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New Versions of Fuzzy-Valued Integral Inclusion over p-Convex Fuzzy Number-Valued Mappings and Related Fuzzy Aumman's Integral Inequalities

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Abstract: It is well known that both concepts of symmetry and convexity are directly connected. Similarly, in fuzzy theory, both ideas behave alike. It is important to note that real and interval-valued mappings are exceptional cases of fuzzy number-valued mappings (FNVMs) because fuzzy theory depends upon the unit interval that make a significant contribution to overcoming the issues that arise in the theory of interval analysis and fuzzy number theory. In this paper, the new class of p-convexity over up and down (UD) fuzzy relation has been introduced which is known as UD-p-convex fuzzy number-valued mappings (UD-p-convex FNVMs). We offer a thorough analysis of Hermite–Hadamard-type inequalities for FNVMs that are UD-p-convex using the fuzzy Aumann integral. Some previous results from the literature are expanded upon and broadly applied in our study. Additionally, we offer precise justifications for the key theorems that Kunt and İşcan first deduced in their article titled "Hermite–Hadamard–Fejer type inequalities for p-convex functions". Some new and classical exceptional cases are also discussed. Finally, we illustrate our findings with well-defined examples.

Keywords: fuzzy number-valued p-convexity; up and down fuzzy relation; fuzzy number-valued mappings; integral inequalities

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1. Introduction

Since the discovery of the first convex inequality, also referred to as the Jensen inequality, convex inequalities have been a hotly debated subject in mathematics. There are many inequalities that are derived using convexity; for example, see the books [1,2]. See the works [3–14] for further information on the applications of inequality to diverse areas of mathematics, such as numerical analysis, probability density functions, and optimization. It should be noted that L'Hospital and Leibniz first proposed the concept of fractional calculus in 1695. Numerous mathematicians, including Riemann, Grunwald, Letnikov, Hadamard, and Weyl, expanded on this idea. These mathematicians contributed significantly to fractional calculus and its many applications. For further information on fractional calculus, see [15–27]. In the modern era, fractional calculus is frequently used to describe a variety of phenomena, such as the fractional conservation of mass, and the fractional Schrodinger equation in quantum theory. One of the inequalities that has garnered the most interest in the mathematical community is the Hermite–Hadamard inequality [28], which was independently proven by Charles Hermite and Jacques Hadamard. Numerous mathematicians have generalized this inequality in numerous ways. For more related results, see [29–44]. The inequality is written as if $\mathcal{T} : I \to \mathbb{R}$ is a convex function on I and $\imath, \jmath \in I$ with $\imath \leq \jmath$ such that:

$$\mathcal{T}\left(\frac{\imath+\jmath}{2}\right) \leq \frac{1}{\jmath-\imath}\int_{\imath}^{\jmath} \mathcal{T}(\omega)d\omega \leq \frac{\mathcal{T}(\imath)+\mathcal{T}(\jmath)}{2}. \tag{1}$$

New variations of these disparities have been obtained in recent years using various creative ways. For instance, the Hermite–Hadamard inequality's first fractional analog was discovered by Sarikaya et al. [45]. Ramik [46] used fuzzy numbers to derive inequalities in 1985 and applied the inequality in fuzzy optimization. The concept of FNVMs was created by the authors in [47] using Jensen-type inequalities. Costa et al.'s [48] computation of fresh integral inequality uses the idea of FNVMs. For more information, see [49–59] and the references therein. In order to look at inequalities of the Jensen and Hermite–Hadamard types, Zhao et al. [60] introduced the concept of generalized interval-valued convexity. Liu et al. used J-inclusions to study the modular inequalities of interval-valued soft sets in [61]. Yang et al.'s [62] formulation of novel Hermite–Hadamard-type inequalities in conjunction with exponential FNVM was published in 2021. The authors of [63] computed Ostrowski-type inequalities and applied them to numerical integration using the concept of interval-valued mappings. Santos–Gomez used fuzzy number-valued pre-invex functions in [64] to study coordinated inequalities. In [65], Khan et al. developed new harmonically FNVM-based Hermite–Hadamard inclusions. According to the authors of [66], some Hermite–Hadamard inequalities and their weighted variants, referred to as Fejer-type inclusions, involve generalized fractional operators with an exponential kernel. See [67–82] for contemporary research and applications involving the Hermite–Hadamard inequality.

This inequality was discovered by many scholars as a result of the generalization employing various types of convexity with fractional operators [83–89]. The works go into further detail concerning the Hermite–Hadamard inequality and UD-p-convex inequality. In the current study, several UD-p-convex inequalities are derived together with fuzzy Aumann integral operators using the fuzzy number-valued settings and newly defined fuzzy UD-convexity. In this study, the recent findings of Kunt and İşcan [90] are generalized and several exceptional cases are discussed.

2. Preliminaries

We will go through the fundamental terminologies and findings in this section, which aid in comprehending the ideas behind our fresh findings.

Let $\mathcal{L}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$, and $\pitchfork \in \mathcal{L}_C$ be defined by

$$\pitchfork = [\pitchfork_*, \pitchfork^*] = \{\omega \in \mathbb{R} | \pitchfork_* \leq \omega \leq \pitchfork^*\}, (\pitchfork_*, \pitchfork^* \in \mathbb{R}). \tag{2}$$

If $\pitchfork_* = \pitchfork^*$, then $\pitchfork$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $\pitchfork_* \geq 0$, then $[\pitchfork_*, \pitchfork^*]$ is called a positive interval. The set of all positive intervals is denoted by $\mathcal{L}_C^+$ and defined as

$$\mathcal{L}_C^+ = \{[\pitchfork_*, \pitchfork^*] : [\pitchfork_*, \pitchfork^*] \in \mathcal{L}_C \; and \; \pitchfork_* \geq 0\}. \tag{3}$$

Let $\tau \in \mathbb{R}$ and $\tau \cdot \pitchfork$ be defined by

$$\tau \cdot \pitchfork = \begin{cases} [\tau\pitchfork_*, \; \tau\pitchfork^*] \; if \; \tau > 0, \\ \{0\} \quad if \; \tau = 0, \\ [\tau\pitchfork^*, \tau\pitchfork_*] \; if \; \tau < 0. \end{cases} \tag{4}$$

Then the Minkowski difference $Ð - \pitchfork$, addition $\pitchfork + Ð$, and $\pitchfork \times Ð$ for $\pitchfork, Ð \in \mathcal{L}_C$ are defined by

$$[Ð_*, Ð^*] + [\pitchfork_*, \pitchfork^*] = [Ð_* + \pitchfork_*, Ð^* + \pitchfork^*], \tag{5}$$

$$[Ð_*, Ð^*] \times [⋒_*, ⋒^*] = \left[min\left\{Ð_*⋒_*, Ð^*⋒_*, Ð_*⋒^*, Ð^*⋒^*\right\}, max\left\{Ð_*⋒_*, Ð^*⋒_*, Ð_*⋒^*, Ð^*⋒^*\right\}\right], \tag{6}$$

$$[Ð_*, Ð^*] - [⋒_*, ⋒^*] = \left[Ð_* - ⋒^*, Ð^* - ⋒_*\right]. \tag{7}$$

Remark 1. *(i) For given $[Ð_*,\ Ð^*]$, $[⋒_*,\ ⋒^*] \in \mathcal{L}_C$, the relation " $\supseteq_I$ " defined on $\mathcal{L}_C$ by $[⋒_*,\ ⋒^*] \supseteq_I [Ð_*,\ Ð^*]$ if and only if $⋒_* \leq Ð_*$, $Ð^* \leq ⋒^*$ for all $[Ð_*,\ Ð^*]$, $[⋒_*,\ ⋒^*] \in \mathcal{L}_C$ is a partial interval inclusion relation. The relation $[⋒_*,\ ⋒^*] \supseteq_I [Ð_*,\ Ð^*]$ is coincident to $[⋒_*,\ ⋒^*] \supseteq [Ð_*,\ Ð^*]$ on $\mathcal{L}_C$. It can be easily seen that "$\supseteq_I$" looks like "up and down" on the real line $\mathbb{R}$, so we call " $\supseteq_I$ " "up and down" (or "UD" order, in short) [80]. For $[Ð_*,\ Ð^*]$, $[⋒_*,\ ⋒^*] \in \mathcal{L}_C$, the Hausdorff–Pompeiu distance between intervals $[Ð_*,\ Ð^*]$ and $[⋒_*,\ ⋒^*]$ is defined by*

$$d_H([Ð_*, Ð^*], [⋒_*, ⋒^*]) = max\left\{\left|Ð_* - ⋒_*\right|, \left|Ð^* - ⋒^*\right|\right\}. \tag{8}$$

It is a familiar fact that $(\mathcal{L}_C, d_H)$ is a complete metric space [74–76].

Noting that we will be using the traditional definitions of fuzzy set and fuzzy numbers, we will only review some fundamental ideas about fuzzy set and fuzzy numbers. Be mindful that we refer to the set of all fuzzy subsets and fuzzy numbers of $\mathbb{R}$ as $\mathcal{L}$, and $\mathcal{L}_C$.

Definition 1 ([78,79]). *Given $\widetilde{f} \in \mathcal{L}_C$, the level sets or cut sets are given by $\left[\widetilde{f}\right]^\tau = \left\{\omega \in \mathbb{R}\middle|\widetilde{f}(\omega) \geq \tau\right\}$ for all $\tau \in (0,\ 1]$ and by $\left[\widetilde{f}\right]^0 = \boldsymbol{cl}\left\{\omega \in \mathbb{R}\middle|\widetilde{f}(\omega) > 0\right\}$, where $\left[\widetilde{f}\right]^0$ is known as support of $\widetilde{f}$. These sets are known as τ-level sets or τ-cut sets of $\widetilde{f}$.*

Definition 2 ([48]). *Let $\widetilde{f}, \widetilde{g} \in \mathcal{L}_C$. Then, relation " $\leq_{\mathbb{F}}$ " is given on $\mathcal{L}_C$ by $\widetilde{f} \leq_{\mathbb{F}} \widetilde{g}$ when and only when $\left[\widetilde{f}\right]^\tau \leq_I [\widetilde{g}]^\tau$, for every $\tau \in [0,\ 1]$, which are left- and right-order relations.*

Definition 3 ([77]). *Let $\widetilde{f}, \widetilde{g} \in \mathcal{L}_C$. Then, relation " $\supseteq_{\mathbb{F}}$ " is given on $\mathcal{L}_C$ by $\widetilde{f} \supseteq_{\mathbb{F}} \widetilde{g}$ when and only when $\left[\widetilde{f}\right]^\tau \supseteq_I [\widetilde{g}]^\tau$ for every $\tau \in [0,\ 1]$, which is the $UD-$order relation on $\mathcal{L}_C$.*

Remember the approaching notions, which are offered in the literature. If $\widetilde{f}, \widetilde{g} \in \mathcal{L}_C$, and $\tau \in \mathbb{R}$, then, for every $\tau \in [0,\ 1]$, the arithmetic operations addition "$\oplus$", multiplication "$\otimes$", and scaler multiplication "$\odot$" are defined by

$$\left[\widetilde{f} \oplus \widetilde{g}\right]^\tau = \left[\widetilde{f}\right]^\tau + [\widetilde{g}]^\tau, \tag{9}$$

$$\left[\widetilde{f} \otimes \widetilde{g}\right]^\tau = \left[\widetilde{f}\right]^\tau \times [\widetilde{g}]^\tau, \tag{10}$$

$$\left[\mathcal{T} \odot \widetilde{f}\right]^\tau = \mathcal{T}\left[\widetilde{f}\right]^\tau, \tag{11}$$

Equations (4)–(6) directly relate to these processes.

Definition 4 ([69]). *Let H be a Hausdorff metric. Then a supremum metric is handled by the space $\mathcal{L}_C$; that is, for each $\widetilde{f},\ \widetilde{g} \in \mathcal{L}_C$, the whole metric space is represented by the formula*

$$d_\infty\left(\widetilde{f},\ \widetilde{g}\right) = \sup_{0 \leq \tau \leq 1} d_H\left(\left[\widetilde{f}\right]^\tau,\ [\widetilde{g}]^\tau\right), \tag{12}$$

Theorem 1 ([75,78]). *$\mathcal{T}$ is an Aumann integrable (IA integrable) over $[\imath,\ \jmath]$ when and only when $\mathcal{T}_*(\omega)$ and $\mathcal{T}^*(\omega)$ both are integrable over $[\imath,\ \jmath]$, such that*

$$(IA)\int_\imath^\jmath \mathcal{T}(\omega)d\omega = \left[\int_\imath^\jmath \mathcal{T}_*(\omega)d\omega,\ \int_\imath^\jmath \mathcal{T}^*(\omega)d\omega\right], \tag{13}$$

where $\mathcal{T} : [\imath, \jmath] \subset \mathbb{R} \to \mathcal{L}_C$ is an interval-valued mapping (IVM) fulfilling that $\mathcal{T}(\omega) = [\mathcal{T}_*(\omega), \mathcal{T}^*(\omega)]$.

The literature supports the following inferences [47,48,70,72,73]:

Definition 5. **([48]).** *A fuzzy interval-valued map $\widetilde{\mathcal{T}} : K \subset \mathbb{R} \to \mathcal{L}_C$ is called FNVM. For each $\tau \in (0, 1]$, its IVMs are classified according to their τ-cuts $\mathcal{T}_\tau : K \subset \mathbb{R} \to \mathcal{L}_C$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau), \mathcal{T}^*(\omega, \tau)]$ for all $\omega \in K$. Here, for each $\tau \in (0, 1]$, the end point real functions $\mathcal{T}_*(., \tau), \mathcal{T}^*(., \tau) : K \to \mathbb{R}$ are called lower and upper functions of $\widetilde{\mathcal{T}}(\omega)$.*

Definition 6. *Let $\widetilde{\mathcal{T}} : [\imath, \jmath] \subset \mathbb{R} \to \mathcal{L}_C$ be a FNVM. Then, fuzzy integral of $\widetilde{\mathcal{T}}$ over $[\imath, \jmath]$, denoted by $(FA)\int_\imath^\jmath \widetilde{\mathcal{T}}(\omega)d\omega$, is given level-wise by*

$$\left[(FA)\int_\imath^\jmath \widetilde{\mathcal{T}}(\omega)d\omega\right]^\tau = (IA)\int_\imath^\jmath \mathcal{T}_\tau(\omega)d\omega = \left\{\int_\imath^\jmath \mathcal{T}(\omega, \tau)d\omega : \mathcal{T}(\omega, \tau) \in \mathcal{R}_{([\imath, \jmath], \tau)}\right\}, \quad (14)$$

for all $\tau \in (0, 1]$, where $\mathcal{R}_{([\imath, \jmath], \tau)}$ denotes the collection of Riemannian integrable functions of IVMs. The FNVM $\widetilde{\mathcal{T}}$ is FA-integrable over $[\imath, \jmath]$ if $(FA)\int_\imath^\jmath \widetilde{\mathcal{T}}(\omega)d\omega \in \mathcal{L}_C$. Note that, if $\mathcal{T}_*(\omega, \tau), \mathcal{T}^*(\omega, \tau)$ are Lebesgue-integrable, then $\mathcal{T}$ is fuzzy Aumann-integrable function over $[\imath, \jmath]$, see [47].

