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New Subclass of Close-to-Convex Functions Defined by Quantum Difference Operator and Related to Generalized Janowski Function

Suha B. Al-Shaikh [1,*], Mohammad Faisal Khan [2], Mustafa Kamal [3] and Naeem Ahmad [4]

1 Faculty of Computer Studies, Arab Open University, Riyadh 11681, Saudi Arabia
2 Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Riyadh 11673, Saudi Arabia; f.khan@seu.edu.sa
3 Department of Basic Sciences, College of Science and Theoretical Studies, Saudi Electronic University, Dammam 32256, Saudi Arabia; m.kamal@seu.edu.sa
4 Department of Civil Engineering, College of Engineering, Qassim University, Unaizah 56452, Saudi Arabia; n.naeem@qu.edu.sa
* Correspondence: s.alshaikh@arabou.edu.sa

Abstract: This work begins with a discussion of the quantum calculus operator theory and proceeds to develop and investigate a new family of close-to-convex functions in an open unit disk. Considering the quantum difference operator, we define and study a new subclass of close-to-convex functions connected with generalized Janowski functions. We prove the necessary and sufficient conditions for functions that belong to newly defined classes, including the inclusion relations and estimations of the coefficients. The Fekete–Szegő problem for a more general class is also discussed. The results of this investigation expand upon those of the previous study.

Keywords: analytic functions; q-derivative (q-difference) operator; close-to-convex functions; Schwarz lemma; subordination; Janowski functions; fractional derivative; Fekete–Szegő problems

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathcal{U}, \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let us define the subclasses of $\mathcal{A}$ ($\mathcal{K}$ and $\mathcal{S}^*$), whose members are almost close-to-convex and starlike in $\mathcal{U}$. The set of all starlike functions of order α, $\alpha \in [0,1)$ is alternatively represented as $\mathcal{S}^*(\alpha)$.

If there is a Schwarz function, w, analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = \eta(w(z))$ for all $z \in \mathcal{U}$, then the function f is subordinate to η, $(f(z) \prec \eta(z))$, where f and η are analytic functions. If the function η is univalent in $\mathcal{U}$, then we have

$$f(z) \prec \eta(z) \Leftrightarrow f(0) = 0 \text{ and } f(\mathcal{U}) \subset \eta(\mathcal{U}).$$

Let an analytic function, p, satisfying the conditions $p(0) = 1$ and $Rep(z) > 0$, $z \in \mathcal{U}$, belong to the class $\mathcal{P}$ [1] and be defined as:

$$p \in \mathcal{P} : p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \Leftrightarrow Rep(z) > 0, \ z \in \mathcal{U}. \tag{2}$$

A function, $p(z)$, of form (2) is in the class $P(M,N)$ if

$$p(z) \prec \frac{1+Mz}{1+Nz},$$

where $-1 \leq N < M \leq 1$.

Kowalczyk and Leś-Bomba [2] recently investigated the subclass $K_s(\alpha)$ of analytic functions connected to the starlike functions. They defined $K_s(\alpha)$ as follows:

Let an analytic function $f \in K_s(\alpha)$, $(0 \leq \alpha < 1)$ if there exists a function $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that

$$Re\left(\frac{-z^2 f'(z)}{\eta(z)\eta(-z)}\right) > \alpha,\ z \in \mathcal{U}$$

or

$$\frac{-z^2 f'(z)}{\eta(z)\eta(-z)} \prec \frac{1+(1-2\alpha)z}{1-z}.$$

Goyal [3] proposed and investigated a new family, $K_s(M,N;t,v)$, of starlike analytic functions connected with Janowski functions, inspired by the work of Kowalczyk and Le-Bomba [2].

Definition 1 ([3]). *Let an analytic function f be in the class $K_s(M,N;t,v)$ if there exists a function $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that*

$$\frac{tvz^2 f'(z)}{\eta(tz)\eta(vz)} \prec \frac{1+Mz}{1+Nz}, \tag{3}$$

where

$$(-1 \leq N < M \leq 1;\ t,v \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ |t| \leq 1 \text{ and } |v| \leq 1).$$

Remark 1. *Let $M = 1-2\alpha$ $(0 \leq \alpha < 1)$ and $N = -1$; we find that*

$$\frac{tvz^2 f'(z)}{\eta(tz)\eta(vz)} \prec \frac{1+(1-2\alpha)z}{(1-z)}$$

implies

$$Re\left(\frac{tvz^2 f'(z)}{\eta(tz)\eta(vz)}\right) > \alpha,\quad z \in \mathcal{U},\quad (0 \leq \alpha < 1), \tag{4}$$

and this subclass is denoted by $K_s(\alpha;t,v)$.

The study of the geometric function theory (GFT) has benefited from the theory of real and complex-order integrals and derivatives, and these concepts have also shown promise in mathematical modeling and analyses of applied scientific problems. Recent research on differential and integral operators from a number of perspectives, including quantum calculus, has yielded outstanding findings with implications for many branches of physics and mathematics. Such examples include research related to the analysis of the transmission dynamics of dengue infection [4] and the new model of the human liver, which uses Caputo–Fabrizio fractional derivatives with the exponential kernel proposed in [5]; for more details, see [6,7]. Since the research study of operators provides solutions for partial differential equations, they are important in the study of differential equations from the perspectives of operator theory and functional analyses. In this line of study, a q-analogous of a differential operator that has been previously presented and investigated is used to study a family of analytic functions.

