I \times H \times H$*;*
(iii) $\|f(t, x)\| \leq M$ *for all* $(t, x) \in I \times H$*, for some constant* $M > 0$.

Let $\nu := dr + Md\mu$.
Assume further that there is $\beta \in]0, 1[$ such that $\forall t \in I$, $0 \leq 2M\frac{d\mu}{d\nu}(t)\nu(\{t\}) \leq \beta < 1$.
Then there is a unique BVRC solution to the problem

$$\begin{cases} u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t) \quad \forall t \in \; I \\ \frac{du}{d\nu} \in L^\infty_H(I, \nu) \\ -\dfrac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t))\dfrac{d\mu}{d\nu}(t), \; \nu \text{ a.e. } t \in I. \end{cases}$$

Proof. Existence follows from Theorem 3. The proof of uniqueness is carried out in a similar way to that of Corollary 2. $\square$

3. Towards the Existence of BVRC Periodic Solution

Proposition 1. *Assume that for every $t \in I = [0, T]$, $A_t : D(A_t) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_2^*)$ and $(\mathcal{H}^*)$.*
Let $f : I \times H \to H$ satisfying:

(i) $f(., x) \in L^1_H(I, \mathcal{L}(I), \lambda)$ *for all* $x \in H$*;*
(ii) $\|f(t, x) - f(t, y)\| \leq M\|x - y\|$ *for all* $(t, x, y) \in I \times H \times H$*;*
(iii) $\|f(t, x)\| \leq M$ *for all* $(t, x) \in I \times H$*, for some constant* $M > 0$.

Let $\nu := dr + M\lambda$.
Assume further that there is $\beta \in]0, 1[$ such that $\forall t \in I$, $0 \leq 2M\frac{d\lambda}{d\nu}(t)\nu(\{t\}) \leq \beta < 1$.
Then there is a **unique** *BVRC periodic solution to the problem*

$$\begin{cases} u(0) = u(T) \\ u(t) \in D(A_t) \quad \forall t \in \; I \\ \frac{du}{d\nu} \in L^\infty_H(I, \nu) \\ -\dfrac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t))\dfrac{d\lambda}{d\nu}(t), \; \nu \text{ a.e. } t \in I. \end{cases}$$

Proof. **Existence** of BVRC solution follows from Theorem 3.

Uniqueness: The demonstration takes place with necessary adaptations similarly to that of Corollary 2.

Periodicity: Let u_a and u_b be two BVRC solutions to the problem under consideration, that is,

$$-\frac{du_a}{d\nu}(t) - f(t, u_a(t))\frac{d\lambda}{d\nu}(t) \in A_t u_a(t), u_a(0) = a \in D(A_0)$$

$$-\frac{du_b}{d\nu}(t) - f(t, u_b(t))\frac{d\lambda}{d\nu}(t) \in A_t u_b(t), u_b(0) = b \in D(A_0)$$

By repeating the previous calculus and applying again Gronwall's Lemma 4, we have

$$||u_a(t) - u_b(t)||^2 \leq ||a-b||^2 \exp\Big(\frac{1}{1-\beta}\int_{]0,t]} 2M\frac{d\lambda}{d\nu}(s)\nu(ds)\Big), \forall t \in I$$

in particular,

$$||u_a(T) - u_b(T)||^2 \leq ||a-b||^2 \exp\Big(\frac{1}{1-\beta}\int_{]0,T]} 2M\frac{d\lambda}{d\nu}(s)\nu(ds)\Big).$$

This shows that the mapping $a \to u_a(T)$ is a Lipschitz mapping from $D(A_0)$ into $D(A_T) \subset D(A_0)$. Since $D(A_T)$ is convex compact, by the Schauder fixed point theorem, there exists at least one $a \in D(A_0)$ such that $u_a(T) = a$. This provides us a BVRC periodic solution to $-Du \in A_t u(t) + f(t, u(t))$. $\square$

There is a direct application to the sweeping process.

Proposition 2. *Let $C : I \to H$ be a closed convex-valued mapping satisfying*

$(\mathcal{H})^*_C$ *$d_H(C(t), C(\tau)) \leq r(t) - r(\tau)$ for all $0 \leq \tau \leq t \leq T$, where $r : I \to \mathbb{R}^+$ is non-decreasing continuous on I with $r(0) = 0, r(T) < \infty$.*

$(\mathcal{H})^{**}_C$ *$C(T)$ is compact and $C(T) \subset C(0)$;*

Let $f : I \times H \to H$ satisfying:

(i) $f(\cdot, x)$ is $\mathcal{L}(I))$-measurable on I;
(ii) $\|f(t,x) - f(t,y)\| \leq M\|x-y\|$ for all $(t,x,y) \in I \times H \times H$;
(iii) $\|f(t,x)\| \leq M$ for all $(t,x) \in I \times H$, for some constant $M > 0$.

Let $\nu := dr + M\lambda$.

Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I$, $0 \leq 2M\frac{d\lambda}{d\nu}(t)\nu(\{t\}) \leq \beta < 1$.

Then there is a unique BVRC periodic solution to the problem

$$\begin{cases} u(0) = u(T) \\ u(t) \in C(t) \quad \forall t \in \ I \\ \frac{du}{d\nu} \in L^\infty_H(I,\nu) \\ -\dfrac{du}{d\nu}(t) \in N_{C(t)}u(t) + f(t,u(t))\dfrac{d\lambda}{d\nu}(t),\ \nu \text{ a.e. } t \in I. \end{cases}$$

The following result deals with another class of time-dependent maximal monotone operator [54].

Proposition 3. *Let $t :\mapsto A_t : D(A_t) \to ccwl(E)$ be a time-dependent maximal monotone operator satisfying $(\mathcal{H}^*_1)$, $(\mathcal{H}^*_4)$, $(\mathcal{H}^*_g)$ and $(\mathcal{H}^A_3)$ and $(\mathcal{H}^A_4)$ of Theorem 2.*

Let $f : I \times H \to H$ such that:

(i) $f(.,x)$ is $\mathcal{L}(I)$-measurable on I for all $x \in H$;
(ii) $||f(t,x) - f(t,y)|| \leq M||x-y||$ for all $t,x,y \in I \times H \times H$;
(iii) $||f(t,x)|| \leq M$ for all $(t,x) \in I \times H$, for some constant $M > 0$.

Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ relative to the measure ν. Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I$, $0 \leq 2M\frac{d\lambda}{d\nu}(t)d\nu(\{t\}) \leq \beta < 1$.

Then, for all $u_0 \in D(A_0)$, the evolution problem

$$-Du(t) \in A_t u(t) + f(t, u(t))$$

admits a unique BVRC periodic solution u with $u(0) = u(T)$, that is, there exists a BVRC function $u : I \to H$ such that

$$\begin{cases} u(0) = u(T) \\ u(t) \in D(A_t), \forall t \in \ I \\ \frac{du}{d\nu}(t) \in L^\infty_H(I, d\nu) \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t)) \frac{d\lambda}{d\nu}(t), \ \nu \ a.e. \end{cases}$$

Proof. Existence of the BVRC solution follows from Theorem 2. We need only to prove the uniqueness.

Uniqueness: Let u and v be two BVRC solutions to the problem under consideration, that is,

$$-\frac{du}{d\nu}(t) - f(t, u(t)) \frac{d\lambda}{d\nu}(t) \in A_t u(t), u(0) = a \in D(A_0)$$

$$-\frac{dv}{d\nu}(t) - f(t, v(t)) \frac{d\lambda}{d\nu}(t) \in A_t v(t), v(0) = a \in D(A_0)$$

By the monotonicity of A_t, we obtain

$$\Big\langle -\frac{du}{d\nu}(t) - f(t, u(t)) \frac{d\lambda}{d\nu}(t) - (-\frac{dv}{d\nu}(t) - f(t, v(t)) \frac{d\lambda}{d\nu}(t)), u(t) - v(t) \Big\rangle \geq 0.$$

Equivalently,

$$\Big\langle \frac{du}{d\nu}(t) - \frac{dv}{d\nu}(t), u(t) - v(t) \Big\rangle \leq -\Big\langle f(t, u(t)) - f(t, v(t)), (u(t) - v(t)) \frac{d\lambda}{d\nu}(t) \Big\rangle$$

By hypothesis (ii),

$$\begin{aligned} \Big\langle \frac{du}{d\nu}(t) - \frac{dv}{d\nu}(t), v(t) - u(t) \Big\rangle \quad &\leq \quad -\Big\langle f(t, u(t)) - f(t, v(t)), (u(t) - v(t)) \frac{d\lambda}{d\nu}(t) \Big\rangle \\ \leq M \frac{d\lambda}{d\nu}(t) \|u(t) - v(t)\|^2. \end{aligned}$$

On the other hand, we know that u and v are BVC and have the densities $\frac{du}{d\nu}$ and $\frac{dv}{d\nu}$ relative to ν; due to a result of Moreau concerning the differential measure [56], $\|u - v\|^2$ is BVC and we have

$$d\|u - v\|^2 \leq 2\Big\langle u(\cdot) - v(\cdot), \frac{du}{d\nu}(\cdot) - \frac{dv}{d\nu}(\cdot) \Big\rangle d\nu \tag{8}$$

such that by integrating on $]0, t]$ with respect to the measure $d\nu$ and using the above estimate, we obtain

$$\|u(t) - v(t)\|^2 = \int_{]0,t]} d\|u - v\|^2$$

$$\leq \int_{]0,t]} 2\langle u(s) - v(s), \frac{du}{d\nu}(s) - \frac{dv}{d\nu}(s) \rangle d\nu(s)$$

$$\leq \int_{]0,t]} 2M \frac{d\lambda}{d\nu}(s) \|u(s) - v(s)\|^2 d\nu(s)$$

that is,

$$||u(t) - v(t)||^2 \leq \int_{]0,t]} 2M \frac{d\lambda}{d\nu}(s) \|u(s) - v(s)\|^2 d\nu(s)$$

According to the assumption $\forall t \in I, 0 \leq 2M \frac{d\lambda}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$ and using Grownwall's Lemma 4, we deduce from the last inequality that $u = v$ in I.

Periodicity: Let u_a and u_b be two BVRC solutions to the problem under consideration, that is,

$$-\frac{du_a}{d\nu}(t) - f(t, u_a(t)) \frac{d\lambda}{d\nu}(t) \in A_t u_a(t), u_a(0) = a \in D(A_0)$$

$$-\frac{du_b}{dv}(t) - f(t, u_b(t))\frac{d\lambda}{dv}(t) \in A_t u_b(t), u_b(0) = b \in D(A_0)$$

By repeating the above argument, we have

$$||u_a(t) - u_b(t)||^2 \leq ||a - b||^2 + \int_{]0,t]} 2M\frac{d\lambda}{dv}(s)\|u_a(s) - u_b(s)\|^2 dv(s))$$

such that again by Gronwall's Lemma 4,

$$||u_a(t) - u_b(t)||^2 \leq ||a - b||^2 \exp\Big(\frac{1}{1-\beta}\int_{]0,t]} 2M\frac{d\lambda}{dv}(s)\nu(ds)\Big), \forall t \in I$$

such that

$$||u_a(T) - u_b(T)||^2 \leq ||a - b||^2 \exp\Big(\frac{1}{1-\beta}\int_{]0,T]} 2M\frac{d\lambda}{dv}(s)\nu(ds)\Big).$$

This shows that the mapping $a \to u_a(T)$ is a Lipschitz mapping from $D(A_0)$ into $D(A_T) \subset D(A_0)$. Since $D(A_T)$ is convex compact, by the Schauder fixed point theorem, there exists at least one $a \in D(A_0)$ such that $u_a(T) = a$. This provides us a BVRC periodic solution to $-Du \in A_t u(t) + f(t, u(t))$. □

Most cases of the BVRC periodic solution given here are new. Several variants dealing with absolutely continuous or BVC periodic solutions are available. For the sake of brevity, we omit the details. However, it is worth mentioning that indealing with the uniqueness of a BVRC solution, a special condition is required.

4. Second-Order Problem with Perturbation of the BVRC Setting

Now we study some second-order evolution inclusions driven by a time- and state-dependent maximal monotone operator in the bounded variation right continuous setting. The interest in studying second-order evolution problems is motivated by their applications; see the large synthesis by Brogliato et al. [5], particularly dry friction in mechanics [8,9].

Let $I = [0, T]$ and let H be a separable Hilbert space. We state the existence of a second-order evolution driven by a time- and state-dependent maximal monotone operator $A_{(t,x)}$ in the bounded variation right continuous setting. In the remainder of the work, $d\rho$ denotes the Stieltjes measure associated with a non-decreasing right continuous function $\rho : I \to \mathbb{R}^+$ with $\rho(0) = 0, \rho(T) < +\infty$. The following assumptions are used for obtaining our results.

$(\mathcal{H}_1)$ $\|A^0_{(t,x)}y\| \leq c(1 + \|x\| + \|y\|)$ for all $(t, x, y) \in I \times E \times D(A_{(t,x)})$, for some positive constant c.

$(\mathcal{H}_2)$ $dis(A_{(t,x)}, A_{(\tau,y)}) \leq r(t) - r(\tau) + \|x - y\|$, for all $0 \leq \tau \leq t \leq T$ and for all $(x, y) \in H \times H$, where $r : I \to [0, +\infty[$ is non-decreasing right continuous on I with $r(0) = 0, r(T) < \infty$.

$(\mathcal{H}_2^c)$ $dis(A_{(t,x)}, A_{(\tau,y)}) \leq r(t) - r(\tau) + \|x - y\|$, for all $0 \leq \tau \leq t \leq T$ and for all $(x, y) \in H \times H$, where $r : I \to [0, +\infty[$ is non-decreasing continuous on I with $r(0) = 0$ and $r(T) < \infty$.

$(\mathcal{H}_3)$ $D(A_{(t,x)})$ is boundedly compactly measurable, in the sense of (i) and (ii):

(i) $D(A_{(t,x)}) \subset X(t) := \gamma(t)\overline{B}_H$ for all $(t, x) \in I \times H$ where γ is a positive $L^1(I, \lambda)$-integrable function;

(ii) for any bounded subset $\mathcal{B} \subset \mathcal{C}_H(I)$, there is a compact-valued Borel-measurable mapping $\Psi_{\mathcal{B}} : I \to H$ such that $D(A_{(t,h(t))}) \subset \Psi_{\mathcal{B}}(t) \subset \gamma(t)\overline{B}_H$ for all $(t, h) \in I \times \mathcal{B}$.