Theorem 2. *Let $\widetilde{\mathcal{T}} : [\imath, \jmath] \subset \mathbb{R} \to \mathcal{L}_C$ be a FNVM, its IVM s are classified according to their τ-cuts $\mathcal{T}_\tau : [\imath, \jmath] \subset \mathbb{R} \to \mathcal{L}_C$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau), \mathcal{T}^*(\omega, \tau)]$ for all $\omega \in [\imath, \jmath]$ and for all $\tau \in (0, 1]$. Then, $\widetilde{\mathcal{T}}$ is FA-integrable over $[\imath, \jmath]$ if and only if, $\mathcal{T}_*(\omega, \tau)$ and $\mathcal{T}^*(\omega, \tau)$ are both A-integrable over $[\imath, \jmath]$. Moreover, if $\widetilde{\mathcal{T}}$ is FA-integrable over $[\imath, \jmath]$, then*

$$\begin{aligned}\left[(FA)\int_\imath^\jmath \widetilde{\mathcal{T}}(\omega)d\omega\right]^\tau &= \left[(A)\int_\imath^\jmath \mathcal{T}_*(\omega, \tau)d\omega, (A)\int_\imath^\jmath \mathcal{T}^*(\omega, \tau)d\omega\right] \\ &= (IA)\int_\imath^\jmath \mathcal{T}_\tau(\omega)d\omega,\end{aligned} \quad (15)$$

for all $\tau \in (0, 1]$. For all $\tau \in (0, 1]$, $\mathcal{FA}_{([\imath, \jmath], \tau)}$ denotes the collection of all FA-integrable FNVMs over $[\imath, \jmath]$.

Definition 7 **([80]).** *Let $[\imath, \jmath]$ be a convex interval. Then, FNVM $\widetilde{\mathcal{T}} : [\imath, \jmath] \to \mathcal{L}_C$ is said to be UD-convex on $[\imath, \jmath]$ if*

$$\widetilde{\mathcal{T}}(\hbar\omega + (1-\hbar)\mathfrak{v}) \supseteq_{\mathbb{F}} \hbar \odot \widetilde{\mathcal{T}}(\omega) \oplus (1-\hbar) \odot \widetilde{\mathcal{T}}(\mathfrak{v}), \quad (16)$$

for all $\omega, \mathfrak{v} \in [\imath, \jmath]$, $\hbar \in [0, 1]$, where $\widetilde{\mathcal{T}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$, for all $\omega \in [\imath, \jmath]$. If inequality (16) is reversed, then $\widetilde{\mathcal{T}}$ is said to be UD-concave FNVM on $[\imath, \jmath]$. The set of all UD-convex (UD-concave) FNVM s is denoted by

$$UDFSX([\imath, \jmath], \mathcal{L}_C), (UDFSV([\imath, \jmath], \mathcal{L}_C)).$$

Definition 8. *Let $[\imath, \jmath]$ be a p-convex interval. Then, FNVM $\widetilde{\mathcal{T}} : [\imath, \jmath] \to \mathcal{L}_C$ is said to be UD-p-convex on $[\imath, \jmath]$ if*

$$\widetilde{\mathcal{T}}\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \hbar \odot \widetilde{\mathcal{T}}(\omega) \oplus (1-\hbar) \odot \widetilde{\mathcal{T}}(\mathfrak{v}), \quad (17)$$

for all $\omega, \mathfrak{v} \in [\imath, \jmath]$, $\hbar \in [0, 1]$, where $\widetilde{\mathcal{T}}(\omega) \geq_{\mathbb{F}} \widetilde{0}$, for all $\omega \in [\imath, \jmath]$. If inequality (17) is reversed, then $\widetilde{\mathcal{T}}$ is said to be UD-p-concave FNVM on $[\imath, \jmath]$. The set of all UD-p-convex (UD-p-concave) FNVM s is denoted by

$$UDFSX([\imath, \jmath], \mathcal{L}_C, p), (UDFSV([\imath, \jmath], \mathcal{L}_C, p)).$$

Remark 2. *If $p \equiv 1$, then UD-p-convex FNVM becomes UD-convex FNVM, see Definition 7.*

When $p \equiv -1$, then inequality (17) is converted into inequality obtained from the definition of harmonically UD-convex FNVMs.

The following results discuss the characterization of definition of UD-convex FNVM

Theorem 3. *Let $[\imath,\ \jmath]$ be a convex set, and $\widetilde{\mathcal{T}} : [\imath,\ \jmath] \to \mathcal{L}_C$ be a FNVM. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to {\mathcal{K}_C}^+ \subset \mathcal{K}_C$ are given by*

$$\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau),\ \mathcal{T}^*(\omega, \tau)],\ \forall\ \omega \in [\imath,\ \jmath] \tag{18}$$

for all $\omega \in [\imath,\ \jmath]$ and for all $\tau \in [0,\ 1]$. Then, $\widetilde{\mathcal{T}}$ is UD-p-convex on $[\imath,\ \jmath]$, if and only if, for all $\tau \in [0,\ 1]$, $\mathcal{T}_*(\omega,\ \tau)$ is p-convex and $\mathcal{T}^*(\omega,\ \tau)$ is p-concave functions.

Proof. Assume that for each $\tau \in [0,\ 1]$, $\mathcal{T}_*(\omega,\ \tau)$ is p-convex and $\mathcal{T}^*(\omega,\ \tau)$ is p-concave functions on $[\imath,\ \jmath]$. Then, from (17) we have

$$\mathcal{T}_*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right) \le \hbar\mathcal{T}_*(\omega,\ \tau) + (1-\hbar)\mathcal{T}_*(\mathfrak{v},\ \tau),\ \forall\ \omega, \mathfrak{v} \in [\imath,\ \jmath],\ \hbar \in [0,\ 1],$$

and

$$\mathcal{T}^*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right) \ge \hbar\mathcal{T}^*(\omega,\ \tau) + (1-\hbar)\mathcal{T}^*(\mathfrak{v},\ \tau), \forall\ \omega, \mathfrak{v} \in [\imath,\ \jmath],\ \hbar \in [0,\ 1].$$

Then, by (9), (11), and (18), we obtain

$$\begin{aligned} \mathcal{T}_\tau\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\right) &= \left[\mathcal{T}_*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right),\ \mathcal{T}^*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}},\ \tau\right)\right], \\ &\supseteq_I [\hbar\mathcal{T}_*(\omega,\ \tau),\ \hbar\mathcal{T}^*(\omega,\ \tau)] + \left[(1-\hbar)\mathcal{T}_*(\mathfrak{v},\ \tau),\ (1-\hbar)\mathcal{T}^*(\mathfrak{v},\ \tau)\right], \end{aligned}$$

that is

$$\widetilde{\mathcal{T}}\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \hbar \odot \widetilde{\mathcal{T}}(\omega) \oplus (1-\hbar) \odot \widetilde{\mathcal{T}}(\mathfrak{v}), \forall\ \omega, \mathfrak{v} \in [\imath,\ \jmath],\ \hbar \in [0,\ 1].$$

Hence, $\widetilde{\mathcal{T}}$ is UD-p-convex FNVM on $[\imath, \jmath]$.

Conversely, let $\widetilde{\mathcal{T}}$ be UD-p-convex FNVM on $[\imath,\ \jmath]$. Then, for all $\omega, \mathfrak{v} \in [\imath,\ \jmath]$, and $\hbar \in [0,\ 1]$, we have

$$\widetilde{\mathcal{T}}\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \hbar \odot \widetilde{\mathcal{T}}(\omega) \oplus (1-\hbar) \odot \widetilde{\mathcal{T}}(\mathfrak{v}).$$

Therefore, from (18), we have

$$\mathcal{T}_\tau\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\right) = \left[\mathcal{T}_*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right),\ \mathcal{T}^*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right)\right].$$

Again, from (9), (11), and (18), we obtain

$$\hbar\mathcal{T}_\tau(\omega) + (1-\hbar)\mathcal{T}_\tau(\mathfrak{v}) = [\hbar\mathcal{T}_*(\omega,\ \tau),\ \hbar\mathcal{T}^*(\omega,\ \tau)] + \left[(1-\hbar)\mathcal{T}_*(\mathfrak{v},\ \tau),\ (1-\hbar)\mathcal{T}^*(\mathfrak{v},\ \tau)\right],$$

for all $\omega, \mathfrak{v} \in [\imath,\ \jmath]$ and $\hbar \in [0,\ 1]$. Then, by UD-p-convexity of $\widetilde{\mathcal{T}}$, we have for all $\omega, \mathfrak{v} \in [\imath,\ \jmath]$ and $\hbar \in [0,\ 1]$ such that

$$\mathcal{T}_*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right) \le \hbar\mathcal{T}_*(\omega,\ \tau) + (1-\hbar)\mathcal{T}_*(\mathfrak{v},\ \tau),$$

and

$$\mathcal{T}^*\left([\hbar\omega^p + (1-\hbar)\mathfrak{v}^p]^{\frac{1}{p}}\ ,\ \tau\right) \ge \hbar\mathcal{T}^*(\omega,\ \tau) + (1-\hbar)\mathcal{T}^*(\mathfrak{v},\ \tau),$$

for each $\tau \in [0,\ 1]$. Hence, the result follows. □

Example 1. *We consider the FNVM* $\mathcal{T} : [0,\ 1] \to \mathcal{L}_C$ *defined by,*

$$\mathcal{T}(\omega)(\lambda) = \begin{cases} \frac{\lambda}{2\omega^2},\ \lambda \in \left[0,\ 2\omega^2\right] \\ \frac{4\omega^2-\lambda}{2\omega^2},\ \lambda \in \left(2\omega^2,\ 4\omega^2\right] \\ 0,\ otherwise, \end{cases}$$

then, for each $\tau \in [0,\ 1]$, *we have* $\mathcal{T}_\tau(\omega) = \left[2\tau\omega^2, (4-2\tau)\omega^2\right]$. *Since end point functions,* $\mathcal{T}_*(\omega,\ \tau)$ *is p-convex and* $\mathcal{T}^*(\omega,\ \tau)$ *is p-concave functions.for each* $\tau \in [0,\ 1]$. *Hence* $\widetilde{\mathcal{T}}(\omega)$ *is UD-p-convex FNVM.*

Remark 3. *If* $\mathcal{T}_*(\omega\ ,\ \tau) = \mathcal{T}^*(\omega\ ,\ \tau)$, *then Definition 6 cuts down to the definition of classical p-convex function,* [90].

If $\mathcal{T}_*(\omega,\ \tau) = \mathcal{T}^*(\omega,\ \tau)$ *and* $p \equiv 1$ *, then definition 6 cuts down to the definition of classical convex function.*

3. Jensen's and Schur's Type Inequalities

We first provide a new ideal inequality known as discrete Jensen's type inequality for UD-*p*-convex FNVM. This is how it is explained.

Theorem 4. *Let* $\hbar_\updownarrow \in \mathbb{R}^+$, $\imath_\updownarrow \in [\imath,\ \jmath]$, $(\updownarrow = 1,\ 2,\ 3,\ \ldots, k,\ k \geq 2)$ *and* $\widetilde{\mathcal{T}} \in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$ *and for all* $\tau \in [0,\ 1]$, *the family of IVMs is defined by* τ*-cuts* $\mathcal{T}_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *are given by* $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ *for all* $\omega \in [\imath,\ \jmath]$. *Then,*

$$\widetilde{\mathcal{T}}\left(\left[\frac{1}{W_k}\sum_{\updownarrow=1}^{k} \hbar_\updownarrow \imath_\updownarrow{}^p\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \sum_{\updownarrow}^{k} \frac{\hbar_\updownarrow}{W_k} \odot \widetilde{\mathcal{T}}\left(\imath_\updownarrow\right), \tag{19}$$

where $W_k = \sum_{\updownarrow=1}^{k} \hbar_\updownarrow$.*If* $\widetilde{\mathcal{T}}$ *is UD-p-concave, then inequality (19) is reversed.*

Proof. When $k = 2$, then inequality (19) is true. Consider inequality (19) is true for $k = n - 1$, then

$$\widetilde{\mathcal{T}}\left(\left[\frac{1}{W_{n-1}}\sum_{\updownarrow=1}^{n-1} \hbar_\updownarrow \imath_\updownarrow{}^p\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \sum_{\updownarrow=1}^{n-1} \frac{\hbar_\updownarrow}{W_{n-1}} \widetilde{\mathcal{T}}\left(\imath_\updownarrow\right).$$

Now, let us prove that inequality (19) holds for $k = n$.

$$\widetilde{\mathcal{T}}\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n} \hbar_\updownarrow \imath_\updownarrow{}^p\right]^{\frac{1}{p}}\right) = \widetilde{\mathcal{T}}\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n-2} \hbar_\updownarrow \imath_\updownarrow{}^p + \frac{\hbar_{n-1} + \hbar_n}{W_n}\left(\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\imath_{n-1}{}^p + \frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\imath_n{}^p\right]^{\frac{1}{p}}\right).$$

Therefore, for each $\tau \in [0,\ 1]$, we have

$$
\begin{aligned}
&\mathcal{T}_*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}},\tau\right)\\
&\mathcal{T}^*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}},\tau\right)\\
&=\mathcal{T}_*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n-2}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}+\frac{\hbar_{n-1}+\hbar_n}{W_n}\left(\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\imath_{n-1}{}^{p}+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\imath_n{}^{p}\right]^{\frac{1}{p}},\tau\right),\\
&=\mathcal{T}^*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n-2}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}+\frac{\hbar_{n-1}+\hbar_n}{W_n}\left(\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\imath_{n-1}{}^{p}+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\imath_n{}^{p}\right]^{\frac{1}{p}},\tau\right),\\
&\le\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right)+\frac{\hbar_{n-1}+\hbar_n}{W_n}\mathcal{T}_*\left(\left[\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\imath_{n-1}{}^{p}+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\imath_n{}^{p}\right]^{\frac{1}{p}},\tau\right),\\
&\ge\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right)+\frac{\hbar_{n-1}+\hbar_n}{W_n}\mathcal{T}^*\left(\left[\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\imath_{n-1}{}^{p}+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\imath_n{}^{p}\right]^{\frac{1}{p}},\tau\right),\\
&\le\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right)+\frac{\hbar_{n-1}+\hbar_n}{W_n}\left[\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\mathcal{T}_*(\imath_{n-1},\tau)+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\mathcal{T}_*(\imath_n,\tau)\right],\\
&\ge\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right)+\frac{\hbar_{n-1}+\hbar_n}{W_n}\left[\frac{\hbar_{n-1}}{\hbar_{n-1}+\hbar_n}\mathcal{T}^*(\imath_{n-1},\tau)+\frac{\hbar_n}{\hbar_{n-1}+\hbar_n}\mathcal{T}^*(\imath_n,\tau)\right],\\
&\le\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right)+\left[\frac{\hbar_{n-1}}{W_n}\mathcal{T}_*(\imath_{n-1},\tau)+\frac{\hbar_n}{W_n}\mathcal{T}_*(\imath_n,\tau)\right],\\
&\ge\sum_{\updownarrow=1}^{n-2}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right)+\left[\frac{\hbar_{n-1}}{W_n}\mathcal{T}^*(\imath_{n-1},\tau)+\frac{\hbar_n}{W_n}\mathcal{T}^*(\imath_n,\tau)\right],\\
&=\sum_{\updownarrow=1}^{n}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right),\\
&=\sum_{\updownarrow=1}^{n}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right).
\end{aligned}
$$