Considering that differential operators are used in so many different areas of mathematics, scholars have focused their emphasis on this field of study. Many branches of mathematics and science have recently benefited from the use of quantum (q-) calculus, including, but not limited to, geometric function theory, q-difference equations, and q-integral equations. According to Jackson [8], who laid the groundwork for q-calculus theory, q-derivative and q-integral operators were first introduced, and later, Ismail et al. [9] used these operators to develop q-starlike functions. There was a flurry of research on the q-analogues of other differential operators after the introduction of the q-difference operator. Significant work has been carried out in both GFT and q-calculus theory by several mathematicians (for details, see [10–17]). Using the q-difference operator and generalized Janowski functions, we construct a new class of q-close-to-convex functions and explore several unique characteristics of the analytic function f that belongs to this class.

The q-analogue of the derivative and integral was initially defined and some of its applications were explained by Jackson [8]. The q-difference operator is defined as follows:

Definition 2 ([8])**.** *For $f \in \mathcal{A}$, the q-derivative operator (q-difference operator) is defined as*

$$\partial_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \in \mathcal{U},\ z \neq 0,\ q \in (0,1). \tag{5}$$

Using (1) together with (5), we have

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{6}$$

For $n \in \mathbb{N}$ and $z \in \mathcal{U}$, we have

$$\partial_q z^n = [n]_q z^{n-1}, \quad \partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

We can observe that

$$\lim_{q \to 1-} \partial_q f(z) = f'(z).$$

Janowski-type generalized functions are defined as follows:

Definition 3. *If a function, $p(z)$, is analytic in $\mathcal{U}$, with $p(0) = 1$ and*

$$\begin{aligned} p(z) \quad &\prec \quad (1-\lambda)\frac{1+Mz}{1+Nz} + \lambda \\ &= \quad \frac{1 + \{(1-\lambda)M + \lambda N\}z}{1+Nz}, \end{aligned}$$

then $p(z) \in P(M; N; \lambda)$, where $0 \leq \lambda < 1$ and $-1 \leq N < M \leq 1$.

Remark 2. *When $\lambda = 0$, we have*

$$p(z) \prec \frac{1+Mz}{1+Nz}.$$

Definition 4. *Let an analytic function, f, be of form (1) and $f \in K_s(M, N; q, t, v, \lambda)$, if there exists a function $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that*

$$\frac{tvz^2 \partial_q f(z)}{\eta(tz)\eta(vz)} \prec \frac{1 + \{(1-\lambda)M + \lambda N\}z}{1+Nz}, \tag{7}$$

or, inequality (7) is equivalent to

$$\left|\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} - 1\right| < \left|N\left(\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)}\right) - \{(1-\lambda)M + \lambda N\}\right|, \quad z \in \mathcal{U}, \tag{8}$$

where

$$0 \leq \lambda < 1,\ 0 < q < 1,\ t, v \in \mathbb{C}^* = \mathbb{C}\backslash\{0\},\ -1 \leq N < M \leq 1;\ |t| \leq 1, \text{ and } |v| \leq 1.$$

Special cases: *(i) When $q \to 1-$ and $\lambda = 0$, $K_s(M, N; q, t, v, \lambda) = K_s(M, N, t, v)$, as proved in [3]. (ii) $K_s(1-2\gamma, -1; q \to 1-, 1, -1, 0) = K_s(\gamma)$, as defined by Kowalczyk and Les-Bomba in [2]. (iii) Obviously, if $q \to 1-$, then $K_s(1, -1, 1, -1, 0) = K_s$, as investigated by Gao and Zhou in [18].*

Remark 3. *If we take $M = 1 - 2\alpha$ and $N = -1$, $(0 \leq \alpha < 1)$, we find that*

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} \prec \frac{1 + \{(1-\lambda)(1-2\alpha) - \lambda\}z}{1-z},$$

which implies that

$$Re\left(\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)}\right) > \frac{1 - ((1-\lambda)(1-2\alpha) - \lambda)}{2}, \tag{9}$$

where

$$0 \leq \alpha < 1; 0 < q < 1; 0 \leq \lambda < 1; t, v \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}; |t| \leq 1; \text{ and } |v| \leq 1.$$

We denote to this class of functions as $K_s(\alpha, q, t, v, \lambda)$.

Remark 4. *When $q \to 1-$, $K_s(1-2\gamma, -1; 1, v, 0) = \mathcal{X}_v(\gamma)$, which was defined by Cho et al. in [2] as*

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} \prec \frac{1 + (1-2\gamma)z}{1-z},\ 0 \leq \gamma < 1,\ z \in \mathcal{U}.$$

Example 1. *The function*

$$f_*(z) = \frac{t + 1 - 2\gamma}{(t-1)^2} \ln \frac{1-z}{1-tz} - \frac{2(1-2\gamma)z}{(1-t)(1-z)},\ z \in \mathcal{U},$$

belongs to the class $\mathcal{X}_v(\gamma)$. Indeed, f_* is analytic in $\mathcal{U}$, and $f_*(0) = 0$. If we put $g_*(z) = \frac{z}{1-z}$, $z \in \mathcal{U}$, then $g_* \in \mathcal{S}^*\left(\frac{1}{2}\right)$ and

$$Re\left(\frac{vz^2\partial_q f(z)}{\eta(z)\eta(vz)}\right) = Re\left(\frac{1 + (1-2\gamma)z}{1-z}\right) > \gamma.$$

For our planned research into the classes $K_s(M, N; q, t, v, \lambda)$ and $K_s(\alpha, q, t, v, \lambda)$, we need the following lemmas.

