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Analysis of Coefficient-Related Problems for Starlike Functions with Symmetric Points Connected with a Three-Leaf-Shaped Domain

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Abstract: The basic aspect of the research on coefficient problems for numerous families of univalent functions is to describe the coefficients of functions in a specific family by the coefficients of the Carathéodory functions. Thus, in utilizing the inequalities that are known for the class of Carathéodory functions, coefficient functionals may be examined. Several coefficient problems will be addressed in this study by utilizing the methodology for the abovementioned functions' family. The family of starlike functions with respect to symmetric points connected to a three-leaf-shaped image domain is the topic of our investigation.

Keywords: starlike functions with symmetrical points; coefficient bounds; Krushkal and Zalcman inequalities; Hankel determinant of order three

MSC: 30C45; 30C50

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1. Introduction and Definitions

To give a complete understanding of the main results given in this paper, the basic terminology that are used throughout in our key findings are outlined, and some preliminary definitions followed by related results are discussed here. We begin with presenting the most basic symbol for a unit disc that is open with $\mathbb{U}_d = \{z \in \mathbb{C} : |z| < 1\}$, and we will use $\mathcal{A}$ to indicate the group of those analytic functions that have been normalized by $g(0) = g'(0) - 1 = 0$. This signifies that $g \in \mathcal{A}$, that is, every function of this group of functions can be written as follows by using the Taylor's series expansion

$$g(z) = z + \sum_{j=1}^{\infty} d_j z^j, \ \ z \in \mathbb{U}_d. \tag{1}$$

To represent the group of univalent functions in $\mathcal{A}$, we use the symbol $\mathcal{S}$. This family of functions was developed by Köebe in 1907.

In 1916, Bieberbach [1] earned the credit of stating one of the most popular and used results of GFT, which is known as the "Bieberbach conjecture". This conjecture states that if $g \in \mathcal{S}$, then $|d_n| \leq n$ for every $n \geq 2$. He gave his contribution by proving this stated problem for one particular value, $n = 2$. It is evident that several well-known researchers kept providing their input to prove interesting theories related to this unproved result, and, for that, they used diversified approaches. This helped in the overall development of GFT to a great extent. We will list the contributions of a few of them here. For example, for $n = 3$, the conjecture was proved in the remarkable work of Löwner [2], who used Löwner differential equations followed by the other two well known researchers, Schaeffer and Spencer [3], who used the variational method. Afterward, Jenkins [4] also proved the same result, that is, the coefficient inequality $|d_3| \leq 3$, but he proved it by using quadratic differentials. Garabedian and Schiffer [5] then continued this chain of proving the related results, and they used the same variational technique but advanced the research by determining the next results, that is, $|d_4| \leq 4$. Pederson and Schiffer [6] were the ones who proved that the fifth coefficient in the aforementioned conjecture is less than or equal to 5 by using the well-known Garabedian–Schiffer inequality ([7] p. 108). This sequence of successful proofs by numerous authors continued, and then Pederson [8] and Ozawa [9,10] gave the next level results that proved the "Bieberbach conjecture", which was stated for all $n \geq 2$ and for $n \geq 6$, that is, $|d_6| \leq 6$. They achieved it by using Grunsky inequality ([7] p. 60). For some time, then, we see that no result was presented in any research paper to show the proof for $n \geq 7$. This conjecture remained unsolved for any other value of n in particular, or as a general proof . Ultimately, it was then that de-Branges [11], who took the credit in 1985, proved this well-known conjecture—which had been unsolved for a longer period of time—for every $n \geq 2$. He completed this remarkable piece of research with the help of one of the special functions known as hypergeometric functions.

In an attempt to solve the above problem between the years 1916 and 1985, many other interesting results were presented by numerous researchers, which ultimately gave a boost to research in GFT. Some of those were the calculations of the estimates of the nth coefficient bounds meant for a number of sub-collections of the family of univalent functions. To name a few, we also had starlike functions represented by $\mathcal{S}^*$, convex functions denoted by $\mathcal{C}$, close-to-convex functions known as $\mathcal{K}$, etc. Some of the fundamental families are defined below:

$$\begin{aligned} \mathcal{S}^* &= \left\{ g \in \mathcal{S} : \Re \frac{zg'(z)}{g(z)} > 0, \ \ (z \in \mathbb{U}_d) \right\}, \\ \mathcal{C} &= \left\{ g \in \mathcal{S} : \Re \frac{(zg'(z))'}{g'(z)} > 0, \ \ (z \in \mathbb{U}_d) \right\}, \\ \mathcal{K} &= \left\{ g \in \mathcal{S} : \Re \frac{zg'(z)}{h(z)} > 0 \text{ with } h \in \mathcal{S}^* \ (z \in \mathbb{U}_d) \right\}. \end{aligned}$$

By choosing special values for these general parameters, we obtained some other sub-collections with interesting geometrical properties. For example, if we select $h(z) = z$,—i.e., the close to convex family, which is represented by $\mathcal{K}$—it becomes the collection of functions for bounded turning. This special group of functions is represented by the symbol $\mathcal{BT}$. The notable contribution by the authors [12] in 1992 was the consideration that a function ϕ, which is univalent in the domain that is an open unit disc and that satisfies the properties $\phi'(0) > 0$, is also $\Re\phi > 0$. The interesting geometric property of the region $\phi(\mathbb{U}_d)$ is that it is star-shaped around the fixed point $\phi(0) = 1$. Its axis of symmetry is the real line. Continuing on the same lines, the authors defined the unified sub-collection of the class $\mathcal{S}$ by using the idea of subordination as follows.

$$\mathcal{S}^*(\phi) = \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec \phi(z), \ \ (z \in \mathbb{U}_d) \right\}.$$

The authors kept their focus on some very basic and important results, all of which were based on the geometrical properties of these functions. Some of them were covering, growth of function, and/or distortion theorems. During the past few years, we have observed in the literature that various sub-collections of the collection of univalent functions $\mathcal{S}$ have been thoroughly studied as specific options for the class $\mathcal{S}^*(\phi)$. Inspired by the remarkable vital research in this direction, we list a few of these subfamilies that have been discovered lately.

(i). $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*(\sqrt{1+z})$ [13], $\mathcal{S}^*_{car} \equiv \mathcal{S}^*\left(1+\frac{2}{3}z+\frac{1}{3}z^2\right)$ [14] , $\mathcal{S}^*_{\exp} \equiv \mathcal{S}^*(\exp(z))$ [15],
(ii). $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos(z))$ [16], $\mathcal{S}^*_{\sin} \equiv \mathcal{S}^*(1+\sin(z))$ [17], $\mathcal{S}^*_{pet} \equiv \mathcal{S}^*\left(1+\sinh^{-1} z\right)$ [18],
(iii). $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh(z))$ [19], $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1+\tanh(z))$ [20], $\mathcal{S}^*_{c} \equiv \mathcal{S}^*(1+z+\frac{1}{2}z^2)$ [21],
(iv). $\mathcal{S}^*_{(n-1)\mathcal{L}} \equiv \mathcal{S}^*(\Psi_{n-1}(z))$ [22] with $\Psi_{n-1}(z) = 1+\frac{n}{n+1}z+\frac{1}{n+1}z^n$ for $n \geq 4$.

We now give a very important determinant denoted by $\mathcal{D}_{\lambda,n}(g)$ with $n,\lambda \in \mathbb{N} = \{1,2,\ldots\}$. This determinant is named after Hankel and consists of the coefficients of the function g, which is an element of $\mathcal{S}$

$$\mathcal{D}_{\lambda,n}(g) = \begin{vmatrix} d_n & d_{n+1} & \ldots & d_{n+\lambda-1} \\ d_{n+1} & d_{n+2} & \ldots & d_{n+\lambda} \\ \vdots & \vdots & \ldots & \vdots \\ d_{n+\lambda-1} & d_{n+\lambda} & \ldots & d_{n+2\lambda-2} \end{vmatrix}.$$

The above equation was provided by Pommerenke [23,24]. The Hankel determinants have extensively been used in many technological studies, especially where mathematical tools come into consideration. They are used in the theory of non-stationary signals in the Hamburger moment problem, the theory of Markov processes, and in many others, and these can be accessed from [25–27].

The first and second determinants mentioned above have been thoroughly utilized by researchers in a number of articles. They have been particularly studied in the perspectives of various sub-collections of univalent functions. It would be unjust not to mention the contributions provided by the researchers [28–31]. This piece of work is important to highlight because, in these articles, the authors calculated the sharp bounds for the second Hankel determinant. More interesting results on this determinant can be seen in the articles of [32–36].