Theorem 4. *Let $(t, x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.*

Let $f : I \times H \times H \to H$ be such that for every $x, y \in H$, the mapping $f(\cdot, x, y) \in L^1_H(I, \mathcal{B}(I), \mu)$ and for every $t \in I$, the mapping $f(t, \cdot, \cdot)$ is continuous on $H \times H$ and satisfies:

(i) $\|f(t,x,y)\| \leq M$ *for all* $(t,x,y) \in I \times H \times H$;

(ii) $\|f(t,z,x) - f(t,z,y)\| \leq M\|x-y\|$ *for all* $(t,z,x,y) \in I \times H \times H \times H$,

for some nonnegative constant M.

Let $\rho(t) = r(t) + \int_0^t \gamma(s)ds, t \in I$ *and* $\nu = d\rho + M\mu$. *Assume further that there is* $\beta \in]0,1[$ *such that* $\forall t \in I$, $0 \leq 2M\frac{d\mu}{d\nu}(t)\nu(\{t\}) \leq \beta < 1$ *where* $\frac{d\mu}{d\nu}(t)$ *is the density of the measure* μ *with respect to the measure* ν.

Then, for any $(x_0, u_0) \in H \times D(A_{(0,x_0)})$ *there exists an absolutely continuous* $x : I \to H$ *and a BVRC* $u : I \to H$ *with density* $\frac{du}{d\nu}$ *with respect to* ν, *such that*

$$\begin{cases} x(t) = x_0 + \int_0^t u(s)ds, \quad \forall t \in I \\ x(0) = x_0, u(0) = u_0 \in D(A_{(0,x_0)}) \\ u(t) \in D(A_{(t,x(t))}), \forall t \in I \\ -\frac{du}{d\nu}(t) \in A_{(t,x(t))}u(t) + f(t,x(t),u(t))\frac{d\mu}{d\nu}(t), \ \nu \text{ a.e. } t \in I \end{cases}$$

Proof. Let

$$\mathcal{X} := \{u_f \in C_H(I : u_f(t) = x_0 + \int_0^t f(s)ds, t \in I, f \in S^1_X\}$$

Then $\mathcal{X}$ is closed convex $\subset C_H(I)$ using the weak compactness of the convex weakly compact-valued integral $\int_0^t X(s)ds$ and equi-absolutely continuous. For each $h \in \mathcal{X}$, the time-dependent maximal monotone operator $A_{(t,h(t))}$ is **equi-BVRC in variation**:

$$dis(A_{(t,h(t))}, A_{(\tau,h(\tau))}) \leq r(t) - r(\tau) + \int_\tau^t \gamma(s)ds = \rho(t) - \rho(\tau)$$

for all $\tau < t \in I$ where $\rho(t) = r(t) + \int_0^t \gamma(s)ds$. Let us set $\nu = d\rho + M\mu$ where $d\rho$ is the Stieljies measure associated to the non-decreasing right continuous function ρ. Let us denote by $\frac{d\mu}{d\nu}$ the density of the measure μ with respect to ν. By applying Corollary 3, where A_t is replaced by $A_{(t,h(t))}$, with $\nu = dr + M\mu$ replaced by $\nu = d\rho + M\mu$, for any $h \in \mathcal{X}$, there is a unique BVRC solution u_h to

$$\begin{cases} u_h(0) = u_0 \\ u_h(t) \in D(A_{(t,h(t))})) \quad \forall t \in I \\ -\dfrac{du_h}{d\nu}(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t))\dfrac{d\mu}{d\nu}(t), \ \nu \text{ a.e. } t \in I. \end{cases}$$

with $u_h(t) = u_0 + \int_{]0,t]} \frac{du_h}{d\nu}(s)d\nu(s)$ for all $t \in I$ and $\|\frac{du_h}{d\nu}(t)\| \leq K$ ν-a.e. The existence and uniqueness of such a solution is ensured by Corollary 3. Indeed, for any fixed $h \in \mathcal{X}$, the mapping $f_h(.,x) = f(.,h(.),x) \in$ satisfies $||f_h(t,x)|| \leq M$ for all $(t,x) \in I \times H$, $||f_h(t,x) - f_h(t,y)|| = ||f(t,h(t),x) - f(t,h(t),y)|| \leq M||x-y||$ for all $(t,x,y) \in I \times H \times H$, $f_h(.,x) \in L^1_H(I, \mathcal{B}(I), \mu)$, for all $x \in H$. Now for each $h \in \mathcal{X}$, let us consider the mapping

$$\Phi(h)(t) := x_0 + \int_0^t u_h(s)ds \quad \forall t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because by $(\mathcal{H}_3)$, $u_h(t) \in D(A_{(t,h(t)})) \subset \Psi_{\mathcal{X}}(t) \subset \gamma(t)\overline{B}_H$ for all $t \in I$. We are going to show the main fact $\Phi(\mathcal{X}) \subset \mathcal{Y} \subset \mathcal{X}$ where $\mathcal{Y}$ is convex compact in $\mathcal{C}_H(I)$ with

$$\Phi(h) \in \mathcal{Y} := \{u_f \in C_H(I : u_f(t) = x_0 + \int_0^t f(s)ds, t \in I, f \in S^1_{\overline{co}\Psi_{\mathcal{X}}}\}$$

But this last set is convex **compact** in $\mathcal{C}_H(I)$, e.g., [59]. Our aim is to prove that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous in order to obtain the existence theorem by a fixed point approach. This needs a careful look using the estimate of the BVRC solution given above. It is enough to show that, if (h_n) converges uniformly to h in $\mathcal{X}$, then the sequence (u_{h_n}) of BVRC solutions associated with (h_n)

$$\begin{cases} u_{h_n}(0) = u_0 \in D(A_{(0,h_n(0))}) = D(A_{(0,x_0)}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}) \quad \forall t \in I \\ -\dfrac{du_{h_n}}{d\nu}(t) \in A_{(t,h_n(t))}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t))\dfrac{d\mu}{d\nu}(t), \quad \nu \quad a.e. \end{cases}$$

converges pointwise to the BVRC solution u_h associated with h

$$\begin{cases} u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,x_0)}) \\ u_h(t) \in D(A_{(t,h(t))}) \quad \forall t \in I \\ -\dfrac{du_h}{d\nu}(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t))\dfrac{d\mu}{d\nu}(t), \quad \nu \quad a.e. \end{cases}$$

As (u_{h_n}) is bounded in variation since $\|u_{h_n}(t) - u_{h_n}(\tau)\| \leq K(d\nu(]\tau,t])$, for $\tau \leq t$ with $u_{h_n}(t) \in D(A_{(t,h_n(t))}) \subset \Psi_{\mathcal{X}}(t)$, for all $t \in I$, it is relatively compact by the Helly principle [58]; we may ensure that (u_{h_n}) converges pointwise to a BV mapping $u : I \to H$. As $\|u(t) - u(\tau)\| \leq K(\nu(]\tau,t])$, for $0 \leq \tau \leq t \leq T$, u is BVRC with $||\frac{du}{d\nu}|| \leq K, \nu$ *a.e.* and $u(t) - u_0 = \int_{]0,t]} \frac{du}{d\nu}(s)d\nu(s), \forall t \in I$. Now, since for all $t \in I$, $u_{h_n}(t) = u_0 + \int_{]0,t]} \frac{du_{h_n}}{d\nu} d\nu$ and $\frac{du_{h_n}}{d\nu}(s) \in K\overline{B}_H$ ν *a.e*, we may assume that $(\frac{du_{h_n}}{d\nu})$ converges weakly in $L^1_H(I,d\nu)$ to $w \in L^1_H(I,d\nu)$ with $w(t) \in K\overline{B}_H$ ν *a.e.*, so that by identifying the limits

$$\langle e, u(t)\rangle = \lim_{n\to\infty} \langle e, u_{h_n}(t)\rangle = \langle e, u_0\rangle + \langle e, \int_{]0,t]} w(s)d\nu(s)\rangle, \quad \forall e \in H, \forall t \in I.$$

we obtain

$$\int_{]0,t]} \frac{du}{d\rho}(s)d\nu(s) = \int_{]0,t]} w(s)d\nu(s) \quad \forall t \in I;$$

hence, $\frac{du}{d\nu} = w$ and $(\frac{du_{h_n}}{d\nu})$ weakly converges to $\frac{du}{d\nu}$ in $L^1_H(I,d\nu)$, so we may assume that it Komlos converges to $\frac{du}{d\nu}$.

It is clear that $z_n(t) := f(t,h_n(t),u_{h_n}(t))\frac{d\mu}{d\nu}(t) \to z(t) := f(t,h(t),u_h(t))\frac{d\mu}{d\nu}(t)$ pointwise. Hence,

$$z_n(.) := f(.,h_n(.),u_{h_n}(.))\frac{d\mu}{d\nu}(.) \to z(.) := f(.,h(.),u_h(.))\frac{d\mu}{d\nu}(.)$$

in $L^1_H(I,\nu)$. Hence, we may assume that $\frac{du_{h_n}}{d\nu} + f(.,h(.),u_h(.))\frac{d\mu}{d\nu}(.) \to \frac{du}{d\nu} + f(.,h(.),u_h(.))\frac{d\mu}{d\nu}(.)$ Further, we note that $u(t) \in D(A_{(t,h(t))})$ for all $t \in I$. Indeed, we have $dis(A_{(t,h_n(t))}, A_{(t,h(t))}) \leq ||h_n(t) - h(t)|| \to 0$. It is clear that $y_n = A^0_{(t,h_n(t))}u_{h_n}(t)$ is bounded, hence relatively weakly compact. By applying Lemma 2 to $u_{h_n}(t) \to u(t)$ and to a convergent subsequence of (y_n) to show that $u(t) \in D(A_{(t,h(t))})$, there is a ν-negligible set N such that

$$-\frac{du_{h_n}}{d\nu}(t) - z_n(t) \in A_{(t,h_n(t))}u_{h_n}(t),\ t \in I \setminus N,$$

$$\lim_n \frac{1}{n}\sum_{j=1}^n (\frac{du_{h_j}}{d\nu}(t) + z_j(t)) = \frac{du}{d\nu}(t) + z(t),\ t \in I \setminus N.$$

Let $t \in I \setminus N$. Let $\eta \in D(A_{(t,h(t))})$. Apply Lemma 3 to $A_{(t,h_n(t))}$ and $A_{(t,h(t))}$ to find a sequence (η_n) such that $\eta_n \in D(A_{(t,h_n(t))})$, $\eta_n \to \eta$, $A^0_{(t,h_n(t))}\eta_n \to A^0_{(t,h(t))}\eta$. From

$$-\frac{du_{h_n}}{d\nu}(t) - z_n(t) \in A_{(t,h_n(t))}u_{h_n}(t)$$

by monotonicity,

$$\langle \frac{du_{h_n}}{dv}(t) + z_n(t), u_{h_n}(t) - \eta_n \rangle \leq \langle A^0_{(t,h_n(t))}\eta_n, \eta_n - u_{h_n}(t) \rangle.$$

From

$$\begin{aligned}&\left\langle \tfrac{du_{h_n}}{dv}(t) + z_n(t), u(t) - \eta \right\rangle \\ &= \left\langle \tfrac{du_{h_n}}{dv}(t) + z_n(t), u_{h_n}(t) - \eta_n \right\rangle + \left\langle \tfrac{du_{h_n}}{dv}(t) + z_n(t), u(t) - u_{h_n}(t) - (\eta - \eta_n) \right\rangle,\end{aligned}$$

let us write

$$\frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{dv}(t) + z_j(t), u(t) - \eta \right\rangle = \frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{dv}(t) + z_j(t), u_{h_j}(t) - \eta_j \right\rangle$$

$$+\frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{dv}(t) + z_j(t), u(t) - u_{h_j}(t) \right\rangle + \frac{1}{n}\sum_{j=1}^{n} \left\langle \frac{du_{h_j}}{dv}(t) + z_j(t), \eta_j - \eta \right\rangle,$$

so that

$$\begin{aligned}&\tfrac{1}{n}\textstyle\sum_{j=1}^{n} \left\langle \tfrac{du_{h_j}}{dv} + z_j(t)(t), u(t) - \eta \right\rangle \\ &\leq \tfrac{1}{n}\textstyle\sum_{j=1}^{n} \left\langle A^0_{(t,h_j(t))}\eta_j, \eta_j - u_{h_j}(t) \right\rangle + (K+M)\tfrac{1}{n}\textstyle\sum_{j=1}^{n} \|u(t) - u_{h_j}(t))\|.\end{aligned}$$

$$+(K+M)\frac{1}{n}\sum_{j=1}^{n} \|\eta_j - \eta\|.$$

Passing to the limit using $n \to \infty$, this last inequality immediately gives

$$\left\langle \frac{du}{dv}(t) + z(t), u(t) - \eta \right\rangle \leq \left\langle A^0_{(t,h(t))}\eta, \eta - u(t) \right\rangle \ \ a.e.$$

As a consequence, by Lemma 1, we obtain $-\frac{du}{dv}(t) \in A_{(t,h(t))}u(t) + z(t)$, ν a.e. with $u(t) \in D(A_{(t,h(t))})$ for all $t \in I$, so that by uniqueness, $u = u_h$. Consequently, for all $t \in I$,

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^t (u_{h_n}(s) - u_h(s))ds,$$

and since $(u_{h_n}(s) - u_h(s)) \to 0$ and is pointwise bounded : $\|u_{h_n}(s) - u_h(s)\| \leq 2\gamma(s)$, we conclude by the Lebesgue theorem that

$$\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^T \|u_{h_n}(s) - u_h(s)\|ds \longrightarrow 0,$$

such that $\Phi(h_n) - \Phi(h) \to 0$ in $\mathcal{C}(I, H)$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous with $\Phi(\mathcal{X}) \subset \mathcal{Y} \subset \mathcal{X}$, by the Schauder theorem, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$\begin{cases} h(t) = \Phi(h)(t) = x_0 + \int_0^t u_h(s)ds, \ \forall t \in \ I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,x_0)}) \\ u_h(t) \in D(A_{(t,h(t))}), \quad \forall t \in \ I \\ -\dfrac{du_h}{dv}(t) \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t))\dfrac{d\mu}{dv}(t), \quad \nu \quad a.e.\ t \in I. \end{cases}$$

□

We present a study of second-order differential equation with m-point boundary conditions coupled with a time-dependent maximal monotone operator. For the sake of completeness, we recall and summarize some results developed in [60].

