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First- and Second-Order Forces in the Asymmetric Dynamical Casimir Effect for a Single $\delta - \delta'$ Mirror

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Abstract: Here, we consider an asymmetric $\delta - \delta'$ mirror undergoing time-dependent interactions with a massless scalar field in 1 + 1 dimensions. Using fluctuation-dissipation theory for a mirror in vacuum, we compute the force on a moving $\delta - \delta'$ mirror with time-dependent material properties. We investigate the first-order forces arising from the two distinct fluctuation sources and calculate the linear susceptibility in each case. We then plot the resulting forces. At the second order, we also find the independent contributions to the total force as well as the force that arises from the interference phenomena between the two fluctuation sources.

Keywords: quantum vacuum; vacuum fluctuation; dynamical Casimir effect; Casimir forces; asymmetry; asymmetric excitations; asymmetric dynamical Casimir effect

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1. Introduction

A mirror subject to time-dependent interactions with the quantum vacuum, in which its position or boundary (material) properties may fluctuate, will experience the dynamical Casimir effect (DCE) and produce real particles. This phenomenon has been thoroughly investigated for numerous theoretical configurations (see [1–3] for several detailed reviews of this topic) and has also been experimentally verified [4]. There has been recent interest in understanding the consequences of modifying DCE systems by introducing asymmetric boundary conditions to a mirror undergoing time-dependent interactions with the quantum vacuum [5–9]. This asymmetry leads to an asymmetric spectrum of produced particles in what is now known as the asymmetric dynamical Casimir effect (ADCE). To better understand the ADCE, it is convenient to investigate the interaction between the quantum vacuum and a partially transparent mirror in a (1 + 1)D (dimensional) spacetime. This is achieved by modeling the mirror as $\delta - \delta'$ potential [5,6,10–16] (δ' being the spatial derivative of the Dirac δ function). Previous literature has explored the ADCE spectrum of a moving $\delta - \delta'$ mirror [5] and a $\delta - \delta'$ mirror with time-dependent material properties [6] as well as when the mirror possesses both of these independent fluctuation sources. In this latter case, there is an interference effect between the two sources that modifies the total asymmetric spectrum of produced particles [9].

An asymmetric production of photons on either side of the mirror leads to an unbalanced force on the mirror due to the imbalance in the number of particles produced by the two sides of the mirror [6–8,17]. Specifically, in the case of a $\delta - \delta'$ mirror with time-dependent properties, the initially stationary mirror will be perturbed in such a way that the imbalance in particle production will induce motion upon the mirror [6]. The quantum vacuum will, in turn, act as a dissipative medium and react to the motion of objects moving through it. For a perfectly reflective mirror moving in (1 + 1)D spacetime, a dissipative reaction force acts on the mirror:

$$F(t) = \frac{\hbar}{6\pi c^2}\frac{d^3}{dt^3}q(t), \tag{1}$$

which is proportional to the third time derivative of the mirror's position, $q(t)$. Here, $\hbar$ is the reduced Planck constant and c is the speed of light. Thus, to fully understand the forces present in the $\delta - \delta'$ system, one must account for both the force from the radiation pressure generated by the asymmetry in particles and the dissipative effects of the object moving through the quantum vacuum.

In this paper, we compute the full spectrum of forces for a moving $\delta - \delta'$ mirror with time-dependent boundary conditions using fluctuation–dissipation theory [18–20]. At first-order, we calculate the independent force contributions from both the motion of the mirror and from its time-dependent material properties by first calculating the linear susceptibility. By prescribing a specific form to the fluctuation sources, we are able to plot the mean force on the mirror for different magnitudes of λ_0, which controls the degree of asymmetry in the $\delta - \delta'$ mirror. In addition to explicit first-order forces, the second-order forces are presented after first deriving the second-order correction to the output field. At second-order, the forces resulting from the two independent fluctuation sources are again found along with the addition of a third force that results from the interference between the motion of the mirror and its changing properties.

The remainder of this paper is organized as follows. In Section 2, we review fluctuation–dissipation theory [18], which is used to calculate the susceptibility and force. Section 3 goes over the scattering formalism, which describes the interaction between the quantum vacuum and the $\delta - \delta'$ mirror. Here, we also derive the necessary second-order corrections to the output field. The first-order forces are then calculated in Section 4, which also includes numerically integrated plots of the resulting forces. Section 5 contains the calculations of the second-order forces, including the additional term arising from the interference of the two fluctuation sources. We conclude with ending remarks in Section 6.

Unless otherwise stated, it is assumed throughout this paper that $\hbar = c = 1$. We also use square brackets on a function $f[\omega]$ to denote that this frequency domain function is the Fourier transform of some function $f(t)$ in the time domain. Additionally, we take $\eta = \mathrm{diag}(1, -1)$. Throughout the paper we will use primes in two distinct ways. A prime on a function is understood to mean the spatial derivative of that function, where as primes on variables are understood to simply index distinct variables.

2. Quantum Fluctuation–Dissipation Theorem

This Section reviews the notation and terminology necessary to understand the fluctuation–dissipation theorem applied to quantum interactions with the vacuum. Following the conventions in Ref. [18], we decompose a (1 + 1)D scalar field into the sum of two counter-propagating fields, which are denoted as $\varphi(t - x)$ and $\psi(t + x)$. We denote the incoming fields with an "in" subscript and the outgoing fields with an "out" subscript. In what follows, we adopt a scattering framework, taking our ingoing field as the initial field which scatters by some interaction and is perturbatively modified into an outgoing field. We then specialize this to the case of the background quantum vacuum scattering off partially reflecting mirrors. We make use of the following column matrix notation to write the field as

$$\Phi(t, x) = \begin{pmatrix} \varphi(t - x) \\ \psi(t + x) \end{pmatrix}. \tag{2}$$

In the frequency domain, the field $\Phi[\omega, x]$ can be expressed in terms of the stationary field $\Phi[\omega, 0]$ at $x = 0$,

$$\Phi[\omega, x] = \begin{pmatrix} \varphi[\omega]e^{i\omega x} \\ \psi[\omega]e^{-i\omega x} \end{pmatrix} = e^{i\eta\omega x}\Phi[\omega, 0], \tag{3}$$

with the frequency ω.

Going forward, we employ a shorthand for this stationary field by taking $\Phi(t,0) = \Phi(t)$ and $\Phi[\omega,0] = \Phi[\omega]$. The two stationary incoming counter-propagating fields can be related to the standard creation and annihilation operators. Explicitly, these are

$$\varphi_{\text{in}}[\omega] = (2|\omega|)^{-1/2}\left[\Theta(\omega)a_L[\omega] + \Theta(-\omega)a_L^\dagger[-\omega]\right] \tag{4}$$

and

$$\psi_{\text{in}}[\omega] = (2|\omega|)^{-1/2}\left[\Theta(\omega)a_R[\omega] + \Theta(-\omega)a_R^\dagger[-\omega]\right]. \tag{5}$$

Here, $a_j[\omega]$ and $a_j^\dagger[\omega]$ $(j = L, R)$ are the annihilation and creation operators for the left (L) and right (R) sides of the mirror, and $\Theta(\omega)$ is the Heaviside function.

Two important quantities we use below are the energy density, $e(t,x)$, and impulsion density, $p(t,x)$. One may write these quantities in terms of the counter-propagating fields:

$$e(t,x) = \varphi'(t-x)^2 + \psi'(t+x)^2, \qquad p(t,x) = \varphi'(t-x)^2 - \psi'(t+x)^2. \tag{6}$$

We show below that the mean (expectation) value of these terms can be used to directly calculate the force on the mirror. In order to calculate these quantities, will use two-point correlation functions, written in terms of the covariance, which are defined as

$$\text{cov}(\Phi(t,x),\Phi(t',x')) \equiv C_{x,x'}(t,t') = \left\langle \Phi(t,x)\Phi(t',x')^{\text{T}} \right\rangle. \tag{7}$$

The flux densities are then

$$\langle e(t,x)\rangle = \left\{\text{Tr}\left[\partial_t\partial_{t'}C_{x,x'}(t,t')\right]\right\}_{t=t'}, \qquad \langle p(t,x)\rangle = \left\{\text{Tr}\left[\eta\partial_t\partial_{t'}C_{x,x'}(t,t')\right]\right\}_{t=t'}, \tag{8}$$

where "Tr" denotes the trace operation.

Using the following expression for the correlator in the frequency domain,

$$C_{x,x'}[\omega,\omega'] = \left\langle \Phi_x[\omega]\Phi_{x'}[\omega']^{\text{T}} \right\rangle = e^{i\eta\omega x}C[\omega,\omega']e^{i\eta\omega' x'}, \tag{9}$$

we implicitly define the Fourier transforms of the energy and impulsion densities as, respectively,

$$\langle e(t,x)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi} e^{i\omega t - i\omega' t} i\omega i\omega' \,\text{Tr}\left[C_{x,x}[\omega,\omega']\right] \tag{10}$$

and

$$\langle p(t,x)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi} e^{i\omega t - i\omega' t} i\omega i\omega' \,\text{Tr}\left[\eta C_{x,x}[\omega,\omega']\right]. \tag{11}$$

One can now compute the outgoing field (Φ_{out}), and the resulting forces, by expressing the outgoing field in terms of the ingoing field (Φ_{in}). The ingoing state corresponds to a stationary state, whose covariance matrices depend only upon one parameter, and whose correlator now becomes

$$C(t,t') = c(t-t'), \qquad C[\omega,\omega'] = 2\pi\delta(\omega+\omega')c[\omega]. \tag{12}$$

For a vacuum ingoing state, we have

$$c_{\text{vac}}[\omega] = \frac{\Theta(\omega)}{2\omega} I_2, \tag{13}$$

where I_2 is the identity matrix.