The most challenging problem to study is the above third-order determinant, especially in finding its sharp bounds. Although there are several papers on the investigation of the non-sharp bounds of this determinant, we cite here a few of them. (See [37–42].) In fact, Babalola was the very first person to study the bounds of the third-order determinant for the $\mathcal{K}$, $\mathcal{S}^*$ and $\mathcal{BT}$ families in a paper [43] that surfaced in 2010. After that, with the use of a novel technique, Zaprawa [44] enhanced Babalola's findings in 2017. He proved the following non-sharp bounds

$$|\mathcal{D}_{3,1}(g)| \leq \begin{cases} \frac{49}{540}, & \text{for} \quad g \in \mathcal{C}, \\ 1, & \text{for} \quad g \in \mathcal{S}^*, \\ \frac{41}{60}, & \text{for} \quad g \in \mathcal{BT}. \end{cases}$$

Following that, certain scientists have worked hard to prove the sharp bounds for these inequalities, and some of them [45,46] were successful in obtaining improved bounds for the class $\mathcal{S}^*$. The sharp bounds of this determinant were finally obtained for classes $\mathcal{C}$, $\mathcal{S}^*$, and $\mathcal{BT}$ in the articles [47,48], and [49], respectively. These sharp bounds are

$$|\mathcal{D}_{3,1}(g)| \leq \begin{cases} \frac{4}{135}, & \text{for} \quad g \in \mathcal{C}, \\ \frac{4}{9}, & \text{for} \quad g \in \mathcal{S}^*, \\ \frac{1}{4}, & \text{for} \quad g \in \mathcal{BT}. \end{cases}$$

The sharp bounds for the abovementioned subclass of starlike functions $\mathcal{S}^*(\phi)$ have been found by many researchers with different values of the function ϕ. Some of the recent developments are listed in Table 1.

Table 1. Sharp bounds on $|\mathcal{D}_{3,1}(g)|$ for some subclasses of $\mathcal{S}^*$.

Author/s	$\phi(z)$	Sharp Bound	Year	Reference
B. Rath et al.	$\frac{1}{1-z}$	$1/9$	2022	[50]
S. Banga and S.S. Kumar	$\sqrt{1+z}$	$1/36$	2020	[51]
K. Ullah et al.	$1+\tanh(z)$	$1/9$	2021	[52]
Shi et al.	$1+\sin(z)$	$1/9$	2022	[53]
Riaz et al.	$\frac{2}{1+e^{-z}}$	$1/36$	2022	[54]
V. Neha and S.S Kumar.	$1+ze^z$	$1/9$	2022	[55]
Z.-G Wang et al.	$1+\sinh^{-1}(z)$	$1/9$	2023	[56]

Using the same methodology, Lecko et al. [57] computed the sharp bounds of $|\mathcal{D}_{3,1}(g)|$ for the functions belonging to the family $\mathcal{S}^*(1/2)$. (We recommend the much appreciated work by[58–65].) In some of these articles, the authors proved the sharp bounds of the third-order Hankel determinant, and they performed this for the various sub-collections of univalent functions.

In [22], a subclass of starlike functions was introduced by Gandhi as follows

$$\mathcal{S}^*_{3l} = \left\{ g \in \mathcal{S} : \frac{zg'(z)}{g(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (z \in \mathbb{U}_d) \right\}.$$

Motivated by the last definition, we now introduce the class $\mathcal{S}^*_{3l,s}$ of starlike functions with respect to the symmetric points associated with the three-leaf-shaped region, which is given by

$$\mathcal{S}^*_{3l,s} = \left\{ g \in \mathcal{S} : \frac{2zg'(z)}{g(z) - g(-z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \qquad (z \in \mathbb{U}_d) \right\}. \tag{2}$$

In this article, our focus is the computation of the sharp estimates of the coefficients d_n with $n = 2, \ldots, 5$, as well as the Fekete-Szegö, Zalcman, and Krushkal inequalities for the class $\mathcal{S}^*_{3l,s}$ with respect to the symmetric points linked with a three-leaf-shaped domain. Furthermore, the estimates of $|\mathcal{D}_{2,2}(g)|$, $|\mathcal{D}_{2,3}(g)|$, and $|\mathcal{D}_{3,1}(g)|$ were also obtained for the same class.

2. A Set of Lemmas

Let $\mathcal{P}$ represent the class of all functions p that are regular in $\mathbb{U}_d$ with $\Re(p(z)) > 0$, and which has the series representation given below

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{U}_d). \tag{3}$$

Lemma 1. *Let $p \in \mathcal{P}$ be given by (3). Then*

$$|c_p| \leq 2 \text{ for } p \geq 1. \tag{4}$$

and

$$|c_{p+q} - \delta c_p c_q| \leq 2\max\{1, |2\delta - 1|\} = \begin{cases} 2 & \text{for} \quad \delta \in [0,1]; \\ 2|2\delta - 1| & \quad \text{otherwise}. \end{cases} \tag{5}$$

Also, If $B \in [0,1]$ with $B(2B-1) \leq D \leq B$, we achieve

$$\left| c_3 - 2Bc_1c_2 + Dc_1^3 \right| \leq 2. \tag{6}$$

The inequalities (4), (5) and (6) are taken from [7,66] and [67] respectively.

Lemma 2 ([68])**.** *If a, γ, α, and β satisfy $a \in (0,1)$ and $\alpha \in (0,1)$ with*

$$\left((-\beta + \alpha(\alpha + a))^2 + (-2\gamma + \beta\alpha)^2 \right) 8(1-a)a + (-2a\alpha + \beta)^2(1-\alpha)\alpha \leq 4\alpha^2(1-\alpha)^2(1-a)a. \tag{7}$$

Let $p \in \mathcal{P}$ be given by (3). Then

$$\left|\gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4\right| \leq 2.$$

Lemma 3. *If $p \in \mathcal{P}$ be given by (3), then for $x, \varsigma, \rho \in \overline{\mathbb{U}_d}$, we have*

$$2c_2 = \left(4 - c_1^2\right)x + c_1^2, \tag{8}$$

$$4c_3 = 2\left(4 - c_1^2\right)xc_1 - x^2\left(4 - c_1^2\right)c_1 + 2\varsigma\left(1 - |x|^2\right)\left(4 - c_1^2\right) + c_1^3, \tag{9}$$

$$\begin{aligned} 8c_4 = {} & \left[c_1^2\left(-3x + x^2 + 3\right) + 4x\right]\left(4 - c_1^2\right)x - 4\left(1 - |x|^2\right)\left(4 - c_1^2\right) \\ & \left[(x-1)\varsigma c + \varsigma^2\overline{x} - \rho\left(1 - |\varsigma|^2\right)\right] + c_1^4. \end{aligned} \tag{10}$$

The formulae c_2, c_3, and c_4 are studied in [7], [69], and [70], respectively.

3. Coefficient Inequalities

First, we can study the upper estimates up to the fifth coefficient d_5 for $g \in \mathcal{S}^*_{3l,s}$.

Theorem 1. *If $g \in \mathcal{S}^*_{3l,s}$ has the series expansion (1), then*

$$|d_2| \leq \frac{2}{5}, \tag{11}$$

$$|d_3| \leq \frac{2}{5}, \tag{12}$$

$$|d_4| \leq \frac{1}{5}, \tag{13}$$

$$|d_5| \leq \frac{1}{5}. \tag{14}$$

These outcomes are sharp.

Proof. Let $g \in \mathcal{S}^*_{3l,s}$, then (2), if written in the form of Schwarz function, has the following form

$$\frac{2zg'(z)}{g(z) - g(-z)} = 1 + \frac{4}{5}w(z) + \frac{1}{5}(w(z))^4, \qquad (z \in \mathbb{U}_d).$$

If a function $p \in \mathcal{P}$, then we can write it in terms of Schwarz function $w(z)$ as

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{15}$$

or, correspondingly, as

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}. \tag{16}$$

Using Equation (1), it follows that

$$\begin{aligned} \frac{2zg'(z)}{g(z) - g(-z)} = {} & 1 + 2d_2 z + 2d_3 z^2 + (4d_4 - 2d_2 d_3)z^3 \\ & + \left(4d_5 - 2d_3^2\right)z^4 + \cdots. \end{aligned} \tag{17}$$

By simplification and using the series expansion of (16) , we obtain

$$
\begin{aligned}
1+\frac{4}{5}w(z)+\frac{1}{5}w(z)^4 &= 1+\left(\frac{2}{5}c_1\right)z+\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right)z^2+\left(\frac{1}{10}c_1^3-\frac{2}{5}c_1c_2+\frac{2}{5}c_3\right)z^3 \\
&\quad+\left(-\frac{3}{80}c_1^4+\frac{3}{10}c_1^2c_2-\frac{1}{5}c_2^2-\frac{2}{5}c_1c_3+\frac{2}{5}c_4\right)z^4+\cdots. \qquad (18)
\end{aligned}
$$

In comparing (17) and (18), we obtain

$$d_2 = \frac{1}{5}c_1, \qquad (19)$$

$$d_3 = \frac{1}{2}\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right), \qquad (20)$$

$$d_4 = \frac{3}{200}c_1^3-\frac{2}{25}c_1c_2+\frac{1}{10}c_3, \qquad (21)$$

$$d_5 = -\frac{3}{100}c_2^2+\frac{11}{200}c_1^2c_2-\frac{7}{1600}c_1^4-\frac{1}{10}c_1c_3+\frac{1}{10}c_4. \qquad (22)$$