From which, we have

$$\left[\mathcal{T}_*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}},\tau\right),\ \mathcal{T}^*\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}},\tau\right)\right]\supseteq_I\left[\sum_{\updownarrow=1}^{n}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right),\ \sum_{\updownarrow=1}^{n}\frac{\hbar_{\updownarrow}}{W_n}\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right)\right],$$

that is

$$\widetilde{\mathcal{T}}\left(\left[\frac{1}{W_n}\sum_{\updownarrow=1}^{n}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}}\right)\supseteq_{\mathbb{F}}\sum_{\updownarrow=1}^{n}\frac{\hbar_{\updownarrow}}{W_n}\odot\widetilde{\mathcal{T}}\left(\imath_{\updownarrow}\right),$$

and the result follows. □

If $\hbar_1=\hbar_2=\hbar_3=\ldots=\hbar_k=1$, then from (19) we obtain following outcome:

Corollary 1. *Let $\imath_{\updownarrow}\in[\imath,\ \jmath]$, $(\updownarrow=1,\ 2,\ 3,\ \ldots,k,\ k\ge 2)$, and $\widetilde{\mathcal{T}}\in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau:[\imath,\ \jmath]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are given by $\mathcal{T}_\tau(\omega)=[\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ for all $\omega\in[\imath,\ \jmath]$, and for all $\tau\in[0,\ 1]$. Then,*

$$\widetilde{\mathcal{T}}\left(\left[\frac{1}{W_k}\sum_{\updownarrow=1}^{k}\hbar_{\updownarrow}\imath_{\updownarrow}{}^{p}\right]^{\frac{1}{p}}\right)\supseteq_{\mathbb{F}}\sum_{\jmath=1}^{k}\frac{1}{k}\odot\widetilde{\mathcal{T}}\left(\imath_{\updownarrow}\right).\tag{20}$$

If $\widetilde{\mathcal{T}}$ is an p-concave, then inequality (20) is reversed.

Here is the generalized form of discrete Schur's type inequality for UD-p-convex FNVM.

Theorem 5. *Let $\widetilde{\mathcal{T}}\in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau:[\imath,\ \jmath]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are given by $\mathcal{T}_\tau(\omega)=[\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ for all $\omega\in[\imath,\ \jmath]$, and for all*

$\tau \in [0,\ 1]$. If $\imath_1,\ \imath_2,\ \imath_3 \in [\imath,\ \jmath]$, such that $\imath_1 < \imath_2 < \imath_3$ and $\imath_3{}^p - \imath_1{}^p, \imath_3{}^p - \imath_2{}^p, \imath_2{}^p - \imath_1{}^p \in [0,\ 1]$, then we have

$$(\imath_3{}^p - \imath_1{}^p) \odot \widetilde{\mathcal{T}}(\imath_2) \supseteq_{\mathbb{F}} (\imath_3{}^p - \imath_2{}^p) \odot \widetilde{\mathcal{T}}(\imath_1) \oplus (\imath_2{}^p - \imath_1{}^p) \odot \widetilde{\mathcal{T}}(\imath_3). \tag{21}$$

If $\widetilde{\mathcal{T}}$ is a UD-p-concave, then inequality (21) is reversed.

Proof. Let $\imath_1,\ \imath_2,\ \imath_3 \in [\imath,\ \jmath]$ and $\imath_3{}^p - \imath_1{}^p > 0$. Consider $\hbar = \frac{\imath_3{}^p - \imath_2{}^p}{\imath_3{}^p - \imath_1{}^p}$, then $\imath_2{}^p = \hbar \imath_1{}^p + (1-\hbar)\imath_3{}^p$. Since $\widetilde{\mathcal{T}}$ is a UD-p-convex FNVM, then by hypothesis, we have

$$\widetilde{\mathcal{T}}(\imath_2) \supseteq_{\mathbb{F}} \left(\frac{\imath_3{}^p - \imath_2{}^p}{\imath_3{}^p - \imath_1{}^p}\right) \odot \widetilde{\mathcal{T}}(\imath_1) \oplus \left(\frac{\imath_2{}^p - \imath_1{}^p}{\imath_3{}^p - \imath_1{}^p}\right) \odot \widetilde{\mathcal{T}}(\imath_3).$$

Therefore, for each $\tau \in [0,\ 1]$, we have

$$\begin{aligned} \mathcal{T}_*(\imath_2, \tau) &\leq \left(\tfrac{\imath_3{}^p - \imath_2{}^p}{\imath_3{}^p - \imath_1{}^p}\right)\mathcal{T}_*(\imath_1, \tau) + \left(\tfrac{\imath_2{}^p - \imath_1{}^p}{\imath_3{}^p - \imath_1{}^p}\right)\mathcal{T}_*(\imath_3, \tau), \\ \mathcal{T}^*(\imath_2, \tau) &\geq \left(\tfrac{\imath_3{}^p - \imath_2{}^p}{\imath_3{}^p - \imath_1{}^p}\right)\mathcal{T}^*(\imath_1, \tau) + \left(\tfrac{\imath_2{}^p - \imath_1{}^p}{\imath_3{}^p - \imath_1{}^p}\right)\mathcal{T}^*(\imath_3, \tau), \end{aligned} \tag{22}$$

$$\begin{aligned} &= \tfrac{(\imath_3{}^p - \imath_2{}^p)}{(\imath_3{}^p - \imath_1{}^p)}\mathcal{T}_*(\imath_1, \tau) + \tfrac{(\imath_2{}^p - \imath_1{}^p)}{(\imath_3{}^p - \imath_1{}^p)}\mathcal{T}_*(\imath_3, \tau), \\ &= \tfrac{(\imath_3{}^p - \imath_2{}^p)}{(\imath_3{}^p - \imath_1{}^p)}\mathcal{T}^*(\imath_1, \tau) + \tfrac{(\imath_2{}^p - \imath_1{}^p)}{(\imath_3{}^p - \imath_1{}^p)}\mathcal{T}^*(\imath_3, \tau). \end{aligned} \tag{23}$$

From (23), we have

$$\begin{aligned} (\imath_3{}^p - \imath_1{}^p)\mathcal{T}_*(\imath_2, \tau) &\leq (\imath_3{}^p - \imath_2{}^p)\mathcal{T}_*(\imath_1, \tau) + (\imath_2{}^p - \imath_1{}^p)\mathcal{T}_*(\imath_3, \tau), \\ (\imath_3{}^p - \imath_1{}^p)\mathcal{T}^*(\imath_2, \tau) &\geq (\imath_3{}^p - \imath_2{}^p)\mathcal{T}^*(\imath_1, \tau) + (\imath_2{}^p - \imath_1{}^p)\mathcal{T}^*(\imath_3, \tau), \end{aligned}$$

that is

$$\begin{aligned} [(\imath_3{}^p - \imath_1{}^p)\mathcal{T}_*(\imath_2, \tau),\ (\imath_3{}^p - \imath_1{}^p)\mathcal{T}^*(\imath_2, \tau)] \supseteq_I [(\imath_3{}^p - \imath_2{}^p)\mathcal{T}_*(\imath_1, \tau) + \\ (\imath_3{}^p - \imath_2{}^p)\mathcal{T}_*(\imath_3, \tau),\ (\imath_2{}^p - \imath_1{}^p)\mathcal{T}^*(\imath_1, \tau) + (\imath_2{}^p - \imath_1{}^p)\mathcal{T}^*(\imath_3, \tau)], \end{aligned}$$

hence

$$(\imath_3{}^p - \imath_1{}^p) \odot \widetilde{\mathcal{T}}(\imath_2) \supseteq_{\mathbb{F}} \imath_3{}^p - \imath_2{}^p \odot \widetilde{\mathcal{T}}(\imath_1) \oplus (\imath_2{}^p - \imath_1{}^p) \odot \widetilde{\mathcal{T}}(\imath_3).$$

□

The following theorem provides a clarification of Jensen's type inequality for UD-p-convex FNVMs.

Theorem 6. *Let $\hbar_\updownarrow \in \mathbb{R}^+$, $\imath_\updownarrow \in [\imath,\ \jmath]$, $(\updownarrow = 1,\ 2,\ 3,\ \ldots, k,\ k \geq 2)$ and $\widetilde{\mathcal{T}} \in UDFSX$ $([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau),\ \mathcal{T}^*(\omega, \tau)]$ for all $\omega \in [\imath,\ \jmath]$ and for all $\tau \in [0,\ 1]$. If $(\imath,\ U) \subseteq [\imath, \jmath]$, then*

$$\sum_{\updownarrow=1}^k \left(\frac{\hbar_\updownarrow}{W_k}\right) \odot \widetilde{\mathcal{T}}\left(\imath_\updownarrow\right) \supseteq_{\mathbb{F}} \sum_{\updownarrow=1}^k \left(\left(\frac{U^p - \imath_\updownarrow{}^p}{U^p - \imath^p}\right)\left(\frac{\hbar_\updownarrow}{W_k}\right) \odot \widetilde{\mathcal{T}}(\imath, \tau) \oplus \left(\frac{\imath_\updownarrow{}^p - \imath^p}{U^p - \imath^p}\right)\left(\frac{\hbar_\updownarrow}{W_k}\right) \odot \widetilde{\mathcal{T}}(U, \tau)\right), \tag{24}$$

where $W_k = \sum_{\updownarrow=1}^k \hbar_\updownarrow$. If $\widetilde{\mathcal{T}}$ is UD-p-concave, then inequality (24) is reversed.

Proof. Consider $\imath = \imath_1,\ \imath_\updownarrow = \imath_2,\ (\updownarrow = 1,\ 2,\ 3,\ \ldots, k)$, $U = \imath_3$. Then, by hypothesis and inequality (24), we have

$$\widetilde{\mathcal{T}}\left(\imath_\updownarrow\right) \supseteq_{\mathbb{F}} \left(\frac{U^p - \imath_\updownarrow{}^p}{U^p - \imath^p}\right) \odot \widetilde{\mathcal{T}}(\imath, \tau) \oplus \left(\frac{\imath_\updownarrow{}^p - \imath^p}{U^p - \imath^p}\right) \odot \widetilde{\mathcal{T}}(U, \tau).$$

Therefore, for each $\tau \in [0,\ 1]$, we have

$$\begin{aligned}\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right) &\leq \left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\mathcal{T}_*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\mathcal{T}_*(U,\tau),\\ \mathcal{T}^*\left(\imath_{\updownarrow},\tau\right) &\geq \left(\frac{U-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\mathcal{T}^*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\mathcal{T}^*(U,\tau).\end{aligned}$$

The above inequality can be written as

$$\begin{aligned}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right) &\leq \left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(U,\tau),\\ \left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right) &\geq \left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(U,\tau).\end{aligned} \tag{25}$$

Taking the sum of all inequalities (25) for $\updownarrow = 1,\ 2,\ 3,\ \ldots,k$, we have

$$\begin{aligned}\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right) &\leq \sum_{\updownarrow=1}^{k}\left(\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(U,\tau)\right),\\ \sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right) &\geq \sum_{\updownarrow=1}^{k}\left(\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(\imath,\tau)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(U,\tau)\right).\end{aligned}$$

That is

$$\begin{aligned}\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}\left(\imath_{\updownarrow}\right) &= \left[\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*\left(\imath_{\updownarrow},\tau\right),\ \sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*\left(\imath_{\updownarrow},\tau\right)\right]\\ &\supseteq_I \left[\sum_{\updownarrow=1}^{k}\left(\begin{array}{c}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(\imath,\tau)\\ +\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}_*(U,\tau)\end{array}\right),\ \sum_{\updownarrow=1}^{k}\left(\begin{array}{c}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(\imath,\tau)\\ +\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}^*(U,\tau)\end{array}\right)\right],\\ &\supseteq_I \sum_{\updownarrow=1}^{k}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\left[\mathcal{T}_*(\imath,\tau),\ \mathcal{T}^*(\imath,\tau)\right]+\sum_{\updownarrow=1}^{k}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\left[\mathcal{T}_*(U,\tau),\ \mathcal{T}^*(U,\tau)\right].\\ &= \sum_{\updownarrow=1}^{k}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}(\imath,\tau)+\sum_{\updownarrow=1}^{k}\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}(U,\tau).\end{aligned}$$

Thus,

$$\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\odot\widetilde{\mathcal{T}}\left(\imath_{\updownarrow}\right)\supseteq_{\mathbb{F}}\sum_{\updownarrow=1}^{k}\left(\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\odot\widetilde{\mathcal{T}}(\imath)\oplus\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\odot\widetilde{\mathcal{T}}(U)\right),$$

completes the proof. □

In the next outcomes, we shall discuss the exceptional cases that are acquired from the Theorems 6 and 7.

If $\mathcal{T}_*(\omega,\tau) = \mathcal{T}_*(\omega,\tau)$ with $\tau = 1$, then Theorems 5 and 6 cuts down to the following results:

Corollary 2 ([81]). *Let $\hbar_{\updownarrow} \in \mathbb{R}^+$, $\imath_{\updownarrow} \in [\imath,\ \jmath]$, ($\updownarrow = 1,\ 2,\ 3,\ \ldots,k,\ k \geq 2$) and let $\mathcal{T} : [\imath,\ \jmath] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathcal{T}$ is a p-convex function, then*

$$\mathcal{T}\left(\left[\frac{1}{W_k}\sum_{\updownarrow=1}^{k}\hbar_{\updownarrow}\imath_{\updownarrow}{}^p\right]^{\frac{1}{p}}\right)\leq\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}\left(\imath_{\updownarrow}\right), \tag{26}$$

where $W_k = \sum_{\updownarrow=1}^{k}\hbar_{\updownarrow}$. If $\mathcal{T}$ is p-concave function, then inequality (26) is reversed.