Lemma 8. *Assume that H is a separable Hilbert space and $I = [0,1]$. Let $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, $\gamma > 0$, $m > 3$ be an integer number, and $\alpha_i \in \mathbb{R}$ $(i = 1, \cdots, m-2)$ satisfying the condition*

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i)) \neq 0.$$

Let $G : I \times I \to \mathbb{R}$ be the function defined by

$$G(t,s) = \begin{cases} \frac{1}{\gamma}(1 - \exp(-\gamma(t-s))), & 0 \le s \le t \le 1 \\ 0, & t < s \le 1 \end{cases} + \frac{A}{\gamma}(1 - \exp(-\gamma t))\phi(s), \tag{9}$$

where

$$\phi(s) = \begin{cases} 1 - \exp(-\gamma(1-s)) - \sum_{i=1}^{m-2} \alpha_i (1 - \exp(-\gamma(\eta_i - s))), & 0 \le s < \eta_1, \\ 1 - \exp(-\gamma(1-s)) - \sum_{i=2}^{m-2} \alpha_i (1 - \exp(-\gamma(\eta_i - s))), & \eta_1 \le s \le \eta_2, \\ \dots\dots\dots & \\ 1 - \exp(-\gamma(1-s)), & \eta_{m-2} \le s \le 1, \end{cases} \tag{10}$$

and

$$A = \left(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma\eta_i) \right)^{-1}. \tag{11}$$

Then the following assertions hold:

(i) *For every fixed $s \in [0,1]$, the function $G(\cdot, s)$ is right derivable on $[0,1[$ and left derivable on $]0,1]$.*

(ii) *$G(\cdot,\cdot)$ and $\frac{\partial G}{\partial t}(\cdot,\cdot)$ satisfies*

$$|G(t,s)| \le M_G \quad \text{and} \quad \left| \frac{\partial G}{\partial t}(t,s) \right| \le M_G \quad \forall (t,s) \in I \times I,$$

where

$$M_G = \max\{\gamma^{-1}, 1\} \left[1 + |A| \left(1 + \sum_{i=1}^{m-2} |\alpha_i| \right) \right].$$

(iii) *If $u \in W_H^{2,1}(I)$ with $u(0) = x$ and $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$, then*

$$u(t) = e_x(t) + \int_0^1 G(t,s)(\ddot{u}(s) + \gamma \dot{u}(s))ds, \quad \forall t \in I,$$

where

$$e_x(t) = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x.$$

(iv) *Let $f \in L_H^1([0,1])$ and let $u_f : [0,1] \to H$ be the function defined by*

$$u_f(t) = e_x(t) + \int_0^1 G(t,s) f(s) ds \quad \forall t \in [0,1].$$

Then we have

$$u_f(0) = x \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i).$$

Further, the function u_f is derivable on $[0,1]$ and its derivative $\dot{u}_f$

$$\dot{u}_f(t) = \dot{e}_x(t) + \int_\tau^1 \frac{\partial G}{\partial t}(t,s)f(s)ds,$$

where

$$\dot{e}_x(t) = \gamma A(1 - \sum_{i=1}^{m-2} \alpha_i) \exp(-\gamma t)x.$$

(v) *If $f \in L^1_H([0,1])$, the function $\dot{u}_f$ is scalarly derivable, and its weak derivative $\ddot{u}_f$ satisfies $\ddot{u}_f(t) + \gamma\dot{u}_f(t) = f(t)$ a.e. $t \in [0,1]$.*

The following is a direct consequence of Lemma 8.

Proposition 4. *Let $f \in L^1_H([0,1])$. The m-point boundary problem*

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) = f(t),\ t \in [0,1] \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

has a unique $W^{2,1}_H([0,1])$-solution u_f, with integral representation formulas

$$\begin{cases} u_f(t) = e_x(t) + \int_0^1 G(t,s)f(s)ds,\ t \in [0,1] \\ \dot{u}_f(t) = \dot{e}_x(t) + \int_0^1 \frac{\partial G}{\partial t}(t,s)f(s)ds,\ t \in [0,1]. \end{cases}$$

where

$$\begin{cases} e_x(t) & = x + A(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \exp(-\gamma t))x \\ \dot{e}_x(t) & = \gamma A\Big(1 - \sum_{i=1}^{m-2} \alpha_i\Big) \exp(-\gamma t)x \\ A & = \Big(\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma(\eta_i))\Big)^{-1}. \end{cases}$$

The following result is crucial for our purposes. For the sake of brevity, we omit the proof; one can find the details in Theorem 5.1 of [60].

Proposition 5. *With the hypotheses and notations of Proposition 4, let H be a separable Hilbert space and let $X : [0,1] \rightrightarrows H$ be a measurable convex compact-valued and integrably bounded mapping. Then the solution set of $W^{2,1}_H([0,1])$-solutions to*

$$\begin{cases} \ddot{u}_f(t) + \gamma\dot{u}_f(t) = f(t),\ t \in [0,1],\ f \in S^1_X \\ u_f(0) = x, \quad u_f(1) = \sum_{i=1}^{m-2} \alpha_i u_f(\eta_i), \end{cases}$$

is bounded, convex, equicontinuous, and compact in $\mathcal{C}_H([0,1])$.

Now comes an existence result with a second-order differential inclusion with m-point boundary condition coupled with a time-dependent maximal monotone operator with Lipschitz perturbation.

Theorem 5. *Let $I := [0,1]$. Let $t \to A_t,: D(A_t) \to 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_2^*)$ and $(\mathcal{H}_3^*)$.*

Let $f : I \times H \times H \to H$ be such that for every $x, y \in H$, the mapping $f(\cdot, x, y) \in L^1_H(I, \mathcal{B}(I), \mu)$ and for every $t \in I$, the mapping $f(t,\cdot,\cdot)$ is continuous on $H \times H$ and satisfies:

(i) $\|f(t,x,y)\| \le M$ *for all* $(t,x,y) \in I \times H \times H$;

(ii) $\|f(t,z,x) - f(t,z,y)\| \le M\|x-y\|$ *for all* $(t,z,x,y) \in I \times H \times H \times H$,

for some nonnegative constant M.

Let $\nu = dr + M\mu$.

Assume further that there is $\beta \in]0,1[$ *such that* $\forall t \in I$, $0 \leq 2M\frac{d\mu}{d\nu}(t)\nu(\{t\}) \leq \beta < 1$ *where* $\frac{d\mu}{d\nu}(t)$ *is the density of the measure* μ *with respect to the measure* ν.

Then there is a $W^{2,1}_H(I)$ *mapping* $u : I \to H$ *and a BVRC mapping* $v : I \to H$ *satisfying*

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) = v(t), \ t \in \ I \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \\ v(0) = v_0 \in D(A_0) \\ -\frac{dv}{d\nu}(t) \in A_t v(t) + f(t, u(t), v(t))\frac{d\mu}{d\nu}(t), \ \nu \text{ a.e. } t \in I. \end{cases}$$

Proof. Let $\mathcal{X} := \{u_f : I \to H, \ u_f(t) = e_x(t) + \int_0^1 G(t,s)f(s)ds, \ t \in I, f \in S^1_X\}$ be the solution set to the second-order differential inclusion with m-point boundary conditions

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in X(t), \ t \in \ I \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

Then by Proposition 5, $\mathcal{X}$ is convex compact in $\mathcal{C}_H(I)$. Let us set $\nu = dr + M\mu$ where dr is the Stieljies measure associated to the nondecreasing right continuous function r. Let us denote by $\frac{d\mu}{d\nu}$ the density of the measure μ with respect to ν. By applying Corollary 3 for any $h \in \mathcal{X}$, there is a unique BVRC solution v_h to

$$\begin{cases} v_h(0) = v_0 \in D(A_0) \\ v_h(t) \in D(A_t), \ \forall t \in \ I \\ -\dfrac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{d\mu}{d\nu}(t), \ \nu \ a.e. \ t \in I. \end{cases}$$

with $v_h(t) = v_0 + \int_{]0,t]} \frac{dv_h}{d\nu}(s)d\nu(s)$ for all $t \in I$ and $\|\frac{dv_h}{d\nu}(t)\| \leq K$ ν-a.e. Now for every $h \in \mathcal{X}$, let us set

$$\Phi(h)(t) = e_x(t) + \int_0^1 G(t,s)v_h(s)ds, \ t \in I,$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because by $(\mathcal{H}_3^*)$ $v_h(t) \in D(A_t) \subset X(t) \subset \kappa(t)\overline{B}_H$ for all $t \in I$. We claim that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. For this purpose, by repeating the arguments given in the proof of Theorem 4 or Lemma 7 via the Komlos argument, we show that if $h_n \to h$ in $\mathcal{X}$, then the BVRC solution v_{h_n} associated with h_n to

$$\begin{cases} -\frac{dv_{h_n}}{d\nu}(t) \in A_t v_{h_n}(t) + f(t, h_n(t), v_{h_n}(t))\frac{d\mu}{d\nu}(t), \ \nu \ a.e. t \in \ I \\ v_{h_n}(0) = v_0 \in D(A_0) \end{cases}$$

converges pointwise to the BVRC solution v_h associated with h to

$$\begin{cases} -\frac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\frac{d\mu}{d\nu}(t), \ \nu \ a.e. t \in \ I \\ v_h(0) = v_0 \in D(A_0) \end{cases}$$

As $\|v_{h_n}(\cdot) - v_h(\cdot)\| \to 0$, we conclude via the estimation in Lemma 8 that $\sup_{t \in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G \|v_{h_n}(\cdot) - v_h(\cdot)\| ds \to 0$ such that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_H(I)$.

Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$h(t) = \Phi(h)(t) = e_x(t) + \int_0^1 G(t,s)v_h(s)ds, \ t \in I,$$

$$\begin{cases} v_h(t) \in D(A_t) \\ -\dfrac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{d\mu}{d\nu}(t), \ \nu \ \text{a.e.} \end{cases}$$

By Lemma 8, this means

$$\begin{cases} \ddot{h}(t) + \gamma\dot{h}(t) = v_h(t),\ t \in I, \\ h(0) = x,\ h(1) = \sum_{i=1}^{m-2} \alpha_i h(\eta_i) \\ v_h(t) \in D(A_t),\ t \in\ I \\ -\dfrac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{d\mu}{d\nu}(t),\ \nu \text{ a.e.} \end{cases}$$

The proof is complete. □

The following is a variant dealing with a new class of time-dependent maximal monotone operator (see Theorem 2).

Theorem 6. *Let $I := [0,1]$. Let $t :\mapsto A_t : D(A_t) \to ccwl(E)$ be a time-dependent maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_3^*)$, $(\mathcal{H}_g^{**})$ and $(\mathcal{H}_3^A)$, $(\mathcal{H}_4^A)$ of Theorem 2.*

Let $f : I \times H \times H \to H$ be such that for every $x, y \in H$, the mapping $f(\cdot, x, y) \in L^1_H(I, \mathcal{L}(I), dt)$ and for every $t \in I$, the mapping $f(t, \cdot, \cdot)$ is continuous on $H \times H$ and satisfies

(i) $\|f(t, x, y)\| \le M$ for all $(t, x, y) \in I \times H \times H$;
(ii) $\|f(t, z, x) - f(t, z, y)\| \le M\|x - y\|$ for all $(t, z, x, y) \in I \times H \times H \times H$,

for some nonnegative constant M. Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ relative to the measure ν.

Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I$, $0 \le 2M\frac{dt}{d\nu}(t)\nu(\{t\}) \le \beta < 1$ where $\frac{dt}{d\nu}(t)$ is the density of the measure dt with respect to the measure ν.

Then there is a $W^{2,1}_H(I)$ mapping $u : I \to H$ and a BVRC mapping $v : I \to H$ satisfying

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) = v(t),\ t \in\ I \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \\ v(0) = v_0 \in D(A_0) \\ -\frac{dv}{d\nu}(t) \in A_t v(t) + f(t, u(t), v(t))\frac{d\lambda}{d\nu}(t),\ \nu \text{ a.e. } t \in I. \end{cases}$$

Proof. We repeat the proof of the preceding theorem with careful modifications.

Let $\mathcal{X} := \{u_f : I \to H,\ u_f(t) = e_x(t) + \int_0^1 G(t,s)f(s)ds,\ t \in I, f \in S^1_X\}$ be the solution set to the second-order differential inclusion with m-point boundary conditions

$$\begin{cases} \ddot{u}(t) + \gamma\dot{u}(t) \in X(t),\ t \in\ I \\ u(0) = x, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{cases}$$

Then by Proposition 5, $\mathcal{X}$ is convex compact in $\mathcal{C}_H(I)$. Let us set $\nu = dr + \lambda$, where dr is the Stieljies measure associated to the nondecreasing right continuous function r and λ is the Lebesgue measure on I. Let us denote by $\frac{dt}{d\nu}$ the density of the measure λ with respect to ν. By applying Corollary 2 for any $h \in \mathcal{X}$, there is a unique BVRC solution v_h to

$$\begin{cases} v_h(0) = v_0 \in D(A_0) \\ v_h(t) \in D(A_t),\ \forall t \in\ I \\ -\dfrac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{dt}{d\nu}(t),\ \nu \text{ a.e.} t \in I. \end{cases}$$

with $v_h(t) = v_0 + \int_{]0,t]} \frac{dv_h}{d\nu}(s)d\nu(s)$ for all $t \in I$ and $\|\frac{dv_h}{d\nu}(t)\| \le K\ \nu$ *a.e* where K is a positive generic constant. Now for every $h \in \mathcal{X}$, let us set

$$\Phi(h)(t) = e_x(t) + \int_0^1 G(t,s)v_h(s)ds,\ t \in I,$$

Then it is clear that $\Phi(h) \in \mathcal{X}$ because by $(\mathcal{H}_3^*)$ $v_h(t) \in D(A_t) \subset X(t) \subset \kappa(t)\overline{B}_H$ for all $t \in I$. We claim that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. For this purpose, by repeating the Komlos arguments, we show that if $h_n \to h$ in $\mathcal{X}$, then the BVRC solution v_{h_n} associated with h_n to

$$\begin{cases} -\frac{dv_{h_n}}{d\nu}(t) \in A_t v_{h_n}(t) + f(t, h_n(t), v_{h_n}(t))\frac{dt}{d\nu}(t), \ \nu \ a.e. t \in \ I \\ v_{h_n}(0) = v_0 \in D(A_0) \end{cases}$$

converges pointwise to the BVRC solution v_h associated with h to

$$\begin{cases} -\frac{dv_h}{d\nu}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\frac{dt}{d\nu}(t), \ \nu \ a.e. t \in \ I \\ v_h(0) = v_0 \in D(A_0) \end{cases}$$