2. A Set of Lemmas

Lemma 1 ([3])**.** *Let*

$$\eta(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*\left(\frac{1}{2}\right). \tag{10}$$

If we set

$$K(z) = \frac{\eta(tz)\eta(vz)}{tvz} = z + \sum_{n=2}^{\infty} C_n(t,v) z^n, \quad z \in \mathcal{U}, \tag{11}$$

then $K \in \mathcal{S}^*$, where

$$C_n(t,v) = \sum_{j=1}^{n} b_j b_{n-j+1} t^{j-1} v^{n-j} \quad (n = 2, 3, \dots), \tag{12}$$

$b_1 = 1$, and $t, v \in \mathbb{C}^*$, with $|t| \leq 1$ and $|v| \leq 1$.

Remark 5. *If we put $t = 1$ and $v = -1$ in (12), we find that*

$$C_n(t,v) = \begin{cases} 0, & \text{if } n = 2k, \\ B_{2k-1}, & \text{if } n = 2k-1, \end{cases}$$

where

$$B_{2k-1} = 2b_{k-1} - 2b_2 - 2b_{2k-2} + \dots + (-1)^k 2b_{k-1}b_{k+1} + (-1)^k b_k^2. \tag{13}$$

We obtain the conclusion for the class of functions that was previously reported by Gao and Zhou [18].

Lemma 2 ([19])**.** *Let p be a function of form (2). If $p \in \mathcal{P}$, then*

$$\left| c_2 - \mu c_1^2 \right| \leq 2 \max\{1; |2\mu - 1|\}, \ \mu \in \mathbb{C},$$

and for the functions provided in (14), we have a sharp result:

$$p(z) = \frac{1+z}{1-z} \ \text{ and } \ \ p(z) = \frac{1+z^2}{1-z^2}. \tag{14}$$

Lemma 3 ([20])**.** *The function $p \in \mathcal{P}$ satisfies $Rep(z) > 0$, $z \in \mathcal{U}$, if and only if*

$$p(z) \neq \frac{x-1}{x+1}, \quad z \in \mathcal{U}$$

for all $|x| = 1$.

Lemma 4. *For $-1 \leq N < M \leq 1$, $0 \leq \lambda < 1$, the function*

$$T(z) = 1 + h_1 z + h_2 z^2 + \dots, \ \ z \in \mathcal{U}$$

is analytic in $\mathcal{U}$. Then, T satisfies the condition

$$\left| \frac{T(z) - 1}{\{(1-\lambda)M + \lambda N\} - NT(z)} \right| < \beta, \ z \in \mathcal{U}$$

for some β $(0 < \beta \leq 1)$, if and only if there exists an analytic function, φ, in $\mathcal{U}$, such that

$$|\varphi(z)| \leq \beta \ \text{ for } \ z \in \mathcal{U}$$

and

$$T(z) = \frac{1 - \{(1-\lambda)M + \lambda N\} z\varphi(z)}{1 - Nz\varphi(z)}, \ \ z \in \mathcal{U}.$$

Proof. We are using an approach similar to Padamanabhan [21]. Let

$$T(z) = 1 + h_1 z + h_2 z^2 + \dots, \quad z \in \mathcal{U},$$

satisfy

$$\left|\frac{T(z)-1}{\{(1-\lambda)M+\lambda N\}-NT(z)}\right|<\beta,\ \ z\in\mathcal{U}.$$

Set

$$h(z)=\frac{1-T(z)}{\{(1-\lambda)M+\lambda N\}-NT(z)}.$$

For $z\in\mathcal{U}$ and $h(0)=0$, we find that the analytic function h in $\mathcal{U}$ fulfills the inequality $|h(z)|<\beta$. By the Schwarz lemma, we know that $h(z)=z\varphi(z)$, where φ is analytic in $\mathcal{U}$ and $|\varphi(z)|\leq\beta$ holds true for every $z\in\mathcal{U}$. Thus, we obtain

$$\begin{aligned}T(z) &= \frac{1-\{(1-\lambda)M+\lambda N\}h(z)}{1-Nh(z)}\\ &= \frac{1-\{(1-\lambda)M+\lambda N\}z\varphi(z)}{1-Nz\varphi(z)}.\end{aligned}$$

Alternatively, if

$$T(z)=\frac{1-\{(1-\lambda)M+\lambda N\}z\varphi(z)}{1-Nz\varphi(z)}$$

and $|\varphi(z)|\leq\beta$ for $z\in\mathcal{U}$, then T is analytic in $\mathcal{U}$. Moreover, since $|z\varphi(z)|\leq\beta|z|<\beta$ for $z\in\mathcal{U}$, we obtain

$$\left|\frac{T(z)-1}{\{(1-\lambda)M+\lambda N\}-NT(z)}\right|=|z\varphi(z)|<\beta,\ z\in\mathcal{U}.$$

Our proof of the lemma is now complete. □

Remark 6. *Lemma 4 yields the well-known lemma proven in [3] when* $\lambda=0$.

Lemma 5. *Let an analytic function be of form (1); then,* $f\in K_s(\alpha,q,t,v,\lambda)$ *if and only if*

$$1+\sum_{n=2}^{\infty}A_nz^{n-1}\neq 0,\quad z\in\mathcal{U},\ |x|=1,$$

where

$$A_n=\frac{\left\{[n]_qa_n+\{(1-\lambda)(1-2\alpha)-\lambda\}C_n(t,v)+x([n]_qa_n-C_n(t,v)\right\}}{2(1-\lambda)(1-\alpha)},\tag{15}$$

where the coefficients $C_n(t,v)$ *are given by (12).*

Proof. In light of Lemma 3, we have $f\in K_s(\alpha,q,t,v,\lambda)$ if and only if

$$\frac{\frac{tvz^2\partial_qf(z)}{\eta(tz)\eta(vz)}-(\alpha(1-\lambda)+\lambda)}{(1-\alpha)(1-\lambda)}\neq\frac{x-1}{x+1},\quad z\in\mathcal{U}\tag{16}$$

for all $|x|=1$.