Corollary 3 ([81]). *Let $\hbar_{\updownarrow} \in \mathbb{R}^+$, $\imath_{\updownarrow} \in [\imath,\ \jmath]$, ($\updownarrow = 1,\ 2,\ 3,\ \ldots,\ k,\ k \geq 2$) and $\mathcal{T} : [\imath,\ \jmath] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathcal{T}$ is a p-convex function and $\imath_1,\ \imath_2,\ldots,\imath_{\updownarrow} \in (\imath,\ U) \subseteq [\imath,\jmath]$ then,*

$$\sum_{\updownarrow=1}^{k}\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}\left(\imath_{\updownarrow}\right)\leq\sum_{\updownarrow=1}^{k}\left(\left(\frac{U^p-\imath_{\updownarrow}{}^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}(\imath)+\left(\frac{\imath_{\updownarrow}{}^p-\imath^p}{U^p-\imath^p}\right)\left(\frac{\hbar_{\updownarrow}}{W_k}\right)\mathcal{T}(U)\right), \tag{27}$$

where $W_k = \sum_{\updownarrow=1}^{k}\hbar_{\updownarrow}$. If $\mathcal{T}$ is a p-concave function, then inequality (27) is reversed.

4. Fuzzy Aumann's Integral Hermite–Hadamard Type Inequalities

Primary goal and focus of this section is to establish a novel version of the H-H-type inequalities in the mode of UD-p-convex FNVMs via fuzzy Aumann's integrals.

Theorem 7. *Let $\widetilde{\mathcal{T}} \in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau),\ \mathcal{T}^*(\omega, \tau)]$ for all $\omega \in [\imath,\ \jmath]$ and for all $\tau \in [0,\ 1]$. If $\widetilde{\mathcal{T}} \in \mathcal{FA}_{([\imath,\ \jmath],\ \tau)}$, then*

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \frac{p}{\jmath^p - \imath^p} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2}. \tag{28}$$

If $\widetilde{\mathcal{T}}(\omega)$ is UD-p-concave FNVM, then

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \subseteq_{\mathbb{F}} \frac{p}{\jmath^p - \imath^p} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega \subseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2} \tag{29}$$

Proof. Let $\widetilde{\mathcal{T}} : [\imath,\ \jmath] \to \mathcal{L}_C$ be an UD-p-convex FNVM. Then, by hypothesis, we have

$$2 \odot \widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \widetilde{\mathcal{T}}\left([\hbar\imath^p + (1-\hbar)v^p]^{\frac{1}{p}}\right) \oplus \widetilde{\mathcal{T}}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right).$$

Therefore, for every $\tau \in [0,\ 1]$, we have

$$\begin{aligned} 2\mathcal{T}_*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) &\le \mathcal{T}_*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right) + \mathcal{T}_*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}},\ \tau\right), \\ 2\mathcal{T}^*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) &\ge \mathcal{T}^*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right) + \mathcal{T}^*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right). \end{aligned}$$

Then

$$\begin{aligned} 2\int_0^1 \mathcal{T}_*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)d\hbar &\le \int_0^1 \mathcal{T}_*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}, \tau\right)d\hbar + \int_0^1 \mathcal{T}_*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)d\hbar, \\ 2\int_0^1 \mathcal{T}^*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right)d\hbar &\ge \int_0^1 \mathcal{T}^*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)d\hbar + \int_0^1 \mathcal{T}^*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right)d\hbar. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{T}_*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) &\le \tfrac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\ \tau)d\omega, \\ \mathcal{T}^*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) &\ge \tfrac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\ \tau)d\omega. \end{aligned}$$

That is

$$\left[\mathcal{T}_*\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right),\ \mathcal{T}^*\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)\right] \supseteq_I \frac{p}{\jmath^p - \imath^p}\left[\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\ \tau)d\omega,\ \int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\ \tau)d\omega\right].$$

Thus,

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \frac{p}{\jmath^p - \imath^p} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega. \tag{30}$$

In a similar way as above, we have

$$\frac{p}{\jmath^p - \imath^p} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2}. \tag{31}$$

Combining (21) and (22), we have

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p} \odot (FA)\int_{\imath}^{\jmath} \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2}.$$

Hence, the required result. □

Remark 4. *If* $p=1$, *then Theorem 7, cuts down to the outcome for UD-convex FNVM, as shown in [77]:*

$$\widetilde{\mathcal{T}}\left(\frac{\imath+\jmath}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\jmath-\imath} \odot (FA)\int_{\imath}^{\jmath} \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2} \tag{32}$$

If $\mathcal{T}_*(\omega,\ \tau) = \mathcal{T}^*(\omega,\ \tau)$ *with* $\tau = 1$, *then Theorem 7, cuts down to the finding for p-convex function, as shown in [90]:*

$$\mathcal{T}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{\jmath^p-\imath^p}(A)\int_{\imath}^{\jmath} \omega^{p-1}\mathcal{T}(\omega)d\omega \le \frac{\mathcal{T}(\imath)+\mathcal{T}(\jmath)}{2}. \tag{33}$$

If $\mathcal{T}_*(\omega,\ \tau) = \mathcal{T}^*(\omega,\ \tau)$ *with* $\tau = 1$ *and* $p = 1$, *then Theorem 7 cuts down to the outcome for classical convex function:*

$$\mathcal{T}\left(\frac{\imath+\jmath}{2}\right) \le \frac{1}{\jmath-\imath}(A)\int_{\imath}^{\jmath} \mathcal{T}(\omega)d\omega \le \frac{\mathcal{T}(\imath)+\mathcal{T}(\jmath)}{2}. \tag{34}$$

Example 2. *Let* p *be an odd number and the FNVM* $\widetilde{\mathcal{T}} : [\imath,\ \jmath] = [2,\ 3] \to \mathcal{L}_C$ *defined by,*

$$\widetilde{\mathcal{T}}(\omega)(\lambda) = \begin{cases} \frac{\lambda-2+\omega^{\frac{p}{2}}}{1-\omega^{\frac{p}{2}}} & \lambda \in \left[2-\omega^{\frac{p}{2}},\ 3\right], \\ \frac{2+\omega^{\frac{p}{2}}-\lambda}{\omega^{\frac{p}{2}}-1} & \lambda \in \left(3,\ 2+\omega^{\frac{p}{2}}\right], \\ 0 \text{ otherwise.} \end{cases} \tag{35}$$

Then, for each $\tau \in [0,\ 1]$, *we have* $\mathcal{T}_\tau(\omega) = \left[(1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau, (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau\right]$. *Since endpoint functions* $\mathcal{T}_*(\omega,\tau) = (1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau$, $\mathcal{T}^*(\omega,\ \tau) = (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau$ *are p-convex functions for each* $\tau \in [0,\ 1]$. *Then,* $\widetilde{\mathcal{T}}(\omega)$ *is UD-p-convex FNVM.*

We now computing the following

$$\begin{aligned}
\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &= (1-\tau)\tfrac{4-\sqrt{10}}{2}+3\tau,\\
\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &= (1-\tau)\tfrac{4+\sqrt{10}}{2}+3\tau,\\
\tfrac{p}{\jmath^p-\imath^p}\int_{\imath}^{\jmath}\omega^{p-1}\mathcal{T}_*(\omega,\ \tau)d\omega &= \int_2^3\left((1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau\right)d\omega \approx \tfrac{843}{2000}(1-\tau)+3\tau,\\
\tfrac{p}{\jmath^p-\imath^p}\int_{\imath}^{\jmath}\omega^{p-1}\mathcal{T}^*(\omega,\ \tau)d\omega &= \int_2^3\left((1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau\right)d\omega \approx \tfrac{179}{50}(1-\tau)+3\tau,\\
\tfrac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2} &= (1-\tau)\left(\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\tau,\\
\tfrac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2} &= (1-\tau)\left(\tfrac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\tau,
\end{aligned}$$

for all $\tau \in [0,\ 1]$. *That means*

$$\begin{aligned}
\left[(1-\tau)\tfrac{4-\sqrt{10}}{2}+3\tau, (1-\tau)\tfrac{4+\sqrt{10}}{2}+3\tau\right] &\supseteq_I \left[\tfrac{843}{2000}(1-\tau)+3\tau, \tfrac{179}{50}(1-\tau)+3\tau\right]\\
&\supseteq_I \left[(1-\tau)\left(\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\tau, (1-\tau)\left(\tfrac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\tau\right],
\end{aligned}$$

for all $\tau \in [0,\ 1]$, *and Theorem 7 has been verified.*

Theorem 8 presents the extended version of Aumann's integral Hermite–Hadamard type inequalities.

Theorem 8. *Let* $\widetilde{\mathcal{T}} \in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. *The family of* $IVMs$ *is defined by* τ*-cuts* $\mathcal{T}_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ *are given by* $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega, \tau),\ \mathcal{T}^*(\omega, \tau)]$ *for all* $\omega \in [\imath,\ \jmath]$ *and for all* $\tau \in [0,\ 1]$. *If* $\widetilde{\mathcal{T}} \in \mathcal{FA}_{([\imath,\ \jmath],\ \tau)}$, *then*

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{p}{\jmath^p - \imath^p} \odot (\mathrm{FA})\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \rhd_1 \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2}, \tag{36}$$

where

$$\rhd_1 = \frac{\frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2} \oplus \widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right)}{2},\ \rhd_2 = \frac{\widetilde{\mathcal{T}}\left(\left[\frac{3\imath^p + \jmath^p}{4}\right]^{\frac{1}{p}}\right) \oplus \widetilde{\mathcal{T}}\left(\left[\frac{\imath^p + 3\jmath^p}{4}\right]^{\frac{1}{p}}\right)}{2}, \text{ and } \rhd_1 = [\rhd_{1*},\ {\rhd_1}^*],\ \rhd_2 = [\rhd_{2*},\ {\rhd_2}^*].$$

Proof. Take $\left[\imath^p,\ \frac{\imath^p + \jmath^p}{2}\right]$, we have

$$2 \odot \widetilde{\mathcal{T}}\left(\frac{\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2} + \frac{\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2}\right) \supseteq_{\mathbb{F}} \widetilde{\mathcal{T}}\left(\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right) \oplus \widetilde{\mathcal{T}}\left(\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}\right).$$

Therefore, for every $\tau \in [0,\ 1]$, we have

$$2\mathcal{T}_*\left(\frac{\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2} + \frac{\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2},\ \tau\right) \le \mathcal{T}_*\left(\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) + \mathcal{T}_*\left(\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right),$$

$$2\mathcal{T}^*\left(\frac{\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2} + \frac{\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}}{2},\ \tau\right) \ge \mathcal{T}^*\left(\left[\hbar\imath^p + (1-\hbar)\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) + \mathcal{T}^*\left(\left[(1-\hbar)\imath^p + \hbar\frac{\imath^p + \jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right).$$

In consequence, we obtain

$$\begin{aligned}
\frac{\mathcal{T}_*\left(\left[\frac{3\imath^p + \jmath^p}{4}\right]^{\frac{1}{p}}, \tau\right)}{2} &\le \frac{p}{\jmath^p - \imath^p} \int_\imath^{\frac{\imath^p + \jmath^p}{2}} \mathcal{T}_*(\omega,\ \tau)d\omega, \\
\frac{\mathcal{T}^*\left(\left[\frac{3\imath^p + \jmath^p}{4}\right]^{\frac{1}{p}}, \tau\right)}{2} &\ge \frac{p}{\jmath^p - \imath^p} \int_\imath^{\frac{\imath^p + \jmath^p}{2}} \mathcal{T}^*(\omega,\ \tau)d\omega.
\end{aligned}$$

That is

$$\frac{\left[\mathcal{T}_*\left(\left[\frac{3\imath^p + \jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right),\ \mathcal{T}^*\left(\left[\frac{3\imath^p + \jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)\right]}{2} \supseteq_I \frac{p}{\jmath^p - \imath^p}\left[\int_\imath^{\frac{\imath^p + \jmath^p}{2}} \mathcal{T}_*(\omega,\ \tau)d\omega,\ \int_\imath^{\frac{\imath^p + \jmath^p}{2}} \mathcal{T}^*(\omega,\ \tau)d\omega\right].$$

It follows that

$$\frac{\widetilde{\mathcal{T}}\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}}\right)}{2} \supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p} \odot (FA)\int_{\imath}^{\frac{\imath^p+\jmath^p}{2}} \widetilde{\mathcal{T}}(\omega)d\omega. \tag{37}$$

In a similar way as above, we have

$$\frac{\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}}\right)}{2} \supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p} \odot (FA)\int_{\frac{\imath^p+\jmath^p}{2}}^{\jmath} \widetilde{\mathcal{T}}(\omega)d\omega. \tag{38}$$

Combining (37) and (38), we have

$$\frac{\left[\widetilde{\mathcal{T}}\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}}\right) \oplus \widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}}\right)\right]}{2} \supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p} \odot (FA)\int_{\imath}^{\jmath} \widetilde{\mathcal{T}}(\omega)d\omega.$$

By using Theorem 7, we have

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) = \widetilde{\mathcal{T}}\left(\left[\frac{1}{2}.\frac{3\imath^p+\jmath^p}{4} + \frac{1}{2}.\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}}\right).$$