We now use this framework to analyze an asymmetric $\delta - \delta'$ mirror. This is a partially transparent mirror whose interaction with the ingoing vacuum state can be linearly related to its modified outgoing field via

$$\Phi_{\text{out}}[\omega] = S[\omega]\Phi_{\text{in}}[\omega], \tag{14}$$

where $S[\omega]$ is the scattering matrix. We see that for partially transparent mirrors, the outgoing correlator can be related to the ingoing correlator as

$$C_{\mathrm{out}}[\omega,\omega'] = S_0[\omega]C_{\mathrm{in}}[\omega,\omega']S_0[\omega']^{\mathrm{T}}, \tag{15}$$

where S_0 is the zeroth-order scattering matrix (see Equation (49)).

Some authors make use of an overbar to denote quantities taken to be comoving with the mirror [18]. We do not make use of this notation, except when introducing the moving mirror in Sections 3.1 and 3.2, as in all other instances we will be able to explicitly work in the laboratory frame.

The perturbed, outgoing fields is eventually expressed as the zeroth, first, and second-order corrections from the scattering matrix for both moving mirrors and stationary mirrors with time-dependent boundary conditions:

$$\Phi_{\mathrm{out}}[\omega] = S_0[\omega]\Phi_{\mathrm{in}}[\omega] + \int \frac{\mathrm{d}\omega'}{2\pi}\delta S[\omega,\omega']\Phi_{\mathrm{in}}[\omega'] + \int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi}\delta S[\omega,\omega',\omega'']\Phi_{\mathrm{in}}[\omega'']. \tag{16}$$

Force on a Mirror

In the (1 + 1)D spacetime considered here, the force on a single, stationary mirror is given by

$$F(t) = \varphi'_{\mathrm{in}}(t)^2 + \psi'_{\mathrm{out}}(t)^2 - \varphi'_{\mathrm{out}}(t)^2 - \psi'_{\mathrm{in}}(t)^2. \tag{17}$$

The resulting force can be interpreted as the difference between the impulsion densities of the ingoing and outgoing fields evaluated at the mirror's position. The force takes the form

$$F(t) = p_{\mathrm{in}} - p_{\mathrm{out}}. \tag{18}$$

This is also related to the T_{xx} component of the stress-energy tensor (radiation pressure) obtained by taking the difference between the energy densities of the left and right half of the mirror

$$F(t) = e_{\mathrm{L}} - e_{\mathrm{R}}. \tag{19}$$

One can freely pull through the time averaging to obtain the mean force relation

$$\langle F(t)\rangle = \langle p_{\mathrm{in}}\rangle - \langle p_{\mathrm{out}}\rangle = \langle e_{\mathrm{L}}\rangle - \langle e_{\mathrm{R}}\rangle. \tag{20}$$

Using Equations (11) and (15) in Equation (20), one can now express the mean force as

$$\langle F(t)\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi} e^{i\omega t - i\omega' t} i\omega i\omega' \,\mathrm{Tr}\big[\mathbb{F}[\omega,\omega']C_{\mathrm{in}}[\omega,\omega']\big], \tag{21}$$

where $\mathbb{F}[\omega,\omega']$ is the matrix

$$\mathbb{F}[\omega,\omega'] = \eta - S_0[\omega']^{\mathrm{T}}\eta S_0[\omega], \tag{22}$$

which possesses the symmetry $\mathbb{F}[\omega,\omega']^T = \mathbb{F}[\omega',\omega]$. Equation (21) allows us to calculate the mean force for any ingoing state. For any stationary ingoing state, whose correlators take the form in Equation (12), the mean force becomes

$$\langle F\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\omega^2\,\mathrm{Tr}[\mathbb{F}[\omega,-\omega]c_{\mathrm{in}}[\omega]], \tag{23}$$

which vanishes in the case of stationary ingoing states [18]. The energy exchange between the field and the mirror is

$$\langle G\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\omega^2\,\mathrm{Tr}[\mathbb{G}[\omega,-\omega]c_{\mathrm{in}}[\omega]], \tag{24}$$

where

$$\mathbb{G}[\omega, \omega'] = I_2 - S[\omega']^{\mathrm{T}} S[\omega]. \tag{25}$$

The energy exchange for any stationary state is zero due to the unitarity of the scattering matrix S.

In general, the perturbed field Φ_{out} takes the form in Equation (16). Using this, one can compute the mean force due to the perturbation δS in the laboratory frame:

$$\langle \delta F(t) \rangle = -\langle \delta p_{\mathrm{out}}(t) \rangle = -\left\{ \partial_t \partial_{t'} \operatorname{Tr}\left[\eta \delta C_{\mathrm{out}}(t, t') \right] \right\}_{t=t'}, \tag{26}$$

which becomes, to first-order,

$$\langle \delta F(t) \rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} e^{-i\omega t - i\omega' t} \omega \omega' \operatorname{Tr}\left[\eta \delta C_{\mathrm{out}}[\omega, \omega'] \right], \tag{27}$$

$$\delta C_{\mathrm{out}}[\omega, \omega'] = \int \frac{\mathrm{d}\omega''}{2\pi} \left(\delta S[\omega, \omega''] C_{\mathrm{in}}[\omega'', \omega'] S_0[\omega']^{\mathrm{T}} + S_0[\omega] C_{\mathrm{in}}[\omega, \omega''] \delta S[\omega', \omega'']^{\mathrm{T}} \right). \tag{28}$$

The force in Equation (27) can be further expressed as

$$\langle \delta F(t) \rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} e^{-i\omega t - i\omega' t} \chi[\omega, \omega'] f[\omega + \omega'], \tag{29}$$

where $f[\omega]$ is the equation that governs the form of time-dependent fluctuations. We eventually use this form to write the force as a linear response to the mirror's perturbation,

$$\langle \delta F[\omega] \rangle = \chi[\omega] \delta f[\omega], \tag{30}$$

expressed in terms of the susceptibility, given by

$$\chi[\omega] = \int \frac{\mathrm{d}\omega'}{2\pi} \chi[\omega', \omega - \omega']. \tag{31}$$

3. The Scattering Matrix

The mirror is initially located at $x = 0$, which allows us to decompose our field as

$$\phi(t, x) = \Theta(x) \phi_+(t, x) + \Theta(-x) \phi_-(t, x), \tag{32}$$

where ϕ_+ (ϕ_-) is the field on the right (left) side of the mirror. In general, we will use "$+$" ("$-$") subscripts to refer to any quantities that pertain to only the right (left) side of the mirror.Using the fact that both $\phi_\pm$ obey the Klein–Gordon equation, One may represent each as the sum of two freely counterpropagating fields. Explicitly, these are

$$\phi_+(t, x) = \int \frac{\mathrm{d}\omega}{\sqrt{2\pi}} \left[\varphi_{\mathrm{out}}[\omega] e^{i\omega x} + \psi_{\mathrm{in}}[\omega] e^{-i\omega x} \right] e^{-i\omega t} \tag{33}$$

and

$$\phi_-(t, x) = \int \frac{\mathrm{d}\omega}{\sqrt{2\pi}} \left[\varphi_{\mathrm{in}}[\omega] e^{i\omega x} + \psi_{\mathrm{out}}[\omega] e^{-i\omega x} \right] e^{-i\omega t}, \tag{34}$$

which depend on the incoming and outgoing fields introduced in Section 2. We assume here that the ingoing and outgoing fields are linearly related as

$$\Phi_{\mathrm{out}}[\omega] = S[\omega] \Phi_{\mathrm{in}}.$$

Thus far, we have not specified any properties of our mirror except that it is partially reflecting. In this case, $S[\omega]$ can be taken to be the most general partially reflecting scattering matrix, which is written explicitly as

$$S[\omega] = \begin{pmatrix} s_+[\omega] & r_+[\omega] \\ r_-[\omega] & s_-[\omega] \end{pmatrix}. \tag{35}$$

Here, $r_\pm[\omega]$ and $s_\pm[\omega]$ are the reflection and transmission coefficients, respectively. Going forward, we consider the mirror interaction to be described by the asymmetric, partially reflected $\delta - \delta'$ mirror, whose potential is given as

$$U(x) = \mu\delta(x) + \lambda\delta'(x). \tag{36}$$

Here, μ is related to the plasma frequency of the mirror and λ is a dimensionless factor. With this, it is now possible to derive explicit forms of the transmission and reflection components [5]:

$$r_\pm[\omega] = \frac{-i\mu_0 \pm 2\omega\lambda_0}{i\mu_0 + \omega(1+\lambda_0^2)} \tag{37}$$

and

$$s_\pm[\omega] = \frac{\omega(1-\lambda_0^2)}{i\mu_0 + \omega(1+\lambda_0^2)}. \tag{38}$$

Here, we introduce the notation μ_0 and λ_0 to denote the zeroth-order terms. This distinction is important when considering perturbative effects due to field interactions with the mirror.

3.1. First-Order Corrections

We start by solving for the ADCE corrections for the $\delta - \delta'$ mirror with time-dependent $\mu(t)$. For this analysis, we assume that the mirror is held at rest. Here, we require that the fluctuations in $\mu(t)$ take the form of small oscillations about a fixed value μ_0. Specifically,

$$\mu(t) = \mu_0[1 + \epsilon f(t)], \tag{39}$$

where $\mu_0 \geq 1$ is a constant and $f(t)$ is an arbitrary function such that $|f(t)| \leq 1$, with $\epsilon \ll 1$.