For $z=0$, the relation stated above holds true since

$$\left.\frac{\frac{tvz^2\partial_qf(z)}{\eta(tz)\eta(vz)}-(\alpha(1-\lambda)+\lambda)}{(1-\alpha)(1-\lambda)}\right|_{z=0}=1\neq\frac{x-1}{x+1},\qquad |x|=1.$$

For $z \neq 0$, the equivalence relation (16) can be written as

$$(x+1)\Big(tvz^2\partial_q f(x) - (\alpha(1-\lambda)+\lambda)\eta(tz)\eta(vz)\Big)$$
$$\neq \quad (x-1)(1-\alpha)(1-\lambda)\eta(tz)\eta(vz)$$

for all $z \in \mathcal{U}\backslash\{0\}$ and $|x| = 1$. Thus, we have

$$2(1-\lambda)(1-\alpha)z + J(q,\alpha,\lambda)z^n \neq 0$$

for $z \in \mathcal{U}\backslash\{0\}$ and $|x| = 1$; equivalently,

$$2(1-\lambda)(1-\alpha)z\left[1 + \frac{1}{2(1-\lambda)(1-\alpha)}(J(q,\alpha,\lambda))z^{n-1}\right] \neq 0, \tag{17}$$

where

$$J(q,\alpha,\lambda) = \sum_{n=2}^{\infty}\left\{\begin{array}{c} [n]_q a_n + \{(1-\lambda)(1-2\alpha)-\lambda\}C_n(t,v) \\ \\ +x([n]_q a_n - C_n(t,v)) \end{array}\right\}.$$

Dividing both sides of (17) by $2(1-\alpha)z$, we obtain

$$\left\{1 + \frac{1}{2(1-\lambda)(1-\alpha)}(J(q,\alpha,\lambda))z^{n-1}\right\} \neq 0$$

for $z \in \mathcal{U}\backslash\{0\}$ and $|x| = 1$, which completes our proof. $\square$

Remark 7. *Lemma 5 yields the well-known lemma proven in [3] when $\lambda = 0$ and $q \to 1-$.*

We divided this study into four sections. In Section 1, first we discussed some known subclasses of close-to-convex functions, the Carathéodory function, and the quantum difference operator. Considering this q-difference operator and generalized Janowski function, we defined a new class of close-to-convex functions as well as a class of close-to-convex functions of order α. In Section 2, we gave some lemmas for our planned research for the classes $K_s(M,N;q,t,v,\lambda)$ and $K_s(\alpha,q,t,v,\lambda)$. Using the aforementioned lemmas, we demonstrated several helpful findings in Section 3. In Section 4, we also created a new class of close-to-convex functions and studied the Fekete–Szegő inequality for the function belonging to these classes.

3. Main Results

Here, we will establish a theorem that offers a sufficient condition for functions f to be members of the class $K_s(M,N;q,t,v,\lambda)$.

Theorem 1. *Let the functions f and η be defined by (1) and (10). If*

$$(1+|N|)\sum_{n=2}^{\infty}[n]_q|a_n| + (1+|\{(1-\lambda)M+\lambda N\}|)\sum_{n=2}^{\infty}|C_n(t,v)| < (1-\lambda)(M-N),$$

where $C_n(t,v)$ is given by (12) and

$$-1 \le N < M \le 1,\ 0 \le \lambda < 1,\ t,v \in \mathbb{C}^*,\ q \in (0,1),\ |t| \le 1,\ |v| \le 1,$$

then $f \in K_s(M,N;q,t,v,\lambda)$.

Proof. This is for the functions f and η defined in (1) and (10). According to the (8), we know that if

$$\left|z\partial_q f(z) - \frac{\eta(tz)\eta(vz)}{tvz}\right| < \left|Nz\partial_q f(z) - \frac{\{(1-\lambda)M + \lambda N\}\eta(tz)\eta(vz)}{tvz}\right|,$$

then $f \in K_s(M, N; q, t, v, \lambda)$. Therefore, we set

$$\left|z\partial_q f(z) - \frac{\eta(tz)\eta(vz)}{tvz}\right| - \left|Nz\partial_q f(z) - \frac{\{(1-\lambda)M + \lambda N\}\eta(tz)\eta(vz)}{tvz}\right| < 0.$$

Let

$$\Delta = \left|z\partial_q f(z) - \frac{\eta(tz)\eta(vz)}{tvz}\right| - \left|Nz\partial_q f(z) - \frac{\{(1-\lambda)M + \lambda N\}\eta(tz)\eta(vz)}{tvz}\right|.$$

Using (6) and (11), we obtain

$$\begin{aligned}\Delta \quad = \quad & \left|\sum_{n=2}^{\infty}[n]_q a_n z^n - \sum_{n=2}^{\infty} C_n(t,v) z^n\right| \\ & -\left|(1-\lambda)(N-M)z + N\sum_{n=2}^{\infty}[n]_q a_n z^n - (\{(1-\lambda)M + \lambda N\})\sum_{n=2}^{\infty} C_n(t,v) z^n\right|.\end{aligned}$$

Applying the triangle inequality, and after some simple calculations, we obtain

$$\begin{aligned}\Delta \quad < \quad & -(1-\lambda)(N-M)|z| + (1+|N|)\sum_{n=2}^{\infty}[n]_q |a_n||z|^n \\ & +(1+|\{(1-\lambda)M + \lambda N\}|)\sum_{n=2}^{\infty}|C_n(t,v)||z|^n.\end{aligned}$$