Therefore, for every $\tau \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) = \mathcal{T}_*\left(\left[\frac{1}{2}.\frac{3\imath^p+\jmath^p}{4} + \frac{1}{2}.\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right),\\
&\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) = \mathcal{T}^*\left(\left[\frac{1}{2}.\frac{3\imath^p+\jmath^p}{4} + \frac{1}{2}.\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right),\\
&\le \left[\frac{1}{2}\mathcal{T}_*\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right) + \frac{1}{2}\mathcal{T}_*\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)\right],\\
&\ge \left[\frac{1}{2}\mathcal{T}^*\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right) + \frac{1}{2}\mathcal{T}^*\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)\right],\\
&= \rhd_{2*},\\
&= \rhd_2{}^*,\\
&\le \frac{p}{\jmath^p-\imath^p}\ \int_{\imath}^{\jmath} \mathcal{T}_*(\omega,\ \tau)d\omega,\\
&\ge \frac{p}{\jmath^p-\imath^p}\ \int_{\imath}^{\jmath} \mathcal{T}^*(\omega,\ \tau)d\omega,\\
&\le \frac{1}{2}\left[\frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2} + \mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)\right],\\
&\ge \frac{1}{2}\left[\frac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2} + \mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)\right],\\
&= \rhd_{1*}, \le \frac{1}{2}\left[\frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2} + \frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2}\right],\\
&= \rhd_1{}^*, \ge \frac{1}{2}\left[\frac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2} + \frac{\mathcal{T}^*(\imath,\tau)+\mathcal{T}^*(\jmath,\tau)}{2}\right],\\
&= \frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2},\\
&= \frac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2},
\end{aligned}$$

that is

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \rhd_2 \supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p} \odot (FA)\int_{\imath}^{\jmath} \widetilde{\mathcal{T}}(\omega)d\omega \supseteq_{\mathbb{F}} \rhd_1 \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{T}}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath)}{2},$$

hence, the result follows. □

Example 3. *Let p be an odd number and the FNVM $\widetilde{\mathcal{T}} : [\imath,\ \jmath] = [2,\ 3] \to \mathcal{L}_C$ defined by, $\mathcal{T}_\tau(\omega) = \left[(1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau,\ (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau\right]$, as in Example 2, then $\widetilde{\mathcal{T}}(\omega)$ is UD-p-convex FNVM and satisfying (38). We have $\mathcal{T}_*(\omega,\ \tau) = (1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau and$ $\mathcal{T}^*(\omega,\ \tau) = (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau$. We now computing the following*

$$\begin{gathered}
\left[\frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2}\right] = \frac{4+2-(1-)\left(\sqrt{2}+\sqrt{3}\right)}{2},\\
\left[\frac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2}\right] = \frac{4+10+(1+)\left(\sqrt{2}+\sqrt{3}\right)}{2},\\
\rhd_{1*} = \frac{\frac{\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)}{2}+\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)}{2} = \frac{8+4-(1-)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4},\\
\rhd_1{}^* = \frac{\frac{\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)}{2}+\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right)}{2} = \frac{8+20+(1+)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4},\\
\rhd_{2*} = \frac{1}{2}\left[\mathcal{T}_*\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)+\mathcal{T}_*\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)\right] = 5+7-\sqrt{11}(1-),\\
\rhd_2{}^* = \frac{1}{2}\left[\mathcal{T}^*\left(\left[\frac{3\imath^p+\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)+\mathcal{T}^*\left(\left[\frac{\imath^p+3\jmath^p}{4}\right]^{\frac{1}{p}},\ \tau\right)\right] = \frac{11+23+\sqrt{11}(1+)}{4},\\
\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) = (1-\tau)\frac{4-\sqrt{10}}{2}+3\tau,\\
\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) = (1-\tau)\frac{4+\sqrt{10}}{2}+3\tau.
\end{gathered}$$

Then, we obtain that

$$\begin{aligned}
&(1-)\tfrac{4-\sqrt{10}}{2}+3 \le \tfrac{5+7-\sqrt{11}(1-)}{4} \le \tfrac{843}{2000}(1-)+3\\
&\quad\le \tfrac{8+4-(1-)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4} \le (1-)\left(\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\\
&(1+)\tfrac{4+\sqrt{10}}{2}+3 \ge \tfrac{11+23+\sqrt{11}(1+)}{4} \ge \tfrac{179}{50}(1+)+3\\
&\quad\ge \tfrac{8+20+(1+)\left(\sqrt{2}+\sqrt{3}+\sqrt{2}\times\sqrt{5}\right)}{4} \ge (1+)\left(\tfrac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3.
\end{aligned}$$

Hence, Theorem 8 is verified.

The next outcomes will give us the Aumann's integral Hermite–Hadamard type inequalities for the product of two UD-p-convex FNVMs

Theorem 9. *Let $\widetilde{\mathcal{T}}, \widetilde{Y} \in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. Then, τ-cuts $\mathcal{T}_\tau,\ Y_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are defined by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ and $Y_\tau(\omega) = [Y_*(\omega,\tau),\ Y^*(\omega,\tau)]$ for all $\omega \in [\imath,\ \jmath]$ and for all $\tau \in [0,\ 1]$. If $\widetilde{\mathcal{T}} \otimes \widetilde{Y} \in \mathcal{FA}_{([\imath,\ \jmath],\ \tau)}$, then*

$$\frac{p}{\jmath^p-\imath^p} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega) \otimes \widetilde{Y}(\omega)d\omega \supseteq_{\mathbb{F}} \frac{\widetilde{\mathcal{M}}(\imath,\jmath)}{3} \oplus \frac{\widetilde{n}(\imath,\jmath)}{6}. \tag{39}$$

where $\widetilde{\mathcal{M}}(\imath,\jmath) = \widetilde{\mathcal{T}}(\imath) \otimes \widetilde{Y}(\imath) \oplus \widetilde{\mathcal{T}}(\jmath) \otimes \widetilde{Y}(\jmath)$, $\widetilde{n}(\imath,\jmath) = \widetilde{\mathcal{T}}(\imath) \otimes \widetilde{Y}(\jmath) \oplus \widetilde{\mathcal{T}}(\jmath) \otimes \widetilde{Y}(\imath)$, and $\mathcal{M}_\tau(\imath,\jmath) = [\mathcal{M}_*((\imath,\jmath),\ \tau),\ \mathcal{M}^*((\imath,\jmath),\ \tau)]$ and $n_\tau(\imath,\jmath) = [n_*((\imath,\jmath),\ \tau),\ n^*((\imath,\jmath),\ \tau)]$.

Example 4. *Let p be an odd number, and UD-p-convex FNVMs $\widetilde{\mathcal{T}}, \widetilde{Y} : [\imath, \jmath] = [2, 3] \to \mathcal{L}_C$ are, respectively, defined by $\mathcal{T}_\tau(\omega) = \left[(1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau,\ (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau\right]$, as in example 3, and taking $Y_\tau(\omega) = [\tau\omega^p, (2-\tau)\omega^p\ ]$. Since $\widetilde{\mathcal{T}}(\omega)$ and $\widetilde{Y}(\omega)$ both are UD-p-convex FNVMs and $\mathcal{T}_*(\omega,\ \tau) = (1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau$, $\mathcal{T}^*(\omega,\ \tau) = (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau$, and $Y_*(\omega,\ \tau) = \tau\omega^p$, $Y^*(\omega,\ \tau) = (2-\tau)\omega^p$, then we computing the following, where $p = 1$*

$$\begin{aligned}
\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\times Y_*(\omega,\tau)d\omega &= \frac{\tau}{10}\left[\left(25-16\sqrt{2}+36\sqrt{3}\right)\tau-36\sqrt{3}+16\sqrt{2}+50\right],\\
\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\times Y^*(\omega,\tau)d\omega &= \frac{(2-\tau)}{10}\left[\left(25+16\sqrt{2}-36\sqrt{3}\right)\tau+36\sqrt{3}-16\sqrt{2}+50\right],\\
\frac{\mathcal{M}_*((\imath,\jmath),\ \tau)}{3} &= \left(5+2\sqrt{2}+3\sqrt{3}\right)\frac{\tau^2}{3}+\frac{\tau}{3}\left(10-2\sqrt{2}-3\sqrt{3}\right),\\
\frac{\mathcal{M}^*((\imath,\jmath),\ \tau)}{3} &= \sqrt{3}\left(\tau-\frac{1}{2}\right)^2+\frac{2\sqrt{2}}{3}\left(\tau-\frac{1}{2}\right)^2-\frac{10}{3}\left(\tau-\frac{3}{4}\right)^2-\frac{\sqrt{3}}{2}-\frac{1}{3\sqrt{2}}+\frac{125}{24},\\
\frac{n_*((\imath,\jmath),\ \tau)}{6} &= \frac{\tau}{6}\left[15\tau+(1-\tau)\left\{10+3\sqrt{2}+2\sqrt{3}\right\}\right]\\
\frac{n^*((\imath,\jmath),\ \tau)}{6} &= \frac{(2-\tau)}{6}\left(10+3\sqrt{2}+2\sqrt{3}-\left(3\sqrt{2}+2\sqrt{3}\right)\tau+15\tau\right),
\end{aligned}$$

for each $\tau \in [0,\ 1]$, that means

$$\begin{aligned}
&\frac{\tau}{10}\left[\left(25-16\sqrt{2}+36\sqrt{3}\right)\tau-36\sqrt{3}+16\sqrt{2}+50\right]\\
&\le \left(5+2\sqrt{2}+3\sqrt{3}\right)\frac{\tau^2}{3}+\frac{\tau}{3}\left(10-2\sqrt{2}-3\sqrt{3}\right)+\frac{\tau}{6}\left[15\tau+(1-\tau)\left\{10+3\sqrt{2}+2\sqrt{3}\right\}\right],\\
&\frac{(2-\tau)}{10}\left[\left(25+16\sqrt{2}-36\sqrt{3}\right)\tau+36\sqrt{3}-16\sqrt{2}+50\right]\\
&\ge \sqrt{3}\left(\tau-\frac{1}{2}\right)^2+\frac{2\sqrt{2}}{3}\left(\tau-\frac{1}{2}\right)^2-\frac{10}{3}\left(\tau-\frac{3}{4}\right)^2-\frac{\sqrt{3}}{2}-\frac{1}{3\sqrt{2}}+\frac{125}{24}\\
&+\frac{(2-\tau)}{6}\left(10+3\sqrt{2}+2\sqrt{3}-\left(3\sqrt{2}+2\sqrt{3}\right)\tau+15\tau\right).
\end{aligned}$$

Hence, Theorem 9 has been verified.

Theorem 10. *Let $\widetilde{\mathcal{T}}, \widetilde{Y} \in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau,\ Y_\tau : [\imath,\ \jmath] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ and $Y_\tau(\omega) = [Y_*(\omega,\tau),\ Y^*(\omega,\tau)]$ for all $\omega \in [\imath,\ \jmath]$ and for all $\tau \in [0,\ 1]$. If $\widetilde{\mathcal{T}} \otimes \widetilde{Y} \in \mathcal{FA}_{([\imath,\ \jmath],\ \tau)}$, then*

$$\begin{aligned}
&2\odot\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right)\otimes\widetilde{Y}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right)\\
&\supseteq_{\mathbb{F}} \frac{p}{\jmath^p-\imath^p}\odot(FA)\int_\imath^\jmath \omega^{p-1}\odot\widetilde{\mathcal{T}}(\omega)\otimes\widetilde{Y}(\omega)d\omega\oplus\frac{\widetilde{\mathcal{M}}(\imath,\jmath)}{6}\oplus\frac{\widetilde{n}(\imath,\jmath)}{3}.
\end{aligned} \tag{40}$$

where $\widetilde{\mathcal{M}}(\imath,\jmath) = \widetilde{\mathcal{T}}(\imath)\otimes\widetilde{Y}(\imath)\oplus\widetilde{\mathcal{T}}(\jmath)\otimes\widetilde{Y}(\jmath)$, $\widetilde{n}(\imath,\jmath) = \widetilde{\mathcal{T}}(\imath)\otimes\widetilde{Y}(\jmath)\oplus\widetilde{\mathcal{T}}(\jmath)\otimes\widetilde{Y}(\imath)$, and $\mathcal{M}_\tau(\imath,\jmath) = [\mathcal{M}_*((\imath,\jmath),\ \tau),\ \mathcal{M}^*((\imath,\jmath),\ \tau)]$ and $n_\tau(\imath,\jmath) = [n_*((\imath,\jmath),\ \tau),\ n^*((\imath,\jmath),\ \tau)]$.

Proof. By hypothesis, for each $\tau \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\\
&\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\\
&\leq \frac{1}{4}\left[\begin{array}{c}\mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&+\frac{1}{4}\left[\begin{array}{c}\mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right],\\
&\geq \frac{1}{4}\left[\begin{array}{c}\mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&+\frac{1}{4}\left[\begin{array}{c}\mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right],\\
&\leq \frac{1}{4}\left[\begin{array}{c}\mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}(\hbar\mathcal{T}_*(\imath,\ \tau)+(1-\hbar)\mathcal{T}_*(\jmath,\ \tau))\\ \times((1-\hbar)Y_*(\imath,\ \tau)+\hbar Y_*(\jmath,\ \tau))\\ +((1-\hbar)\mathcal{T}_*(\imath,\ \tau)+\hbar\mathcal{T}_*(\jmath,\ \tau))\\ \times(\hbar Y_*(\imath,\ \tau)+(1-\hbar)Y_*(\jmath,\ \tau))\end{array}\right],\\
&\geq \frac{1}{4}\left[\begin{array}{c}\mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&\quad+\frac{1}{4}\left[\begin{array}{c}(\hbar\mathcal{T}^*(\imath,\ \tau)+(1-\hbar)\mathcal{T}^*(\jmath,\ \tau))\\ \times((1-\hbar)Y^*(\imath,\ \tau)+\hbar Y^*(\jmath,\ \tau))\\ +((1-\hbar)\mathcal{T}^*(\imath,\ \tau)+\hbar\mathcal{T}^*(\jmath,\ \tau))\\ \times(\hbar Y^*(\imath,\ \tau)+(1-\hbar)Y^*(\jmath,\ \tau))\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}\mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&\quad+\frac{1}{2}\left[\begin{array}{c}\left\{\hbar^2+(1-\hbar)^2\right\}\mathcal{N}_*((\imath,\jmath),\ \tau)\\ +\{\hbar(1-\hbar)+(1-\hbar)\hbar\}\mathcal{M}_*((\imath,\jmath),\ \tau)\end{array}\right],\\
&= \frac{1}{4}\left[\begin{array}{c}\mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\ \tau\right)\\ +\mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\times Y^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\ \tau\right)\end{array}\right]\\
&\quad+\frac{1}{2}\left[\begin{array}{c}\left\{\hbar^2+(1-\hbar)^2\right\}\mathcal{N}^*((\imath,\jmath),\ \tau)\\ +\{\hbar(1-\hbar)+(1-\hbar)\hbar\}\mathcal{M}^*((\imath,\jmath),\ \tau)\end{array}\right],
\end{aligned}$$

A-integrating over $[0,\ 1]$, we have

$$\begin{aligned} 2\,\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &\leq \frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\times Y_*(\omega,\tau)d\omega+\frac{\mathcal{M}_*((\imath,\jmath),\,\tau)}{6} \\ &+\frac{\mathcal{N}_*((\imath,\jmath),\,\tau)}{3}, \\ 2\,\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &\geq \frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\times Y^*(\omega,\tau)d\omega+\frac{\mathcal{M}^*((\imath,\jmath),\,\tau)}{6} \\ &+\frac{\mathcal{N}^*((\imath,\jmath),\,\tau)}{3}, \end{aligned}$$

that is

$$2\odot\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right)\otimes\widetilde{Y}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right)\supseteq_{\mathbb{F}}\frac{p}{\jmath^p-\imath^p}\odot(FA)\int_\imath^\jmath \omega^{p-1}\odot\widetilde{\mathcal{T}}(\omega)\otimes\widetilde{Y}(\omega)d\omega\oplus\frac{\widetilde{\mathcal{M}}(\imath,\jmath)}{6}\oplus\frac{\widetilde{\mathcal{N}}(\imath,\jmath)}{3}.$$