As (v_{h_n}) is bounded in variation, since $\|v_{h_n}(t) - v_{h_n}(\tau)\| \leq K(d\nu(]\tau, t])$, for $\tau \leq t$ with $v_{h_n}(t) \in D(A_t) \subset \Psi_{\mathcal{X}}(t)$, for all $t \in I$, it is relatively compact by the Helly principle [58], and we may ensure that (v_{h_n}) converges pointwise to a BV mapping $v : I \to H$. As $\|v(t) - v(\tau)\| \leq K(\nu(]\tau, t])$, for $0 \leq \tau \leq t \leq T$, u is BVRC with $||\frac{du}{d\nu}|| \leq K, \nu$ a.e. and $v(t) - u_0 = \int_{]0,t]} \frac{du}{d\nu}(s) d\nu(s), \forall t \in I$. Now, since for all $t \in I$, $v_{h_n}(t) = u_0 + \int_{]0,t]} \frac{dv_{h_n}}{d\nu} d\nu$ and $\frac{dv_{h_n}}{d\nu}(s) \in K\overline{B}_H \ \nu \ a.e$, we may assume that $(\frac{dv_{h_n}}{d\nu})$ converges weakly in $L^1_H(I, d\nu)$ to $w \in L^1_H(I, d\nu)$ with $w(t) \in K\overline{B}_H \ \nu$ a.e. so that by identifying the limits,

$$\langle e, v(t) \rangle = \lim_{n \to \infty} \langle e, v_{h_n}(t) \rangle = \langle e, v_0 \rangle + \langle e, \int_{]0,t]} w(s) d\nu(s) \rangle, \quad \forall e \in H, \forall t \in I.$$

we obtain

$$\int_{]0,t]} \frac{dv}{d\nu}(s) d\nu(s) = \int_{]0,t]} w(s) d\nu(s) \quad \forall t \in I;$$

hence, $\frac{dv}{d\nu} = w$ and $(\frac{dv_{h_n}}{d\nu})$ weakly converges to $\frac{dv}{d\nu}$ in $L^1_H(I, d\nu)$, so we may assume that it Komlos converges to $\frac{dv}{d\nu}$. It is clear that $z_n(t) := f(t, h_n(t), v_{h_n}(t))\frac{dt}{d\nu}(t) \to z(t) := f(t, h(t), u_h(t))\frac{dt}{d\nu}(t)$ pointwise. Hence,

$$z_n(.) := f(., h_n(.), u_{h_n}(.))\frac{dt}{d\nu}(.) \to z(.) := f(., h(.), u_h(.))\frac{dt}{d\nu}(.)$$

weakly in $L^1_H(I, \nu)$. Hence, we may assume that
$\frac{dv_{h_n}}{d\nu}(t) + f(., h(.), u_h(.))\frac{dt}{d\nu}(.) \to \frac{du}{d\nu} + f(., h(.), u_h(.))\frac{dt}{d\nu}(.)$ Komlos. Further, we note that $v(t) \in D(A_t)$ for all $t \in I$. There is a ν-negligible set N such that

$$-\frac{dv_{h_n}}{d\nu}(t) - z_n(t) \in A_t v_{h_n}(t), \ t \in I \setminus N,$$

$$\lim_n \frac{1}{n} \sum_{j=1}^n (\frac{dv_{h_j}}{d\nu}(t) + z_j(t)) = \frac{dv}{d\nu}(t) + z(t), \ t \in I \setminus N.$$

Let $t \in I \setminus N$. Let $\eta \in D(A_t)$. From

$$-\frac{dv_{h_n}}{d\nu}(t) - z_n(t) \in A_t v_{h_n}(t)$$

by monotonicity

$$\langle \frac{dv_{h_n}}{d\nu}(t) + z_n(t), v_{h_n}(t) - \eta \rangle \leq \langle A_t^0 \eta, \eta - v_{h_n}(t) \rangle.$$

From
$$\langle \tfrac{dv_{h_n}}{dv}(t) + z_n(t), v(t) - \eta \rangle$$
$$= \langle \tfrac{dv_{h_n}}{dv}(t) + z_n(t), v_{h_n}(t) - \eta \rangle + \langle \tfrac{dv_{h_n}}{dv}(t) + z_n(t), v(t) - v_{h_n}(t) \rangle,$$
let us write
$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{dv_{h_j}}{dv}(t) + z_j(t), v(t) - \eta \rangle = \frac{1}{n}\sum_{j=1}^{n} \langle \frac{dv_{h_j}}{dv}(t) + z_j(t), v_{h_j}(t) - \eta \rangle$$
$$+\frac{1}{n}\sum_{j=1}^{n} \langle \frac{dv_{h_j}}{dv}(t) + z_j(t), v(t) - v_{h_j}(t) \rangle$$
so that
$$\frac{1}{n}\sum_{j=1}^{n} \langle \frac{dv_{h_j}}{dv} + z_j(t)(t), v(t) - \eta \rangle \le \frac{1}{n}\sum_{j=1}^{n} \langle A_t^0\eta, \eta - v_{h_j}(t) \rangle + (K+M)\frac{1}{n}\sum_{j=1}^{n} \|v(t) - v_{h_j}(t))\|.$$
Passing to the limit using $n \to \infty$, this last inequality immediately gives
$$\langle \frac{dv}{dv}(t) + z(t), u(t) - \eta \rangle \le \langle A_t^0\eta, \eta - v(t) \rangle \ a.e.$$
As a consequence, by Lemma 1, we obtain $-\frac{dv}{dv}(t) \in A_{(t,h(t))}u(t) + z(t)$, ν a.e. with $v(t) \in D(A_t)$ for all $t \in I$ so that by uniqueness $v = v_h$. As $\|v_{h_n}(\cdot) - v_h(\cdot)\| \to 0$ we conclude via the estimation in Lemma 8 that $\sup_{t\in I} \|\Phi(h_n)(t) - \Phi(h)(t)\| \le \int_0^T M_G\|v_{h_n}(\cdot) - v_h(\cdot)\|ds \to 0$ so that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_H(I)$.

Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means
$$h(t) = \Phi(h)(t) = e_x(t) + \int_0^1 G(t,s)v_h(s)ds,\ t \in I,$$
$$\begin{cases} v_h(t) \in D(A_t) \\ -\dfrac{dv_h}{dv}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{d\mu}{dv}(t),\ \nu \ a.e. \end{cases}$$
By Lemma 8, this means
$$\begin{cases} \ddot{h}(t) + \gamma\dot{h}(t) = v_h(t),\ t \in I, \\ h(0) = x,\ h(1) = \displaystyle\sum_{i=1}^{m-2} \alpha_i h(\eta_i) \\ v_h(t) \in D(A_t),\ t \in\ I \\ -\dfrac{dv_h}{dv}(t) \in A_t v_h(t) + f(t, h(t), v_h(t))\dfrac{dt}{dv}(t),\ \nu \ a.e. \end{cases}$$
The proof is complete. □

A variant of Theorem 5 dealing with continuous bounded variation (BVC) solutions is available.

Theorem 7. *Let H be a separable Hilbert space. Let, for every $t \in I = [0,T]$, $A_t : D(A_t) \subset H \to 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_2^{*c})$ and $(\mathcal{H}_4^*)$.*

Let $f : I \times H \times H \to H$ be a continuous mapping satisfying

(i) $||f(t,x,y)|| \le M(1 + ||x||),\ \forall t,x,y \in I \times H \times H;$

(ii) $||f(t,x,z) - f(t,y,z)|| \le M||x - y||,\ \forall t,x,y,z \in I \times H \times H \times H,$

for some positive constant M.

Then for $u_0 \in D(A_0), y_0 \in H$, there is a BVC mapping $u : I \to H$, and a BVC mapping $y : I \to H$ satisfying

$$\begin{cases} y(t) = y_0 + \int_0^t u(s)dr(s), \quad t \in I \\ -\frac{du}{dr}(t) \in A_t u(t) + f(t, u(t), y(t)) \; dr\text{-a.e } t \in I \\ u(0) = u_0 \end{cases}$$

with the property $\|u(t) - u(\tau)\| \le K|r(t) - r(\tau)|$ for all $t, \tau \in I$ for some constant $K \in [0, \infty]$.

Proof. This is similar to the proof of Theorem 5, using Theorem 3.1 of [50]. □

5. On Fractional Differential Inclusions

5.1. On a Riemann–Liouville Fractional Differential Inclusion Coupled with Time- and State-Dependent Maximal Monotone Operators

In this subsection, we present a concrete version of the existence of solutions to a fractional differential inclusion (FDI) coupled with a time- and state-dependent maximal monotone operator in the vein of [10,32]. We begin with some preliminary facts.

Definition 1. *(Fractional Bochner integral) Let $f : [0, 1] \to H$. The fractional Bochner integral of order $\alpha > 0$ of the function f is defined by*

$$I^{\alpha}_{a^+} f(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \; t > a.$$

In the above definition, the sign "$\int$" denotes the Bochner integral.

Definition 2. *Let $f \in L^1_H([0, 1])$. We define the Riemann–Liouville fractional derivative of order $\alpha > 0$ of f by*

$$D^{\alpha} f(t) := D^{\alpha}_{0^+} f(t) = \frac{d^n}{dt^n} I^{n-\alpha}_{0^+} f(t) = \frac{d^n}{dt^n} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f(s)ds,$$

where $n = [\alpha] + 1$.

We refer to [38,39,42] for the general theory of Fractional Calculus and Fractional Differential Equations.

We denote by $W^{\alpha,1}_H([0, 1])$ the space of all continuous functions in $\mathcal{C}_H([0, 1])$ such that their Riemann–Liouville fractional derivatives of order $\alpha - 1$ are continuous and their Riemann–Liouville fractional derivatives of order α are Bochner-integrable.

We recall and summarize some useful results in [32].

Lemma 9. *Let $\alpha \in]1, 2]$, $b \in H$ and $f \in L^1_H([0, 1])$. Then the mapping $u_f : [0, 1] \to H$ defined by*

$$u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1}, \; t \in [0, 1]$$

is the unique $W^{\alpha,1}_H([0, 1])$-solution to the (FDI)

$$\begin{cases} D^{\alpha} u(t) = f(t), \; t \in [0, 1] \\ u(0) = 0, D^{\alpha} u(0) = b \\ D^{\alpha-1} u(t) = \int_0^t f(s)ds + b. \end{cases}$$

Lemma 10. *Let $b \in H$. Let $X : [0, 1] \rightrightarrows H$ be a convex compact-valued measurable and integrably bounded multimapping. Then the $W^{\alpha,1}_H([0, 1])$-solution set to the fractional differential inclusion (FDI)*

$$\begin{cases} D^{\alpha} u(t) \in X(t), \; t \in [0, 1] \\ u(0) = 0, D^{\alpha} u(0) = b \end{cases}$$

is bounded, equicontinuous, compact in $\mathcal{C}_H([0,1])$ endowed with the topology of uniform convergence. Furthermore the $W_H^{\alpha,1}([0,1])$-solution set $\mathcal{X}$ is characterized by

$$\mathcal{X} = \{u_f : [0,1] \to H,\ u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},\ f \in S_X^1,\ t \in [0,1]\}.$$

Now comes an existence result to an FDI coupled with a time and state dependent maximal monotone operator.

Theorem 8. *Let $I := [0,1]$ and $b \in H$. Assume that for any $(t,x) \in I \times H$, $A_{(t,x)} : D(A_{(t,x)}) \subset H \rightrightarrows H$ is a maximal monotone operator satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.*
Let $f : I \times H \times H \to H$ be such that for every $x, y \in H$, the mapping $f(\cdot, x, y) \in L_H^1(I, \mathcal{B}(I), \mu)$ and for every $t \in I$, the mapping $f(t, \cdot, \cdot)$ is continuous on $H \times H$ and satisfies

(i) $\|f(t,x,y)\| \leq M$ *for all* $(t,x,y) \in I \times H \times H$;
(ii) $\|f(t,z,x) - f(t,z,y)\| \leq M\|x-y\|$ *for all* $(t,z,x,y) \in I \times H \times H \times H$,
for some nonnegative constant M.

(A) Then the bounded closed convex subset $\mathcal{X}$ in the Banach space $\mathcal{C}_H(I)$ defined by

$$\mathcal{X} = \{u_f : I \to H, u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},$$
$$||f(s)|| \leq \gamma(s)\ a.e\ \ t \in I\}.$$

is equi-K-Lipschitz.

(B) Let $\rho(t) = r(t) + Kt$, for all $t \in I$ and let $\nu = d\rho + M\mu$.
Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I$, $0 \leq 2M\frac{d\mu}{d\nu}(t) d\nu(\{t\}) \leq \beta < 1$.
Then there is a $W_H^{\alpha,1}(I)$ mapping $x : I \to H$ and a BVRC mapping $v : I \to H$ satisfying

$$\begin{cases} D^\alpha x(t) = v(t) \in D(A_{(t,x(t))}),\ t \in\ I \\ x(0) = 0, D^\alpha x(0) = b \\ D^{\alpha-1} x(t) = \int_0^t v(s)ds + b \\ -\frac{dv}{d\nu}(t) \in A_{(t,x(t))} v(t) + f(t, x(t), v(t)\frac{d\mu}{d\nu}(t),\ \nu\ \ a.e.\ t \in I. \end{cases}$$

Proof. (A) Let us consider the bounded closed convex subset $\mathcal{X}$ in the Banach space $\mathcal{C}_H(I)$ defined by

$$\mathcal{X} = \{u_f : [0,1] \to H, u_f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},$$
$$||f(s)|| \leq \gamma(s)\ a.e\ t \in [0,1]\}.$$

We note that $\mathcal{X}$ is equi-Lipschitz (cf. Lemma 4.5 [32]). Indeed, for any $f \in S^1_{\gamma\overline{B}_H}$, $\gamma\overline{B}_H$ denotes the convex weakly compact-valued integrably bounded mapping $t \rightrightarrows \gamma(t)\overline{B}_H$, and for any $0 \leq \tau < t \leq 1$, we have

$$||u_f(t) - u_f(\tau)|| \leq \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} ||b|| + \frac{|t-\tau|^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 \gamma(s)ds,$$

with $\alpha - 1 \in]0,1]$ such that

$$\begin{aligned} ||u_f(t) - u_f(\tau)|| &\leq [\frac{||b||}{\Gamma(\alpha)} + \frac{\int_0^1 \gamma(s)ds}{\Gamma(\alpha)}] |t-\tau|^{\alpha-1} \\ &\leq [\frac{||b||}{\Gamma(\alpha)} + \frac{\int_0^1 \gamma(s)ds}{\Gamma(\alpha)}] |t-\tau| = |L(t) - L(\tau)|, \end{aligned}$$

where $L(t) = \int_0^t K ds, t \in [0,1]$ and $K = \frac{||b||}{\Gamma(\alpha)} + \frac{\int_0^1 \gamma(s)ds}{\Gamma(\alpha)}$.