To find the modified outgoing field, we apply the field equation of the system, determined by the potential in Equation (36), to Equations (33) and (34). From here, the matching conditions can be solved to the first order by following Ref. [5], where the final form becomes

$$\Phi_{\text{out}}[\omega] = S_0[\omega]\Phi_{\text{in}}[\omega] + \epsilon \int \frac{d\omega'}{2\pi} \delta S_\mu^{(1)}[\omega,\omega']\Phi_{\text{in}}[\omega']. \tag{40}$$

The first-order correction to the scattering matrix due to the introduction of $f(t)$ takes the form

$$\delta S_\mu^{(1)}[\omega,\omega'] = -i\mu_0 h(\omega) f[\omega-\omega']\mathbb{S}_\mu[\omega'], \tag{41}$$

where $h(\omega) = [i\mu_0 + \omega(1+\lambda_0^2)]^{-1}$ and

$$\mathbb{S}_\mu[\omega'] = J_2 + S_0[\omega'] = \begin{pmatrix} s_+[\omega'] & 1 + r_+[\omega'] \\ 1 + r_-[\omega'] & s_-[\omega'] \end{pmatrix}. \tag{42}$$

Here, J_2 is the column-reversed identity matrix. This is in agreement with Ref. [6]. Throughout this paper, we use the superscripts "(1)" and "(2)" to denote the first- and second-order contributions, respectively. Additionally, the subscript μ represents the contribution from the time-varying material properties.

Let us now calculate the first-order corrections due to the $\delta - \delta'$ mirror undergoing mechanical oscillations about $x = 0$. Scattering is still linear with

$$\overline{\Phi}_{\text{out}}[\omega] = S[\omega]\overline{\Phi}_{\text{in}}, \tag{43}$$

in the co-moving frame (denoted by the overbar in this Section only). In this frame, the mirror is instantaneously at rest. The movement is assumed to be nonrelativistic ($|\dot{q}(t)| \ll 1$, where the dot denotes the time-derivative) and limited by a small amplitude, such that

$$q(t) = \epsilon g(t), \tag{44}$$

with $|g(t)| \leq 1$ and $\epsilon \ll 1$. To solve this in the laboratory frame, we use the relation

$$\overline{\Phi}(t', 0) = \Phi(t, \epsilon g(t)) = [1 - \epsilon g(t)\eta\partial_t]\Phi(t, 0) + \mathcal{O}(\epsilon^2). \tag{45}$$

Taking advantage of thefact that $\mathrm{d}\bar{t} = \mathrm{d}t$ at the first order, Equation (45) can be rewritten as

$$\overline{\Phi}(t, 0) = [1 - \epsilon g(t)\eta\partial_t]\Phi(t, 0). \tag{46}$$

One finds that applying this transform to Equation (43) in the frequency domain yields

$$\Phi_{\text{out}}[\omega] = S_0[\omega]\Phi_{\text{in}}[\omega] + \epsilon \int \frac{\mathrm{d}\omega'}{2\pi} \delta S_q^{(1)}[\omega, \omega']\Phi_{\text{in}}[\omega'], \tag{47}$$

where the subscript q denotes the motion of the mirror. The first-order S-matrix perturbation, $\delta S_q^{(1)}[\omega, \omega']$, takes the form

$$\delta S_q^{(1)}[\omega, \omega'] = i\omega' g[\omega - \omega']\mathbb{S}_q^{(1)}[\omega, \omega'], \tag{48}$$

where

$$\mathbb{S}_q^{(1)}[\omega, \omega'] = S_0[\omega]\eta - \eta S_0[\omega'] \tag{49}$$

and S_0 is the zeroth-order scattering matrix found from Equations (37) and (38). This is in agreement with Ref. [5].

3.2. Second-Order Corrections

The second-order perturbation due to the time dependence of $\mu(t)$ can be found by carrying through the derivation of the first-order term to second-order in the matching conditions. With this in mind, one finds that the expression for Φ_{out} in Equation (40) to the second-order term is now

$$\Phi_{\text{out}}[\omega] = S_0[\omega]\Phi_{\text{in}}[\omega] + \epsilon \int \frac{\mathrm{d}\omega'}{2\pi} \delta S_\mu^{(1)}[\omega, \omega']\Phi_{\text{in}}[\omega'] + \epsilon^2 \int \frac{\mathrm{d}\omega'}{2\pi} \int \frac{\mathrm{d}\omega''}{2\pi} \delta S_\mu^{(2)}[\omega, \omega', \omega'']\Phi_{\text{in}}[\omega''], \tag{50}$$

with the first-order perturbation term, $\delta S_\mu^{(1)}[\omega, \omega']$, given in Equation (41), and the second-order term,

$$\delta S_\mu^{(2)}[\omega, \omega', \omega''] = -\mu_0^2 h(\omega) h(\omega') f[\omega - \omega'] f[\omega' - \omega''] \mathbb{S}_\mu[\omega''], \tag{51}$$

which agrees with Ref. [6].

The second-order correction due to the motion of the mirror is more complicated. Let us start by evaluating the fields at the time-dependent position of the mirror. This is the frame in which the mirror is instantaneously at rest whereby the field and its Fourier transform can be written as, respectively

$$\overline{\Phi}(\tau) = \Phi_{q_t}(t) = \left\{ e^{-x\eta\partial_t}\Phi(t) \right\}_{x=q_t}, \qquad \overline{\Phi}(\tau) = \int \frac{\mathrm{d}\omega}{2\pi} \Phi'[\omega] e^{-i\omega\tau}, \tag{52}$$

where τ is the mirror's proper time and $q_t \equiv q(t)$. The proper time and laboratory time are related by

$$d\tau = \sqrt{1 - \dot{q_t}^2}\, dt. \tag{53}$$

The first- and second-order expansions in q_t of the mirror's trajectory around $q = 0$, or $\bar{\Phi}(\tau, 0) = \Phi(\tau, \epsilon q(t))$, lead to

$$\bar{\Phi}(\tau) = \Phi(t) - q_t \eta \partial_t \Phi(t) + \frac{1}{2} q_t^2 \partial_t^2 \Phi(t). \tag{54}$$

Now, unlike in the first-order expansion when the mirror's proper time and laboratory time coincide, $d\tau$ is no longer equal to dt. One can see for the second-order time correction, in the nonrelativistic limit ($|\dot{q}(t)| \ll 1$):

$$d\tau \approx \left(1 - \frac{1}{2}\delta \dot{q_t}^2\right) dt. \tag{55}$$

Therefore,

$$\tau = t - \frac{1}{2} \int dt\, \delta \dot{q_t}^2. \tag{56}$$

Using Equation (44), we obtain the explicit result:

$$\bar{\Phi}(\tau, 0) = \left[1 - \epsilon g(t) \eta \partial_t + \frac{1}{2} \epsilon^2 g(t)^2 \partial_t^2 \right] \Phi(t, 0) + \mathcal{O}(\epsilon^3) \tag{57}$$

and

$$\tau = t - \frac{1}{2}\epsilon^2 \int dt\, \dot{g}(t)^2 + \mathcal{O}(\epsilon^3). \tag{58}$$

To find the field in the frequency domain, we substitute the new form of τ in Equation (58) into the field's Fourier transform from Equation (52). We find its second-order approximation to be

$$\bar{\Phi}(\tau) = \int \frac{d\omega}{2\pi} \Phi'[\omega] e^{-i\omega\tau} \approx \int \frac{d\omega}{2\pi} \left[1 + \frac{i\omega\epsilon^2}{2} \int dt\, \dot{g}(t)^2 \right] \Phi'[\omega] e^{-i\omega t}. \tag{59}$$

We can equate this quantity to the Fourier transform of the right-hand side of Equation (54), where we now arrive at the following relationship

$$\left[1 + \frac{i\omega\epsilon^2}{2} \int dt\, \dot{g}(t)^2 \right] \Phi'[\omega] = \Phi[\omega] + i\epsilon\eta \int \frac{d\omega'}{2\pi} \omega' g[\omega - \omega'] \Phi[\omega'] - \frac{\epsilon^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \omega''^2 g[\omega - \omega'] g[\omega' - \omega''] \Phi[\omega'']. \tag{60}$$

Solving for $\bar{\Phi}[\omega]$ in Equation (60), and using $(1 + x)^{-1} \approx 1 - x$, leads to the second-order correction to the field in the laboratory frame,

$$\bar{\Phi}^{(2)}[\omega] = -\frac{i\omega\epsilon^2}{2} \int dt\, \dot{g}(t)^2 \Phi[\omega] - \frac{\epsilon^2}{2} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \omega''^2 g[\omega - \omega'] g[\omega' - \omega''] \Phi[\omega'']. \tag{61}$$