Hence,

$$\begin{aligned}\Delta < \Bigg(& -(1-\lambda)(N-M) + (1+|N|)\sum_{n=2}^{\infty}[n]_q a_n \\ & +(1+|\{(1-\lambda)M + \lambda N\}|)\sum_{n=2}^{\infty} C_n(t,v)\Bigg), \quad z \in \mathcal{U}.\end{aligned}$$

Since $\Delta < 0$,

$$\begin{aligned} & (1+|N|)\sum_{n=2}^{\infty}[n]_q a_n + (1+|\{(1-\lambda)M + \lambda N\}|)\sum_{n=2}^{\infty} C_n(t,v)\Bigg) \\ < \quad & (1-\lambda)(N-M), \quad z \in \mathcal{U},\end{aligned}$$

and so we have

$$\left|z\partial_q f(z) - \frac{\eta(tz)\eta(vz)}{tvz}\right| - \left|Nz\partial_q f(z) - \frac{\{(1-\lambda)M + \lambda N\}\eta(tz)\eta(vz)}{tvz}\right|, \quad z \in \mathcal{U}.$$

Hence, from (8), we obtain $f \in K_s(M, N; q, t, v, \lambda)$. □

Remark 8. *The well-known result proven in [3] is obtained for $\lambda = 0$ and $q \to 1-$ in Theorem 1.*

Remark 9. *Taking $\lambda = 0$, $q \to 1-$, $t = 1$, $v = -1$, $M = 1 - 2\gamma$, $(0 \le \gamma < 1)$, and $N = -1$ in Theorem 1, we have the result proven in [22].*

The estimated coefficients are provided by the following theorem.

Theorem 2. *Let $n \geq 2$ and $-1 \leq N < M \leq 1$. Let an analytic function f be of form (1), $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$, as given by (10), and $f \in K_s(M,N;q,t,v,\lambda)$. Then,*

$$\begin{aligned} &\left|[n]_q a_n - C_n(t,v)\right|^2 - \left|(1-\lambda)(M-N)\right|^2 \\ < &\sum_{k=2}^{n-1} \left(\begin{array}{c} \left|N^2-1\right|[k]_q^2|a_k|^2 + \left|\{(1-\lambda)M+\lambda N\}^2-1\right||C_k(t,v)|^2 + \\ 2[k]_q|a_k C_k(t,v)||1-\{(1-\lambda)M+\lambda N\}N|, \end{array} \right) \end{aligned} \tag{18}$$

where the coefficients $C_n(t,v)$ are defined by (12).

Proof. Since $f \in K_s(M,N;q,t,v,\lambda)$, for some $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$, inequality (8) holds. From Lemma 4, we have

$$\frac{z\partial_q f(z)}{K(z)} = \frac{1-\{(1-\lambda)M+\lambda N\}z\varphi(z)}{1-Nz\varphi(z)} \quad z \in \mathcal{U}, \tag{19}$$

where φ is an analytic function in $\mathcal{U}$, $|\varphi(z)| \leq 1$ for $z \in \mathcal{U}$ and K is given by (11). Using the definitions of f and K given in (1) and (11), from (19), we are able to deduce that

$$\begin{aligned} &\left\{ -N\left(z + \sum_{n=2}^{\infty}[n]_q a_n z^n\right) \right. \\ &\left. +\{(1-\lambda)M+\lambda N\}\left(z+\sum_{n=2}^{\infty} C_n(t,v)z^n\right)\right\} z\phi(z) \\ = &\sum_{n=2}^{\infty} C_n(t,v)z^n - \sum_{n=2}^{\infty}[n]_q a_n z^n \quad z \in \mathcal{U}. \end{aligned} \tag{20}$$

Since the expansion of the function $z\varphi(z)$ is

$$z\varphi(z) = \sum_{n=1}^{\infty} t_n z^n, \qquad z \in \mathcal{U},$$

we can deduce from (20) that

$$\begin{aligned} &\left((1-\lambda)(M-N)z - N\sum_{n=2}^{\infty}[n]_q a_n z^n + \right. \\ &\left. \{(1-\lambda)M+\lambda N\}\sum_{n=2}^{\infty} C_n(t,v)z^n\right)\sum_{n=1}^{\infty} t_n z^n \\ = &\sum_{n=2}^{\infty} C_n(t,v)z^n - \sum_{n=2}^{\infty}[n]_q a_n z^n, \quad z \in \mathcal{U}. \end{aligned} \tag{21}$$

We can now obtain the following by equating the coefficient of z^n in (21):

$$\begin{aligned} &C_n(t,v) - [n]_q a_n \\ = &(1-\lambda)(M-N)t_{n-1} + (-[2]_q N a_2 + \{(1-\lambda)M+\lambda N\}C_2(t,v))t_{n-2} \\ &+(-[3]_q N a_3 + \{(1-\lambda)M+\lambda N\}C_3(t,v))t_{n-3} + \cdots \\ &+(-[n-1]_q N a_{n-1} + \{(1-\lambda)M+\lambda N\}C_{n-1}(t,v))t_1. \end{aligned}$$

Thus, the coefficient combination on the right hand side of (21) depends only upon the coefficient combinations

$$(-[2]_q N a_2 + \{(1-\lambda)M + \lambda N\}C_2(t,v)), \ldots, (-[n-1]_q N a_{n-1}$$

$$+\{(1-\lambda)M + \lambda N\}C_{n-1}(t,v)).$$

Therefore, we can write that, for $n \geq 2$,

$$\begin{aligned} &\left[(1-\lambda)(M-N)z + \sum_{k=2}^{n-1}(-N[k]_q a_k + \{(1-\lambda)M + \lambda N\}C_k(t,v))z^k\right] z\varphi(z) \\ = &\sum_{k=2}^{n}(C_k(t,v) - [k]_q a_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k, \;\; z \in \mathcal{U}. \end{aligned}$$