(B) For any $h \in \mathcal{X}$, the time-dependent maximal monotone operator $A_{(t,h(t))}$ is **equi-BVRC in variation**: For all $\tau < t$, we have by $(\mathcal{H}_2)$

$$\begin{cases} dis(A_{(t,h(t))}, A_{(\tau,h(\tau))}) \\ \leq r(t) - r(\tau) + ||h(t) - h(\tau)|| \\ \leq r(t) - r(\tau) + K(t-\tau) \\ = \rho(t) - \rho(\tau) \end{cases}$$

where $\rho(t) = r(t) + Kt$, $\forall t \in I$. So ρ is non-decreasing right continuous on I with $\rho(0) = 0, \rho(1) < +\infty$. Further, by $(\mathcal{H}_1)$, we have

$$\begin{cases} \|A^0_{(t,h(t))}y\| \leq c(1 + \|h(t)\| + \|y\|) \\ \leq d(1 + \|y\|) \end{cases}$$

for all $y \in D(A_{(t,h(t))})$, where d is a positive generic constant, because $h \in \mathcal{X}$, which is is uniformly bounded. Further, each $f_h(t,x) := f(t,h(t),x)\ \forall (t,x) \in I \times H$ satisfies $\|f(t,h(t),x)\| \leq M$ for all $(t,x) \in I \times H$, and $\|f(t,h(t),x) - f(t,h(t),y)\| \leq M\|x-y\|$ for all $(t,x,y) \in I \times H \times H$. So by virtue of Corollary 3, for every $h \in \mathcal{X}$, there is a unique BVRC solution u_h to

$$\begin{cases} -\frac{du_h}{d\nu}(t) \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t))\frac{d\mu}{d\nu}(t)\ \nu\ a.e.\ t \in\ I \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in\ I \\ u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,0)}) \end{cases}$$

where $\frac{du_h}{d\nu}$ is the density of the differential measure du_h with respect to the measure ν. For each h, let us set

$$\Phi(h)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u_h(s)ds + \frac{b}{\Gamma(\alpha)}t^{\alpha-1},\ t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$, because by $(\mathcal{H}_3)$ $u_h(t) \in D(A_{(t,h(t))}) \subset \Psi_{\mathcal{X}}(t) \subset \gamma(t)\overline{B}_H$ for all $t \in I$ where $\Psi_{\mathcal{X}}$ is a compact-valued Borel-measurable mapping. We note that $\Phi(\mathcal{X}) \subset \mathcal{Y} \subset \mathcal{X}$, where $\mathcal{Y}$ is convex compact in $\mathcal{C}_H([0,1])$:

$$\mathcal{Y} := \{u : [0,1] \to H : \frac{1}{\Gamma(\alpha)}\int_0^1 (t-s)^{\alpha-1}f(s)ds + \frac{b}{\Gamma(\alpha)}t^{\alpha-1}, f \in S^1_{\overline{co}\Psi_{\mathcal{X}}}, t \in I\}.$$

Now we check that Φ is continuous. It is sufficient to show that, if (h_n) uniformly converges to h in $\mathcal{X}$, then the BVRC solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) = u_0 \in D(A_{(0,h_n(0))}) = D(A_{(0,0)}) \\ u_{h_n}(t) \in D(A_{(t,h_n(t))}), \forall t \in\ I \\ -\frac{du_{h_n}}{d\rho} \in A_{(t,h_n(t))}u_{h_n}(t) + f(t,h_n(t),u_{h_n}(t))\frac{d\mu}{d\nu}(t),\ \nu\ a.e.\ t \in I, \end{cases}$$

converges pointwise to the BVRC solution u_h associated with h

$$\begin{cases} u_h(0) = u_0 \in D(A_{(0,h(0))}) = D(A_{(0,0)}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in\ I \\ -\frac{du_h}{d\rho} \in A_{(t,h(t))}u_h(t) + f(t,h(t),u_h(t))\frac{d\mu}{d\nu}(t),\ \nu\ a.e.\ t \in I, \end{cases}$$

by repeating the machinery given in the proof of Theorem 4 via the Komlos argument. As (u_{h_n}) is bounded in variation, since $\|u_{h_n}(t) - u_{h_n}(\tau)\| \leq R(d\nu(]\tau,t])$, for $0 \leq \tau \leq t \leq 1$ where R is a positive generic constant, with $u_{h_n}(t) \in D(A_{(t,h_n(t))}) \subset \overline{co}\Psi_{\mathcal{X}}(t)$, for all $t \in I$, it is relatively compact by the Helly principle [58], and we may assume that (u_{h_n}) converges pointwise to a BV mapping $u : I \to H$. As $\|u(t) - u(\tau)\| \leq R(dv(]\tau, t])$, for $0 \leq \tau \leq t \leq 1$, and u is BVRC with $||\frac{du}{dv}|| \leq R, \nu$ *a.e.* and $u(t) = u_0 + \int_{]0,t]} \frac{du}{d\rho}(s)d\rho(s), \forall t \in I$. Now, since for all $t \in I$, $u_{h_n}(t) = u_0 + \int_{]0,t]} \frac{du_{h_n}}{dv} dv$ and $\frac{du_{h_n}}{dv}(s) \in R\overline{B}_H \; d\rho$ *a.e.*, we may assume that $(\frac{du_{h_n}}{dv})$ converges weakly in $L^1_H(I, \mathcal{B}(I)dv)$ to $w \in L^1_H(I, \mathcal{B}(I), \nu)$ with $w(t) \in R\overline{B}_H \; \nu$ a.e. so that by identifying the limits

$$\langle e, u(t)\rangle = \lim_{n\to\infty} \langle e, u_{h_n}(t)\rangle = \langle e, u_0\rangle + \langle e, \int_{]0,t]} w(s)dv(s)\rangle, \quad \forall e \in H, \forall t \in I.$$

we obtain

$$\int_{]0,t]} \frac{du}{dv}(s)dv(s) = \int_{]0,t]} w(s)dv(s) \quad \forall t \in I;$$

hence, $\frac{du}{dv} = w$ and $(\frac{du_{h_n}}{dv})$ weakly converges to $\frac{du}{dv}$ in $L^1_H(I, dv)$ and so $z_n(.) := f(., h_n(.), u_{h_n}(.))\frac{d\mu}{dv}(.)$ weakly converges to $z(.) := f(., h(.), u_h(.))\frac{d\mu}{dv}(.)$ in $L^1_H(I, dv)$, so by repeating the monotonicity and Komlos arguments given in Theorem 4, we have $u(t) \in D(A_{(t,h(t))}), \forall t \in I$ and $-\frac{du}{d\rho} \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t))\frac{d\mu}{dv}(t)$, ν *a.e.* $t \in I$, so that $u = u_h$ by uniqueness. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous with $\Phi(\mathcal{X}) \subset \mathcal{Y}$, by the Schauder theorem, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u_h(s)ds + \frac{b}{\Gamma(\alpha)} t^{\alpha-1},$$

with

$$\begin{cases} u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in \;\; I \\ -\frac{du_h}{d\rho}(t) \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t))\frac{d\mu}{dv}(t), \; \nu \; a.e. \; t \in I. \end{cases}$$

Coming back to Lemma 10 and applying the above notations, this means that we have just shown that there exists a mapping $h \in W^{\alpha,1}_H(I)$ satisfying

$$\begin{cases} D^\alpha h(t) = u_h(t), \\ h(0) = 0, D^\alpha h(0) = b \\ D^{\alpha-1}h(t) = \int_0^t u_h(s)ds + b \\ u_h(0) \in D(A_{(0,h(0))}) \\ u_h(t) \in D(A_{(t,h(t))}), \forall t \in \;\; I \\ -\frac{du_h}{dv}(t) \in A_{(t,h(t))}u_h(t) + f(t, h(t), u_h(t))\frac{d\mu}{dv}(t), \; \nu \; a.e. \; t \in I. \end{cases}$$

□

Our tools allow us to treat other variants by considering other classes of FDI given in [10,30–32,60].

5.2. On a Caputo Fractional Differential Inclusion Coupled with Time- and State-Dependent Maximal Monotone Operators

We study an example of a Caputo fractional differential inclusion coupled with a time- and state-dependent maximal monotone operator. For the sake of completeness, we recall some needed properties for the fractional calculus and provide a series of lemmas on the fractional integral. Throughout, we assume $\alpha \in [1, 2]$.

Definition 3. *The Caputo fractional derivative of order $\gamma > 0$ of a function $h : I \to H$, ${}^cD^\gamma h : I \to H$ is defined by*

$${}^cD^\gamma h(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{h^{(n)}(s)}{(t-s)^{1-n+\gamma}} ds.$$

Here $n = [\gamma] + 1$ and $[\gamma]$ denotes the integer part of γ.

Denote by

$$\mathcal{C}^1_H(I) = \{u \in \mathcal{C}_H(I) : \frac{du}{dt} \in \mathcal{C}_H(I)\},$$

where $\frac{du}{dt}$ is the derivative of u,

$$W^{\alpha,\infty}_{B,H}(I) = \{u \in C^1_H(I) : {}^cD^{\alpha-1}u \in \mathcal{C}_H(I);\ {}^cD^{\alpha}u \in L^{\infty}_H(I)\},$$

where ${}^cD^{\alpha-1}u$ and ${}^cD^{\alpha}u$ are the fractional Caputo derivatives of order $\alpha - 1$ and α of u, respectively. We recall and summarize some properties of a Green function given in [30] that is used in the statement of the problem under consideration.

Lemma 11. *Let $G : [0,T] \times [0,T] \to \mathbb{R}$ be a function defined by*

$$G(t,s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{1+t}{T+2}\left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right], & \text{if } 0 \le s < t, \\ \\ -\frac{1+t}{T+2}\left[\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] & \text{if } t \le s < T. \end{cases}$$

Then the following assertions hold:

(i) *Let $f \in L^{\infty}_H(I)$ and let $u_f : [0,T] \to H$ be a function defined by*

$$u_f(t) = \int_0^T G(t,s)f(s)ds,\ \forall t \in [0,T].$$

Then the following hold:

$$u_f(0) - \frac{du_f}{dt}(0) = 0,$$

$$u_f(T) + \frac{du_f}{dt}(T) = 0,$$

$${}^cD^{\alpha-1}u_f(t) = \int_0^t f(s)ds - \frac{I^{\alpha}f(T) + I^{\alpha-1}f(T)}{(T+2)\Gamma(3-\alpha)}t^{2-\alpha},\ \forall t \in [0,T],$$

$${}^cD^{\alpha}u_f(t) = f(t),\ \forall t \in [0,T].$$

(ii) *Assume that u is a $W^{\alpha,\infty}_{B,H}(I)$-solution to*

$$\begin{cases} {}^cD^{\alpha}u(t) = \sigma(t),\ t \in\ I \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0 \end{cases}$$

where $\sigma \in L^{\infty}_H(I)$; then $u(t) = \int_0^T G(t,s)\sigma(s)ds$, $\forall t \in [0,T]$ with $|G(t,s)| \le M_G := \frac{2T^{\alpha-1}+(\alpha-1)T^{\alpha-2}}{\Gamma(\alpha)}$.

We recall and summarize a crucial lemma (Lemma 3.5 [30]).

Lemma 12. *Let $X : I = [0,T] \rightrightarrows H$ be a convex weakly compact-valued measurable mapping such that $|X(t)| \le \gamma < +\infty$, $\forall t \in I$. Then the $W^{\alpha,\infty}_{B,H}(I)$-solution set $\mathcal{X}$ to the FDI*

$$\begin{cases} {}^cD^{\alpha}u(t) \in X(t),\ t \in\ I \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0, \end{cases}$$

is bounded, convex, equicontinuous, weakly compact in the Banach space $\mathcal{C}_H(I)$ and equi-Lipschitz.

Now comes an existence result with a Caputo fractional differential inclusion coupled with a time- and state-dependent maximal monotone operator.

Theorem 9. *Let $I = [0, T]$. Let $(t, x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.*

Let $f : I \times H \times H \to H$ be such that for every $x, y \in H$, the mapping $f(\cdot, x, y) \in L^\infty_H(I, \mu)$ and for every $t \in I$, the mapping $f(t, \cdot, \cdot)$ is continuous on $H \times H$ and satisfies:

(i) $\|f(t, x, y)\| \le M$ *for all* $(t, x, y) \in I \times H \times H$;

(ii) $\|f(t, z, x) - f(t, z, y)\| \le M\|x - y\|$ *for all* $(t, z, x, y) \in I \times H \times H \times H$,

for some nonnegative constant M.

(A) *Then the bounded closed convex subset $\mathcal{X}$ of $W^{\alpha,\infty}_{B,H}(I)$-solutions to the FDI:*

$$\begin{cases} {}^cD^\alpha u(t) \in \gamma \overline{B}_H, \ t \in \ I \\ u(0) - \frac{du}{dt}(0) = 0 \\ u(T) + \frac{du}{dt}(T) = 0, \end{cases}$$

is equi-K-Lipschitz in the Banach space $\mathcal{C}_H(I)$.

(B) *Let $\rho(t) = r(t) + Kt$, for all $t \in I$ and let $\nu = d\rho + M\mu$.*
Assume further that there is $\beta \in]0, 1[$ such that $\forall t \in I$, $0 \le 2M\frac{d\mu}{d\nu}(t)d\nu(\{t\}) \le \beta < 1$.
Then there is a $W^{\alpha,\infty}_{B,H}(I)$ mapping $x : I \to H$ and a BVRC mapping $v : I \to H$ satisfying

$$\begin{cases} D^\alpha x(t) = v(t) \in D(A_{(t,x(t))}), \ t \in \ I \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(T) + \frac{dx}{dt}(T) = 0 \\ -\frac{dv}{d\nu}(t) \in A_{(t,x(t))}v(t) + f(t, x(t), v(t))\frac{d\mu}{d\nu}(t), \ \nu \text{ a.e. } t \in I. \end{cases}$$

Proof. The proof is omitted. It is sufficient to repeat the proof of the previous theorem with **careful modifications** using the properties of the Caputo fractional inclusion. □

6. Skorohod Problem

By using the above techniques we obtain a fairly general version of Skorohod problem involving time et state dependent maximal monotone operator in the BVC setting.