With Equation (61), which describes the relationship between the field in the instantaneous frame of mirror with the field in the laboratory frame for the second order, one can now calculate the output field as a function of the input field using Equation (43). The full first- and second-order corrections to the outgoing field due to the motion of the mirror are

$$\Phi_{\text{out}}[\omega] = S_0[\omega] \Phi_{\text{in}}[\omega] + \epsilon \int \frac{d\omega'}{2\pi} \delta S_q^{(1)}[\omega, \omega'] \Phi_{\text{in}}[\omega'] + \epsilon^2 \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \delta S_q^{(2)}[\omega, \omega', \omega''] \Phi_{\text{in}}[\omega''], \tag{62}$$

with the definition of $\delta S_q^{(1)}$ from Equation (48) and

$$\delta S_q^{(2)}[\omega, \omega', \omega''] = \frac{1}{2} \omega''^2 g[\omega - \omega'] g[\omega' - \omega''] \mathbb{S}_q^{(2)}[\omega, \omega''], \tag{63}$$

where

$$\mathbb{S}_q^{(2)}[\omega,\omega''] = S_0[\omega''] - S_0[\omega]. \tag{64}$$

4. First-Order Forces

The first-order (in ϵ) contribution to the mean force due to the modification of $\delta S^{(1)}$ is (see Equations (27) and (28))

$$\left\langle \delta F^{(1)}(t) \right\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} e^{-i\omega t - i\omega' t} \omega\omega' \operatorname{Tr}\left[\eta \delta C_{\mathrm{out}}^{(1)}[\omega,\omega']\right],$$
$$\delta C_{\mathrm{out}}^{(1)}[\omega,\omega'] = \int \frac{\mathrm{d}\omega''}{2\pi} \left(\delta S^{(1)}[\omega,\omega''] C_{\mathrm{in}}[\omega'',\omega'] S_0[\omega']^{\mathrm{T}} + S_0[\omega] C_{\mathrm{in}}[\omega,\omega''] \delta S^{(1)}[\omega',\omega'']^{\mathrm{T}} \right).$$

With a stationary ingoing state (see Equations (12) and (13)), $\delta C_{\mathrm{out}}^{(1)}$ reads

$$\delta C_{\mathrm{out}}^{(1)}[\omega,\omega'] = \delta S^{(1)}[\omega,-\omega'] c_{\mathrm{in}}[-\omega'] S_0[\omega']^{\mathrm{T}} + S_0[\omega] c_{\mathrm{in}}[\omega] \delta S^{(1)}[\omega',-\omega]^{\mathrm{T}}. \tag{65}$$

Recall that the force appears as a linear response to the mirror's perturbation,

$$\langle \delta F[\omega] \rangle = \chi[\omega] \delta f[\omega],$$

which is expressed in terms of the susceptibility,

$$\chi[\omega] = \int \frac{\mathrm{d}\omega'}{2\pi} \chi[\omega',\omega-\omega'].$$

4.1. Moving Mirror

Here, we calculate the force on a moving $\delta - \delta'$ mirror whose position $q(t)$ fluctuates about $x = 0$ with a small amplitude $\epsilon g(t)$. Using the first-order correction to the scattering matrix $\delta S_q^{(1)}$ from Equation (48) into Equation (65), $\delta C_q^{(1)}$ becomes

$$\begin{aligned} \delta C_q^{(1)}[\omega,\omega'] = & - i\omega' g[\omega+\omega'] (S_0[\omega]\eta - \eta S_0[-\omega']) c_{\mathrm{in}}[-\omega'] S_0[\omega']^{\mathrm{T}} \\ & - i\omega g[\omega+\omega'] S_0[\omega] c_{\mathrm{in}}[\omega] (\eta S_0[\omega']^{\mathrm{T}} - S_0[-\omega]^{\mathrm{T}} \eta). \end{aligned} \tag{66}$$

Applying the properties of the trace, Equation (66) can be used to find

$$\operatorname{Tr}\left[\eta \delta C_q^{(1)}[\omega,\omega']\right] = g[\omega+\omega'] \operatorname{Tr}\left[\mathbb{F}[\omega,\omega'](i\omega c_{\mathrm{in}}[\omega]\eta + i\omega'\eta c_{\mathrm{in}}[-\omega'])\right], \tag{67}$$

with the matrix $\mathbb{F}[\omega,\omega']$ from Equation (22). Under a double integral over the full domain of ω and ω', one may freely swap these variables. This allows us to modify certain quantities in a way that enables simplification of the integrand without changing the result of the integral. Explicitly, we perform the following swap:

$$g[\omega+\omega'] \operatorname{Tr}\left[\mathbb{F}[\omega,\omega'] i\omega c_{\mathrm{in}}[\omega]\eta\right] \implies g[\omega'+\omega] \operatorname{Tr}\left[\mathbb{F}[\omega',\omega] i\omega' c_{\mathrm{in}}[\omega']\eta\right], \tag{68}$$

Then, one may use the argument swapping symmetry on $\mathbb{F}$ and the definition of c_{in} (where $c_{\mathrm{in}} = c_{\mathrm{vac}}$ from Equation (13)) to re-express Equation (67) as

$$\operatorname{Tr}\left[\eta \delta C_q^{(1)}[\omega,\omega']\right] \implies \frac{i}{2} \operatorname{sgn}(\omega') g[\omega+\omega'] \operatorname{Tr}\left[\mathbb{F}[\omega,\omega']\eta\right]. \tag{69}$$

In Equations (68) and (69), we use the arrow instead of the equation sign to indicate that, while these expressions are not equivalent, they lead to the same final result when integrating. Additionally, when simplifying Equation (69), we used the following definition for the sign function:

$$\operatorname{sgn}(\omega) = \Theta(\omega) - \Theta(-\omega). \tag{70}$$

With Equation (69), we write the first-order motional force in terms of $\chi[\omega,\omega']$ in the following manner:

$$\left\langle \delta F_q^{(1)}(t) \right\rangle = \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} e^{-i\omega t - i\omega' t} \chi_q^{(1)}[\omega,\omega'] g[\omega+\omega'], \tag{71}$$

$$\chi_q^{(1)}[\omega,\omega'] = \frac{i\omega\omega'}{2} \mathrm{sgn}(\omega') \,\mathrm{Tr}\big[\mathbb{F}[\omega,\omega']\eta\big], \tag{72}$$

where Equation (72) can be rewritten as

$$\chi_q^{(1)}[\omega,\omega'] = \mathrm{sgn}(\omega')\omega\omega' h(\omega)h(\omega')\Big[-2i\mu_0^2 + 8i\lambda_0^2\omega\omega' - \mu_0(1+\lambda_0^2)(\omega+\omega')\Big]. \tag{73}$$

The susceptibility is then

$$\begin{aligned} \chi_q^{(1)}[\omega] &= \int \frac{\mathrm{d}\omega'}{2\pi} \chi[\omega', \omega-\omega'] \\ &= \frac{i}{2} \int \frac{\mathrm{d}\omega'}{2\pi} \mathrm{sgn}(\omega-\omega')\omega'(\omega-\omega') \,\mathrm{Tr}\big[\mathbb{F}[\omega',\omega-\omega']\eta\big]. \end{aligned} \tag{74}$$

Taking $\beta = (1+\lambda_0^2)/\mu_0$, we can determine various expansions when the term $\beta\omega$ is assumed to be large or small. The real and imaginary parts of the susceptibility in Equation (74) are,

$$\mathrm{Re}\,\chi_q^{(1)}[\omega] = \frac{1}{2\pi\beta^3(1+\lambda_0^2)^2} \frac{2\mathcal{A}(\omega)\arctan(\beta\omega) - \mathcal{C}_\mathrm{R}(\omega) - \mathcal{B}(\omega)\log\big(1+(\beta\omega)^2\big)}{(4+(\beta\omega)^2)}, \tag{75}$$

$$\mathrm{Im}\,\chi_q^{(1)}[\omega] = \frac{1}{2\pi\beta^3(1+\lambda_0^2)^2} \frac{2\mathcal{B}(\omega)\arctan(\beta\omega) + \mathcal{C}_\mathrm{I}(\omega) + \mathcal{A}(\omega)\log\big(1+(\beta\omega)^2\big)}{(4+(\beta\omega)^2)}, \tag{76}$$

where

$$\begin{aligned} \mathcal{A}(\omega) &= \beta\omega\Big[4\Big(1-4\lambda_0^2+\lambda_0^4\Big) + (\beta\omega)^2\Big(1-6\lambda_0^2+\lambda_0^4\Big)\Big], \\ \mathcal{B}(\omega) &= 4\Big(\lambda_0^2-1\Big)^2 + (\beta\omega)^2\Big(1+\lambda_0^2\Big)^2, \\ \mathcal{C}_\mathrm{R}(\omega) &= (\beta\omega)^2\Big(4+(\beta\omega)^2\Big)\Big[1-6\lambda_0^2+\lambda_0^4\Big], \\ \mathcal{C}_\mathrm{I}(\omega) &= \frac{2}{3}\beta\omega\Big(4+(\beta\omega)^2\Big)\Big[2(\beta\omega)^2\lambda_0^2 - 3\Big(\lambda_0^2-1\Big)^2\Big]. \end{aligned} \tag{77}$$

For $\beta\omega \ll 1$:

$$\begin{aligned} \mathrm{Re}\,\chi_q^{(1)}[\omega] &= -\frac{1}{2\pi\beta^3(1+\lambda_0^2)^2}\left[\frac{(\beta\omega)^4(1+\lambda_0^2)^2}{6} - \frac{(\beta\omega)^6(1+6\lambda_0^2+\lambda_0^4)}{15} + \mathcal{O}\Big[(\beta\omega)^8\Big]\right], \\ \mathrm{Im}\,\chi_q^{(1)}[\omega] &= \frac{1}{6\pi\beta^3(1+\lambda_0^2)^2}\left[(\beta\omega)^3(1+\lambda_0^2)^2 - \frac{(\beta\omega)^5(3+14\lambda_0^2+3\lambda_0^5)}{10} + \mathcal{O}\Big[(\beta\omega)^7\Big]\right], \end{aligned} \tag{78}$$

and for $\beta\omega \gg 1$:

$$\begin{aligned}
\mathrm{Re}\,\chi_q^{(1)}[\omega] &= -\frac{1}{2\pi\beta^3(1+\lambda_0^2)^2}\Big[(\beta\omega)^2(1-6\lambda_0^2+\lambda_0^4)+\beta\omega\pi(-1+6\lambda_0^2-\lambda_0^4)+2(1-6\lambda_0^2+\lambda_0^4) \\
&\quad + 2\log(\beta\omega)(1+\lambda_0^2)^2-\frac{8\pi\lambda_0^2}{\beta\omega}+\frac{1+66\lambda_0^2+\lambda_0^4-96\lambda_0^2\log(\beta\omega)}{3(\beta\omega)^2}+\mathcal{O}\Big[(\beta\omega)^{-3}\Big]\Big], \\
\mathrm{Im}\,\chi_q^{(1)}[\omega] &= \frac{1}{6\pi\beta^3(1+\lambda_0^2)^2}\Big[4(\beta\omega)^3\lambda_0^3-6\beta\omega(1-2\lambda_0^2+\lambda_0^4-\log(\beta\omega)(-1+6\lambda_0^2+\lambda_0^4)) \\
&\quad + 3\pi(1+\lambda_0^2)^2-\frac{3(1+10\lambda_0^2-\lambda_0^4+16\lambda_0^2\log(\beta\omega))}{\beta\omega}-\frac{48\pi\lambda_0^2}{(\beta\omega)^2}+\mathcal{O}\Big[(\beta\omega)^{-3}\Big]\Big].
\end{aligned} \tag{79}$$

The limits in Equations (78) and (79) correspond to the low- and high-frequency limits of the susceptibility, respectively. One should exercise caution though, as using these to produce time-domain quantities can lead to misleading results as the inverse Fourier transform requires an integral over the entire frequency domain.