For d_2, implementing (4) in (19), we obtain

$$|d_2| \le \frac{2}{5}.$$

For d_3, by reordering (20), we obtain

$$d_3 = \frac{1}{5}\left(c_2-\frac{1}{2}c_1c_1\right).$$

Using (5), we have

$$|d_3| \le \frac{2}{5}.$$

For d_4, we can write (21) as

$$|d_4| = \frac{1}{10}\left|\left(c_3-2\left(\frac{2}{5}\right)c_1c_2+\frac{3}{20}c_1^3\right)\right|.$$

From (6), let

$$B=\frac{2}{5} \quad \text{and} \quad D=\frac{3}{20}.$$

It is clear that $0 \le B \le 1$, and $B \ge D$ with

$$B(2B-1) = -\frac{2}{25} \le D.$$

Thus, all the conditions of (6) are satisfied. Hence, we have

$$|d_4| \le \frac{1}{5}.$$

For d_5, we can rewrite (22) as

$$
\begin{aligned}
d_5 &= -\frac{1}{10}\left(\frac{7}{160}c_1^4+\left(\frac{3}{10}\right)c_2^2+2\left(\frac{1}{2}\right)c_1c_3-\frac{3}{2}\left(\frac{11}{30}\right)c_1^2c_2-c_4\right) \\
&= -\frac{1}{10}\left(\gamma c_1^4+dc_2^2+2\alpha c_1c_3-\frac{3}{2}\beta c_1^2c_2-c_4\right), \qquad (23)
\end{aligned}
$$

where

$$\gamma=\frac{7}{160}, \quad a=\frac{3}{10}, \quad \alpha=\frac{1}{2}, \quad \beta=\frac{11}{30},$$

are such that

$$\left((-\beta+\alpha(\alpha+a))^2+(-2\gamma+\beta\alpha)^2\right)8(1-a)a+(-2a\alpha+\beta)^2(1-\alpha)\alpha\leq 4a\alpha^2(1-\alpha)^2(1-a),$$

$a\in(0,1)$, $\alpha\in(0,1)$; therefore, by (7) and (23), we have

$$|d_5|\leq\frac{1}{5}.$$

These results are sharp and equality is achieved from the following functions

$$\begin{aligned}\frac{2zg'(z)}{g(z)-g(-z)} &= 1+\frac{4}{5}z+\frac{1}{5}z^4+\cdots,\\ \frac{2zg'(z)}{g(z)-g(-z)} &= 1+\frac{4}{5}z^2+\frac{1}{5}z^8+\cdots,\\ \frac{2zg'(z)}{g(z)-g(-z)} &= 1+\frac{4}{5}z^3+\frac{1}{5}z^{12}+\cdots,\\ \frac{2zg'(z)}{g(z)-g(-z)} &= 1+\frac{4}{5}z^4+\frac{1}{5}z^{16}+\cdots.\end{aligned}$$

The required proof is thus accomplished. □

Theorem 2. *If $g\in\mathcal{S}^*_{3l,s}$, then*

$$\left|d_3-\delta d_2^2\right|\leq\max\left\{\frac{2}{5},\frac{2|\delta|}{25}\right\},\ \textit{for}\ \delta\in\mathbb{C}.$$

The outcome is sharp.

Proof. By putting (19) and (20), we obtain

$$\left|d_3-\delta d_2^2\right|=\left|\frac{1}{5}c_2-\frac{1}{10}c_1^2-\delta\frac{1}{25}c_1^2\right|.$$

The application of (5) leads us to

$$\left|d_3-\delta d_2^2\right|\leq\frac{1}{5}\max\left\{2,2\left|\left(\frac{5+2\delta}{5}\right)-1\right|\right\}.$$

After the simplification, we obtain

$$\left|d_3-\delta d_2^2\right|\leq\max\left\{\frac{2}{5},\frac{2|\delta|}{25}\right\}.$$

This outcome is best possible and is obtained by

$$\frac{2zg'(z)}{g(z)-g(-z)}=1+\frac{4}{5}z^2+\frac{1}{5}z^8+\cdots.$$

□

Theorem 3. *If g belongs to $\mathcal{S}^*_{3l,s}$ and is given by (1). Then*

$$|d_2d_3-d_4|\leq\frac{1}{5}.$$

This inequality is the best possible.

Proof. By putting (19)–(21), we have

$$|d_2d_3 - d_4| = \frac{1}{10}\left|c_3 - 2\left(\frac{3}{5}\right)c_1c_2 + \frac{7}{20}c_1^3\right|.$$

From (6), we have

$$0 \leq B = \frac{3}{5} \leq 1, B = \frac{3}{5} \geq D = \frac{7}{20}$$

and

$$B(2B-1) = \frac{3}{25} \leq D = \frac{7}{20}.$$

Using (6), we obtain

$$|d_2d_3 - d_4| \leq \frac{1}{5}.$$

This outcome is sharp. Equality is achieved from

$$\frac{2zg'(z)}{g(z) - g(-z)} = 1 + \frac{4}{5}z^3 + \frac{1}{5}z^{12} + \cdots.$$

□

We can now calculate the determinant $\mathcal{D}_{2,2}(g)$ for $g \in \mathcal{S}^*_{3l,s}$.

Theorem 4. *If $g \in \mathcal{S}^*_{3l,s}$ and has the form (1), then*

$$|\mathcal{D}_{2,2}(g)| = \left|d_2d_4 - d_3^2\right| \leq \frac{4}{25}.$$

This outcome is the best possible.

Proof. From (19)—(21), we have

$$\mathcal{D}_{2,2}(g) = \frac{3}{125}c_1^2c_2 + \frac{1}{50}c_1c_3 - \frac{7}{1000}c_1^4 - \frac{1}{25}c_2^2.$$

By applying (8) and (9) to write c_2 and c_3 in terms of c_1 and observing that we can write $c_1 = c$, we achieve

$$\begin{aligned} |\mathcal{D}_{2,2}(g)| &= \left|\frac{1}{500}\left(4-c^2\right)c^2x - \frac{1}{200}\left(4-c^2\right)c^2x^2 - \frac{1}{100}\left(4-c^2\right)^2x^2\right. \\ &\quad \left. + \frac{1}{100}c\left(4-c^2\right)\left(1-|x|^2\right)\varsigma\right|, \end{aligned}$$

By invoking $|x| = t$, $|\varsigma| \leq 1$ with $t \leq 1$ we have the following form if we use triangular inequality to simplify

$$\begin{aligned} |\mathcal{D}_{2,2}(g)| &\leq \left|\frac{1}{500}\left(4-c^2\right)c^2t + \frac{1}{200}\left(4-c^2\right)c^2t^2 + \frac{1}{100}\left(4-c^2\right)^2t^2\right. \\ &\quad \left. + \frac{1}{100}c\left(4-c^2\right)\left(1-t^2\right)\right| := \varphi(c,t). \end{aligned}$$

It is now a straightforward task to illustrate that $\varphi'(c,t) \geq 0$ on $[0,1]$, and hence $\varphi(c,t) \leq \varphi(c,1)$. Thus,

$$|\mathcal{D}_{2,2}(g)| \leq \frac{7}{1000}c^2\left(4-c^2\right) + \frac{1}{100}\left(4-c^2\right)^2 := \varphi(c,1).$$

Without many complicated calculations, it follows that $\varphi(c,1)$ obtains its maxima at 0. Hence,

$$|\mathcal{D}_{2,2}(g)| \leq \frac{4}{25}.$$

The required $\mathcal{D}_{2,2}(g)$ is sharp, and equality is achieved from

$$\frac{2zg'(z)}{g(z)-g(-z)} = 1 + \frac{4}{5}z^2 + \frac{1}{5}z^8 + \cdots.$$

□

Theorem 5. *If $g \in \mathcal{S}^*_{3l,s}$, then*

$$|d_5 - d_2 d_4| \leq \frac{1}{5}.$$

The outcome is sharp.

