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Certain Results on Subclasses of Analytic and Bi-Univalent Functions Associated with Coefficient Estimates and Quasi-Subordination

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Abstract: The purpose of the present paper is to introduce and investigate new subclasses of analytic function class of bi-univalent functions defined in open unit disks connected with a linear q-convolution operator, which are associated with quasi-subordination. We find coefficient estimates of $|h_2|$, $|h_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out. There is symmetry between the results of the subclass $\mathfrak{f}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$ and the results of the subclass $\aleph^{q,\delta}_{\Sigma}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$.

Keywords: analytic function; univalent function; convolution (q-derivatives); quasi-subordination; coefficient estimate

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1. Introduction

The theory of q-calculus plays an important role in many areas of mathematical physical and engineering sciences. Jackson (see [1,2]) was the first to perform some applications of the q-calculus and introduced the q-analogue of the classical derivative and integral operators (see also [3]).

Let $\mathcal{A}$ be the class of analytic functions $\mathcal{T}$ in an open unit disk $\mathfrak{U} = \{\varepsilon \in \mathbb{C}\colon |\varepsilon| < 1\}$ of the form:

$$\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{+\infty} a_j \varepsilon^j, \ (\varepsilon \in \mathfrak{U}). \tag{1}$$

and satisfying the normalization conditions (see [4]): $\mathcal{T}(0) = \mathcal{T}'(0) - 1 = 0$.

Assume that $\sum_{\mathfrak{U}}$ denotes the class of all functions in $\mathcal{A}$ defined by Equation (1), which are univalent in $\mathfrak{U}$.

The well-known Koebe One-Quarter Theorem [5] states that the range of every function of class $\sum_{\mathfrak{U}}$ contains the disk $\{w\colon |w| < \frac{1}{4}\}$. Thus, every univalent function $\mathcal{T}$ has an inverse $\mathcal{T}^{-1}$, such that

$$\mathcal{T}^{-1}(\mathcal{T}(\varepsilon)) = \varepsilon, \ \ (\varepsilon \in \mathfrak{U}),$$

and

$$\mathcal{T}(\mathcal{T}^{-1}(\varsigma)) = \varsigma \ (|\varsigma| < r_0(\mathcal{T}); r_0(\mathcal{T}) \geq \frac{1}{4}), \ (r_0 \text{ is radius}).$$

In fact, the inverse function $\xi = \mathcal{T}^{-1}$ is given by

$$\begin{aligned} \xi(\varsigma) &= \varsigma - a_2\varsigma^2 + (2a_2^2 - a_3)\varsigma^3 - (5a_2^2 - 5a_2a_3 + a_4)\varsigma^4 + \ldots \\ &= \varsigma + \textstyle\sum_{n=2}^{\infty} A_n \varsigma^n. \end{aligned} \quad (2)$$

The function $\mathcal{T} \in \mathcal{A}$ is said to be bi-univalent in $\mathfrak{U}$ if both $\mathcal{T}$ and its inverse $\mathcal{T}^{-1}$ are univalent functions in $\mathfrak{U}$ given by Equation (1).

The class of bi-univalent functions was introduced by Lewin [6] and proved that $|a_2| \leq 1.51$ for the function of the form Equation (1). Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [8] proved that $\max\limits_{\mathcal{T}\in\sum}|a_2| = \frac{4}{3}$. Also, several authors studied classes of bi-univalent analytic functions and found estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in these classes [For two analytic functions $\mathcal{T}$ and ξ, $\mathcal{T}$ is quasi-subordinate to ξ, written as follows:

$$\mathcal{T}(\varepsilon) \prec_q \xi(\varepsilon) \ \ (\varepsilon \in \mathfrak{U}) \quad (3)$$

if there exist analytic functions $h(\varepsilon)$ and $\|(\varepsilon)$, with $|h(z)| \leq 1$, $\|(0) = 0$ and $|\|(\varepsilon)| < 1$, $(\varepsilon \in \mathfrak{U})$, such that

$$\mathcal{T}(\varepsilon) = h(\varepsilon)\xi(\|(\varepsilon)),\ (\varepsilon \in \mathfrak{U}).$$