When $\lambda_0 = 1$, which corresponds to the spectrum of a perfectly reflective $\delta-\delta'$ mirror, the relationship $\beta = 2/\mu_0$ holds, where β is now the Robin parameter. In the limits from Equations (78) and (79), which contain the corrections to the Dirichlet ($\beta \to 0$) and Neumann ($\beta \to \infty$) limits of a moving Robin mirror, respectively, we find that the correct leading order linear susceptibility,

$$\chi[\omega] = i\frac{\omega^3}{6\pi}, \tag{80}$$

is recovered, which leads to the dissipative force in Equation (1).

Notice that when $\lambda_0 = 0$, which corresponds to a perfectly reflective δ mirror with no asymmetry in particle production (the spectrum is identical for both sides of the mirror), the susceptibility does not completely vanish. This is due to the fact that there is still a reaction force from the vacuum onto the mirror originating from the motion of the mirror itself. This is not the case, as one sees in Section 4.2 below, for the stationary mirror with time-dependent $\mu(t)$.

The expression in Equation (71) can be directly computed when an appropriate form of the motion, $g(t)$, is introduced. Here, we use

$$g(t) = \cos(\omega_0 t)\exp(-|t|/\mathcal{T}), \tag{81}$$

where ω_0 is the characteristic frequency of the oscillation and $\mathcal{T}$ is the effective time of the oscillation. Integrals of the type present in Equation (71) do not have analytic solutions to the best of our knowledge. Thus, we numerically integrate this expression; see Figure 1, where we plot with different values of λ_0 and compare these results to the force on a moving Dirichlet mirror using Equation (80). One sees that as the asymmetry between the two sides of the $\delta-\delta'$ mirror grows larger ($\lambda_0 \to \infty$), the magnitude of the force on the mirror grows along with it. Along with this increase in magnitude, the force becomes more sharply peaked. The increase in force arises from the increase in the magnitude of the asymmetric dynamical Casimir effect; the larger imbalance of generated particles leads to an increase in the force on the mirror due to increasingly asymmetric radiation reaction forces.

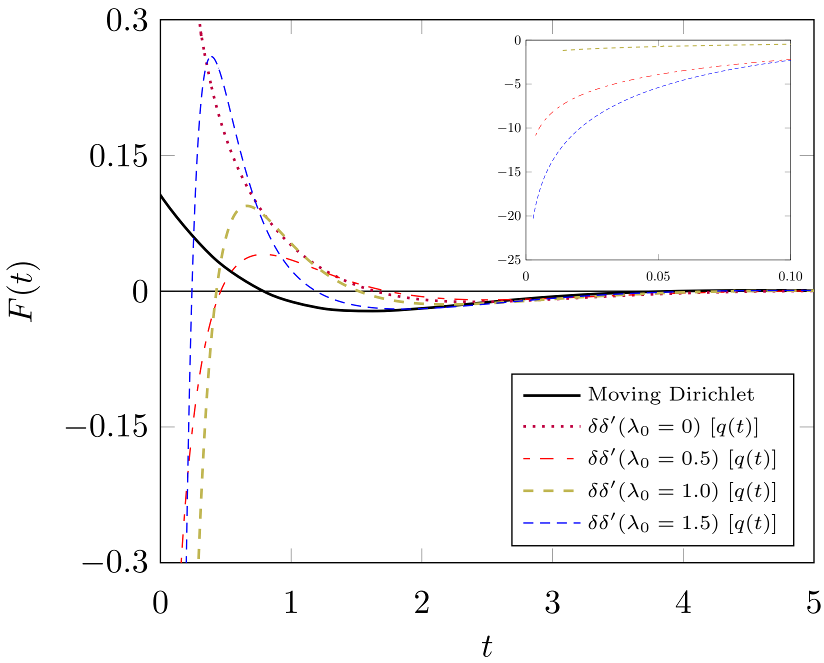

Figure 1. The force (see Equation (71)) on a moving $\delta - \delta'$ mirror (indicated as $\delta\delta'$) as a function of time, presented in natural units, for some values of λ_0, where $\mu_0 = \omega_0 = \mathcal{T} = 1$. The force on a moving Dirichlet mirror is presented for comparison. Inset: the behavior of the moving $\delta - \delta'$ mirror's forces near zero. See text for details.

Compared to the dissipative Dirichlet mirror, the signature of the asymmetry present in the force is apparent. Instead of a positive dissipative force acting on the mirror as it begins to move, there is an initially negative force that corresponds to non-zero dynamical Casimir effect forces that arise due to the asymmetry in particle production. This behavior becomes more obvious when we consider the force on the moving $\delta - \delta'$ mirror for $\lambda_0 = 0$. The force plot now resembles that of the Dirichlet mirror, where the force is once again positive near zero, albeit greater.

Presented in Figure 2, the force as a function of time is plotted for two different values of ω_0. As the frequency of oscillation increases, the number of peaks increases along with it. Compared to the analogous plot for the $\delta - \delta'$ mirror with time-dependent properties, there is a larger increase in the magnitude of the force for the moving mirror. This is expected, as the asymmetry in particle production for the moving $\delta - \delta'$ mirror scales as ω_0^2 when compared to that of the $\delta - \delta'$ mirror with time-dependent material properties [7,8]. Thus, there is an accompanying increase in net force on the moving mirror as the frequency of oscillation increases.

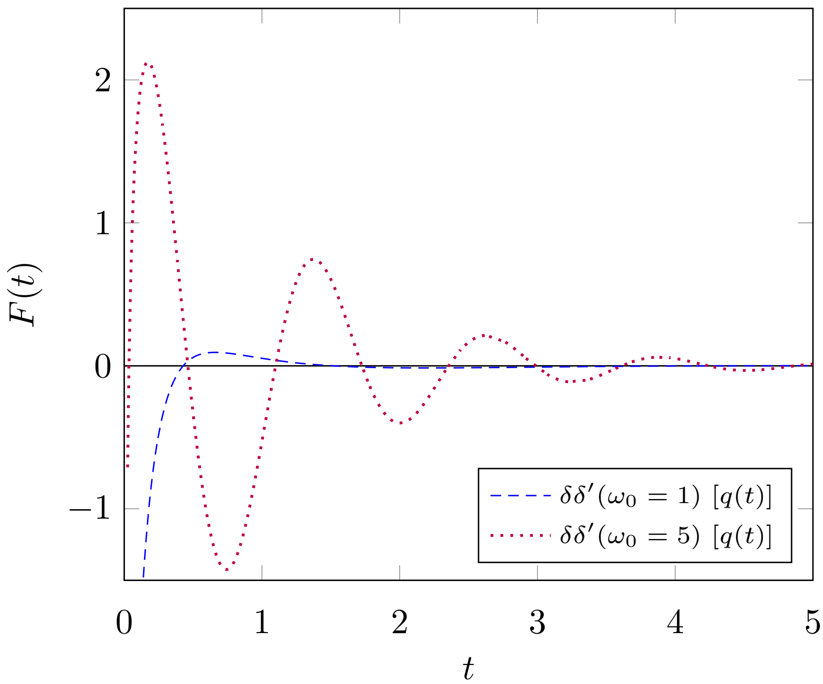

Figure 2. The force (see Equation (71)) on a moving $\delta - \delta'$ mirror as a function of time, presented in natural units, for two values of ω_0, where $\mu_0 = \lambda_0 = \mathcal{T} = 1$. See text for details.