Proof. From (19), (21), and (22), we obtain

$$|d_5 - d_2 d_4| = \left| \frac{71}{1000}c_1^2 c_2 - \frac{3}{25}c_1 c_3 - \frac{59}{8000}c_1^4 - \frac{3}{100}c_2^2 + \frac{1}{10}c_4 \right|.$$

After simplifying, we have

$$|d_5 - d_2 d_4| = \frac{1}{10}\left| \frac{59}{800}c_1^4 + \frac{3}{10}c_2^2 + 2\left(\frac{3}{5}\right)c_1 c_3 - \frac{3}{2}\left(\frac{71}{150}\right)c_1^2 c_2 - c_4 \right|. \tag{24}$$

Comparing the right side of (24) with

$$\left| \gamma c_1^4 + a c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2}\beta c_1^2 c_2 - c_4 \right|,$$

we obtain

$$\gamma = \frac{59}{800}, \quad a = \frac{3}{10}, \quad \alpha = \frac{3}{5}, \quad \beta = \frac{71}{150}.$$

Thus, it follows that

$$\left((-\beta + \alpha(\alpha + a))^2 + (-2\gamma + \beta\alpha)^2 \right) 8(1-a)a + (-2a\alpha + \beta)^2 (1-\alpha)\alpha = 0.04185$$

and

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.048384.$$

From (7), we deduce that

$$|d_5 - d_2 d_4| \leq \frac{1}{5}.$$

This outcome is sharp and equality is attained from

$$\frac{2zg'(z)}{g(z)-g(-z)} = 1 + \frac{4}{5}z^4 + \frac{1}{5}z^{16} + \cdots.$$

□

Theorem 6. *If $g \in \mathcal{S}^*_{3l,s}$ be given by (1), then*

$$\left| d_5 - d_3^2 \right| \leq \frac{1}{5}.$$

This is the finest possible inequality.

Proof. Using (20) and (22), we obtain

$$\left|d_5 - d_3^2\right| = \left|-\frac{7}{100}c_2^2 + \frac{19}{200}c_1^2c_2 - \frac{23}{1600}c_1^4 - \frac{1}{10}c_1c_3 + \frac{1}{10}c_4\right|.$$

After simplifying, we have

$$\left|d_5 - d_3^2\right| = \frac{1}{10}\left|\frac{23}{160}c_1^4 + \frac{7}{10}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{19}{30}\right)c_1^2c_2 - c_4\right|. \tag{25}$$

In comparing the right side of (25) with

$$\left|\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right|,$$

where

$$\gamma = \frac{23}{160}, \quad a = \frac{7}{10}, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{19}{30},$$

it follows that

$$\left((-\beta + \alpha(\alpha + a))^2 + (-2\gamma + \beta\alpha)^2\right)8(1-a)a + (-2a\alpha + \beta)^2(1-\alpha)\alpha = 0.004406$$

and

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.05250.$$

From (7), we deduce that

$$\left|d_5 - d_3^2\right| \le \frac{1}{5}.$$

This inequality is sharp and is attained by

$$\frac{2zg'(z)}{g(z) - g(-z)} = 1 + \frac{4}{5}z^4 + \frac{1}{5}z^{16} + \cdots.$$

□

4. Krushkal Inequalities

This section contains an important result where we give a direct proof of the following result

$$\left|d_n^p - d_2^{p(n-1)}\right| \le 2^{p(n-1)} - n^p,$$

particularly for the class $\mathcal{S}^*_{3l,s}$ with the forthcoming values of parameters, i.e., $n = 4$, $p = 1$, etc., for $n = 5$ and $p = 1$. Krushkal discussed this abovementioned result along with its proof for the whole collection of univalent functions in his article [71].

Theorem 7. *If $g \in \mathcal{S}^*_{3l,s}$ and is given by (1), then*

$$\left|d_4 - d_2^3\right| \le \frac{1}{5}.$$

The outcome of this is sharp.

Proof. By putting (19) and (21), we have

$$\left|d_4 - d_2^3\right| = \frac{1}{10}\left|c_3 - 2\left(\frac{2}{5}\right)c_1c_2 + \frac{7}{100}c_1^3\right|.$$

From (6), let

$$B = \frac{2}{5} \quad \text{and} \quad D = \frac{7}{100},$$

and let $0 \leq B \leq 1$ and $B \geq D$ be with

$$B(2B-1) = -\frac{2}{25} \leq D.$$

Thus, all the conditions of (6) are satisfied. Hence, we have

$$\left|d_4 - d_2^3\right| \leq \frac{1}{5}.$$

This equality is attained from

$$\frac{2zg'(z)}{g(z)-g(-z)} = 1 + \frac{4}{5}z^3 + \frac{1}{5}z^{12} + \cdots.$$

□

Theorem 8. *If $g \in \mathcal{S}^*_{3l,s}$ and is given by (1), then*

$$\left|d_5 - d_2^4\right| \leq \frac{1}{5}.$$

This outcome is sharp.

Proof. From (19) and (22), we obtain

$$\left|d_5 - d_2^4\right| = \left|-\frac{239}{40000}c_1^4 - \frac{3}{100}c_2^2 + \frac{11}{200}c_1^2c_2 - \frac{1}{10}c_1c_3 + \frac{1}{10}c_4\right|.$$

After simplifying, we have

$$\left|d_5 - d_2^4\right| = \frac{1}{10}\left|\frac{239}{4000}c_1^4 + \frac{3}{10}c_2^2 + 2\left(\frac{1}{2}\right)c_1c_3 - \frac{3}{2}\left(\frac{11}{30}\right)c_1^2c_2 - c_4\right|. \tag{26}$$

Comparing the right side of (26) with

$$\left|\gamma c_1^4 + ac_2^2 + 2\alpha c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right|,$$

we obtain

$$\gamma = \frac{239}{4000}, \quad a = \frac{3}{10}, \quad \alpha = \frac{1}{2}, \quad \beta = \frac{11}{30}.$$

Thus, it follows that

$$\left((-\beta + \alpha(\alpha + a))^2 + (-2\gamma + \beta\alpha)^2\right)8(1-a)a + (-2a\alpha + \beta)^2(1-\alpha)\alpha = 0.009823$$

and

$$4a\alpha^2(1-\alpha)^2(1-a) = 0.0525.$$

From (7), we deduce that

$$\left|d_5 - d_2^4\right| \leq \frac{1}{5}.$$

This equality is attained from

$$\frac{2zg'(z)}{g(z)-g(-z)} = 1 + \frac{4}{5}z^4 + \frac{1}{5}z^{16} + \cdots.$$

□

5. Third Hankel Determinant

Finally, we can calculate the determinant $\mathcal{D}_{3,1}(g)$ for $g \in \mathcal{S}^*_{3l,s}$.

Theorem 9. *If $g \in \mathcal{S}^*_{3l,s}$, then*

$$|\mathcal{D}_{3,1}(g)| \leq 0.047.$$

Proof. The determinant $\mathcal{D}_{3,1}(g)$ is described as follows

$$\mathcal{D}_{3,1}(g) = 2d_2d_3d_4 - d_3^3 - d_4^2 + d_3d_5 - d_2^2d_5.$$

Plugging (19)–(22) with $c_1 = c$ we obtain

$$\begin{aligned} \mathcal{D}_{3,1}(g) \quad = \quad & \frac{1}{80000}\Big(63c^6 - 622c^4c_2 + 560c^3c_3 + 1152c^2c_2^2 - 1120c^2c_4 + 320cc_2c_3 \\ & -1120c_2^3 + 1600c_2c_4 - 800c_3^2\Big). \end{aligned} \tag{27}$$

Let $s = 4 - c^2$ in (8)–(10). Now, using these lemmas, we obtain

$$\begin{aligned}
622c^4c_2 &= 311\Big(c^6 + c^4sx\Big), \\
560c^3c_3 &= -140c^4sx^2 + 280c^3s\Big(1 - |x|^2\Big)\varsigma + 280c^4sx + 140c^6, \\
1152c^2c_2^2 &= 288c^6 + 576c^4sx + 288c^2s^2x^2, \\
1120c^2c_4 &= -560\Big(1 - |x|^2\Big)c^2\overline{x}\varsigma^2s - 420c^4sx^2 + 560\Big(1 - |x|^2\Big)c^3\varsigma s + 560c^2sx^2 \\
&\quad + 140c^4sx^3 + 140c^6 + 560\Big(1 - |\varsigma|^2\Big)\Big(1 - |x|^2\Big)c^2\rho s + 420c^4sx \\
&\quad - 560\Big(1 - |x|^2\Big)c^3s\varsigma x, \\
320cc_2c_3 &= -40c^2s^2x^3 - 40c^4sx^2 + 80cxs^2\Big(1 - |x|^2\Big)\varsigma + 80c^2x^2s^2 + 80c^3s \\
&\quad \Big(1 - |x|^2\Big)\varsigma + 120c^4sx + 40c^6, \\
1120c_2^3 &= 140c^6 + 420c^4sx + 420c^2s^2x^2 + 140s^3x^3, \\
1600c_2c_4 &= 100c^6 + 100c^4sx^3 + 400c^4sx - 400c^2sx^2 - 400\Big(1 - |x|^2\Big)s\overline{x}\varsigma^2c^2 \\
&\quad -400\Big(1 - |x|^2\Big)c^3s\varsigma x - 300c^4sx^2 + 400\Big(1 - |\varsigma|^2\Big)\Big(1 - |x|^2\Big)c^2\rho\, s \\
&\quad + 100c^2s^2x^4 - 300c^2s^2x^3 + 400\Big(1 - |x|^2\Big)c^3\varsigma s + 300c^2s^2x^2 \\
&\quad + 400s^2x^3 - 400cs^2x^2\Big(1 - |x|^2\Big)\varsigma - 400xs^2\overline{x}\Big(1 - |x|^2\Big)\varsigma^2 \\
&\quad + 400\Big(1 - |\varsigma|^2\Big)\Big(1 - |x|^2\Big)s^2\rho x + 400\Big(1 - |x|^2\Big)s^2x\varsigma c, \\
800c_3^2 &= 200\Big(1 - |x|^2\Big)^2s^2\varsigma^2 + 50c^2s^2x^4 - 200\Big(1 - |x|^2\Big)s^2x^2\varsigma c - 100c^4sx^2 \\
&\quad +200c^2s^2x^2 - 200c^2s^2x^3 + 400\Big(1 - |x|^2\Big)s^2x\varsigma c + 200c^4sx \\
&\quad + 50c^6 + 200\Big(1 - |x|^2\Big)c^3\varsigma s.
\end{aligned}$$