Theorem 10. *Let $I := [0, 1]$ and $H = \mathbb{R}^e$. Let $(t, x) \to A_{(t,x)} : D(A_{(t,x)}) \to 2^H$ be a maximal monotone operator satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}^c_2)$.*

Let $z \in C^{1-var}([0, 1], \mathbb{R}^d)$ be the space of continuous functions of bounded variation defined on I with values in $\mathbb{R}^d$. Let $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ be the space of linear mappings f from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|f| := \sup_{x \in \mathbb{R}^d, ||x||_{\mathbb{R}^d} = 1} \|f(x)\|_{\mathbb{R}^e}.$$

Let us consider a class of continuous integrand operators $b : I \times \mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)$ satisfying

(a) $|b(t, x)| \le M, \quad \forall (t, x) \in I \times \mathbb{R}^e$;

(b) $|b(t, x) - b(t, y)| \le M||x - y||_{\mathbb{R}^e}, \quad \forall (t, x, y) \in [0, 1] \times \mathbb{R}^e \times \mathbb{R}^e$,

where M is > 0. We note $\int_0^t b(\tau, x(\tau))dz_\tau$ the perturbed Riemann–Stieljies integral of $b(., x(.))$ against z with $x \in C([0, 1], \mathbb{R}^e)$.

Let $g : I \times I \times \mathbb{R}^e \to \mathbb{R}^e$ be a continuous mapping satisfying:

(c) $||g(t, s, x)|| \le M$ *for all* $(t, s, x) \in I \times I \times \mathbb{R}^e$;

(d) $||g(t, s, x) - g(t, s, y)|| \le M$,
for all $(t, s) \in I \times I, x, y \in \mathbb{R}^e \times \mathbb{R}^e$.

Let us set $\rho(t) = r(t) + M|z|_{1-var:[0,t]}, \forall t \in I$ and let $d\rho$ be the Stieltjes measure associated with ρ. Let $a \in D(A_{(0,0)})$.

Then there exists a BVC function $x : I \to H$ and a BVC function $u : I \to H$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t),\ \forall t \in\ I \\ h(t) = \int_0^t b(\tau, x(\tau))dz_\tau,\ \forall t \in\ I \\ k(t) = \int_0^t g(t, s, x(s))ds,\ \forall t \in\ I \\ u(t) \in D(A_{(t,h(t))}),\ \forall t \in\ I \\ -\frac{du}{d\rho}(t) \in A_{(t,h(t))}u(t) + k(t),\ d\rho \text{ a.e.},\ t \in\ I \end{cases}$$

where $\frac{du}{d\rho}$ is the density of the differential measure du with respect to the measure $d\rho$.

Proof. Let $a \in D(A_{(0,0)})$. Let us set for all $t \in I = [0, T]$

$$x^0(t) = a,\ h^1(t) = \int_0^t b(\tau, a)dz_\tau$$

then by Proposition 2.2 in Friz-Victoir [61], we have

$$|\int_0^t b(\tau, a)dz_\tau| \le |b(., a)|_{\infty:I}|z|_{1-var;[0,t]}. \tag{12}$$

Moreover

$$\int_0^t b(\tau, a)dz_\tau - \int_0^s b(\tau, a)dz_\tau = \int_s^t b(\tau, a)dz_\tau$$

so that by condition (a)

$$||h^1(t) - h^1(s)|| \le M|z|_{1-var;[s,t]}, \tag{13}$$

for all $0 \le s \le t \le 1$ and in particular

$$||h^1(t)|| \le M|z|_{1-var;[0,t]} \le M|z|_{1-var;[0,T]}$$

for all $t \in I$. Further, we have

$$dis(A_{(t,h_1(t))}, A_{(s,h_1(s))}) \le r(t) - r(s) + ||h^1(t) - h^1(s)|| \le r(t) - r(s) + M|z|_{1-var:[s,t]}$$

so that with our notation

$$dis(A_{(t,h_1(t))}, A_{(s,h_1(s))}) \le \rho(t) - \rho(s), \tag{14}$$

for all $0 \le s \le t \le 1$ where $\rho : I \to \mathbb{R}^+$ is a non-decreasing continuous function with $\rho(0) = 0$. Let us set for all $t \in I = [0, T]$

$$x^0(t) = a,\ k^1(t) = \int_0^t g(s, x^0(s))ds,$$

then k^1 is continuous with $\|k^1(t)\| \le MT$ for all $t \in I$. By an easy computation, $||k^1(t) - k^1(\tau)|| \le M|t - \tau|$, for all $\tau, t \in I$. Taking account of (14), by Theorem 3.1 ([50]), there is a unique BVC mapping $u^1 : I \to H$ solution of the problem

$$\begin{cases} u^1(0) = a,\ u^1(t) \in D(A_{(t,h^1(t))}),\ \forall t \in I; \\ -\dfrac{du^1}{d\rho}(t) \in A_{(t,h^1(t))}u^1(t) + k^1(t),\ d\rho\ a.e. \end{cases}$$

with$||u^1(t) - u^1(\tau)|| \leq K(\rho(t) - \rho(\tau))$ for all $\tau \leq t \in I$ where K is positive constant depending on the data (cf. Theorem 3.1 in [50] for details). Set

$$x^1(t) = h^1(t) + k^1(t) + u^1(t) = \int_0^t b(\tau, x^0(\tau)dz_\tau + \int_0^t g(t, s, x^0(s))ds + u^1(t).$$

Then x^1 is BVC with $x^1(0) = a$. Now we construct x^n by induction as follows. Let for all $t \in I$

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau$$

$$k^n(t) = \int_0^t g\big(t, s, x^{n-1}(s)\big)ds).$$

By Proposition 2.2 in Friz-Victoir [61], we have the estimate

$$||h^n(t) - h^n(s)|| \leq M|z|_{1-var;[s,t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$||h^n(t)|| \leq M|z|_{1-var;[0,t]} \leq M|z|_{1-var;[0,T]}$$

for all $0 \leq t \leq 1$. Further, we have

$$dis(A_{(t,h^n(t))}A_{(s,h^n(s))}) \leq r(t) - r(s) + \|h^n(t) - h^n(s)\| \leq r(t) - r(s) + M|z|_{1-var;[s,t]}$$

so that with our notation

$$dis(A_{(t,h^n(t))}, A_{(s,h^n(s))}) \leq \rho(t) - \rho(s),$$

for all $0 \leq s \leq t \leq 1$. Further, k^n satisfies $||k^n(t) - k^n(\tau)|| \leq M|t - \tau|$, for all $\tau, t \in I$ with $\|k^n(t)\| \leq MT$ for all $t \in I$. Again, by Theorem 3.1 ([50]), there is a unique BVC mapping $u^n : I \to H$ solution of the problem

$$\begin{cases} u^n(0) = a, \ u^n(t) \in D(A_{(t,h^n(t))}), \ \forall t \in I; \\ -\dfrac{du^n}{d\rho}(t) \in A_{(t,h^n(t))}u^n(t) + k^n(t), \ d\rho \ a.e. \end{cases}$$

with $||u^n(t) - u^n(\tau)|| \leq K(\rho(t) - \rho(\tau))$ for all $\tau \leq t \in I$. Set for all $t \in I$

$$x^n(t) = h^n(t) + k^n(t) + u^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \int_0^t g\big(t, s, x^{n-1}(s)\big)ds + u^n(t),$$

so that x^n is BVC, and

$$-\frac{du^n}{d\rho}(t) \in A_{(t,h^n(t))}u^n(t) + k^n(t), \ d\rho \ a.e. \tag{15}$$

As (u^n) is equicontinuous and for all $t \in I$, $u^n(t) \in D(A_{(t,h_n(t))})$, we may assume that (u^n) converges uniformly to a BVC mapping $u : I \to \mathbb{R}^e$ with $u(t) \in D(A_{(t,h(t))}), \forall t \in I$ and $||u(t) - u(\tau)|| \leq K(\rho(t) - \rho(\tau))$ for all $\tau \leq t \in I$. Now, recall that

$$||h^n(t) - h^n(s)|| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq T$. So (h^n) is bounded and equicontinuous. By the Ascoli theorem, we may assume that h^n converges uniformly to a continuous mapping h. Similarly, (k^n) is bounded and equicontinuous: $||k^n(t) - k^n(\tau)|| \leq M|r(t) - r(\tau)|$, for all $\tau, t \in I$. By the Ascoli theorem, we may assume that k^n converges uniformly to a continuous mapping k. Hence, $x^n(t) = h^n(t) + k^n(t) + u^n(t)$ converges uniformly to $x(t) := h(t) + k(t) + u(t)$, and $b(., x^{n-1}(.))$ converges uniformly to $b(., x(.))$ using the Lipschitz condition (b). Then by Friz-

Victoir [61] (Proposition 2.7), $\int_0^t b(\tau, x^{n-1}(\tau))dz_\tau$ converges uniformly to $\int_0^t b(\tau, x(\tau))dz_\tau$. By hypothesis (i), $g(t,s,x^{n-1}(s))$ converges pointwise to $g(t,s,x(s))$. Hence, $\int_0^t g(t,s, x^{n-1}(s))ds \to \int_0^t g(t,s,x(s))dr(s)$ for each $t \in I$ by the Lebesgue theorem. So by identifying the limit

$$\lim_{n\to\infty} x^n(t) = \lim_{n\to\infty}\int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \lim_{n\to\infty}\int_0^t g(t,s,x^{n-1}(s))ds + \lim_{n\to\infty} u^n(t)$$

$$= h(t) + k(t) + u(t) = \int_0^t b(\tau, x(\tau))dz_\tau + \int_0^t g(t,s,x(s))ds + u(t) = x(t).$$

Further, we note that $u(t) \in D(A_{(t,h(t))})$ for all $t \in I$. Indeed we have $dis(A_{(t,h_n(t))}, A_{(t,h(t))}) \le ||h_n(t) - h(t)|| \to 0$. It is clear that $(y_n = A^0_{(t,h_n(t)}u_{h_n}(t))$ is bounded. By applying Lemma 2 to $u_{h_n}(t) \to u(t)$ and to a convergent subsequence of (y_n) to show that $u(t) \in D(A_{(t,h(t))})$, it remains to check that

$$-\frac{du}{d\rho}(t) \in A_{(t,h(t))}u(t) + k(t),\ d\rho\ a.e.\ t \in I.$$

As $||u^n(t) - u^n(\tau)|| \le K(\rho(t) - \rho(\tau))$ for all $\tau \le t \in I$, we have $\frac{du^n}{d\rho}(t) \in K\overline{B}_H$ such that $u^n(t) = a + \int_{]0,t]}\frac{du^n}{d\rho}(s)d\rho(s) \to u(t) := a + \int_{]0,t]}\frac{du}{d\rho}(s)d\rho(s)$ with $\frac{du^n}{d\rho} \to \frac{du}{d\rho}$ weakly in $L^1_H(I, d\rho)$. We use Komlos's trick to finish. For convenient notation, let

$$z_n(t) = -\frac{du^n}{d\rho}(t) - k^n(t) \quad \text{and} \quad z(t) = -\frac{du}{d\rho}(t) - k(t)$$

Then $\{z_n\}$ weakly converges in $L^1_H(I, d\rho)$ to z.

We will show that

$$z(t) = -\frac{du}{d\rho}(t) - k(t) \in A_{(t,h(t))}u(t),\ d\rho\ a.e$$

Since $z_n(.) \to z(.)$ weakly in $L^1_{\mathbb{R}^e}(I, d\rho)$, we may assume that $z_n = \frac{du_{h_n}}{d\rho} + k^n$ Komlos converges to $z = \frac{du}{d\rho} + k$. Further, we note that $u(t) \in D(A_{(t,h(t))})$ for all $t \in I$. Indeed we have $dis(A_{(t,h_n(t))}, A_{(t,h(t))}) \le ||h_n(t) - h(t)|| \to 0$. It is clear that $(y_n = A^0_{(t,h_n(t))}u_{h_n}(t))$ is bounded, hence relatively compact. By applying Lemma 2 to $u_{h_n}(t) \to u(t)$ and to a convergent subsequence of (y_n) to show that $u(t) \in D(A_{(t,h(t))})$, there is a $d\rho$-negligible set N such that

$$-\frac{du_{h_n}}{dv}(t) - k^n(t) \in A_{(t,h_n(t))}u_{h_n}(t),\ t \in I \setminus N,$$

$$\lim_n \frac{1}{n}\sum_{j=1}^n\left(\frac{du_{h_j}}{d\rho}(t) + k^j(t)\right) = \frac{du}{d\rho}(t) + k(t),\ t \in I \setminus N.$$

Let $t \in [0,1] \setminus N$. Let $\eta \in D(A_{(t,h(t))})$. Apply Lemma 3 to $A_{(t,h_n(t))}$ and $A_{(t,h(t))})$ to find a sequence (η_n) such that $\eta_n \in D(A_{(t,h_n(t))})$, $\eta_n \to \eta$, $A^0_{(t,h_n(t))}\eta_n \to A^0_{(t,h(t))}\eta$. From

$$-\frac{du_{h_n}}{d\rho}(t) - k^n(t) \in A_{(t,h_n(t))}u_{h_n}(t)$$

by monotonicity

$$\langle\frac{du_{h_n}}{d\rho}(t) + k^n(t), u_{h_n}(t) - \eta_n\rangle \le \langle A^0_{(t,h_n(t))}\eta_n, \eta_n - u_{h_n}(t)\rangle.$$

From

$$\begin{array}{l}\langle \frac{du_{h_n}}{d\rho}(t)+k^n(t),u(t)-\eta\rangle\\ =\langle \frac{du_{h_n}}{d\rho}(t)+k^n(t),u_{h_n}(t)-\eta_n\rangle+\langle \frac{du_{h_n}}{d\rho}(t)+k^n(t),u(t)-u_{h_n}(t)-(\eta-\eta_n)\rangle,\end{array}$$

let us write

$$\frac{1}{n}\sum_{j=1}^{n}\langle \frac{du_{h_j}}{d\rho}(t)+k^j(t),u(t)-\eta\rangle=\frac{1}{n}\sum_{j=1}^{n}\langle \frac{du_{h_j}}{d\rho}(t)+k^j(t),u_{h_j}(t)-\eta_j\rangle$$

$$+\frac{1}{n}\sum_{j=1}^{n}\langle \frac{du_{h_j}}{d\rho}(t)+k^j(t),u(t)-u_{h_j}(t)\rangle+\frac{1}{n}\sum_{j=1}^{n}\langle \frac{du_{h_j}}{d\rho}(t)+k^j(t),\eta_j-\eta\rangle,$$

so that

$$\begin{array}{c}\frac{1}{n}\sum_{j=1}^{n}\langle \frac{du_{h_j}}{dv}+k^j(t)(t),u(t)-\eta\rangle\\ \leq \frac{1}{n}\sum_{j=1}^{n}\langle A^0_{(t,h_j(t))}\eta_j,\eta_j-u_{h_j}(t)\rangle+(K+M)\frac{1}{n}\sum_{j=1}^{n}\|u(t)-u_{h_j}(t))\|.\\ +(K+M)\frac{1}{n}\sum_{j=1}^{n}\|\eta_j-\eta\|.\end{array}$$

Passing to the limit using $n\to\infty$, this last inequality immediately gives

$$\langle \frac{du}{d\rho}(t)+k(t),u(t)-\eta\rangle\leq\langle A^0_{(t,h(t))}\eta,\eta-u(t)\rangle$$

As a consequence, by Lemma 3 we obtain $-\frac{du}{d\rho}(t)\in A_{(t,h(t))}u(t)+k(t)$, $d\rho$ a.e. with $u(t)\in D(A_{(t,h(t))})$ for all $t\in I$. The proof is therefore complete. □

In Theorem 10, we present a new result for the Skorohod problem (SKP) driven by a time- and state-dependent operator $A_{(t,x)}$ under rough signal $\int_0^t b(s,x(s))dz_s$ and Volterra integral perturbation $\int_0^t g(t,s,x(s))ds$ in the BVC setting. So it has several novelties and our tools allow us to state several variants of Theorem 10 according to the nature of the perturbation and the operator. It is a challenge to obtain the uniqueness. Nevertheless, some uniqueness results are discussed below. In this setting, our result is quite new by comparison with some classical integral equations existing in the literature.