Applying the fact that $|z\varphi(z)| \leq |z| < 1$, we have

$$\begin{aligned} &\Bigg|(1-\lambda)(M-N)z + \sum_{k=2}^{n-1}(-N[k]_q a_k \\ &+\{(1-\lambda)M + \lambda N\}C_k(t,v))z^k\Bigg| \\ > &\left|\sum_{k=2}^{n}(C_k(t,v) - [k]_q a_k)z^k + \sum_{k=n+1}^{\infty} d_k z^k, \quad z \in \mathcal{U}\right|. \end{aligned}$$

Integration along $|z| = r$ $(0 < r < 1)$, after squaring the aforementioned inequality, we obtain

$$\begin{aligned} &\int_0^{2\pi}\Bigg|(1-\lambda)(M-N)re^{i\theta} + \\ &\sum_{k=2}^{n-1}(-N[k]_q a_k + \{(1-\lambda)M + \lambda N\}C_k(t,v))r^k e^{ik\theta 2}\Bigg| d\theta \\ > &\int_0^{2\pi}\left|\sum_{k=2}^{n}(C_k(t,v) - [k]_q a_k)r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} d_k r^k e^{ik\theta}\right|^2 d\theta. \end{aligned}$$

By using the Parseval inequality, we obtain

$$\begin{aligned} &|(1-\lambda)(M-N)|^2 r^2 + \sum_{k=2}^{n-1}\left|-N[k]_q a_k + \{(1-\lambda)M + \lambda N\}C_k(t,v)\right|^2 r^{2k} \\ > &\sum_{k=2}^{n}\left|[k]_q a_k - C_k(t,v)\right|^2 r^{2k} + \sum_{k=n+1}^{\infty}|d_k|^2 r^{2k}. \end{aligned}$$

Letting $r \to 1$, we obtain

$$\begin{aligned} &|(1-\lambda)(M-N)|^2 + \sum_{k=2}^{n-1}\left|-N[k]_q a_k + \{(1-\lambda)M + \lambda N\}C_k(t,v)\right|^2 \\ \geq &\sum_{k=2}^{n}\left|[k]_q a_k - C_k(t,v)\right|^2 + \sum_{k=n+1}^{\infty}|d_k|^2, \end{aligned}$$

which implies

$$
\begin{aligned}
& |(1-\lambda)(M-N)|^2 + \sum_{k=2}^{n-1} \left| -N[k]_q a_k + \{(1-\lambda)M + \lambda N\} C_k(t,v) \right|^2 \\
> \; & \sum_{k=2}^{n} \left| [k]_q a_k - C_k(t,v) \right|^2 .
\end{aligned}
$$

Hence, we deduce that

$$
\begin{aligned}
& |[n]_q a_n - C_n(t,v)|^2 - |(1-\lambda)(M-N)|^2 \\
< \; & \sum_{k=2}^{n-1} \left(\begin{array}{c} |N^2-1| [k]_q^2 |a_k|^2 + \left| \{(1-\lambda)M+\lambda N\}^2 - 1 \right| |C_k(t,v)|^2 + \\ 2[k]_q |a_k C_k(t,v)| |1 - \{(1-\lambda)M + \lambda N\} N| \end{array} \right).
\end{aligned}
$$

This leads us to inequality (18), which concludes our proof. □

Remark 10. *In Theorem 2, where $\lambda = 0$ and $q \to 1-$, we get the well-established result proven in [3].*

Remark 11. *Taking $q \to 1-$, $\lambda = 0$, $t = 1$, $v = -1$, $M = 1 - 2\gamma$, $(0 \le \gamma < 1)$, and $N = -1$ in Theorem 2, we obtain the well-established result proven in [22].*

Theorem 3. *If the function $f \in \mathcal{A}$ has form (1) and satisfies the condition*

$$
\begin{aligned}
& \sum_{n=2}^{\infty} \left| \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{l} ([k]_q a_k + \{(1-\lambda)(1-2\alpha) - \lambda\} C_k(t,v))(-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \\
& + \left| \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{l} ([k]_q a_k - C_k(t,v))(-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \\
< \; & 2(1-\alpha)(1-\lambda),
\end{aligned}
$$

where $0 \le \alpha < 1$, $0 \le \lambda < 1$, $q \in (0,1)$, $\gamma, \delta \in \mathbb{N}$, and the coefficients $C_n(t,v)$ are given by (12), then $f \in K_s(\alpha, q, t, v, \lambda)$.

Proof. In light of Lemma 5, to prove that $1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0$ for all $z \in \mathcal{U}$ and $|x| = 1$, where A_n is defined by (15), it is sufficient to show that

$$
\begin{aligned}
& \left(1 + \sum_{n=2}^{\infty} A_n z^{n-1}\right) (1-z)^{\gamma} (1+z)^{\delta} \\
= \; & 1 + \sum_{n=2}^{\infty} \left[\sum_{l=1}^{n} \left\{ \sum_{k=1}^{l} A_k (-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right] z^{n-1} \neq 0
\end{aligned}
$$

for all $z \in \mathcal{U}$ and $|x| = 1$, where $A_0 = 0$, $A_1 = 1$, and $\gamma, \delta \in \mathbb{N}$. Thus, if the function f satisfies

$$
\sum_{n=2}^{\infty} \left| \sum_{l=1}^{n} \left\{ \sum_{k=1}^{l} A_k (-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| < 1, \quad |x| = 1,
$$