Hence, the required result. □

Example 5. *Let p be an odd number, and UD-p-convex FNVMs $\widetilde{\mathcal{T}},\widetilde{Y}:[\imath,\ \jmath]=[2,\ 3]\to\mathcal{L}_C$ are, respectively, defined by $\mathcal{T}_\tau(\omega)=\left[(1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau,\ (1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau\right]$, as in Example 3, and $Y_\tau(\omega)=[\tau\omega^p,(2-\tau)\omega^p\ ]$. Since $\widetilde{\mathcal{T}}(\omega)$ and $\widetilde{Y}(\omega)$ both are UD-p-convex FNVMs and $\mathcal{T}_*(\omega,\ \tau)=(1-\tau)\left(2-\omega^{\frac{p}{2}}\right)+3\tau$, $\mathcal{T}^*(\omega,\ \tau)=(1-\tau)\left(2+\omega^{\frac{p}{2}}\right)+3\tau$, and $Y_*(\omega,\ \tau)=\tau\omega^p$, $Y^*(\omega,\ \tau)=(2-\tau)\omega^p$, then we computing the following, where $p=1$*

$$\begin{aligned} 2\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &= \frac{5}{2}\left[\left(2+\sqrt{10}\right)\tau^2+\left(4-\sqrt{10}\right)\tau\right], \\ 2\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right)\times Y^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\tau\right) &= \frac{5}{2}(2-\tau)\left[4+\sqrt{10}+\left(2-\sqrt{10}\right)\tau\right], \\ \frac{\mathcal{M}_*((\imath,\jmath),\,\tau)}{6} &= \left(5+2\sqrt{2}+3\sqrt{3}\right)\frac{\tau^2}{6}+\frac{\tau}{6}\left(10-2\sqrt{2}-3\sqrt{3}\right), \\ \frac{\mathcal{M}^*((\imath,\jmath),\,\tau)}{6} &= \frac{\sqrt{3}}{2}\left(\tau-\frac{1}{2}\right)^2+\frac{2\sqrt{2}}{6}\left(\tau-\frac{1}{2}\right)^2-\frac{5}{3}\left(\tau-\frac{3}{4}\right)^2-\frac{\sqrt{3}}{4}-\frac{1}{6\sqrt{2}}+\frac{125}{48}, \\ \frac{\mathcal{N}_*((\imath,\jmath),\,\tau)}{3} &= \frac{\tau}{3}\left[15\tau+(1-\tau)\left\{10+3\sqrt{2}+2\sqrt{3}\right\}\right] \\ \frac{\mathcal{N}^*((\imath,\jmath),\,\tau)}{3} &= \frac{(2-\tau)}{3}\left(10+3\sqrt{2}+2\sqrt{3}-\left(3\sqrt{2}+2\sqrt{3}\right)\tau+15\tau\right), \end{aligned}$$

for each $\tau\in[0,\ 1]$, that means

$$\begin{aligned} &\frac{5}{2}\left[\left(2+\sqrt{10}\right)\tau^2+\left(4-\sqrt{10}\right)\tau\right] \\ &\leq \frac{\tau}{10}\left[\left(25-16\sqrt{2}+36\sqrt{3}\right)\tau-36\sqrt{3}+16\sqrt{2}+50\right] \\ &+\left(5+2\sqrt{2}+3\sqrt{3}\right)\frac{\tau^2}{6}+\frac{\tau}{6}\left(10-2\sqrt{2}-3\sqrt{3}\right) \\ &+\frac{\tau}{3}\left[15\tau+(1-\tau)\left\{10+3\sqrt{2}+2\sqrt{3}\right\}\right], \\ &\frac{5}{2}(2-\tau)\left[4+\sqrt{10}+\left(2-\sqrt{10}\right)\tau\right] \\ &\geq \frac{(2-\tau)}{10}\left[\left(25+16\sqrt{2}-36\sqrt{3}\right)\tau+36\sqrt{3}-16\sqrt{2}+50\right] \\ &+\frac{\sqrt{3}}{2}\left(\tau-\frac{1}{2}\right)^2+\frac{2\sqrt{2}}{6}\left(\tau-\frac{1}{2}\right)^2-\frac{5}{3}\left(\tau-\frac{3}{4}\right)^2-\frac{\sqrt{3}}{4}-\frac{1}{6\sqrt{2}}+\frac{125}{48} \\ &+\frac{(2-\tau)}{3}\left(10+3\sqrt{2}+2\sqrt{3}-\left(3\sqrt{2}+2\sqrt{3}\right)\tau+15\tau\right), \end{aligned}$$

hence, Theorem 10 is verified.

H-H Fejér inequality for UD-p-convex FNVM

Theorem 11. *(H-H Fejér inequality for UD-p-convex FNVM) Let $\widetilde{\mathcal{T}}\in UDFSX([\imath,\ \jmath],\ \mathcal{L}_C,\ p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau:[\imath,\ \jmath]\subset\mathbb{R}\to{\mathcal{K}_C}^+$ are given by $\mathcal{T}_\tau(\omega)=$*

$[\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ *for all* $\omega\in[\imath,\ \jmath]$ *and for all* $\tau\in[0,\ 1]$. *If* $\widetilde{\mathcal{T}}\in\mathcal{FA}_{([\imath,\ \jmath],\ \tau)}$ *and* $\mathrm{N}:[\imath,\ \jmath]\to\mathbb{R},\mathrm{N}(\omega)\geq 0,$ *symmetric with respect to* $\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}$, *then*

$$\frac{p}{\jmath^p-\imath^p}\odot(FA)\int_\imath^\jmath \omega^{p-1}\odot\widetilde{\mathcal{T}}(\omega)\mathrm{N}(\omega)d\omega\supseteq_{\mathbb{F}}\left[\widetilde{\mathcal{T}}(\imath)\oplus\widetilde{\mathcal{T}}(\jmath)\right]\odot\int_0^1\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar. \tag{41}$$

If $\widetilde{\mathcal{T}}$ *is UD-p-concave FNVM, then inequality (41) is reversed.*

Proof. Let $\widetilde{\mathcal{T}}$ be an UD-p-convex FNVM. Then, for each $\tau\in[0,\ 1]$, we have

$$\begin{gathered}\mathcal{T}_*\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)\leq\left(\hbar\mathcal{T}_*(\imath,\ \tau)+(1-\hbar)\mathcal{T}_*(\jmath,\ \tau)\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right),\\ \mathcal{T}^*\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)\geq\left(\hbar\mathcal{T}^*(\imath,\ \tau)+(1-\hbar)\mathcal{T}^*(\jmath,\ \tau)\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right).\end{gathered}\tag{42}$$

And

$$\begin{gathered}\mathcal{T}_*\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\\ \leq\left((1-\hbar)\mathcal{T}_*(\imath,\ \tau)+\hbar\mathcal{T}_*(\jmath,\ \tau)\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right),\\ \mathcal{T}^*\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\\ \geq\left((1-\hbar)\mathcal{T}^*(\imath,\ \tau)+\hbar\mathcal{T}^*(\jmath,\ \tau)\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right).\end{gathered}\tag{43}$$

After adding (42) and (43), and integrating over $[0,\ 1]$, we get

$$\begin{gathered}\int_0^1\mathcal{T}_*\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)d\hbar\\ +\int_0^1\mathcal{T}_*\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar\\ \leq\int_0^1\left[\begin{array}{c}\mathcal{T}_*(\imath,\ \tau)\left\{\hbar\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)+(1-\hbar)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\right\}\\ +\mathcal{T}_*(\jmath,\ \tau)\left\{(1-\hbar)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)+\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\right\}\end{array}\right]d\hbar,\\ \int_0^1\mathcal{T}^*\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar\\ +\int_0^1\mathcal{T}^*\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}},\ \tau\right)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)d\hbar\\ \geq\int_0^1\left[\begin{array}{c}\mathcal{T}^*(\imath,\ \tau)\left\{\hbar\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)+(1-\hbar)\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\right\}\\ +\mathcal{T}^*(\jmath,\ \tau)\left\{(1-\hbar)\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)+\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)\right\}\end{array}\right]d\hbar,\\ =2\mathcal{T}_*(\imath,\ \tau)\int_0^1\hbar\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)d\hbar+2\mathcal{T}_*(\jmath,\ \tau)\int_0^1\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar,\\ =2\mathcal{T}^*(\imath,\ \tau)\int_0^1\hbar\mathrm{N}\left(\left[\hbar\imath^p+(1-\hbar)\jmath^p\right]^{\frac{1}{p}}\right)d\hbar+2\mathcal{T}^*(\jmath,\ \tau)\int_0^1\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar.\end{gathered}$$

Since N is symmetric, then

$$\begin{gathered}=2\left[\mathcal{T}_*(\imath,\ \tau)+\mathcal{T}_*(\jmath,\ \tau)\right]\int_0^1\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar,\\ =2\left[\mathcal{T}^*(\imath,\ \tau)+\mathcal{T}^*(\jmath,\ \tau)\right]\int_0^1\hbar\mathrm{N}\left(\left[(1-\hbar)\imath^p+\hbar\jmath^p\right]^{\frac{1}{p}}\right)d\hbar.\end{gathered}\tag{44}$$

Since

$$\begin{gathered}
\int_0^1 \mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\,\tau\right)\mathrm{N}\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
=\int_0^1 \mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\,\tau\right)\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
=\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega \\
\int_0^1 \mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\,\tau\right)\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
=\int_0^1 \mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\,\tau\right)\mathrm{N}\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
=\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\,\tau)\mathrm{N}(\omega)d\omega.
\end{gathered} \tag{45}$$

Then, from (44), we have

$$\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega \le [\mathcal{T}_*(\imath,\,\tau)+\mathcal{T}_*(\jmath,\,\tau)]\int_0^1 \hbar\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar,$$

$$\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega s \ge [\mathcal{T}^*(\imath,\,\tau)+\mathcal{T}^*(\jmath,\,\tau)]\int_0^1 \hbar\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar,$$

that is

$$\left[\frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega,\ \frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega\right]$$

$$\supseteq_I [\mathcal{T}_*(\imath,\,\tau)+\mathcal{T}_*(\jmath,\,\tau),\ \mathcal{T}^*(\imath,\,\tau)+\mathcal{T}^*(\jmath,\,\tau)]\int_0^1 \hbar\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar,$$

hence

$$\frac{p}{\jmath^p-\imath^p}\odot(FA)\int_\imath^\jmath \omega^{p-1}\odot\widetilde{\mathcal{T}}(\omega)\mathrm{N}(\omega)d\omega \supseteq_{\mathbb{F}} \left[\widetilde{\mathcal{T}}(\imath)\oplus\widetilde{\mathcal{T}}(\jmath)\right]\odot\int_0^1 \hbar\mathrm{N}\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar.$$

□

Theorem 12. *(H-H Fejér inequality for UD-p-convex FNVM)* *Let $\widetilde{\mathcal{T}} \in UDFSX([\imath,\,\jmath],\,\mathcal{L}_C,\,p)$. The family of IVMs is defined by τ-cuts $\mathcal{T}_\tau : [\imath,\,\jmath] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{T}_\tau(\omega) = [\mathcal{T}_*(\omega,\tau),\ \mathcal{T}^*(\omega,\tau)]$ for all $\omega \in [\imath,\,\jmath]$ and for all $\tau \in [0,\,1]$. If $\widetilde{\mathcal{T}} \in \mathcal{FA}_{([\imath,\,\jmath],\,\tau)}$ and $\mathrm{N} : [\imath,\,\jmath] \to \mathbb{R}, \mathrm{N}(\omega) \ge 0,$ symmetric with respect to $\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}$, and $\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega > 0$, then*

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega}\odot(FA)\int_\imath^\jmath \omega^{p-1}\odot\widetilde{\mathcal{T}}(\omega)\mathrm{N}(\omega)d\omega. \tag{46}$$

If $\widetilde{\mathcal{T}}$ is UD-p-concave FNVM, then inequality (46) is reversed.

Proof. Since $\widetilde{\mathcal{T}}$ is an UD-convex, then for $\tau \in [0,\,1]$, we have

$$\begin{aligned}
&\mathcal{T}_*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\,\tau\right) \le \tfrac{1}{2}\left(\mathcal{T}_*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\,\tau\right)+\mathcal{T}_*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\,\tau\right)\right), \\
&\mathcal{T}^*\left(\left[\tfrac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\,\tau\right) \ge \tfrac{1}{2}\left(\mathcal{T}^*\left([\hbar\imath^p+(1-\hbar)\jmath^p]^{\frac{1}{p}},\,\tau\right)+\mathcal{T}^*\left([(1-\hbar)\imath^p+\hbar\jmath^p]^{\frac{1}{p}},\,\tau\right)\right).
\end{aligned} \tag{47}$$

Since $\mathrm{N}\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}\right) = \mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)$, then by multiplying (47) by $\mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)$ and integrate it with respect to $\hbar$ over $[0,1]$, we obtain

$$\begin{aligned} &\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right)\int_0^1 \mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &\leq \frac{1}{2}\left(\begin{array}{c} \int_0^1 \mathcal{T}_*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ +\int_0^1 \mathcal{T}_*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right)d\hbar\mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right) \end{array}\right), \\ &\mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right)\int_0^1 \mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &\geq \frac{1}{2}\left(\begin{array}{c} \int_0^1 \mathcal{T}^*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ +\int_0^1 \mathcal{T}^*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \end{array}\right). \end{aligned} \tag{48}$$

Since

$$\begin{aligned} &\int_0^1 \mathcal{T}_*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &= \int_0^1 \mathcal{T}_*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &= \frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega, \\ &\int_0^1 \mathcal{T}^*\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &= \int_0^1 \mathcal{T}^*\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}, \tau\right)\mathrm{N}\left([\hbar\imath^p + (1-\hbar)\jmath^p]^{\frac{1}{p}}\right)d\hbar \\ &= \frac{p}{\jmath^p-\imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega. \end{aligned} \tag{49}$$

Then, from (49) we have

$$\begin{aligned} \mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right) &\leq \frac{1}{\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega, \\ \mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right) &\geq \frac{1}{\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega, \end{aligned}$$

from which, we have

$$\begin{aligned} &\left[\mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right), \mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}, \tau\right)\right] \\ &\supseteq_I \frac{1}{\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega}\left[\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega, \int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega\right], \end{aligned}$$

that is

$$\widetilde{\mathcal{T}}\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}}\right) \supseteq_{\mathbb{F}} \frac{1}{\int_\imath^\jmath \omega^{p-1}\mathrm{N}(\omega)d\omega} \odot (FA)\int_\imath^\jmath \omega^{p-1} \odot \widetilde{\mathcal{T}}(\omega)\mathrm{N}(\omega)d\omega,$$

this completes the proof. □

Remark 5. *If in Theorems 11 and 12, $p = 1$, then we obtain the appropriate theorems for UD-convex fuzzy-IVMs [77].*

If in the Theorems 11 and 12, $\mathcal{T}_*(\omega,\gamma) = \mathcal{T}^*(\omega,\gamma)$ with $\gamma = 1$, then we obtain the appropriate theorems for p-convex function [84].