Plugging the above expressions in (27), we obtain

$$\begin{aligned}\mathcal{D}_{3,1}(g) &= \frac{1}{80000}\Big\{48c^2x^2s^2 - 140c^2x^3s^2 + 40c^4x^2s - 160c^2x^2s - 40c^4x^3s \\ &+50c^2x^4s^2 - 200s^2\Big(1-|x|^2\Big)^2\varsigma^2 - 140x^3s^3 + 160c^3xs\Big(1-|x|^2\Big)\varsigma \\ &+160\Big(1-|x|^2\Big)s\overline{x}\varsigma^2c^2 - 200\Big(1-|x|^2\Big)x^2s^2c\varsigma + 80\Big(1-|x|^2\Big)cxs^2\varsigma \\ &-400\Big(1-|x|^2\Big)xs^2\overline{x}\varsigma^2 + 400x^3s^2 + 400\Big(1-|\varsigma|^2\Big)\Big(1-|x|^2\Big)s^2x\rho \\ &+25c^4xs - 160\Big(1-|\varsigma|^2\Big)\Big(1-|x|^2\Big)c^2\rho s - 10c^6\Big\}.\end{aligned}$$

Since $s = 4 - c^2$, then

$$\mathcal{D}_{3,1}(g) = \frac{1}{80000}\Big(I_0(c,x) + I_1(c,x)\varsigma + I_2(c,x)\varsigma^2 + \varrho(c,x,\varsigma)\rho\Big),$$

where $\varsigma, x, \rho \in \overline{\mathbb{U}_d}$, and

$$\begin{aligned}I_0(c,x) &= -10c^6 + \Big(4-c^2\Big)\Big[\Big(-160x^3 + 50c^2x^4 + 48c^2x^2\Big)\Big(4-c^2\Big) \\ &\qquad +25c^4x - 160c^2x^2 - 40c^4x^3 + 40c^4x^2\Big], \\ I_1(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)\Big[\Big(80cx - 200cx^2\Big)\Big(4-c^2\Big) + 160c^3x\Big], \\ I_2(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)\Big[\Big(-200|x|^2 - 200\Big)\Big(4-c^2\Big) + 160c^2\overline{x}\Big], \\ \varrho(c,x,\varsigma) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)\Big(1-|\varsigma|^2\Big)\Big[-160c^2 + 400x\Big(4-c^2\Big)\Big].\end{aligned}$$

By replacing $|x|$ with x, and $|\varsigma|$ with y,, if we apply the statement $|\rho| \leq 1$, it follows that

$$\begin{aligned}|\mathcal{D}_{3,1}(g)| &\leq \frac{1}{80000}\Big(|I_0(c,x)| + |I_1(c,x)|y + |I_2(c,x)|y^2 + |\varrho(c,x,\varsigma)|\Big). \\ &\leq \frac{1}{80000}(T(c,x,y)),\end{aligned} \tag{28}$$

where

$$T(c,x,y) = v_0(c,x) + v_1(c,x)y + v_2(c,x)y^2 + v_3(c,x)\Big(1-y^2\Big),$$

with

$$\begin{aligned}v_0(c,x) &= 10c^6 + \Big(4-c^2\Big)\Big[\Big(160x^3 + 50c^2x^4 + 48c^2x^2\Big)\Big(4-c^2\Big) \\ &\qquad +25c^4x + 160c^2x^2 + 40c^4x^3 + 40c^4x^2\Big], \\ v_1(c,x) &= \Big(1-x^2\Big)\Big(4-c^2\Big)\Big[\Big(80cx + 200cx^2\Big)\Big(4-c^2\Big) + 160c^3x\Big], \\ v_2(c,x) &= \Big(1-x^2\Big)\Big(4-c^2\Big)\Big[\Big(200x^2 + 200\Big)\Big(4-c^2\Big) + 160c^2x\Big], \\ v_3(c,x) &= \Big(1-x^2\Big)\Big(4-c^2\Big)\Big[160c^2 + 400\Big(4-c^2\Big)x\Big].\end{aligned}$$

Now, our aim is to find the maximum of $T(c,x,y)$ in a very particular domain, i.e., a closed cuboid $\Xi : [0,2] \times [0,1] \times [0,1]$.

To achieve the required result, we have to enact this proof for $T(c,x,y)$ in three regions, i.e., in the interior of the domain Ξ, as well as in its faces and then on the edges.

1. Interior points of the cuboid Ξ :

Suppose $(c, x, y) \in (0,2) \times (0,1) \times (0,1)$. Then, on differentiating $T(c, x, y)$ partially about the parameter y, we obtain

$$\begin{aligned}\frac{\partial T}{\partial y} &= \left(4-c^2\right)(1-x^2)\left[400y(x-1)\left(\left(4-c^2\right)(x-1)+\frac{4}{5}c^2\right)\right.\\ &\qquad \left.+40c\left(x\left(4-c^2\right)(2+5x)+4c^2x\right)\right].\end{aligned}$$

Taking $\frac{\partial T}{\partial y} = 0$, gives

$$y = \frac{40c\left(x(4-c^2)(2+5x)+4c^2x\right)}{400(x-1)\left((4-c^2)(1-x)-\frac{4}{5}c^2\right)} = y^*.$$

If y^* should belong to $(0,1)$, then it is possible only if

$$160c^3x + 40cx\left(4-c^2\right)(2+5x) + 400(1-x)^2\left(4-c^2\right) < 320(1-x)c^2 \tag{29}$$

and

$$c^2 > \frac{20(1-x)}{9-5x}. \tag{30}$$

Now, only a solution that can meet both the inequalities (29) and (30) will be accepted as a critical point.

Suppose $g(x) = \frac{20(1-x)}{9-5x}$. Thus, $g(x)$ decreases over $(0,1)$. Thus, $c^2 > 0$, and a straightforward task illustrates that (29) will not hold for all values of $x \in (0,1)$. This implies that we have not found a critical point for T in $(0,2) \times (0,1) \times (0,1)$.

2. Interior of all the six faces of the cuboid Ξ **:**

(i) In choosing $c = 0$, we achieve

$$q_1(x,y) = 640\left(4x^3 + (10x + (x-1)(5x-5)y^2)(1-x^2)\right) = T(0,x,y).$$

When partially differentiating $q_1(x,y)$ about the parameter y, we obtain

$$\frac{\partial q_1}{\partial y} = 1280y(1-x^2)(5x-5)(x-1).$$

But $\frac{\partial q_1}{\partial y} \neq 0$ for $x, y \in (0,1)$. Hence, the final result is that there is no maximum value for $T(0,x,y)$ in $(0,1) \times (0,1)$.

(ii) In setting $c = 2$, we have

$$T(2,x,y) \leq 640.$$

(iii) By taking $x = 0$, we obtain

$$T(c,0,y) = q_2(c,y) = 10c^6 + (4-c^2)\left(-360c^2y^2 + 800y^2 + 160c^2\right).$$

When partially differentiating $q_2(c,y)$ about the parameter y and the parameter c, we obtain

$$\frac{\partial q_2}{\partial y} = (4-c^2)\left(-720c^2y + 1600y\right)$$

and

$$\frac{\partial q_2}{\partial c} = 60c^5 - 320c^3 + \left(4-c^2\right)\left(-720cy^2 + 320c\right) + 720c^3y^2 - 1600cy^2.$$

Another outcome followed by a simple calculation is that no optimal solution is attained for $T(c,0,y)$ in $(0,2) \times (0,1)$.