that is, if

$$\begin{aligned}
& \frac{1}{2(1-\alpha)(1-\lambda)} \sum_{n=2}^{\infty} \left| \sum_{l=1}^{n} \left\{ \sum_{k=1}^{l} \left\{ [k]_q a_k + \{(1-\lambda)(1-2\alpha) - \lambda\} C_k(t,v) \right) \right. \right. \right. \\
& \left. \left. + x([k]_q a_k - C_k(t,v))(-1)^{l-k} \right\} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \Bigg| \\
\leq \ & \frac{1}{2(1-\alpha)(1-\lambda)} \sum_{n=2}^{\infty} \left| \sum_{l=1}^{n} \left\{ \sum_{k=1}^{l} \left\{ [k]_q a_k + \{(1-\lambda)(1-2\alpha) - \lambda\} C_k(t,v) \right) \right. \right. \right. \\
& \left. \left. (-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \\
& + |x| \left| \sum_{l=1}^{n} \left\{ \sum_{k=1}^{l} ([k]_q a_k - C_k(t,v))(-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \\
< \ & 1,
\end{aligned}$$

then $f \in K_s(\alpha, q, t, v, \lambda)$. The proof is finished. □

Remark 12. *The well-established conclusion proven in [3] for $\lambda = 0$ and $q \to 1-$ is obtained in Theorem 3.*

Letting $\gamma = \delta = 0$ in Theorem 3, we have a new result:

Corollary 1. *If the function $f \in \mathcal{A}$ has form (1) and satisfies the condition*

$$\begin{aligned}
& \sum_{n=2}^{\infty} \left(\left| [n]_q a_n + \{(1-\lambda)(1-2\alpha) - \lambda\} C_n(t,v) \right| + \left| [n]_q a_n + C_n(t,v) \right| \right) \\
< \ & 2(1-\lambda)(1-\alpha)
\end{aligned}$$

for some α $(0 \leq \alpha < 1)$, then $f \in K_s(\alpha, q, t, v, \lambda)$.

Remark 13. *Letting $\lambda = 0$, $q \to 1-$, and $\gamma = \delta = 0$ in Theorem 3, we obtain the known corollary, as proven in [3].*

A new result is found when $t = 1$ and $v = -1$ are substituted into Theorem 3.

Corollary 2. *If an analytic function of form (1) satisfies the condition*

$$\begin{aligned}
& \sum_{n=2}^{\infty} \left| \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{l} ([k]_q a_k + \{(1-\lambda)(1-2\alpha) - \lambda\} \right. \right. \\
& \left. \left. B_{2k-1})(-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| + \\
& \left| \sum_{l=1}^{\infty} \left\{ \sum_{k=1}^{l} ([k]_q a_k - B_{2k-1})(-1)^{l-k} \binom{\gamma}{l-k} \right\} \binom{\delta}{n-l} \right| \\
< \ & 2(1-\lambda)(1-\alpha),
\end{aligned}$$

where for some α $(0 \leq \alpha < 1), 0 \leq \lambda < 1$, $q \in (0,1)$, $\gamma, \delta \in \mathbb{N}$, $B_1 = 0$, and $B_{2k-1}(k = 2,3,4\ldots)$ are given by (13), then $f \in K_s(\alpha, q, \lambda) := K_s(\alpha; q, 1, -1, \lambda)$.

Remark 14. *In Theorem 2, if we fix $\lambda = 0$, $q \to 1-$, $t = 1$, and $v = -1$, we have a well-established corollary that was first proven in [3].*

We obtain a new result when we fix $\alpha = 0$ in Theorem 2

Corollary 3. *If the function $f \in \mathcal{A}$ has form (1) and satisfies the condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}([k]_q a_k + (1-2\lambda)B_{2k-1})(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right| + \right.$$

$$\left.\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}([k]_q a_k - B_{2k-1})(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right) < 2(1-\lambda),$$

where $\gamma, \delta \in \mathbb{N}$, $B_1 = 0$, and $B_{2k-1}(k = 2,3,4\ldots)$ are given by (13), then $f \in K_s(0; q, 1, -1, \lambda) = K_s(q, \lambda)$.

4. Fekete–Szegő Inequality

Here, we suppose that it is a positive-real-part analytic function, φ, that maps the unit disk $\mathcal{U}$ onto a starlike region, which is symmetric along the real axis and normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. In such a case, the function φ has an expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, B_1 > 0.$$

Definition 5. *Let an analytic function $f \in K_s(\varphi; q, t, v)$ if there exist $\eta \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that*

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} \prec \varphi(z)$$

where

$$0 < q < 1,\ t, v \in \mathbb{C}^*,\ |t| \le 1 \text{ and } |v| \le 1.$$

Theorem 4. *Let an analytic function f of form (1) is in the class $K_s(\varphi; q, t, v)$; then,*

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| \quad &< \quad \frac{1}{[3]_q}\max\left(B_1, \left|B_2 - \frac{B_1^2[3]_q\mu}{[2]_q^2}\right|\right) - \frac{tv}{[3]_q} \\ &\quad + B_1 c_1 b_2(t+v)\left(\frac{1}{2[3]_q} - \frac{\mu}{[2]_q^2}\right) \\ &\quad + (t+v)^2\left(\frac{b_3}{[3]_q} - \frac{\mu b_2^2}{[2]_q^2}\right),\end{aligned} \tag{22}$$

where

$$t, v \in \mathbb{C}^*,\ |t| \le 1 \text{ and } |v| \le 1.$$

Proof. Let $f \in K_s(\varphi; q, t, v)$, then

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} \prec \varphi(z), \quad z \in \mathcal{U}.$$

By using the definition of the subordination, we have

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)} = \varphi(w(z)), \quad z \in \mathcal{U}. \tag{23}$$