If in the Theorems 11 and 12, $\mathcal{T}_*(\omega,\gamma) = \mathcal{T}^*(\omega,\gamma)$ with $\gamma = 1$ and $p = 1$, then we obtain the appropriate theorems for convex function [90].

If $\daleth(\omega) = 1$, then combining Theorems 11 and 12, we get Theorem 7.

Example 6. *We consider the FNVM $\widetilde{\mathcal{T}} : [0,\ 2] \to \mathcal{L}_C$ defined by,*

$$\widetilde{\mathcal{T}}(\omega)(\theta) = \begin{cases} \frac{\theta - 2 + \omega^{\frac{p}{2}}}{\frac{3}{2} - 2 - \omega^{\frac{p}{2}}}\ \theta \in \left[2 - \omega^{\frac{p}{2}},\ \frac{3}{2}\right], \\ \frac{2 + \omega^{\frac{p}{2}} - \theta}{2 + \omega^{\frac{p}{2}} - \frac{3}{2}}\ \theta \in \left(\frac{3}{2},\ 2 + \omega^{\frac{p}{2}}\right], \\ 0 \text{ otherwise,} \end{cases}$$

Then, for each $\tau \in [0,\ 1]$, we have $\mathcal{T}_\tau(\omega) = [(1-\tau)\left(2 - \omega^{\frac{p}{2}}\right) + \frac{3}{2}\tau, (1+\tau)\left(2 + \omega^{\frac{p}{2}}\right)$ $+\frac{3}{2}\tau]$. Since end point mappings $\mathcal{T}_*(\omega,\tau)$, and $\mathcal{T}^*(\omega,\tau)$ are convex and concave mappings, respectively, for each $\tau \in [0,\ 1]$, then $\widetilde{\mathcal{T}}(\omega)$ is U$\mathcal{D}$-convex FNVM. If

$$\daleth(\omega) = \begin{cases} \omega^{\frac{p}{2}},\ \sigma \in [0,1], \\ (2-\omega)^{\frac{p}{2}},\ \sigma \in (1,\ 2], \end{cases}$$

then $\daleth\left((2-\omega)^p\right) = \daleth(\omega) \geq 0$, for all $\omega \in [0,\ 2]$.

Since $\mathcal{T}_*(\omega,\tau) = (1-\tau)\left(2 - \omega^{\frac{p}{2}}\right) + \frac{3}{2}\tau$ and $\mathcal{T}^*(\omega,\tau) = (1+\tau)\left(2 + \omega^{\frac{p}{2}}\right) + \frac{3}{2}\tau$.

Now we compute the following:

$$\begin{aligned}
\frac{p}{\jmath^p - \imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\daleth(\omega)d\omega &= \frac{1}{2}\int_0^2 \omega^{p-1}\mathcal{T}_*(\omega,\tau)\daleth(\omega)d\omega \\
&= \frac{1}{2}\int_0^1 \omega^{p-1}\mathcal{T}_*(\omega,\tau)\daleth(\omega)d\omega + \frac{1}{2}\int_1^2 \omega^{p-1}\mathcal{T}_*(\omega,\tau)\daleth(\omega)d\omega, \\
\frac{p}{\jmath^p - \imath^p}\int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\daleth(\omega)d\omega &= \frac{1}{2}\int_0^2 \omega^{p-1}\mathcal{T}^*(\omega,\tau)\daleth(\omega)d\omega \\
&= \frac{1}{2}\int_0^1 \omega^{p-1}\mathcal{T}^*(\omega,\tau)\daleth(\omega)d\omega + \frac{1}{2}\int_1^2 \omega^{p-1}\mathcal{T}^*(\omega,\tau)\daleth(\omega)d\omega, \\
= \frac{1}{2}\int_0^1 \left[(1-\tau)\left(2 - \omega^{\frac{1}{2}}\right) + \frac{3}{2}\tau\right]\left(\sqrt{\omega}\right)d\omega &+ \frac{1}{2}\int_1^2 \left[(1-\tau)\left(2 - \omega^{\frac{1}{2}}\right) + \frac{3}{2}\tau\right]\left(\sqrt{2-\omega}\right)d\omega \\
&= \frac{1}{4}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \tau\left[\frac{\pi}{8} - \frac{1}{12}\right], \\
= \frac{1}{2}\int_0^1 \left[(1+\tau)\left(2 + \omega^{\frac{1}{2}}\right) + \frac{3}{2}\tau\right]\left(\sqrt{\omega}\right)d\omega &+ \frac{1}{2}\int_1^2 \left[(1+\tau)\left(2 + \omega^{\frac{1}{2}}\right) + \frac{3}{2}\tau\right]\left(\sqrt{2-\omega}\right)d\omega \\
&= \frac{1}{4}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \tau\left[\frac{\pi}{8} + \frac{31}{12}\right].
\end{aligned} \tag{50}$$

And

$$\begin{aligned}
&[\mathcal{T}_*(\imath,\ \tau) + \mathcal{T}_*(\jmath,\ \tau)]\int_0^1 \hbar\daleth\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
&= \left[4(1-\tau) - \sqrt{2}(1-\tau) + 3\tau\right]\left[\int_0^{\frac{1}{2}} \hbar\sqrt{2\hbar}d\hbar + \int_{\frac{1}{2}}^1 \hbar\sqrt{2(1-\hbar)}d\hbar\right] \\
&= \frac{1}{3}\left(4(1-\tau) - \sqrt{2}(1-\tau) + 3\tau\right), \\
&[\mathcal{T}^*(\imath,\ \tau) + \mathcal{T}^*(\jmath,\ \tau)]\int_0^1 \hbar\daleth\left([(1-\hbar)\imath^p + \hbar\jmath^p]^{\frac{1}{p}}\right)d\hbar \\
&= \left[4(1+\tau) + \sqrt{2}(1+\tau) + 3\tau\right]\left[\int_0^{\frac{1}{2}} \hbar\sqrt{2\hbar}d\hbar + \int_{\frac{1}{2}}^1 \hbar\sqrt{2(1-\hbar)}d\hbar\right] \\
&= \frac{1}{3}\left(4(1+\tau) + \sqrt{2}(1+\tau) + 3\tau\right).
\end{aligned} \tag{51}$$

From (50) and (51), we have

$$\begin{aligned}
&\left[\frac{1}{4}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \tau\left[\frac{\pi}{4} - \frac{7}{6}\right],\ \frac{1}{4}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \tau\left[\frac{\pi}{4} + \frac{25}{6}\right]\right] \\
&\supseteq_I \left[\frac{1}{3}\left(4(1-\tau) - \sqrt{2}(1-\tau) + 3\tau\right),\ \frac{1}{3}\left(4(1+\tau) + \sqrt{2}(1+\tau) + 3\tau\right)\right], \text{ for all } \tau \in [0,\ 1].
\end{aligned}$$

Hence, Theorem 11 is verified.

For Theorem 12, we have

$$\begin{aligned} \mathcal{T}_*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) = \mathcal{T}_*(1,\ \tau) = \tfrac{2+\tau}{2}, \\ \mathcal{T}^*\left(\left[\frac{\imath^p+\jmath^p}{2}\right]^{\frac{1}{p}},\ \tau\right) = \mathcal{T}^*(1,\ \tau) = \tfrac{3(2+3\tau)}{2}, \end{aligned} \tag{52}$$

$$\int_\imath^\jmath \mathrm{N}(\omega)d\omega = \int_0^1 \sqrt{\omega}d\omega + \int_1^2 \sqrt{2-\omega}d\omega = \frac{4}{3},$$

$$\begin{aligned} \frac{p}{\int_\imath^\jmath \mathrm{N}(\omega)d\omega} \int_\imath^\jmath \omega^{p-1}\mathcal{T}_*(\omega,\tau)\mathrm{N}(\omega)d\omega = \tfrac{3}{8}\left[\tfrac{13}{3} - \tfrac{\pi}{2}\right] + \tfrac{3\tau}{2}\left[\tfrac{\pi}{8} - \tfrac{1}{12}\right], \\ \frac{p}{\int_\imath^\jmath \mathrm{N}(\omega)d\omega} \int_\imath^\jmath \omega^{p-1}\mathcal{T}^*(\omega,\tau)\mathrm{N}(\omega)d\omega = \tfrac{3}{8}\left[\tfrac{19}{3} + \tfrac{\pi}{2}\right] + \tfrac{3\tau}{2}\left[\tfrac{\pi}{8} + \tfrac{31}{12}\right]. \end{aligned} \tag{53}$$

From (52) and (53), we have

$$\left[\frac{2+\tau}{2},\ \frac{3(2+3\tau)}{2}\right] \supseteq_I \left[\frac{3}{8}\left[\frac{13}{3} - \frac{\pi}{2}\right] + \frac{3\tau}{2}\left[\frac{\pi}{8} - \frac{1}{12}\right],\ \frac{3}{8}\left[\frac{19}{3} + \frac{\pi}{2}\right] + \frac{3\tau}{2}\left[\frac{\pi}{8} + \frac{31}{12}\right]\right].$$

Hence, Theorem 12 has been verified.

5. Conclusions

This work examines the new class of p-convexity over up and down fuzzy relation which is known as UD-p-convex FNVMs. The usage of FNVMs in probability density functions and numerical integration makes the subject intriguing. Kunt and İşcan's (see, [90]) indings are generalized in the context of convex FNVMs. We obtain both a novel Hermite–Hadamard-type inequality and a Hermite–Hadamard–Fejer-type inequality. There are no doubts regarding the feasibility of generalizing the fuzzy Aumann integral type inequalities found in this article because we have given some exceptional cases that can be viewed as applications of main outcomes. Some new examples have been provided to discuss the validity of main results.

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References

1. Pěcaríc, J.; Proschan, F.; Tong, Y. (Eds.) *Convex Functions, Partial Orderings, and Statistical Applications*; Academic Press: London, UK, 1992.
2. Mitrinovíc, D.S. *Analytic Inequalities*; Springer: Berlin/Heidelberg, Germany, 1970.
3. Aljaaidi, T.A.; Pachpatte, D.B. The Minkowski's inequalities via Riemann-Liouville fractional integral operators. *Rend. Circ. Mat. Palermo Ser. B* **2021**, *70*, 893–906. [CrossRef]
4. Awan, M.U.; Talib, S.; Chu, Y.M.; Noor, M.A.; Noor, K.I. Some new refinements of Hermite-Hadamard-type inequalities Involving-Riemann-Liouville fractional integrals and applications. *Math. Probl. Eng.* **2020**, *2020*, 3051920. [CrossRef]
5. Chen, H.; Katugampola, U.N. Hermite-Hadamard and Hermite-Hadamard-Fejr type inequalities for generalized fractional integrals. *J. Math. Anal. Appl.* **2017**, *446*, 1274–1291. [CrossRef]
6. Sarikaya, M.Z.; Yildirim, H. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals. *Miskolc Math. Notes* **2016**, *17*, 1049–1059. [CrossRef]
7. Chu, Y.-M.; Wang, G.-D.; Zhang, X.-H. The Schur multiplicative and harmonic convexities of the complete symmetric function. *Math. Nachr.* **2011**, *284*, 653–663. [CrossRef]
8. Chu, Y.-M.; Xia, W.-F.; Zhang, X.-H. The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications. *J. Multivar. Anal.* **2012**, *105*, 412–421. [CrossRef]
9. Cheng, J.-F.; Chu, Y.-M. Solution to the linear fractional differential equation using Adomian decomposition method. *Math. Probl. Eng.* **2011**, *2011*, 587068. [CrossRef]
10. Cheng, J.-F.; Chu, Y.-M. On the fractional difference equations of order (2, q). *Abstr. Appl. Anal.* **2011**, *2011*, 497259. [CrossRef]
11. Khan, M.B.; Althobaiti, A.; Lee, C.-C.; Soliman, M.S.; Li, C.-T. Some New Properties of Convex Fuzzy-Number-Valued Mappings on Coordinates Using Up and Down Fuzzy Relations and Related Inequalities. *Mathematics* **2023**, *11*, 2851. [CrossRef]
12. Khan, M.B.; Macías-Díaz, J.E.; Althobaiti, A.; Althobaiti, S. Some New Properties of Exponential Trigonometric Convex Functions Using Up and Down Relations over Fuzzy Numbers and Related Inequalities through Fuzzy Fractional Integral Operators Having Exponential Kernels. *Fractal Fract.* **2023**, *7*, 567. [CrossRef]
13. Khan, M.B.; Guiro, J.L.G. Riemann Liouville fractional-like integral operators, convex-like real-valued mappings and their applications over fuzzy domain. *Chaos Solitons Fractals* **2023**, *177*, 114196. [CrossRef]
14. Khan, M.B.A.; Othman, H.A.; Santos-García, G.; Saeed, T.; Soliman, M.S. On fuzzy fractional integral operators having exponential kernels and related certain inequalities for exponential trigonometric convex fuzzy-number valued mappings. *Chaos Solitons Fractals* **2023**, *169*, 113274. [CrossRef]
15. Hermann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific: Singapore, 2011.
16. Oldham, K.B.; Spanier, J. (Eds.) *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Academic Press: London, UK, 1974.
17. Yang, X.J. *General Fractional Derivatives: Theory, Methods and Applications*; Chapman & Hall: London, UK, 2019.
18. Khan, M.B.; Santos-García, G.; Budak, H.; Treanță, S.; Soliman, M.S. Some new versions of Jensen, Schur and Hermite-Hadamard type inequalities for $(p, \mathfrak{J})$-convex fuzzy-interval-valued functions. *AIMS Math.* **2023**, *8*, 7437–7470. [CrossRef]
19. Khan, M.B.; Othman, H.A.; Voskoglou, M.G.; Abdullah, L.; Alzubaidi, A.M. Some Certain Fuzzy Aumann Integral Inequalities for Generalized Convexity via Fuzzy Number Valued Mappings. *Mathematics* **2023**, *11*, 550. [CrossRef]
20. Khan, M.B.; Rakhmangulov, A.; Aloraini, N.; Noor, M.A.; Soliman, M.S. Generalized Harmonically Convex Fuzzy-Number-Valued Mappings and Fuzzy Riemann–Liouville Fractional Integral Inequalities. *Mathematics* **2023**, *11*, 656. [CrossRef]
21. Khan, M.B.; Catas, A.; Aloraini, N.; Soliman, M.S. Some Certain Fuzzy Fractional Inequalities for Up and Down $\hbar$-Pre-Invex via Fuzzy-Number Valued Mappings. *Fractal Fract.* **2023**, *7*, 171. [CrossRef]
22. Cheng, J.-F.; Chu, Y.-M. Fractional difference equations with real variable. *Abstr. Appl. Anal.* **2012**, *2012*, 918529. [CrossRef]
23. Hu, X.-M.; Tian, J.-F.; Chu, Y.-M.; Lu, Y.-X. On Cauchy–Schwarz inequality for N-tuple diamond-alpha integral. *J. Inequal. Appl.* **2020**, *2020*, 1–15. [CrossRef]
24. Zhao, T.-H.; Chu, Y.-M.; Wang, H. Logarithmically complete monotonicity properties relating to the gamma function. *Abstr. Appl. Anal.* **2011**, *2011*, 896483. [CrossRef]
25. Ashpazzadeh, E.; Chu, Y.-M.; Hashemi, M.S.; Moharrami, M.; Inc, M. Hermite multiwavelets representation for the sparse solution of nonlinear Abel's integral equation. *Appl. Math. Comput.* **2022**, *427*, 127171. [CrossRef]
26. Chu, Y.-M.; Ullah, S.; Ali, M.; Tuzzahrah, G.F.; Munir, T. Numerical investigation of Volterra integral equations of second kind using optimal homotopy asymptotic method. *Appl. Math. Comput.* **2022**, *430*, 127304. [CrossRef]
27. Chu, Y.-M.; Inc, M.; Hashemi, M.S.; Eshaghi, S. Analytical treatment of regularized Prabhakar fractional differential equations by invariant subspaces. *Comput. Appl. Math.* **2022**, *41*, 271. [CrossRef]
28. Hadamard, J. Étude sur les propriétés des fonctions entières en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
29. Khan, M.B.; Santos-García, G.; Noor, M.A.; Soliman, M.S. New Class of Preinvex Fuzzy Mappings and Related Inequalities. *Mathematics* **2022**, *10*, 3753. [CrossRef]
30. Khan, M.B.; Macías-Díaz, J.E.; Treanță, S.; Soliman, M.S. Some Fejér-Type Inequalities for Generalized Interval-Valued Convex Functions. *Mathematics* **2022**, *10*, 3851. [CrossRef]