4.2. Mirror with Time-Dependent Properties

To find the first-order force on a stationary $\delta - \delta'$ mirror with time-dependent $\mu(t)$ (see Equation (39)), let us start by using $\delta S_\mu^{(1)}[\omega, \omega']$ from Equation (41) in Equation (65) to find

$$\delta C_\mu^{(1)}[\omega, \omega'] = -i\mu_0 f[\omega + \omega'] \Big(h(\omega)\mathbb{S}_\mu[-\omega'] c_{\text{in}}[-\omega'] S_0[\omega']^{\text{T}} + h(\omega') S_0[\omega] c_{\text{in}}[\omega] \mathbb{S}_\mu[-\omega]^{\text{T}} \Big). \tag{82}$$

Again using the definition of $c_{\text{in}}[\omega]$ from Equation (13), and implementing a change of variables in the second term as done in Equation (68), $\delta C_\mu^{(1)}$ becomes

$$\delta C_\mu^{(1)}[\omega, \omega'] = -\frac{i\mu_0}{2\omega'} h(\omega) f[\omega + \omega'] \Big(\Theta(\omega') S_0[\omega'] \mathbb{S}_\mu[-\omega']^{\text{T}} - \Theta(-\omega') \mathbb{S}_\mu[-\omega'] S_0[\omega']^{\text{T}} \Big), \tag{83}$$

which yields

$$\text{Tr}\Big[\eta \delta C_\mu^{(1)}[\omega, \omega']\Big] = -\frac{i\mu_0}{2\omega'} \text{sgn}(\omega') h(\omega) f[\omega + \omega'] \, \text{Tr}\Big[\eta S_0[\omega'] \mathbb{S}_\mu[-\omega']^{\text{T}}\Big]. \tag{84}$$

Performing the direct calculation

$$\text{Tr}\Big[\eta S_0[\omega'] \mathbb{S}_\mu[-\omega']^{\text{T}}\Big] = 4\lambda_0 \omega' h(\omega'), \tag{85}$$

one sees that

$$\text{Tr}\Big[\eta \delta C_\mu^{(1)}[\omega, \omega']\Big] = -2i\lambda_0\mu_0 \text{sgn}(\omega') h(\omega) h(\omega') f[\omega + \omega']. \tag{86}$$

From Equation (86), we write the motional force in terms of $\chi[\omega, \omega']$ in the following manner

$$\Big\langle \delta F_\mu^{(1)}(t) \Big\rangle = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} e^{-i\omega t - i\omega' t} \chi_\mu^{(1)}[\omega, \omega'] f[\omega + \omega'], \tag{87}$$

$$\chi_\mu^{(1)}[\omega, \omega'] = -2i\lambda_0\mu_0 \text{sgn}(\omega') \omega\omega' h(\omega) h(\omega'). \tag{88}$$

Plugging in Equation (88) into Equation (31) gives the explicit first-order susceptibility for the time-dependent $\delta - \delta'$ mirror,

$$\chi_{\mu}^{(1)}[\omega] = \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[\frac{2+(\beta\omega)^2 - 2i\beta\omega}{4+(\beta\omega)^2}\left[2i\arctan(\beta\omega) - \log\left(1+\beta^2\omega^2\right)\right] - i\beta\omega\right], \tag{89}$$

which is in agreement with Ref. [6], up to discrepancy of a factor of 2 on the entire term. Unlike in the motional case presented in Equation (74), when the asymmetry is no longer present ($\lambda_0 = 0$), the susceptibility in Equation (89) completely vanishes. This is expected, as there are no longer any time-dependent interactions occurring between the mirror and the vacuum, and thus there is no force present.

The susceptibility's real and imaginary components are

$$\mathrm{Re}\,\chi_{\mu}^{(1)}[\omega] = \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[\frac{2\beta\omega\arctan(\beta\omega) - (2+\beta^2\omega^2)\log(1+\beta^2\omega^2)}{4+\beta^2\omega^2}\right], \tag{90}$$

$$\mathrm{Im}\,\chi_{\mu}^{(1)}[\omega] = \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[\frac{-4\beta\omega - \beta^3\omega^3 + (2+\beta^2\omega^2)2\arctan(\beta\omega) + \beta\omega\log(1+\beta^2\omega^2)}{4+\beta^2\omega^2}\right]. \tag{91}$$

For $\beta\omega \ll 1$:

$$\begin{aligned}
\mathrm{Re}\,\chi_{\mu}^{(1)}[\omega] &= \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[-\frac{(\beta\omega)^4}{6} + \frac{(\beta\omega)^6}{10} + \mathcal{O}\left[(\beta\omega)^8\right]\right], \\
\mathrm{Im}\,\chi_{\mu}^{(1)}[\omega] &= \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[\frac{(\beta\omega)^3}{6} - \frac{2(\beta\omega)^5}{15} + \mathcal{O}\left[(\beta\omega)^7\right]\right].
\end{aligned} \tag{92}$$

For $\beta\omega \gg 1$:

$$\begin{aligned}
\mathrm{Re}\,\chi_{\mu}^{(1)}[\omega] &= \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[-2\log(\beta\omega) + \frac{\pi}{\beta\omega} + \frac{-3+4\log(\beta\omega)}{(\beta\omega)^2} - \frac{4\pi}{(\beta\omega)^3} + \mathcal{O}\left[(\beta\omega^{-4})\right]\right], \\
\mathrm{Im}\,\chi_{\mu}^{(1)}[\omega] &= \frac{\lambda_0}{\pi\beta^2(1+\lambda_0^2)}\left[-\beta\omega + \pi - \frac{2(1-\log(\beta\omega))}{\beta\omega} - \frac{2\pi}{(\beta\omega)^2} + \frac{17-24\log(\beta\omega)}{3(\beta\omega)^3} + \mathcal{O}\left[(\beta\omega)^{-4}\right]\right].
\end{aligned} \tag{93}$$

Similar to the moving mirror case of Section 4.1, the limits in Equations (92) and (93) correspond to the low- and high-frequency limits of the susceptibility, respectively. One should still exercise caution, as again, using these to produce time-domain quantities can lead to misleading results as the inverse Fourier transform requires an integral over the entire frequency domain.

As in the first-order motional case, we again plot the force that arises from the time-dependent perturbation, now due to $\mu(t)$, in Figure 3. The behavior is similar to that of the moving mirror; the magnitude of the force increases as the asymmetry grows. Compared to the moving mirror, the positive force peaks have been shifted toward zero slightly but still dies off just as quickly. While the positive force peaks due to the time-dependent material properties are of the same order as the force from the motion of the mirror, the initial negative force is approximately an order of magnitude lower. In Figure 4, the behavior of the $\delta - \delta'$ mirror with time-dependent material properties is plotted. As in the moving mirror case there is an increase in the number of peaks and the magnitude of the peaks, although the increase is not as dramatic as the moving case.

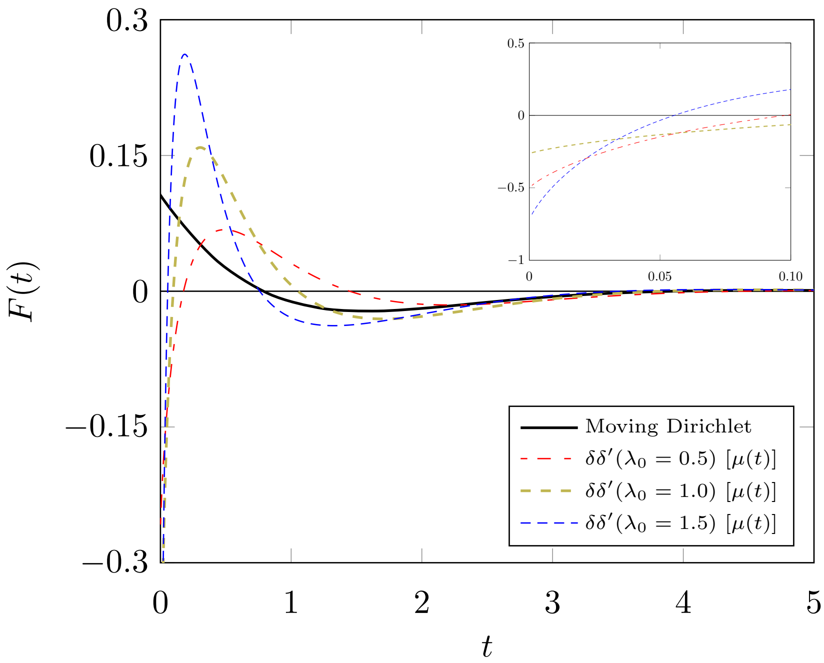

Figure 3. The force (see Equation (87)) on a $\delta - \delta'$ mirror with time-dependent $\mu(t)$ as function of time, presented in natural units, for some values of λ_0, where $\mu_0 = \omega_0 = \mathcal{T} = 1$. The force on a moving Dirichlet mirror is presented for comparison. Inset: the behavior of the time-dependent $\delta - \delta'$ mirror's forces near zero. See text for details.

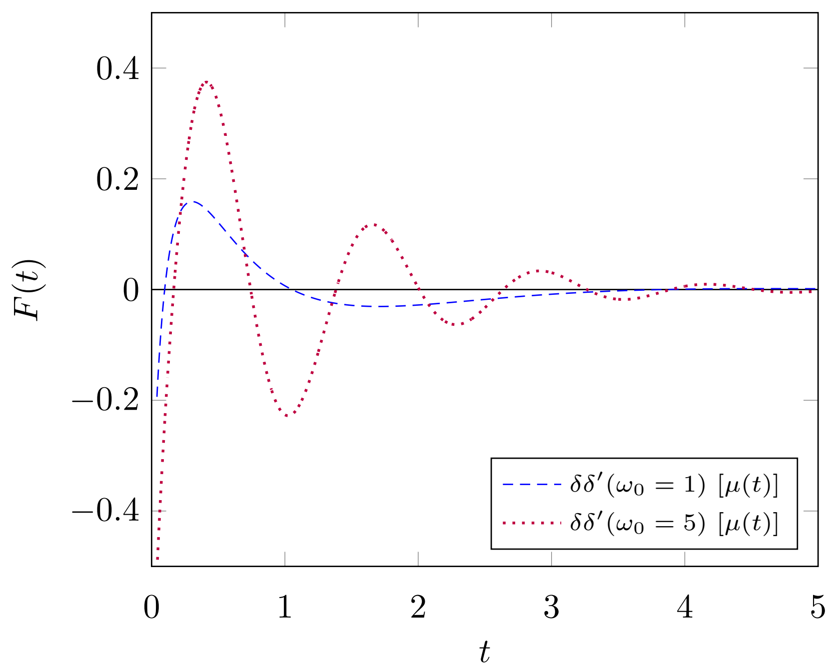

Figure 4. The force (see Equation (87)) on a $\delta - \delta'$ mirror with time-dependent $\mu(t)$ (see Equation (39)) as a function of time, presented in natural units, for two values of ω_0, where $\mu_0 = \lambda_0 = \mathcal{T} = 1$. See text for details.