If

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\ldots,z\in\mathcal{U}, \tag{24}$$

then p is analytic and $Re(p(z)>0$ with $p(0)=1$, and from (24), we obtain

$$w(z)=c_1z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots,\ z\in\mathcal{U}. \tag{25}$$

Solving the L.H.S of (23), we have

$$\frac{tvz^2\partial_q f(z)}{\eta(tz)\eta(vz)}=1+d_1z+d_2z^2+\cdots,\ z\in\mathcal{U}. \tag{26}$$

This gives

$$d_1=[2]_q a_2-b_2(t+v),$$

$$d_2=[3]_q a_3-[2]_q a_2 b_2(t+v)-b_3(t^2+v^2)-b_2^2 tv+b_2^2(t+v)^2. \tag{27}$$

Taking the R.H.S of (23), and by using (25), we obtain

$$\varphi(w(z))=1+\frac{B_1c_1}{2}z\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2\right]z^2+\cdots,\ z\in\mathcal{U}. \tag{28}$$

Now, from (23), (26)–(28), we obtain

$$\frac{B_1c_1}{2}=[2]_q a_2-b_2(t+v)$$

and

$$\begin{aligned}&\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2\\ =\ &[3]_q a_3-[2]_q a_2 b_2(t+v)-b_3(t^2+v^2)-b_2^2 tv+b_2^2(t+v)^2.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}a_3-\mu a_2^2 \;=\;& \frac{1}{2[3]_q}B_1(c_2-vc_1^2)-\frac{2tv}{[3]_q}\left(b_3-\frac{b_2^2}{2}\right)\\ &+B_1c_1b_2(t+v)\left(\frac{1}{2[3]_q}-\frac{\mu}{[2]_q^2}\right)+(t+v)^2\left(\frac{b_3}{[3]_q}-\frac{\mu b_2^2}{[2]_q^2}\right),\end{aligned}$$

where

$$v=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{B_1[3]_q\mu}{[2]_q^2}\right).$$

Using Lemma 2 and the estimate that $\left|b_3-\frac{b_2^2}{2}\right|\le\frac{1}{2}$, for any analytic function $\eta(z)=z+b_2z^2+b_3z^3+\ldots,z\in\mathcal{U}$, which is starlike of the order $\frac{1}{2}$ (see [2]), we have

$$\left|a_3 - \mu a_2^2\right| < \frac{1}{[3]_q} \max\left(B_1, \left|B_2 - \frac{B_1^2[3]_q\mu}{[2]_q^2}\right|\right) - \frac{tv}{[3]_q} + B_1c_1b_2(t+v)\left(\frac{1}{2[3]_q} - \frac{\mu}{[2]_q^2}\right) + (t+v)^2\left(\frac{b_3}{[3]_q} - \frac{\mu b_2^2}{[2]_q^2}\right).$$

Hence, our result is proved. □

Remark 15. *The well-established result is obtained when $q \to 1-$, as shown in [3].*

Remark 16. *Cho et al. [2] derived a result by putting $q \to 1-$, $t = 1$, and $v = -1$ into the aforementioned theorem.*

Setting $\mu = 0$ in Theorem 4, we obtain the third coefficient bounds of the function $f \in K_s(\varphi; q, t, v)$:

Theorem 5. *Let $f \in K_s(\varphi; q, t, v)$; then,*

$$|a_3| < \frac{B_1}{[3]_q} \max\left\{1, \left|\frac{B_2}{B_1}\right|\right\} - \frac{tv}{[3]_q} + \frac{B_1c_1b_2(t+v)}{2[3]_q} + (t+v)^2\frac{b_3}{[3]_q}. \quad (29)$$

Proof. A similar technique can be used to prove this theorem as in Theorem 4. □

Remark 17. *Cho et al. [2] derived a result by putting $q \to 1-$, $t = 1$, and $v = -1$ in (29).*

Remark 18. *If we take $\mu \to \infty$ in (22), we obtain*

$$|a_2| < \sqrt{\frac{B_1^2}{[2]_q^2} - \frac{b_2(t+v)}{[2]_q^2}[B_1c_1 + (t+v)b_2]}. \quad (30)$$

Remark 19. *Cho et al. [2] derived a result by putting $q \to 1-$, $t = 1$, and $v = -1$ in (30).*

5. Conclusions

The introduction of new subclasses of analytic functions and the investigation of their properties, such as convexity, distortion properties, close-to-convexity, and coefficient bounds, are extremely important and necessary due to the wide range of fields in which they are applied, including mathematics, physics, electronics, mechanics, and many others. Recently, the field of geometric function theory has garnered more interest due to the q-analogous nature of operator theory, and many researchers have begun to use the q-calculus operator theory for a variety of subclasses of analytic functions. Therefore, in this article, we studied new applications of the q-difference operator for new subclasses of close-to-convex functions. We identified and examined a novel subclass of close-to-convex functions associated with generalized Janowski functions while taking the quantum difference operator into consideration. We established the inclusion relation and coefficient estimates, as well as the necessary and sufficient criteria for functions that are members of recently established classes. We also explored the Fekete–Szegő problem for a more general class.

Researchers can create new q-analogous differential operators, consider these operators to define new subclauses of analytic functions, and explore these findings using those newly defined operators as potential avenues for future study. Furthermore, the recently presented classes may exhibit intriguing convexity, starlikeness, and related features. Furthermore, the concepts investigated in this work offer potential for extending to other operators such as the symmetric quantum difference operator. Researchers can consider the symmetric quantum difference operator to define new classes of close-to-convex functions, and the same type of results can be investigated through the use of a symmetric q-difference operator.

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