(iv) Considering $x = 1$, we have

$$q_3(c, y) = 10c^6 + (4 - c^2)\Big((4 - c^2)(160 + 98c^2) + 160c^2 + 105c^4\Big) = T(c, 1, y).$$

Then

$$\frac{\partial q_3}{\partial c} = 18c^5 - 1456c^3 + 1856c.$$

By taking $\frac{\partial q_3}{\partial c} = 0$, we achieve $c \approx 1.138$, at which $q_3(c, y)$ attains its maxima, which is

$$q_3(c, y) \leq 3157.83.$$

(v) If we choose $y = 0$, we find that

$$\begin{aligned} q_4(c, x) \quad = \quad & 50c^6x^4 - 40c^6x^3 + 8c^6x^2 - 400c^4x^4 - 25c^6x - 80c^4x^3 \\ & + 10c^6 - 224c^4x^2 + 800c^2x^4 + 500c^4x + 1920c^2x^3 - 160c^4 \\ & + 768c^2x^2 - 3200c^2x - 3840x^3 + 640c^2 + 6400x = T(c, x, 0). \end{aligned}$$

Now, by partially differentiating about the parameter c, and parameter x, as well as simplifying, we have

$$\begin{aligned} \frac{\partial q_4}{\partial c} \quad = \quad & 300c^5x^4 - 240c^5x^3 + 48c^5x^2 - 1600c^3x^4 - 150c^5x - 320c^3x^3 \\ & + 60c^5 - 896c^3x^2 + 1600cx^4 + 2000c^3x + 3840cx^3 - 640c^3 \\ & + 1536cx^2 - 6400cx + 1280c \end{aligned}$$

and

$$\begin{aligned} \frac{\partial q_4}{\partial x} \quad = \quad & 200c^6x^3 - 120c^6x^2 + 16c^6x - 1600c^4x^3 - 25c^6 - 240c^4x^2 \\ & - 448c^4x + 3200c^2x^3 + 500c^4 + 5760c^2x^2 + 1536c^2x \\ & - 3200c^2 - 11520x^2 + 6400. \end{aligned}$$

From computation, we can conclude that no solution exists for the abovementioned system of equations:

$$\frac{\partial q_4}{\partial c} = 0 \text{ and } \frac{\partial q_4}{\partial x} = 0,$$

and in $(0, 2) \times (0, 1)$.

(vi) By taking $y = 1$, the following result is obtained:

$$\begin{aligned} q_5(c, x) \quad = \quad & 50c^6x^4 - 40c^6x^3 - 200c^5x^4 + 8c^6x^2 + 80c^5x^3 \\ & - 600c^4x^4 - 25c^6x + 200c^5x^2 + 480c^4x^3 + 1600c^3x^4 \\ & + 10c^6 - 80c^5x - 384c^4x^2 + 2400c^2x^4 - 60c^4x - 1600c^3x^2 \\ & - 1920c^2x^3 - 3200cx^4 + 200c^4 + 1408c^2x^2 - 1280cx^3 \\ & - 3200x^4 + 640c^2x + 3200cx^2 + 2560x^3 - 1600c^2 \\ & + 1280cx + 3200 = T(c, x, 1). \end{aligned}$$

With the partial derivative of $q_5(c,x)$ about the parameter c and parameter x, we have

$$\begin{aligned}\frac{\partial q_5}{\partial c} = {} & 300c^5x^4 - 240c^5x^3 - 1000c^4x^4 + 48c^5x^2 + 400c^4x^3 - 2400c^3x^4 \\ & -150c^5x + 1000c^4x^2 + 1920c^3x^3 + 4800c^2x^4 + 60c^5 - 400c^4x \\ & -1536c^3x^2 + 4800cx^4 - 240c^3x - 4800c^2x^2 - 3840cx^3 + 800c^3 \\ & - 3200x^4 + 2816cx^2 - 1280x^3 + 1280cx + 3200x^2 - 3200c \\ & + 1280x\end{aligned}$$

and

$$\begin{aligned}\frac{\partial q_5}{\partial x} = {} & 200c^6x^3 - 120c^6x^2 - 800c^5x^3 + 16c^6x + 240c^5x^2 - 2400c^4x^3 \\ & - 25c^6 + 400c^5x + 1440c^4x^2 + 6400c^3x^3 - 80c^5 - 768c^4x \\ & + 9600c^2x^3 - 60c^4 - 3200c^3x - 5760c^2x^2 - 12800cx^3 \\ & + 2816c^2x - 3840cx^2 - 12800x^3 + 640c^2 + 6400cx \\ & + 7680x^2 + 1280c.\end{aligned}$$

The result that a unique solution $(c,x) \approx (0.689, 0.720)$ exists is followed by simple calculations for the abovementioned system of equations. As such,

$$\frac{\partial q_5}{\partial c} = 0 \quad \text{and} \quad \frac{\partial q_5}{\partial x} = 0,$$

and in $(0,2) \times (0,1)$. Hence,

$$T(c,x,1) = q_5(c,x) \leq 3790.225.$$

3. On the Edges of the Cuboid Ξ **:**

(i) By selecting $x = 0$ and $y = 0$, we find that

$$T(c,0,0) = 10c^6 - 160c^4 + 640c^2 = q_6(c).$$

When differentiating $q_6(c)$ about the parameter c, we have

$$q_6'(c) = 60c^5 - 640c^3 + 1280c.$$

We note that $q_6'(c) = 0$ for the critical point $c \approx 1.632$, at which $q_6(c)$ obtains its maxima. Thus,

$$q_6(c) \leq 758.51.$$

(ii) By substituting $x = 0$ and $y = 1$, we obtain

$$T(c,0,1) = 10c^6 + 200c^4 - 1600c^2 + 3200 = q_7(c).$$

When differentiating $q_7(c)$ about the parameter c, we have

$$q_7'(c) = 60c^5 + 800c^3 - 3200c.$$

We can see that $q_7'(c) < 0$ for $[0,2]$ indicates that $q_7(c)$ is a decreasing function and obtains its maxima at 0. Therefore,

$$T(c,0,1) = q_7(c) \leq 3200.$$

(iii) By choosing $c = 0$ and $x = 0$, we obtain

$$q_8(y) = 3200y^2 = T(0,0,y).$$

It follows that $q_8'(y) > 0$ for $[0,1]$ shows that $q_8(y)$ is an increasing function and that the maxima is attained at 1. Therefore,

$$q_8(y) \leq 3200.$$

(iv) We note that $T(c,1,y)$ is free of y. Thus, it follows that

$$q_9(c) = T(c,1,1) = T(c,1,0).$$

$$q_9(c) = 3c^6 - 364c^4 + 928c^2 + 2560.$$

When partially differentiating $q_9(c)$ about the parameter c, we obtain

$$q_9'(c) = 18c^5 - 1456c^3 + 1856c.$$

By taking $q_9'(c) = 0$, we achieve $c \approx 1.138$, at which $q_9(c)$ achieves its maxima. Thus,

$$q_9(c) \leq 3157.83.$$

(v) By selecting $c = 0$ and $x = 1$, we achieve

$$T(0,1,y) \leq 2560.$$

(vi) By taking $c = 2$, we obtain

$$T(2,x,y) \leq 640.$$

We see that $T(2,x,y)$ is free of y, x, c. Thus, it follows that

$$T(2,1,y) = T(2,x,1) = T(2,x,0) = T(2,0,y) \leq 640.$$

(vii) By substituting $c = 0$ and $y = 1$, we find that

$$q_{10}(x) = -3200x^4 + 2560x^3 + 3200 = T(0,x,1).$$

Thus, it follows that

$$q_{10}'(x) = -12800x^3 - 7680x^2.$$

For the critical point $\frac{\partial q_{10}}{\partial x} = 0$, we achieve $x \approx 0.60$, at which $q_{10}(x)$ achieves its maxima. Hence,

$$q_{10}(x) \leq 3338.24.$$

(viii) By taking $c = 0$ and $y = 0$, we have

$$T(0,x,0) = -3840x^3 + 6400x = q_{11}(x).$$

Clearly,

$$q_{11}'(x) = -11520x^2 + 6400.$$

Thus, we know that $q_{11}'(x) = 0$ gives $x \approx 0.745$, at which $q_{11}(x)$ obtain its maximum value, which is given by

$$T(0,x,0) = q_{11}(x) \leq \frac{12800}{9}\sqrt{5}.$$

Hence, from the above situations, we achieve

$$T(c,x,y) \leq 3790.225 \quad \text{on} \quad [0,2] \times [0,1] \times [0,1].$$

By using Equation (28), it follows that

$$|\mathcal{D}_{3,1}(g)| \leq \frac{1}{80000}(T(c,x,y)) \leq 0.047.$$

Thus, we have completed the proof. □

Remark 1. *The sharp bound on the third Hankel determinant for the class of symmetric points with respect to three-leaf type domain is* $\frac{1}{25}$. *Equality, for the class* $\mathcal{S}^*_{3l,s}$, *holds in the case of the function* g *which is defined by*

$$\frac{2zg'(z)}{g(z)-g(-z)} = 1 + \frac{4}{5}z^3 + \frac{1}{5}z^{12} + \cdots.$$

Theorem 10. *If* $g \in \mathcal{S}^*_{3l,s}$ *and has the form* (1), *then*

$$|\mathcal{D}_{2,3}(g)| = \left|d_3 d_5 - d_4^2\right| \leq 0.044.$$

Proof. By plugging (20)–(22) with $c_1 = c$, we obtain

$$\begin{aligned}\mathcal{D}_{2,3}(g) &= \frac{1}{80000}\Big(17c^6 - 318c^4c_2 + 560c^3c_3 + 608c^2c_2^2 - 800c^2c_4 - 320cc_2c_3 \\ &\qquad -480c_2^3 + 1600c_2c_4 - 800c_3^2\Big).\end{aligned} \tag{31}$$