Proposition 6. *Let $I:=[0,T]$ and $H=\mathbb{R}^e$. Let $C:[0,T)]\to H$ be a closed convex-valued mapping satisfying $d_H(C(t),C(\tau)\leq r(t)-r(\tau)$, for all $0\leq\tau<t\leq T$, where $r:I\to\mathbb{R}^+$ is non-decreasing continuous with $r(0)=0$.*

Let $g:I\times\mathbb{R}^e\to\mathbb{R}^e$ be a continuous mapping satisfying $||g(t,x)||\leq 1$ for all $(t,x)\in I\times\mathbb{R}^e$ and $||g(t,x)-g(t,y)||\leq||x-y||$, for all $(t,x,y)\in I\times\mathbb{R}^e\times\mathbb{R}^e$.

Let μ be a probability nonatomic Radon measure on I and let $\nu=dr+\mu$. Let $a\in C(0)$. Then there exists a BVC function $x:I\to H$ and a BVC function $u:I\to H$ satisfying

$$\begin{cases} x(0)=u(0)=a\\ x(t)=h(t)+u(t),\ \forall t\in\ I\\ h(t)=\int_0^t g(s,x(s))\mu(ds),\ \forall t\in\ I\\ u(t)\in C(t)-h(t),\ \forall t\in\ I\\ -\frac{du}{dv}(t)\in N_{C(t)-h(t)}u(t),\ dv\ a.e.,\ t\in I\end{cases}$$

where $\frac{du}{dv}$ is the density of of the differential measure du with respect to the measure ν. The BVC solution (x,u) is **unique***.*

Proof. From $h(t)=\int_0^t g(s,x(s))\mu(ds)$, we have $||h(t)-h(\tau)||\leq\mu([\tau,t])$ for all $\tau\leq t\in I$. Let us set $C_h(t)=C(t)-h(t)$. Then $d_H(C_h(t),C_h(\tau))\leq r(t)-r(\tau)+||h(t)-$

$h(\tau)|| \leq \nu([\tau,t])$. Hence, the existence follows by repetition of the arguments given in Theorem 10, with $d\rho$ replaced by $d\nu$. Now we prove unicity of the BVC solution (x,u). Assume that (x,u) and (x^*,u^*) are two solutions with

$$\begin{cases} x(t) = h(t) + u(t) = \int_0^t g(s,x(s))\mu(ds) + u(t) \\ x^*(t) = h^*(t) + u^*(t) = \int_0^t g(s,x^*(s))\mu(ds) + u^*(t) \\ -\frac{du}{d\nu}(t) \in N_{C(t)-h(t)}u(t) = N_{C(t)}(u(t)+h(t)) = N_{C(t)}x(t) \\ -\frac{du^*}{d\nu}(t) \in N_{C(t)-h^*(t)}u^*(t) = N_{C(t)}(u^*(t)+h^*(t)) = N_{C(t)}x^*(t) \end{cases}$$

By our construction, it is easily seen that h, u, h^*, u^* are BVC. By monotonicity, we have

$$\langle -\frac{du}{d\nu}(t) + \frac{du^*}{d\nu}(t), x(t) - x^*(t)\rangle \geq 0. \tag{16}$$

On the other hand, since x and x^* are BVC and have the densities $\frac{dx}{d\nu}$ and $\frac{dx^*}{d\nu}$ relative to the measure $d\nu$, by a result of Moreau concerning the differential measure [56], $\|x-x^*\|^2$ is BVC and we have

$$d\|x-x^*\|^2 \leq 2\Big\langle x(\cdot) - x^*(\cdot), \frac{dx}{d\nu}(\cdot) - \frac{dx^*}{d\nu}(\cdot)\Big\rangle d\nu$$

so that by integrating on $[0,t]$ we obtain

$$\|x(t)-x^*(t)\|^2 \quad = \quad \int_0^t d\|x-x^*\|^2 \leq \int_0^t 2\langle x(s)-x^*(s), \frac{dx}{d\nu}(s) - \frac{dx^*}{d\nu}(s)\rangle d\nu(s).$$

We have

$$\begin{aligned} &\int_0^t \langle x(s)-x^*(s), \frac{dx}{d\nu}(s) - \frac{dx^*}{d\nu}(s)\rangle d\nu(s) \\ &= \int_0^t [\langle x(s)-x^*(s), \frac{du}{d\nu}(s) - \frac{du^*}{d\nu}(s)\rangle + \langle x(s)-x^*(s), \frac{dh}{d\nu}(s) - \frac{dh^*}{d\nu}(s)\rangle] d\nu(s) \\ &\leq \int_0^t \langle x(s)-x^*(s), \frac{dh}{d\nu}(s) - \frac{dh^*}{d\nu}(s)\rangle d\nu(s) \quad \text{(using (16))} \\ &= \int_0^t \langle x(s)-x^*(s), g(s,x(s)) - g(s,x^*(s))\rangle \frac{d\mu}{d\nu}(s) d\nu(s) \\ &\leq \int_0^t ||x(s)-x^*(s)||^2 \frac{d\mu}{d\nu}(s) d\nu(s) \end{aligned}$$

so that

$$\|x(t)-x^*(t)\|^2 \leq \int_0^t 2\frac{d\mu}{d\nu}(s)||x(s)-x^*(s)||^2 d\nu(s)$$

By applying Gronwall's Lemma 5, we conclude that $x = x^*$. Then $h = h^*$ and $u = u^*$ and the proof is complete. □

In this vein, some more uniqueness of solutions is available using specific Gronwall lemmas. However, the uniqueness solutions to the sweeping process with perturbation $h(t) = \int_0^t b(\tau, x(\tau))dz_\tau$ is an open question, although existence is ensured. We refer to [2] for some problems of uniqueness related to the sweeping process perturbed by rough signal. Also, related (SKP) problems for the sweeping process are developed in [1,62] with the existence and uniqueness of solution.

7. Fractional Differential Inclusion/Evolution Inclusion under Rough Signals and Young Integrals: The BVRC Setting

Let $z \in C^{1-var}([0,T],\mathbb{R}^d)$ be the space of bounded variation continuous mappings defined on $[0,T]$ with values in $\mathbb{R}^d$. We recall some notations. By $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ we denote the space of linear mappings from $\mathbb{R}^d$ to $\mathbb{R}^e$ endowed with the operator norm

$$|\Lambda| := \sup_{x\in\mathbb{R}^d, ||x||_{\mathbb{R}^d}=1} |\Lambda(x)|_{\mathbb{R}^e}.$$

Let $A_T := \{(s,t) : 0 \le s \le t \le T\}$. A map $\omega : A_T \to [0,\infty[$ is a control function on [0, T] if ω is continuous, superadditive [61] and $\omega(s,s) = 0$ for $0 \le s \le T$. An example of a control function is $(s,t) \to |t-s|^\theta$ for $\theta \ge 1$, or $(s,t) \to \int_s^t \rho(\tau)d\tau$ where ρ is a positive Lebesgue integrable function.

Let us consider the class $\mathfrak{B}$ of continuous integrand operator $b : [0,T]\times\mathbb{R}^e \to \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ satisfying the conditions

$$\begin{array}{ll} (\mathcal{B}_1) & |b(t,x)| \le M, \quad \forall x \in \mathbb{R}^e \\ (\mathcal{B}_2) & |b(s,x) - b(t,x)| \le \omega(s,t), \quad 0 \le s \le t \le T,\ \forall x \in \mathbb{R}^e, \\ (\mathcal{B}_3) & |b(t,x) - b(t,y)| \le M||x-y||, \quad \forall t \in [0,T],\ \forall x \in \mathbb{R}^e, \end{array}$$

where ω is a control function on $[0,T]$ and M is a positive constant. If $\mathcal{X} \subset C([0,T],\mathbb{R}^e)$ a set of continuous mappings from $[0,T]$ into $\mathbb{R}^e$ is controlled by a control function $\alpha(s,t)$: $||x(s)-x(t)|| \le \alpha(s,t)$ for all $x \in \mathcal{X}$, for all $s < t$, then the set of mappings $\{b(.,x(.)); x \in \mathcal{X}\}$ from $[0,T]$ into $\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)$ is uniformly bounded and uniformly bounded in variation, in particular $b(.,x(.)) \in C^{1-var}([0,T],\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e))$. Indeed, we have

$$|b(s,x(s)) - b((t,x(t))| \le |b(s,x(s)) - b(t,x(s))| + |b(t,x(s)) - b(t,x(t))|$$

with $|b(s,x(s)) - b(t,x(s))| \le \omega(s,t)$ and $|b(t,x(s)) - b(t,x(t)| \le M||x(s)-x(t)|| \le M\alpha(s,t)$ for all $s < t \le T$ so that using Proposition 1.11 [61], $|b(.,x(.))|_{1-var;[s,t]} < \infty$, $0 \le s \le t \le T$. Consequently the Young integral $\int_0^t b(s,x(s))dz_s$ along z is well-defined and belongs to $C^{1-var}([0,T],\mathbb{R}^e)$, and according to Friz-Victoir Theorem 6.8 [61], we have the following estimates:

$$\begin{aligned} ||\int_s^t b(\tau,x(\tau))dz_\tau|| &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|b(.,x(.))|_{1-var;[s,t]} + |b(.,x(.))|||z(t)-z(s)||_{\mathbb{R}^d} \\ &\le \frac{1}{1-2^{1-\theta}}|z|_{1-var;[s,t]}|b(.,x(.))|_{1-var;[s,t]} + M||z(t)-z(s)||_{\mathbb{R}^d} \end{aligned}$$

for all $0 \le s \le t \le T$ with $\theta = 2$ and

$$|\int_0^{\cdot} b(\tau,x(\tau))dz_\tau|_{1-var;[s,t]} \le C(1,1)|z|_{1-var;[s,t]}(|b(.,x(.))|_{1-var;[s,t]} + |b(.,x(.))|_{\infty;[s,t]})$$

for all $0 \le s \le t \le T$. As consequence, we see that the set $\mathcal{Y}$ of mappings

$$\mathcal{Y} := \{\int_0^{\cdot} b(\tau,x(\tau))dz_\tau; x \in \mathcal{X}\}$$

in $C^{1-var}([0,T],\mathbb{R}^e)$ is uniformly bounded, and by the continuity of $t \mapsto |z|_{1-var;[0,t]}$, since $z \in C^{1-var}([0,T],\mathbb{R}^d)$ and $\mathcal{Y}$ is also equicontinuous; further, it is additionally uniformly bounded in variation. Altogether, $\mathcal{Y}$ is uniformly bounded, equicontinuous, and uniformly bounded in variation. When $\mathcal{X}$ is compact $\subset C([0,T],\mathbb{R}^e)$ and equi-Lipchitz, then $\mathcal{Y}$ is **compact** with respect to the topology uniform convergence.

Theorem 11. *Let $I := [0,1]$. Assume that for every $t \in I = [0,1]$, $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ is a maximal monotone operator satisfying $(\mathcal{H}_1^*)$ and $(\mathcal{H}_2^*)$.*

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and $b \in \mathfrak{B}$.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a continuous mapping satisfying:

(i) $\|f(t,x,z) - f(t,y,z)\| \leq M\|x-y\|$ *for all* $(t,x,y,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$.

(ii) $\|f(t,x,z)\| \leq M$ *for all* $(t,x,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.

Let $\nu := dr + \lambda$. Assume further that there is $\delta \geq 0$ such that $\forall t \in I, 0 \leq 2M\frac{dt}{d\nu}(t)\nu(\{t\}) \leq \delta < 1$.

Assume that $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0, \gamma > 0$.

Then for any $u_0 \in D(A_0)$, there exists a $W^{\alpha,1}_{B,\mathbb{R}^e}([0,1])$ mapping $x : I \to \mathbb{R}^e$ and a BVRC mapping $u : I \to \mathbb{R}^e$ satisfying the dynamic with rough signal

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = u(t), \ t \in \ I \\ I^\beta_{0^+}x(t)|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+}x(1) \\ u(t) \in D(A_t), \ t \in \ I \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s,x(s))dz_s)\frac{dt}{d\nu}(t) \quad a.e.\ t \in \ I \\ u(0) = a \in D(A_0) \end{cases}$$

Let $L : [0,1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0,\infty[$ be a lower semicontinuous integrand such that $L(t,x,y,.)$ is convex on $\mathbb{R}^e$ for every $(t,x,y) \in [0,1] \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function $\int_0^1 L(t, \int_0^t b(s,x(s))dz_s, u(t), \frac{du}{d\nu}(t))d\nu$ subject to

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = u(t), \ t \in [0,1] \\ I^\beta_{0^+}x(t)|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+}x(1) \\ -\frac{du}{d\nu}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s,x(s))dz_s)\frac{dt}{d\nu}(t) \quad a.e.\ t \in \ I \\ u(0) = a \in D(A_0) \end{cases}$$

has an optimal solution.