5. Second-Order Forces

The second-order contribution (of order ϵ^2) to the non-vanishing mean force that arises due to the terms $\delta S^{(1)}$ and $\delta S^{(2)}$ (see Equations (41) and (51), respectively) is

$$\left\langle \delta F^{(2)}(t) \right\rangle = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} e^{-i\omega t - i\omega' t} \omega\omega' \,\mathrm{Tr}\left[\eta \delta C^{(2)}_{\mathrm{out}}[\omega, \omega']\right], \tag{94}$$

where

$$\begin{aligned}\delta C^{(2)}_{\mathrm{out}}[\omega, \omega'] = \int \frac{d\omega''}{2\pi} \int \frac{d\omega'''}{2\pi} \Big(&\delta S^{(1)}[\omega, \omega''] C_{\mathrm{in}}[\omega'', \omega'''] \delta S^{(1)}[\omega', \omega''']^{\mathrm{T}} \\ + S_0[\omega] C_{\mathrm{in}}[\omega, \omega'''] \delta S^{(2)}[\omega', \omega'', \omega''']^{\mathrm{T}} &+ \delta S^{(2)}[\omega, \omega'', \omega'''] C_{\mathrm{in}}[\omega''', \omega'] S_0[\omega']^{\mathrm{T}} \Big).\end{aligned} \tag{95}$$

With a stationary ingoing state (see Equations (12) and (13)), this becomes

$$\begin{aligned}\delta C^{(2)}_{\mathrm{out}}[\omega, \omega'] = \int \frac{d\omega''}{2\pi} \Big(&\delta S^{(1)}[\omega, \omega''] c_{\mathrm{in}}[\omega''] \delta S^{(1)}[\omega', -\omega'']^{\mathrm{T}} \\ &+ S_0[\omega] c_{\mathrm{in}}[\omega] \delta S^{(2)}[\omega', \omega'', -\omega]^{\mathrm{T}} + \delta S^{(2)}[\omega, \omega'', -\omega'] c_{\mathrm{in}}[-\omega'] S_0[\omega']^{\mathrm{T}} \Big).\end{aligned} \tag{96}$$

It is helpful to simplify $\delta C^{(2)}_{\mathrm{out}}$ here using the definition $c_{\mathrm{in}}[\omega]$ from Equation (13). This becomes

$$\begin{aligned}\delta C^{(2)}_{\mathrm{out}}[\omega, \omega'] = \int \frac{d\omega''}{2\pi} \Big(&\frac{\Theta(\omega'')}{2\omega''} \delta S^{(1)}[\omega, \omega''] \delta S^{(1)}[\omega', -\omega'']^{\mathrm{T}} \\ &+ \frac{\Theta(\omega)}{2\omega} S_0[\omega] \delta S^{(2)}[\omega', \omega'', -\omega]^{\mathrm{T}} - \frac{\Theta(-\omega')}{2\omega'} \delta S^{(2)}[\omega, \omega'', -\omega'] S_0[\omega']^{\mathrm{T}} \Big).\end{aligned} \tag{97}$$

Again, as in Section 4.2, using the fact that exchanging ω and ω' under the double integral does not change the result of the integral, we may modify the expression (97) to

$$\begin{aligned}\mathrm{Tr}\left[\eta \delta C^{(2)}_{\mathrm{out}}[\omega, \omega']\right] \implies \int \frac{d\omega''}{2\pi} \Big(&\frac{\Theta(\omega'')}{2\omega''} \mathrm{Tr}\left[\eta \delta S^{(1)}[\omega, \omega''] \delta S^{(1)}[\omega', -\omega'']^{\mathrm{T}}\right] \\ &+ \frac{\mathrm{sgn}(\omega')}{2\omega'} \mathrm{Tr}\left[\eta S_0[\omega'] \delta S^{(2)}[\omega, \omega'', -\omega']^{\mathrm{T}}\right] \Big).\end{aligned} \tag{98}$$

5.1. Moving Mirror

Now, using the definition of the first-order perturbation from Equation (48) and the second-order perturbation from Equation (63) for a moving $\delta - \delta'$ mirror one sees that Equation (98) becomes

$$\begin{aligned}\mathrm{Tr}\left[\eta \delta C^{(2)}_{q}[\omega, \omega']\right] = \int \frac{d\omega''}{2\pi} \Big(&\frac{\Theta(\omega'')}{2} \omega'' g[\omega - \omega''] g[\omega' + \omega''] \,\mathrm{Tr}\left[\eta \mathbb{S}^{(1)}_q[\omega, \omega''] \mathbb{S}^{(1)}_q[\omega', -\omega'']^{\mathrm{T}}\right] \\ &+ \frac{\mathrm{sgn}(\omega')}{4} \omega' g[\omega - \omega''] g[\omega' + \omega''] \,\mathrm{Tr}\left[\eta S_0[\omega'] \mathbb{S}^{(2)}_q[\omega, \omega'', -\omega']^{\mathrm{T}}\right] \Big).\end{aligned} \tag{99}$$

We find

$$\mathrm{Tr}\left[\eta \mathbb{S}^{(1)}_q[\omega, \omega''] \mathbb{S}^{(1)}_q[\omega', -\omega'']^{\mathrm{T}}\right] = 8i\lambda_0 \mu_0^3 \mathcal{Q}[\omega, \omega', \omega''] h(\omega) h(\omega') h(\omega'') h(-\omega''), \tag{100}$$

$$\mathrm{Tr}\left[\eta S_0[\omega'] \mathbb{S}^{(2)}_q[\omega, \omega'', -\omega']^{\mathrm{T}}\right] = 4i\lambda_0 \mu_0 (\omega + \omega') h(\omega) h(\omega'), \tag{101}$$

where

$$\mathcal{Q}[\omega, \omega', \omega''] = \left[1 + (\beta\omega'')^2\right](\omega + \omega') - i\beta(\omega - \omega'')(\omega' + \omega''). \tag{102}$$

Let us now express the second-order force from Equation (94) in terms of $\chi_q^{(2)}$:

$$\left\langle \delta F_q^{(2)}(t) \right\rangle = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i\omega t - i\omega' t} \chi_q^{(2)}[\omega, \omega', \omega''] g[\omega - \omega''] g[\omega' + \omega''], \tag{103}$$

where

$$\begin{aligned} \chi_q^{(2)}[\omega, \omega', \omega''] = {} & i\omega\omega' \lambda_0 \mu_0 h(\omega) h(\omega') \\ & \times \left[4\mu_0^2 \Theta(\omega'')\omega'' \mathcal{Q}[\omega, \omega', \omega''] h(\omega'') h(-\omega'') + \mathrm{sgn}(\omega')\omega'(\omega + \omega') \right], \end{aligned} \tag{104}$$

and thus

$$\left\langle \delta F_q^{(2)}[\omega] \right\rangle = \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \chi_q^{(2)}[\omega', \omega - \omega', \omega''] g[\omega' - \omega''] g[\omega - \omega' + \omega'']. \tag{105}$$

5.2. Mirror with Time-Dependent Properties

Now, using the definition of the first-order perturbation from Equation (41) and the second-order perturbation from Equation (51) for a $\delta - \delta'$ mirror with time dependent material properties Equation (98) becomes

$$\begin{aligned} \mathrm{Tr}\left[\eta \delta C_\mu^{(2)}[\omega, \omega'] \right] = & - \int \frac{d\omega''}{2\pi} \frac{\Theta(\omega'')}{2\omega''} \mu_0^2 h(\omega) h(\omega') f[\omega - \omega''] f[\omega' + \omega''] \,\mathrm{Tr}\left[\eta \mathbb{S}_\mu[\omega''] \mathbb{S}_\mu[-\omega'']^{\mathrm{T}} \right] \\ & - \int \frac{d\omega''}{2\pi} \frac{\mathrm{sgn}(\omega')}{2\omega'} \mu_0^2 h(\omega) h(\omega'') f[\omega - \omega''] f[\omega' + \omega''] \,\mathrm{Tr}\left[\eta S_0[\omega'] \mathbb{S}_\mu[-\omega']^{\mathrm{T}} \right] \end{aligned} \tag{106}$$

We find:

$$\mathrm{Tr}\left[\eta \mathbb{S}_\mu[\omega''] \mathbb{S}_\mu[-\omega'']^{\mathrm{T}} \right] = \frac{8\lambda_0 (1 + \lambda_0^2) \omega''^2}{\mu_0^2 + \omega''^2 (1 + \lambda_0^2)} = -8\lambda_0 (1 + \lambda_0^2) \omega''^2 h(\omega'') h(-\omega''), \tag{107}$$

$$\mathrm{Tr}\left[\eta S_0[\omega'] \mathbb{S}_\mu[-\omega']^{\mathrm{T}} \right] = \frac{4\lambda_0 \omega'}{i\mu_0 + \omega' (1 + \lambda_0^2)^2} = 4\omega' \lambda_0 h(\omega'), \tag{108}$$

Expressing the second-order force in terms of $\chi_\mu^{(2)}$:

$$\left\langle \delta F_\mu^{(2)}(t) \right\rangle = \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} e^{-i\omega t - i\omega' t} \chi_\mu^{(2)}[\omega, \omega', \omega''] f[\omega - \omega''] f[\omega' + \omega''], \tag{109}$$

where

$$\chi_\mu^{(2)}[\omega, \omega', \omega''] = 2\omega\omega' \lambda_0 \mu_0^2 h(\omega) h(\omega') h(\omega'') \left[2(1 + \lambda_0^2) \Theta(\omega'') \omega'' h(-\omega'') - \mathrm{sgn}(\omega') \right], \tag{110}$$

leads to

$$\left\langle \delta F_\mu^{(2)}[\omega] \right\rangle = \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} \chi_\mu^{(2)}[\omega', \omega - \omega', \omega''] f[\omega' - \omega''] f[\omega - \omega' + \omega'']. \tag{111}$$

While the first-order force calculation (see Equation (71)) agrees with that from Ref. [6], the corresponding second-order force calculation (see Equation (111)) does not. The susceptibility in Equation (110) differs substantially and includes an additional term dependant on $(1 + \lambda_0^2)$. We believe our derivation of the second-order force term is correct and indicates an issue in the corresponding calculation in Ref. [6].