Let $s = 4 - c^2$ in (8)–(10). Now, through using these lemmas, we obtain

$$\begin{aligned}
318c^4c_2 &= 159\Big(c^6 + c^4sx\Big), \\
560c^3c_3 &= -140c^4sx^2 + 280c^3s\Big(1-|x|^2\Big)\varsigma + 280c^4sx + 140c^6, \\
608c^2c_2^2 &= 152c^6 + 304c^4sx + 152c^2s^2x^2, \\
800c^2c_4 &= 100c^6 + 100c^4sx^3 - 300c^4sx^2 + 300c^4sx + 400c^2sx^2 - 400c^3sx \\
&\quad \Big(1-|x|^2\Big)\varsigma - 400c^2s\overline{x}\Big(1-|x|^2\Big)\varsigma^2 + 400c^2s\Big(1-|x|^2\Big)\Big(1-|\varsigma|^2\Big)\rho \\
&\quad + 400c^3s\Big(1-|x|^2\Big)\varsigma, \\
320cc_2c_3 &= -40c^2s^2x^3 - 40c^4sx^2 + 80cs^2x\Big(1-|x|^2\Big)\varsigma + 80c^2s^2x^2 \\
&\quad +80c^3s\Big(1-|x|^2\Big)\varsigma + 120c^4sx + 40c^6, \\
480c_2^3 &= 60c^6 + 180c^4sx + 180c^2s^2x^2 + 60s^3x^3, \\
1600c_2c_4 &= 100c^6 + 100c^4sx^3 - 300c^4sx^2 + 400c^4sx + 400c^2sx^2 - 400c^3sx \\
&\quad \Big(1-|x|^2\Big)\varsigma - 400c^2s\overline{x}\Big(1-|x|^2\Big)\varsigma^2 + 400c^2s\Big(1-|x|^2\Big)\Big(1-|\varsigma|^2\Big)\rho \\
&\quad + 400c^3s\Big(1-|x|^2\Big)\varsigma + 100c^2s^2x^4 - 300c^2s^2x^3 + 300c^2s^2x^2 \\
&\quad + 400s^2x^3 - 400cs^2x^2\Big(1-|x|^2\Big)\varsigma - 400s^2x\overline{x}\Big(1-|x|^2\Big)\varsigma^2 \\
&\quad + 400s^2x\Big(1-|x|^2\Big)\Big(1-|\varsigma|^2\Big)\rho + 400cs^2x\Big(1-|x|^2\Big)\varsigma, \\
800c_3^2 &= 50c^2s^2x^4 - 200cs^2x^2\Big(1-|x|^2\Big)\varsigma - 200c^2s^2x^3 - 100c^4sx^2 \\
&\quad +200s^2\Big(1-|x|^2\Big)^2\varsigma^2 + 400cs^2x\Big(1-|x|^2\Big)\varsigma + 200c^2s^2x^2 \\
&\quad + 200c^3s\Big(1-|x|^2\Big)\varsigma + 200c^4sx + 50c^6.
\end{aligned}$$

By plugging the above expressions in (31), we obtain

$$\begin{aligned}\mathcal{D}_{2,3}(g) &= \frac{1}{80000}\Big\{-60x^3s^3+400x^3s^2-80cxs^2\Big(1-|x|^2\Big)\varsigma-200cx^2s^2\Big(1-|x|^2\Big)\varsigma \\ &\quad -400xs^2\Big(1-|x|^2\Big)\overline{x}\varsigma^2+400xs^2\Big(1-|x|^2\Big)\Big(1-|\varsigma|^2\Big)\rho+25c^4xs \\ &\quad -8c^2x^2s^2-60c^2x^3s^2+50c^2s^2x^4-200s^2\Big(1-|x|^2\Big)^2\varsigma^2\Big\}.\end{aligned}$$

Since $s=4-c^2$, we have

$$\mathcal{D}_{2,3}(g)=\frac{1}{80000}\Big(J_0(c,x)+J_1(c,x)\varsigma+J_2(c,x)\varsigma^2+J_3(c,x,\varsigma)\rho\Big),$$

where

$$\begin{aligned}J_0(c,x) &= \Big(4-c^2\Big)\Big[\Big(4-c^2\Big)\Big(160x^3-8c^2x^2+50c^2x^4\Big)+25c^4x\Big],\\ J_1(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)^2\Big(-80cx-200cx^2\Big),\\ J_2(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)^2\Big(-200|x|^2-200\Big),\\ J_3(c,x,\varsigma) &= 400x\Big(1-|x|^2\Big)\Big(4-c^2\Big)^2\Big(1-|\varsigma|^2\Big).\end{aligned}$$

By replacing $|x|$ with x, and $|\varsigma|$ with y, if we apply the statement $|\rho|\leq 1$, it follows that

$$\begin{aligned}|\mathcal{D}_{2,3}(g)| &\leq \frac{1}{80000}\Big(|J_0(c,x)|+|J_1(c,x)|y+|J_2(c,x)|y^2+|J_3(c,x,\varsigma)|\Big).\\ &\leq \frac{1}{80000}(K(c,x,y)),\end{aligned} \tag{32}$$

where

$$K(c,x,y)=O_0(c,x)+O_1(c,x)y+O_2(c,x)y^2+O_3(c,x)\Big(1-y^2\Big),$$

with

$$\begin{aligned}O_0(c,x) &= \Big(4-c^2\Big)\Big[\Big(4-c^2\Big)\Big(160x^3+8c^2x^2+50c^2x^4\Big)+25c^4x\Big],\\ O_1(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)^2\Big(80cx+200cx^2\Big),\\ O_2(c,x) &= \Big(1-|x|^2\Big)\Big(4-c^2\Big)^2\Big(200x^2+200\Big),\\ O_3(c,x) &= 400x\Big(1-|x|^2\Big)\Big(4-c^2\Big)^2.\end{aligned}$$

Again, our aim is to find the maximum value of $K(c,x,y)$ in a particular domain; in this case, the closed cuboid is as follows $\Xi:[0,2]\times[0,1]\times[0,1]$.

To achieve the above stated goal, we need to first calculate the maximum value of $K(c,x,y)$ in the interior of the domain Ξ, as well as in its faces and then on the edges.

1. Interior points of cuboid Ξ

Suppose $(c,x,y)\in(0,2)\times(0,1)\times(0,1)$. Then, when partially differentiating $K(c,x,y)$ about the parameter y, we obtain

$$\frac{\partial K}{\partial y}=\frac{1}{40}\Big(1-x^2\Big)\Big(4-c^2\Big)\Big[y(x-1)(10x-10)\Big(4-c^2\Big)+cx\Big(4-c^2\Big)(5x+2)\Big].$$

In taking $\frac{\partial K}{\partial y} = 0$, we obtain

$$y = \frac{cx(4-c^2)(5x+2)}{(4-c^2)(x-1)(10-10x)} = y_1.$$

If y_1 should belong to $(0,1)$, then it is possible only if

$$cx\left(4-c^2\right)(5x+2) < \left(4-c^2\right)(x-1)(10-10x) \tag{33}$$

and

$$c^2 > 4. \tag{34}$$

Now, only a solution that meets both the inequalities (33) and (34) will be accepted as a critical point.

Thus, $c^2 > 4$ and a straightforward task illustrates that (33) does not hold for all values of $x \in (0,1)$. This implies that we have found no critical point for K in $(0,2) \times (0,1) \times (0,1)$.

2. Interior of all the six faces of cuboid Ξ

(i) In taking $c = 0$, we find that

$$t_1(x,y) = 640\left[4x^3 + 5\left(1-x^2\right)\left(y^2(x-1)^2 + 2x\right)\right] = K(0,x,y).$$

When differentiating $t_1(x,y)$ about the parameter y, we have

$$\frac{\partial t_1}{\partial y} = 6400y(1-x^2)(x-1)^2.$$

But $\frac{\partial t_1}{\partial y} \neq 0$ for $x, y \in (0,1)$. Hence, we have found no critical point for $K(0,x,y)$ in $(0,1) \times (0,1)$.

(ii) When choosing $c = 2$, we achieve

$$K(2,x,y) \leq 0.$$

(iii) When substituting $x = 0$, we obtain

$$t_2(c,y) = 200y^2(4-c^2)^2 = K(c,0,y).$$

When differentiating $t_2(c,y)$ about the parameter y and parameter c, we have

$$\frac{\partial t_2}{\partial y} = 400y(4-c^2)^2$$

and

$$\frac{\partial t_2}{\partial c} = -800cy^2(4-c^2).$$

A calculation shows that $t_2(c,y)$ has no optimal solution in $(0,2) \times (0,1)$.