31. Khan, M.B.; Santos-García, G.; Treanţă, S.; Soliman, M.S. New Class up and down Pre-Invex Fuzzy Number Valued Mappings and Related Inequalities via Fuzzy Riemann Integrals. *Symmetry* **2022**, *14*, 2322. [CrossRef]
32. Khan, M.B.; Macías-Díaz, J.E.; Soliman, M.S.; Noor, M.A. Some New Integral Inequalities for Generalized Preinvex Functions in Interval-Valued Settings. *Axioms* **2022**, *11*, 622. [CrossRef]
33. Khan, M.B.; Zaini, H.G.; Santos-García, G.; Noor, M.A.; Soliman, M.S. New Class up and down λ-Convex Fuzzy-Number Valued Mappings and Related Fuzzy Fractional Inequalities. *Fractal Fract.* **2022**, *6*, 679. [CrossRef]
34. Khan, M.B.; Santos-García, G.; Treanţă, S.; Noor, M.A.; Soliman, M.S. Perturbed Mixed Variational-Like Inequalities and Auxiliary Principle Pertaining to a Fuzzy Environment. *Symmetry* **2022**, *14*, 2503. [CrossRef]
35. Khan, M.B.; Zaini, H.G.; Macías-Díaz, J.E.; Soliman, M.S. Up and Down h-Pre-Invex Fuzzy-Number Valued Mappings and Some Certain Fuzzy Integral Inequalities. *Axioms* **2023**, *12*, 1. [CrossRef]
36. Khan, M.B.; Othman, H.A.; Santos-García, G.; Noor, M.A.; Soliman, M.S. Some new concepts in fuzzy calculus for up and down λ-convex fuzzy-number valued mappings and related inequalities. *AIMS Math.* **2023**, *8*, 6777–6803. [CrossRef]
37. Wang, M.-K.; Chu, Y.-M.; Qiu, Y.-F.; Qiu, S.-L. An optimal power mean inequality for the complete elliptic integrals. *Appl. Math. Lett.* **2011**, *24*, 887–890. [CrossRef]
38. Wang, M.-K.; Chu, Y.-M.; Zhang, W. Monotonicity and inequalities involving zero-balanced hypergeometric function. *Math. Inequal. Appl.* **2019**, *22*, 601–617. [CrossRef]
39. Wang, M.-K.; Chu, Y.-M.; Zhang, W. Precise estimates for the solution of Ramanujan's generalized modular equation. *Ramanujan J.* **2019**, *49*, 653–668. [CrossRef]
40. Wang, M.-K.; He, Z.-Y.; Chu, Y.-M. Sharp Power Mean Inequalities for the Generalized Elliptic Integral of the First Kind. *Comput. Methods Funct. Theory* **2020**, *20*, 111–124. [CrossRef]
41. Abbas Baloch, I.; Chu, Y.-M. Petrovic-type inequalities for harmonic h-convex functions. *J. Funct. Spaces* **2020**, *2020*, 3075390. [CrossRef]
42. Chu, Y.-M.; Long, B.-Y. Sharp inequalities between means. *Math. Inequal. Appl.* **2011**, *14*, 647–655. [CrossRef]
43. Chu, Y.-M.; Qiu, Y.-F.; Wang, M.-K. Hölder mean inequalities for the complete elliptic integrals. *Integral Transform. Spec. Funct.* **2012**, *23*, 521–527. [CrossRef]
44. Chu, Y.-M.; Wang, M.-K. Inequalities between arithmetic geometric, Gini, and Toader means. *Abstr. Appl. Anal.* **2012**, *2012*, 830585. [CrossRef]
45. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Basak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
46. Ramk, J. Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets Syst.* **1985**, *16*, 123–138. [CrossRef]
47. Costa, T. Jensen's inequality type integral for fuzzy-interval-valued functions. *Fuzzy Sets Syst.* **2017**, *327*, 31–47. [CrossRef]
48. Costa, T.M.; Román-Flores, H. Some integral inequalities for fuzzy-interval-valued functions. *Inf. Sci.* **2017**, *420*, 110–125. [CrossRef]
49. Chu, Y.M.; Zhao, T.H. Concavity of the error function with respect to Hölder means. *Math. Inequal. Appl.* **2016**, *19*, 589–595. [CrossRef]
50. Zhao, T.H.; Shi, L.; Chu, Y.M. Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means. *RACSAM Rev. R Acad. A* **2020**, *114*, 1–14. [CrossRef]
51. Zhao, T.H.; Zhou, B.C.; Wang, M.K.; Chu, Y.M. On approximating the quasi-arithmetic mean. *J. Inequal. Appl.* **2019**, *2019*, 42. [CrossRef]
52. Zhao, T.H.; Wang, M.K.; Zhang, W.; Chu, Y.M. Quadratic transformation inequalities for Gaussian hyper geometric function. *J. Inequal. Appl.* **2018**, *2018*, 251. [CrossRef]
53. Qian, W.M.; Chu, H.H.; Wang, M.K.; Chu, Y.M. Sharp inequalities for the Toader mean of order -1 in terms of other bivariate means. *J. Math. Inequal.* **2022**, *16*, 127–141. [CrossRef]
54. Zhao, T.H.; Chu, H.H.; Chu, Y.M. Optimal Lehmer mean bounds for the nth power-type Toader mean of n = -1, 1, 3. *J. Math. Inequal.* **2022**, *16*, 157–168. [CrossRef]
55. Wang, M.-K.; Chu, Y.-M. Refinements of transformation inequalities for zero-balanced hypergeometric functions. *Acta Math. Sci.* **2017**, *37B*, 607–622. [CrossRef]
56. Wang, M.-K.; Chu, Y.-M. Landen inequalities for a class of hypergeometric functions with applications. *Math. Inequal. Appl.* **2018**, *21*, 521–537. [CrossRef]
57. Wang, M.-K.; Chu, H.-H.; Chu, Y.-M. Precise bounds for the weighted Holder mean of the complete p-elliptic integrals. *J. Math. Anal. Appl.* **2019**, *480*, 123388. [CrossRef]
58. Wang, M.-K.; Chu, Y.-M.; Jiang, Y.-P. Ramanujan's cubic transformation inequalities for zero-balanced hypergeometric functions. *Rocky Mt. J. Math.* **2016**, *46*, 679–691. [CrossRef]
59. Wang, M.-K.; Chu, H.-H.; Li, Y.-M.; Chu, Y.-M. Answers to three conjectures on convexity of three functions involving complete elliptic integrals of the first kind. *Appl. Anal. Discret. Math.* **2020**, *14*, 255–271. [CrossRef]
60. Zhao, D.; An, T.; Ye, G.; Liu, W. New Jensen and Hermite-Hadamard type inequalities for *h*-convex interval-valued functions. *J. Inequalities Appl.* **2018**, *2018*, 302. [CrossRef]

61. Liu, X.; Feng, F.; Yager, R.R.; Davvaz, B.; Khan, M. On modular inequalities of interval-valued fuzzy soft sets characterized by soft *J*-inclusions. *J. Inequalities Appl.* **2014**, *2014*, 360. [CrossRef]
62. Yang, Y.; Saleem, M.S.; Nazeer, W.; Shah, A.F. New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus via exponentially convex fuzzy interval-valued function. *AIMS Math.* **2021**, *6*, 12260–12278. [CrossRef]
63. Chalco-Cano, Y.; Lodwick, W.A.; Condori-Equice, W. Ostrowski type inequalities and applications in numerical integration for interval-valued functions. *Soft. Comput.* **2015**, *19*, 3293–3300. [CrossRef]
64. Osuna-Gomez, R.; Jimenez-Gamero, M.D.; Chalco-Cano, Y.; Rojas-Medar, M.A. Hadamard and Jensen inequalities for *s*-convex fuzzy processes. In *Soft Methodology and Random Information Systems*; Springer: Berlin/Heidelberg, Germany, 2004; pp. 645–652.
65. Khan, M.B.; Noor, M.A.; Noor, K.I.; Chu, Y.M. New Hermite-Hadamard type inequalities for -convex fuzzy-interval-valued functions. *Adv. Differ. Equ.* **2021**, *2021*, 6–20. [CrossRef]
66. Dragomir, S.S.; Pearce, C.E.M. Selected topics on Hermite–Hadamard inequalities and applications. In *RGMIA Monographs*; Victoria University: Footscray, Australia, 2000.
67. Nanda, S.; Kar, K. Convex fuzzy mappings. *Fuzzy Sets Syst.* **1992**, *48*, 129–132. [CrossRef]
68. Syau, Y.R. On convex and concave fuzzy mappings. *Fuzzy Sets Syst.* **1999**, *103*, 163–168. [CrossRef]
69. Puri, M.L.; Ralescu, D.A. Fuzzy random variables. *Math. Anal. Appl.* **1986**, *114*, 409–422. [CrossRef]
70. Goetschel, R., Jr.; Voxman, W. Elementary fuzzy calculus. *Fuzzy Sets Syst.* **1986**, *18*, 31–43. [CrossRef]
71. Yan, H.; Xu, J. A class of convex fuzzy mappings. *Fuzzy Sets Syst.* **2002**, *129*, 47–56. [CrossRef]
72. Romn-Flores, H.; Chalco-Cano, Y.; Lodwick, W. Some integral inequalities for interval-valued functions. *Comput. Appl. Math.* **2018**, *37*, 1306–1318. [CrossRef]
73. Romn-Flores, H.; Chalco-Cano, Y.; Silva, G.N. A note on Gronwall type inequality for interval-valued functions. In Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 24–28 June 2013; pp. 1455–1458.
74. Moore, R.E. *Interval Analysis*; Prentice Hall: Englewood Cliffs, NJ, USA, 1966.
75. Kaleva, O. Fuzzy Differential Equations. *Fuzzy Sets Syst.* **1987**, *24*, 301–317. [CrossRef]
76. Rainville, E.D. *Special Functions*; Chelsea Publ. Co.: Bronx, NY, USA, 1971.
77. Khan, M.B.; Santos-García, G.; Noor, M.A.; Soliman, M.S. Some new concepts related to fuzzy fractional calculus for up and down convex fuzzy-number valued functions and inequalities. *Chaos Solitons Fractals* **2022**, *164*, 112692. [CrossRef]
78. Diamond, P.; Kloeden, P. *Metric Spaces of Fuzzy Sets: Theory and Applications*; World Scientific: Singapore, 1994.
79. Bede, B. *Mathematics of Fuzzy Sets and Fuzzy Logic, Volume 295 of Studies in Fuzziness and Soft Computing*; Springer: Berlin/Heidelberg, Germany, 2013.
80. Zhang, D.; Guo, C.; Chen, D.; Wang, G. Jensen's inequalities for set-valued and fuzzy set-valued functions. *Fuzzy Sets Syst.* **2020**, *404*, 178–204. [CrossRef]
81. Fang, Z.B.; Shi, R.J. On the (p, h)-convex function and some integral inequalities. *J. Inequal. Appl.* **2014**, *45*, 1–6. [CrossRef]
82. Fejér, L. Über die Bestimmung des Sprunges der Funktion aus ihrer Fourierreihe. *J. Reine Angew. Math.* **1913**, *142*, 165–188. [CrossRef]
83. An, Y.; Ye, G.; Zhao, D.; Liu, W. Hermite-Hadamard type inequalities for interval (h1; h2)-convex functions. *Mathematics* **2019**, *7*, 436. [CrossRef]
84. Chandola, A.; Agarwal, R.; Pandey, M.R. Some new Hermite-Hadamard, Hermite-Hadamard Fejer and weighted Hardy type inequalities involving (k-p) Riemann-Liouville fractional integral operator. *Appl. Math. Inf. Sci.* **2022**, *16*, 287–297.
85. Mubeen, S.; Habibullah, G. k-Fractional integrals and application. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 89–94.
86. Sarikaya, M.; Dahmani, Z.; Kiris, M.; Ahmed, F. (k; s)-Riemann-Liouville fractional integral and applications. *Hacet. J. Math. Stat.* **2016**, *45*, 77–89. [CrossRef]
87. Srivastava, H.M.; Mehrez, S.; Sitnik, S.M. Hermite-Hadamard-type integral inequalities for convex functions and their applications. *Mathematics* **2022**, *10*, 3127. [CrossRef]
88. Stojiljkovíc, V. Hermite Hadamard type inequalities involving (k-p) fractional operator with (α, h-m)-p convexity. *Eur. J. Pure Appl. Math.* **2023**, *16*, 503–522. [CrossRef]
89. Stojiljkovíc, V.; Ramaswamy, R.; Abdelnaby, O.A.A.; Radenovic, S. Some novel inequalities for LR-(k,h-m)-p convex interval-valued functions by means of pseudo order relation. *Fractal Fract.* **2022**, *6*, 726. [CrossRef]
90. Kunt, M.; İşcan, İ. Hermite–Hadamard–Fejer type inequalities for *p*-convex functions. *Arab J. Math. Sci.* **2017**, *23*, 215–230. [CrossRef]