Proof. For any continuous mapping $g : I \to \mathbb{R}^e$, $u_0 \in D(A_0)$, by Corollary 3, there is a **unique** BVRC solution u_g to the differential inclusion

$$\begin{cases} u_g(0) = a \in D(A_0) \\ u_g(t) \in D(A_t), \ \forall t \in \ I \\ -\frac{du_g}{d\nu}(t) \in A_t u_g(t) + f(t, u_g(t), g(t))\frac{dt}{d\nu}(t) \quad a.e.\ t \in I \end{cases}$$

with u_g uniformly bounded and equi-BVRC:

$$||\frac{du_g}{d\nu}(t)|| \leq \eta$$

for some constant $\eta > 0$. So one has $\|u_g(t)\| \leq K$ for all $t \in I$ for some constant K for all continuous functions g. Now let us consider the set $\mathcal{X}$ defined by

$$\mathcal{X} := \{\xi_f : I \to \mathbb{R}^e : \ f \in S^1_{K\overline{B}_{\mathbb{R}^e}}\},$$

each mapping ξ_f being given for every $t \in I$ by

$$\xi_f(t) = \int_0^1 G(t,s)f(s)\,ds,$$

where G is the Green function ([10], Lemma 8). We note that $\mathcal{X}$ is convex compact and equi-Lipschitz ([10], Theorem 3): $h \in \mathcal{X}$, $||h(t) - h(s)|| \leq N|t-s|^{\alpha-1} \leq N|t-s|$ where N is a positive constant. Then for any $h \in \mathcal{X}$, using $(\mathcal{B}_2)$ and $(\mathcal{B}_3)$,

$$|b(t,h(t)) - b(s,h(s))| \leq |b(t,h(t)) - b(s,h(t))| + |b(s,h(t)) - b(s,h(s))| \leq \omega(s,t) + N|t-s|$$

so that for any $h \in \mathcal{X}$, $b(.,h(.)) \in C^{1-var}([0,1], \mathcal{L}(\mathbb{R}^e, \mathbb{R}^e))$ with $|b(.,h(.))| \le M$ by $(\mathcal{B}_1)$. In particular, the integral $\int_0^{\cdot} b(s,h(s))dz_s$ has a meaning for all $h \in \mathcal{X}$ with $b(.,h(.))$ uniformly bounded in variation. As consequence, it was stated that

$$\mathcal{Y} := \{\int_0^{\cdot} b(s,h(s))dz_s : h \in \mathcal{X}\}$$

is **compact** in $C([0,1], \mathbb{R}^e)$. For each $h \in \mathcal{X}$, let us set (again with the above Green function G)

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds, \quad \text{for all } t \in I.$$

where u_h is a **unique** BVRC solution to the differential inclusion

$$\begin{cases} u_h(0) = a \in D(A_0) \\ u_h(t) \in D(A_t),\ \forall t \in\ I \\ -\frac{du_h}{dv}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s,h(s))dz_s)\frac{dt}{dv}(t) \ \ a.e.\ t \in I \end{cases}$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. Now we check that Φ is continuous relative to $\mathcal{X}$. It is enough to show that, if $(h_n)_n$ converges uniformly to h in $\mathcal{X}$, then the sequence $(u_{h_n})_n$, where each u_{h_n} is the unique BVRC solution of the differential inclusion

$$\begin{cases} u_{h_n}(0) = u_0 \in D(A_0) \\ u_{h_n}(t) \in D(A_t),\ \forall t \in\ I \\ -\frac{du_{h_n}}{dv}(t) \in A_t u_{h_n}(t) + f(t, u_{h_n}(s), \int_0^t b(s,h_n(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I, \end{cases}$$

converges pointwise to the unique BVRC solution u_h of the differential inclusion

$$\begin{cases} u_h(0) = a \in D(A_0) \\ u_h(t) \in D(A_t),\ \forall t \in\ I \\ -\frac{du_h}{dv}(t) \in A_t u_h(t) + f(t, u_h(s), \int_0^t b(s,h(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in\ I. \end{cases}$$

This requires careful examination. Since $(u_{h_n})_n$ is equi-BVRC for each $n \in \mathbb{N}$, the estimate

$$||\frac{du_{h_n}}{dv}(t)|| \le \eta\ a.e.\ t \in I,$$

we may suppose that $(u_{h_n})_n$ converges pointwise to a BVRC mapping $w : I \to \mathbb{R}^e$: $w(t) = u_0 + \int_0^t \frac{dw}{dv}(s)dv(s)$ and we may assume that $(\frac{du_{h_n}}{dv})$ weakly converges to $\frac{dw}{dv}$ in $L^1([0,1], dv, \mathbb{R}^e)$ with $||\frac{dw}{dv}(t)|| \le \eta$, so for every $t \in I$ we have, as $n \to \infty$,

$$k_n(t) := f(t, u_{h_n}(t), \int_0^t b(s,h_n(s))dz_s)\frac{dt}{dv}(t) \to k(t) := f(t, w(t), \int_0^t b(s,h(s))dw(s))\frac{dt}{dv}(t).$$

Keeping in mind that $||f(t, u_{h_n}(t), \int_0^t b(s,h_n(s))dz_s)|| \le M$ for all $t \in I$, we show that w is the solution of the differential inclusion

$$\begin{cases} w(0) = u_0 \\ w(t) \in D(A_t),\ \forall t \in\ I \\ -\frac{dw}{dv}(t) \in A_t w(t) + f(t, w(t), \int_0^t b(s,h(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

by applying the Komlos argument.

Now let us write by ([10], Lemma 8)

$$\begin{aligned}\Phi(h_n)(t) - \Phi(h)(t) &= \int_0^1 G(t,s)u_{h_n}(s)\,ds - \int_0^1 G(t,s)u_h(s)\,ds \\ &= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]\,ds \\ &\leq \int_0^1 M_G\|u_{h_n}(s) - u_h(s)\|\,ds.\end{aligned}$$

Since $\|u_{h_n}(\cdot) - u_h(\cdot)\| \to 0$ on I as $n \to \infty$, we deduce that

$$\sup_{t\in I}\|\Phi(h_n)(t) - \Phi(h)(t)\| \leq \int_0^1 M_G\|u_{h_n}(\cdot) - u_h(\cdot)\|\,ds \to 0,$$

which entails that $\Phi(h_n) \to \Phi(h)$ uniformly on I, as desired. Then $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous; hence, by the Schauder theorem, Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that for every $t \in I$

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)\,ds,$$

with

$$\begin{cases} u_h(0) = u_0 \in D(A_0) \\ u_h(t) \in D(A_t),\ \forall t \in\ I \\ -\frac{du_h}{dv}(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s,h(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

According to ([10], Lemma 9), this means that we have just shown that there exists a mapping $h \in W^{\alpha,1}_{B,\mathbb{R}^e}(I)$ satisfying

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u_h(t), \\ I^\beta_{0^+}h(t)|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+}h(1) \\ u_h(0) = u_0 \in D(A_0) \\ u_h(t) \in D(A_t), \forall t \in [0,1] \\ -\frac{du_h}{dv}(t)\dot{u}_h(t) \in A_t u_h(t) + f(t, u_h(t), \int_0^t b(s,h(s))dz_s) \quad a.e.\ t \in I. \end{cases}$$

Let (h_n, u_n) be a mimimizing sequence in this FDI/EVI, namely

$$\lim_n \int_0^T L(t, \int_0^t b(s,h_n(s))dz_s), u_n(t), \dot{u}_n(t))dv = \inf_{(k,v)} [\int_0^T L(t, \int_0^t b(s,k(s))dz_s, v(t), \dot{v}(t))dv]$$

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1}h_n(t) = u_n(t), \\ I^\beta_{0^+}h_n(t)|_{t=0} = 0, \quad h_n(1) = I^\gamma_{0^+}h_n(1) \\ u_n(0) = u_0 \in D(A_0) \\ u_n(t) \in D(A_t), \forall t \in [0,1] \\ -\frac{du_n}{dv}(t) \in A_t u_n(t) + f(t, u_n(t), \int_0^t b(s,h_n(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

At first, by compactness of the solution set in the FDI, there is a subsequence not relabelled (h_n) in $\mathcal{X}$ converging uniformly to $h \in \mathcal{X}$. Second, by compactness of the solution set in the evolution inclusion

$$\begin{cases} u_n(0) = u_0 \in D(A_0 \\ u_n(t) \in D(A_t), \forall t \in [0,1] \\ -\frac{du_n}{dv}(t) \in A_t u_n(t) + f(t, u_n(t), \int_0^t b(s,h_n(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

there is a subsequence not relabelled (u_n) such that u_n converges pointwise to a BVRC mapping u with $\frac{du_n}{dv} \to \frac{du}{dv}$ weakly in $L^1([0,1]), dv, \mathbb{R}^e)$. By compactness of

$$\mathcal{Y} := \{\int_0^{\cdot} b(s, x(s))dz_s : x \in \mathcal{X}\}$$

we may ensure that $\int_0^t b(s, h_n(s))dz_s \to \int_0^t b(s, h(s))dz_s$ uniformly. So by repeating the above argument, we are ensured that u satisfies the inclusion

$$\begin{cases} u(0) = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in [0,1] \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, h(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

From

$$\begin{cases} D^\alpha h_n(t) + \lambda D^{\alpha-1} h_n(t) = u_n(t), \\ I_{0^+}^\beta h_n(t)|_{t=0} = 0, \quad h_n(1) = I_{0^+}^\gamma h_n(1) \end{cases}$$

this inclusion is equivalent to

$$h_n(t) = \int_0^1 G(t,s) u_n(s)\, ds,$$

again with the Green function considered before. Therefore, by passing to the limit, in this equality, we obtain

$$h(t) = \int_0^1 G(t,s) u(s)\, ds$$

Altogether, we see that (h, u) satisfies the dynamic

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1} h(t) = u(t), \\ I_{0^+}^\beta h(t)|_{t=0} = 0, \quad h(1) = I_{0^+}^\gamma h(1) \\ u(0 = u_0 \in D(A_0) \\ u(t) \in D(A_t), \forall t \in [0,T] \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, h(s))dz_s)\frac{dt}{dv}(t) \quad a.e.\ t \in I. \end{cases}$$

According to the lower semicontinuity of the integral functional (see Theorem 8.16 [63]), we obtain

$$\liminf_n \int_0^1 L(t, \int_0^t b(s, h_n(s))dz_s), u_n(t), \frac{du_n}{dv}(t))\, dv \ge \int_0^1 L(t, \int_0^t b(s, h(s))dz_s), u(t), \frac{du}{dv}(t))\, dv.$$

We see that the pair (h, u) is an optimal solution. □

There is a great novelty in dealing with the dynamic system R.L fractional differential inclusion/evolution inclusion with rough signal in the BVRC setting. In case of the dynamic Caputo fractional differential inclusion/evolution inclusion with rough signal, we provide the variant below.

Theorem 12. *Let $I := [0,1]$. Assume that for every $t \in I = [0,1]$, $A_t : D(A_t) \subset \mathbb{R}^e \rightrightarrows \mathbb{R}^e$ is a maximal monotone operator satisfying $(\mathcal{H}_1^*)$, $(\mathcal{H}_2^*)$ and $(\mathcal{H}_5^*)$ $D(A_t)$ is closed.*

Let $z \in C^{1-var}(I, \mathbb{R}^d)$ and $b \in \mathfrak{B}$.

Let $f : I \times \mathbb{R}^e \times \mathbb{R}^e \to \mathbb{R}^e$ be a continuous mapping satisfying:

(i) $\|f(t,x,z) - f(t,y,z)\| \le M\|x - y\|$ *for all* $(t,x,y,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e$.

(ii) $\|f(t,x,z)\| \le M$ *for all* $(t,x,z) \in I \times \mathbb{R}^e \times \mathbb{R}^e$.

Let $v := dr + dt$.

Assume further that there is $\delta \ge 0$ such that $\forall t \in I$, $0 \le 2M\frac{dt}{dv}(t)v(\{t\}) \le \delta < 1$.

Assume that $\alpha \in]1, 2]$.

Then for any $u_0 \in D(A_0)$, there exists a $W^{\alpha,\infty}_{B,\mathbb{R}^e}([0,1])$ mapping $x : I \to \mathbb{R}^e$ and a BVRC mapping $u : I \to \mathbb{R}^e$ satisfying to the dynamic FDI/ EVI with rough signal

$$\begin{cases} D^\alpha x(t) = u(t), \ t \in [0,1] \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(1) + \frac{dx}{dt}(1) = 0 \\ u(t) \in D(A_t), \, t \in \ I \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, x(s)) dz_s) \frac{dt}{dv}(t) \quad a.e. \ t \in \ I \\ u(0) = a \in D(A_0) \end{cases}$$

Let $L : [0,1] \times \mathbb{R}^e \times \mathbb{R}^e \times \mathbb{R}^e \to [0,\infty[$ be a lower semicontinuous integrand such that $L(t,x,y,.)$ is convex on $\mathbb{R}^e$ for every $(t,x,y) \in [0,1] \times \mathbb{R}^e \times \mathbb{R}^e$. Then the problem of minimizing the cost function $\int_0^1 L(t, \int_0^t b(s,x(s)) dz_s, u(t), \frac{du}{dv}(t)) dv$ subject to

$$\begin{cases} D^\alpha x(t) = u(t), \ t \in [0,1] \\ x(0) - \frac{dx}{dt}(0) = 0 \\ x(1) + \frac{dx}{dt}(1) = 0 \\ u(t) \in D(A_t), \, t \in \ I \\ -\frac{du}{dv}(t) \in A_t u(t) + f(t, u(t), \int_0^t b(s, x(s)) dz_s) \frac{dt}{dv}(t) \quad a.e. \ t \in \ I \\ u(0) = a \in D(A_0) \end{cases}$$

has an optimal solution.

Proof. The proof is omitted. It is sufficient to repeat the proof of the previous theorem with suitable modifications using the properties of the Caputo fractional inclusion given in Theorem 9. □

Direct applications to the convex sweeping process are available.

8. Conclusions

We have established, in the BV frames, existence and uniqueness results for dynamical systems of fractional equations coupled with time- and state-dependent maximal monotone operators, in particular the BV solution for a second order of evolution inclusion with application to the convex sweeping process. The existence of BVRC periodic solutions is stated for first time in the literature. Our results are strong and contain novelties. However, there remain several issues that require further development, for instance, the Skorohod problems, by considering the case when the moving set $C(t,x)$ is not convex. We also have to develop the study of evolution inclusions in the context of unbounded perturbations. In most of the presented settings, the existence of solutions is established, but the question of uniqueness is an open question, particularly with unbounded perturbations in Skorohod, rough signal, Volterra, or Young integral settings. An extension to the stochastic framework could also be considered.

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