(iv) When selecting $x = 1$, we have

$$t_3(c,y) = (4-c^2)((4-c^2)\left(58c^2 + 160\right) + 25c^4) = K(c,1,y).$$

Thus, it is clear that

$$\frac{\partial t_3}{\partial c} = 198c^5 - 816c^3 - 704c.$$

We see that $t_3'(c) < 0$ for $[0,2]$ illustrates that $t_3(c)$ is a decreasing function and achieves its maxima at 0. Thus,

$$t_3(c) \leq 2560.$$

(v) If we choose $y = 0$, we find that

$$\begin{aligned} t_4(c,x) &= 50c^6x^4 + 8c^6x^2 - 400c^4x^4 - 25c^6x - 240c^4x^3 - 64c^4x^2 + 800c^2x^4 \\ &+500c^4x + 1920c^2x^3 + 128c^2x^2 - 3200c^2x - 3840x^3 + 6400x = K(c,x,0). \end{aligned}$$

Now, when partially differentiating about the parameter c, and parameter x, as well as simplifying, we have

$$\begin{aligned} \frac{\partial t_4}{\partial c} &= 300c^5x^4 + 48c^5x^2 - 1600c^3x^4 - 150c^5x - 960c^3x^3 - 256c^3x^2 \\ &\quad + 1600cx^4 + 2000c^3x + 3840cx^3 + 256cx^2 - 6400cx \end{aligned}$$

and

$$\begin{aligned} \frac{\partial t_4}{\partial x} &= 200c^6x^3 + 16c^6x - 1600c^4x^3 - 25c^6 - 720c^4x^2 - 128c^4x + 3200c^2x^3 \\ &\quad + 500c^4 + 5760c^2x^2 + 256c^2x - 3200c^2 - 11520x^2 + 6400. \end{aligned}$$

A numerical calculation shows that a solution does not exist for the system of equations

$$\frac{\partial t_4}{\partial c} = 0 \quad \text{and} \quad \frac{\partial t_4}{\partial x} = 0,$$

and in $(0,2) \times (0,1)$.

(vi) When choosing $y = 1$, we obtain

$$\begin{aligned} t_5(c,x) &= 50c^6x^4 - 200c^5x^4 + 8c^6x^2 - 80c^5x^3 - 600c^4x^4 + 80c^5x - 64c^4x^2 \\ &-25c^6x + 200c^5x^2 + 160c^4x^3 + 1600c^3x^4 - 1600c^3x^2 - 1280c^2x^3 \\ &\quad + 640c^3x^3 + 2400c^2x^4 + 100c^4x - 640c^3x + 128c^2x^2 - 1280cx^3 \\ &\quad - 3200cx^4 + 200c^4 - 3200x^4 + 3200cx^2 + 2560x^3 - 1600c^2 \\ &\quad + 1280cx + 3200 = K(c,x,1). \end{aligned}$$

When partially deriving $t_5(c,x)$ about the parameter c and parameter x, we have

$$\begin{aligned} \frac{\partial t_5}{\partial c} &= 300c^5x^4 - 1000c^4x^4 + 48c^5x^2 - 400c^4x^3 - 2400c^3x^4 - 150c^5x \\ &\quad + 1000c^4x^2 + 640c^3x^3 + 4800c^2x^4 + 400c^4x - 256c^3x^2 \\ &\quad + 1920c^2x^3 + 4800cx^4 + 400c^3x - 4800c^2x^2 - 2560cx^3 \\ &\quad - 3200x^4 + 800c^3 - 1920c^2x + 256cx^2 - 1280x^3 + 3200x^2 \\ &\quad - 3200c + 1280x. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial t_5}{\partial x} &= 200c^6x^3 - 800c^5x^3 + 16c^6x - 240c^5x^2 - 2400c^4x^3 - 25c^6 + 400c^5x \\ &+480c^4x^2 + 6400c^3x^3 + 80c^5 - 128c^4x + 1920c^3x^2 + 9600c^2x^3 \\ &+100c^4 - 3200c^3x - 3840c^2x^2 - 12800cx^3 - 640c^3 + 256c^2x \\ &-3840cx^2 - 12800x^3 + 6400cx + 7680x^2 + 1280c. \end{aligned}$$

A simple computation illustrates that there exists a unique solution $(c,x) \approx (0.358, 0.647)$ for the system of equations

$$\frac{\partial t_5}{\partial c} = 0 \quad \text{and} \quad \frac{\partial t_5}{\partial x} = 0,$$

and in $(0,2) \times (0,1)$. Thus, we have

$$K(c,x,1) \leq 3569.49.$$

3. On the Edges of the Cuboid Ξ

(i) By setting $x = 0$ and $y = 0$, we obtain

$$K(c,0,0) \leq 0.$$

(ii) By choosing $x = 0$ and $y = 1$, we find that

$$t_6(c) = 200c^4 - 1600c^2 + 3200 = K(c,0,1).$$

When differentiating $t_6(c)$ about the parameter c, we have

$$t_6'(c) = 800c^3 - 3200c.$$

Via a simple computation, it is indicated that $t_6(c)$ achieves its maxima at 0. Thus,

$$t_6(c) \leq 3200.$$

(iii) By selecting $c = 0$ and $x = 0$, we obtain

$$t_7(y) = 3200y^2 = K(0,0,y).$$

It follows that $t_7'(y) > 0$ for $[0,1]$ shows that $t_7(y)$ is an increasing function and that its maxima is attained at 1. Therefore,

$$K(0,0,y) = t_7(y) \leq 3200.$$

(iv) We note that $K(c,1,y)$ is free of y; as such, we obtain

$$t_8(c) = K(c,1,1) = K(c,1,0).$$

$$t_8(c) = 33c^6 - 204c^4 - 352c^2 + 2560.$$

It follows that

$$t_8'(c) = 198c^5 - 816c^3 - 704c.$$

Via a simple computation, it is indicated that $t_8(c)$ achieves its maxima at 0. Thus,

$$t_8(c) \leq 2560.$$

(v) By taking $c = 0$ and $x = 1$, we obtain

$$K(0,1,y) \leq 2560.$$

(vi) By choosing $c = 2$, it becomes

$$K(2,x,y) \leq 0.$$

We can see that $K(2,x,y)$ is free of y, x, c. Thus, it follows that

$$K(2,1,y) = K(2,x,1) = K(2,x,0) = K(2,0,y) \leq 0.$$

(vii) By setting $c = 0$ and $y = 1$, we achieve

$$K(0,x,1) = -3200x^4 + 2560x^3 + 3200 = t_9(x).$$

It is clear that

$$t_9'(x) = -12800x^3 + 7680x^2.$$

For the critical point, $\frac{\partial t_9}{\partial x} = 0$, we obtain $x \approx 0.60$, at which the maximum value is attained for $t_9(x)$. Therefore,

$$K(0, x, 1) \leq \frac{83456}{25}.$$

(viii) By substituting $c = 0$ and $y = 0$, we find that

$$K(0, x, 0) = -3840x^3 + 6400x = t_{10}(x).$$

When differentiating $t_{10}(x)$ about the parameter x, we have

$$t_{10}'(x) = -11520x^2 + 6400.$$

For the critical point, $\frac{\partial t_{10}}{\partial x} = 0$, we obtain $x \approx 0.745$, at which maximum value is attained for $t_{10}(x)$. Therefore,

$$K(0, x, 0) \leq \frac{12800}{9}\sqrt{5}.$$

Hence, from the above situations, we achieve

$$K(c, x, y) \leq 3569.497 \quad \text{on} \quad [0, 2] \times [0, 1] \times [0, 1].$$

By using Equation (32), it follows that

$$|\mathcal{D}_{2,3}(g)| \leq \frac{1}{80000}(K(c, x, y)) \leq 0.044.$$

The required proof is thus completed. □

Remark 2. *The sharp bound on the Hankel determinant $H_{2,3}(g)$ for the class of symmetric points with respect to a three-leaf type domain is $\frac{1}{25}$. Equality, for the class $\mathcal{S}^*_{3l,s}$, holds in the case of the function g which is defined by*

$$\frac{2zg'(z)}{g(z) - g(-z)} = 1 + \frac{4}{5}z^3 + \frac{1}{5}z^{12} + \cdots.$$

6. Conclusions

In this study, we investigated starlike functions that are associated with three-leaf-shaped geometrical regions with respect to symmetric points. We have estimated the sharp coefficient inequalities for the said functions. The discussed coefficient inequalities include the first five sharp coefficient bounds, the sharp bound for the third-order Hankel determinant, as well as the Zalcman and Krushkal inequalities. Based on our estimated results, we have also proposed certain conjectures that are strongly supported by our results. These conjectures and the sharpness of all the results distinguish this work from the already known results. The newly defined class $\mathcal{S}^*_{3l,s}$ can be studied further in more investigations, such as in the analysis of coefficient problems for their inverse functions and logarithmic coefficients.

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