5.3. Force from the Interference Effect

A system that possesses two distinct sources of time-dependent fluctuations experiences an interference effect due to the interaction between these two sources [9,21]. This interaction occurs as a second-order effect, as there is no first-order mixing term present. Thus, in addition to the independent force terms that are present at the second order, there is a mixing of the first-order perturbation terms, $\delta S_\mu^{(1)}[\omega,\omega']$ (see Equation (41)) and $\delta S_q^{(1)}[\omega,\omega']$ (see Equation (48)):

$$\left\langle \delta F_{\rm int}^{(2)}(t)\right\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi} e^{-i\omega t - i\omega' t}\omega\omega' \,\mathrm{Tr}\left[\eta \delta C_{\rm int}^{(2)}[\omega,\omega']\right], \tag{112}$$

where

$$\begin{aligned}\delta C_{\rm int}^{(2)}[\omega,\omega'] = \int \frac{\mathrm{d}\omega''}{2\pi}\int \frac{\mathrm{d}\omega'''}{2\pi}\big(&\delta S_\mu^{(1)}[\omega,\omega'']C_{\rm in}[\omega'',\omega''']\delta S_q^{(1)}[\omega',\omega''']^{\rm T} \\ &+\delta S_q^{(1)}[\omega,\omega'']C_{\rm in}[\omega'',\omega''']\delta S_\mu^{(1)}[\omega',\omega''']^{\rm T}\big).\end{aligned} \tag{113}$$

With $C_{\rm in}[\omega,\omega'] = 2\pi\delta(\omega+\omega')c_{\rm in}[\omega]$, as in Equation (12), this becomes

$$\delta C_{\rm int}^{(2)}[\omega,\omega'] = \int \frac{\mathrm{d}\omega''}{2\pi}\left(\delta S_\mu^{(1)}[\omega,\omega'']c_{\rm in}[\omega'']\delta S_q^{(1)}[\omega',-\omega'']^{\rm T} + \delta S_q^{(1)}[\omega,\omega'']c_{\rm in}[\omega'']\delta S_\mu^{(1)}[\omega',-\omega'']^{\rm T}\right), \tag{114}$$

which can be further simplified with the definition of $c_{\rm in}[\omega]$ as in Equation (13) to obtain

$$\delta C_{\rm int}^{(2)}[\omega,\omega'] = \int \frac{\mathrm{d}\omega''}{2\pi}\frac{\Theta(\omega'')}{2\omega''}\left(\delta S_\mu^{(1)}[\omega,\omega'']\delta S_q^{(1)}[\omega',-\omega'']^{\rm T} + \delta S_q^{(1)}[\omega,\omega'']\delta S_\mu^{(1)}[\omega',-\omega'']^{\rm T}\right). \tag{115}$$

Now, to simplify this expression, we make a change of variables in the second term of Equation (115). We take $\omega'' \to -\omega''$ and swap $\omega \leftrightarrow \omega'$ to arrive at

$$\delta C_{\rm int}^{(2)}[\omega,\omega'] = \int \frac{\mathrm{d}\omega''}{2\pi}\left(\frac{\Theta(\omega'')}{2\omega''}\delta S_\mu^{(1)}[\omega,\omega'']\delta S_q^{(1)}[\omega',-\omega'']^{\rm T} + \frac{\Theta(-\omega'')}{2\omega''}\delta S_q^{(1)}[\omega',-\omega'']\delta S_\mu^{(1)}[\omega,\omega'']^{\rm T}\right) \tag{116}$$

Using the properties of the trace, we find:

$$\mathrm{Tr}\left[\eta\delta C_{\rm int}^{(2)}[\omega,\omega']\right] = \int \frac{\mathrm{d}\omega''}{2\pi}\frac{1}{2\omega''}\,\mathrm{Tr}\left[\eta \delta S_\mu^{(1)}[\omega,\omega'']\delta S_q^{(1)}[\omega',-\omega'']^{\rm T}\right], \tag{117}$$

where the identity $\Theta(\omega)+\Theta(-\omega) = 1$ is used. Now, using the definition of the two first-order perturbation terms $\delta S_\mu^{(1)}[\omega,\omega']$ (see Equation (41)) and $\delta S_q^{(1)}[\omega,\omega']$ (see Equation (48)) we find:

$$\mathrm{Tr}\left[\eta\delta C_{\rm int}^{(2)}[\omega,\omega']\right] = -\int \frac{\mathrm{d}\omega''}{2\pi}\frac{\mu_0}{2}h(\omega)f[\omega-\omega'']g[\omega'+\omega'']\,\mathrm{Tr}\left[\eta\mathbb{S}_\mu[\omega'']\mathbb{S}_q^{(1)}[\omega',-\omega'']^{\rm T}\right]. \tag{118}$$

The trace term becomes

$$\mathrm{Tr}\left[\eta\mathbb{S}_\mu[\omega'']\mathbb{S}_q^{(1)}[\omega',-\omega'']^{\rm T}\right] = 4i\omega'' h(\omega')h(\omega'')h(-\omega'')\mathcal{I}[\omega,\omega',\omega''], \tag{119}$$

where

$$\mathcal{I}[\omega,\omega',\omega''] = i\mu_0^2(1+\lambda_0^2) + \mu_0(1-\lambda_0^2)^2\omega' - 4i\lambda_0^2(1+\lambda_0^2)\omega'\omega'' \tag{120}$$

We can now express the second-order force due to the interference of the two sources (see Equation (112)) in terms of $\chi_{\rm int}^{(2)}$:

$$\left\langle \delta F_{\rm int}^{(2)}(t)\right\rangle = \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi} e^{-i\omega t - i\omega' t}\chi_{\rm int}^{(2)}[\omega,\omega',\omega'']f[\omega-\omega'']g[\omega'+\omega''], \tag{121}$$

where

$$\chi_{\rm int}^{(2)}[\omega,\omega',\omega''] = 2i\mu_0\omega\omega'\omega'' h(\omega)h(\omega')h(\omega'')h(-\omega'')\mathcal{I}[\omega,\omega',\omega'']. \tag{122}$$

The force for the interference term now reads

$$\left\langle \delta F_{\rm int}^{(2)}[\omega]\right\rangle = \int \frac{\mathrm{d}\omega'}{2\pi}\int \frac{\mathrm{d}\omega''}{2\pi}\chi_{\rm int}^{(2)}[\omega',\omega-\omega',\omega'']f[\omega'-\omega'']g[\omega-\omega'+\omega'']. \tag{123}$$

6. Conclusions

When a mirror in a vacuum undergoes time-dependent fluctuations, it produces real particles via the dynamical Casimir effect. In the case when such a mirror possesses asymmetric boundary properties, its spectrum of particles is also be asymmetric. This asymmetry in particle production results in a perturbation in the position of the mirror; that is, the imbalance in radiation reaction forces results in induced motion of the mirror [6–8]. The vacuum, in turn, acts as a dissipative medium and resists the motion of the mirror, which is described in part by fluctuation–dissipation theory [18]. Here, we have used fluctuation–dissipation theory to calculate the vacuum-induced response to the time-dependent fluctuations of an asymmetric $\delta - \delta'$ mirror which is both moving and possesses time-dependent material properties. We find that the resulting forces are both dissipative and motion inducing, since the asymmetry in particle production generates a secondary force in addition to the dissipative force of the vacuum, which seeks to suppress the motion of the mirror.

The linear susceptibility, used to calculate the mean force, is calculated to the first- and second-order for both the contribution from the motion of the moving mirror and from the time-dependent boundary conditions. For the first-order, we are able to provide exact results for the susceptibility as well as expansions in the limits $\beta\omega \gg 1$ and $\beta\omega \ll 1$. We plot the resulting force numerically for different values of λ_0 and compare them to the purely dissipative force of a moving Dirichlet mirror. Additionally, we have also looked at the resulting changes to the force when the fluctuation oscillation frequency is increased. The resulting second-order forces are calculated, which also include a mixed interference term in addition to the second-order contributions from the two separate fluctuation sources.

Thus far, our numerical analysis has been restricted to first-order forces with only a single type of oscillation. In the future, we wish to extend these numerical methods to a higher order in force and study other physically interesting types of fluctuations, which would cause novel interactions with the vacuum. Additionally, these methods could be used to analyze work and impulse delivered to the Casimir system, allowing us to determine optimal system configurations along with parameters for generating motion.

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