Note that if $(h(\varepsilon) = 1)$, then $\mathcal{T}(\varepsilon) = \xi(\|(\varepsilon))$; hence, $\mathcal{T}(\varepsilon) \prec \xi(\varepsilon)$ $(z \in \mathfrak{U})$. If ξ is univalent in $\mathfrak{U}$, then $\mathcal{T} \prec \xi$ if and only if $\mathcal{T}(0) = \xi(0)$ and $\mathcal{T}(\mathfrak{U}) \subset \xi(\mathfrak{U})$].

For the functions $\mathcal{T}, \rho \in \sum_{\mathfrak{U}}$ defined by $\mathcal{T}(\varepsilon) = \sum_{j=1}^{+\infty} a_j \varepsilon^j$ and $\rho(\varepsilon) = \sum_{j=1}^{+\infty} h_j \varepsilon^j (\varepsilon \in \mathfrak{U})$, the convolution of $\mathcal{T}$ and ρ denoted by $\mathcal{T} * \rho$ is

$$(\mathcal{T} * \rho)(\varepsilon) = \sum_{j=1}^{+\infty} a_j h_j \varepsilon^j = (\rho * \mathcal{T})(\varepsilon) \ \ (\varepsilon \in \mathfrak{U}).$$

To start with, we recall the following differential and integral operators. For $0 < q < 1$, El-Deeb et al. [9,10], and others [11] defined the q-convolution operator (see also [1]) for $\mathcal{T} * \rho$ by

$$\mathfrak{Q}_q(\mathcal{T} * \rho)(\varepsilon) = \mathfrak{Q}_q\left(\varepsilon + \sum_{j=2}^{+\infty} a_j h_j \varepsilon^j\right)$$

$$\frac{(\mathcal{T} * \rho)(\varepsilon) - (\mathcal{T} * \rho)(q\varepsilon)}{\varepsilon(1-q)} = 1 + \sum_{j=2}^{+\infty} [j]_q\, a_j h_j \varepsilon^{j-1}, \varepsilon \in \mathfrak{U},$$

where

$$[j]_q = \frac{1-q^j}{1-q} = 1 + \sum_{j=1}^{j-1} q^j, [0]_q = 0. \quad (4)$$

We used the linear operator $\mathcal{Y}_{\rho}^{\zeta,q}$: $\mathcal{A} \to \mathcal{A}$ according to El-Deeb [9] (see also [12]) for and $\zeta > -1$, $0 < q < 1$. If

$$\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon) * \mathbb{I}_q^{\zeta+1}(\varepsilon) = \varepsilon\mathfrak{Q}_q(\mathcal{T} * \rho)(\varepsilon), \varepsilon \in \mathfrak{U},$$

where $\mathbb{I}_q^{\zeta+1}$ is given by

$$\mathbb{I}_q^{\zeta+1}(\varepsilon) = \varepsilon + \sum_{j=2}^{+\infty} \frac{[\zeta+1]_{q,\varepsilon-1}}{[\varepsilon-1]_q!}\, \varepsilon^j, \varepsilon \in \mathfrak{U},$$

then,

$$\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon) = \varepsilon + \sum_{j=2}^{+\infty} \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} a_j h_j\, \varepsilon^j (\zeta > -1, 0 < q < 1, \varepsilon \in \mathfrak{U}). \quad (5)$$

Using the operator $\mathcal{Y}_{\rho}^{\zeta,q}$, we define a new operator as follows:

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,0}\mathcal{T}(\varepsilon)=\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon)$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,1}\mathcal{T}(\varepsilon)=(\sigma-\vartheta)\varepsilon^{3}\left(\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon)\right)'''+(1+2(\sigma-\vartheta))\varepsilon^{2}\left(\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon)\right)''+\varepsilon\left(\mathcal{Y}_{\rho}^{\zeta,q}\mathcal{T}(\varepsilon)\right)'$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)= \tag{6}$$

$$(\sigma-\vartheta)\varepsilon^{3}\left(\mathcal{Y}_{\rho}^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)'''+(1+2(\sigma-\vartheta))\varepsilon^{2}\left(\mathcal{Y}_{\rho}^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)''+\varepsilon\left(\mathcal{Y}_{\rho}^{\zeta,q,n-1}\mathcal{T}(\varepsilon)\right)'$$

$$=\varepsilon+\sum_{j=2}^{\infty}j^{2n}((\sigma-\vartheta)(j-1)+1)^{n}\frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}}a_j h_j\,\varepsilon^{j}$$

$$\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)=\varepsilon+\sum_{j=2}^{\infty}\psi_j h_j\,\varepsilon^{j}\left(\begin{array}{c}\zeta>-1,0<q<1,\ \vartheta\geq 0\,,\ \sigma>0\,,\sigma\neq\vartheta\,,\\ n\in\mathbb{N}_0=N\cup\{0\}\text{ and }\varepsilon\in U\end{array}\right), \tag{7}$$

where

$$\psi_j=j^{2n}((\sigma-\vartheta)(j-1)+1)^{n}\frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}}a_j,$$

and by [1], let 0 < q < 1 and $[j]_q$ be defined by $[j]_q=\frac{1-q^j}{1-q}=1+\sum_{j=1}^{j-1}q^j,[0]_q=0.$

The $q-number$ shift factorial is given by

$$[j]_q!=\begin{cases}[j]_q\,[j-1]_q\ldots[2]_q\,[1]_q, & \text{if } j=1,2,3,\ldots,\\ 1, & \text{if } j=0\,.\end{cases}$$

From the definition relation Equation (5), we obtain

$$(i)[\zeta+1]_q\ \mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)=[\zeta]_q\ \mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta+1,q,n}\mathcal{T}(\varepsilon)+q^{\zeta}\varepsilon\mathfrak{D}_q\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta+1,q,n}\mathcal{T}(\varepsilon)\right),\varepsilon\in\mathfrak{U}; \tag{8}$$

$$(ii)\mathcal{R}_{\rho,\sigma,\vartheta}^{\zeta,n}\mathcal{T}(\varepsilon)=\lim_{q\to 1^-}\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)=\varepsilon+\sum_{j=2}^{+\infty}j^{2n}((\sigma-\vartheta)(j-1)+1)^{n}\frac{[j]_q!}{[\zeta]_{q,\epsilon-1}}a_j h_j\varepsilon^{j}. \tag{9}$$

The $q-generalized$ Pochhammer symbol is defined by $[\zeta]_{q,\epsilon-1}=\frac{\Gamma_q(\zeta+\epsilon-1)}{\Gamma_q(\zeta)}$, $\epsilon-1\in\mathbb{N},\zeta\in\mathbb{N}$.

For $q\to 1^-$, $[\zeta]_{q,\epsilon-1}$ reduces to $(\zeta)_{\epsilon-1}=\frac{\Gamma(\zeta+\epsilon-1)}{\Gamma(\zeta)}$.

Remark 1. *We find the following special cases for the operator $\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}$ by considering several particular cases for the coefficients a_j and n:*

1. *Putting $a_j=1,\vartheta=0$ and $n=0$ into this operator, we obtain the operator $QTRcalB_q^{\alpha}$ defined by Srivastava et al.* [13];
2. Putting $a_j=\frac{(-1)^j\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)}$ $(\rho>0),\vartheta=0$ and n = 0 in this operator, we obtain the operator $\mathcal{N}_{p,q}^{\sigma}$ defined by El-Deeb and Bulboacă [10] and El-Deeb [9];
3. Putting $a_j=\left(\frac{\tau+1}{\tau+j}\right)^{r}(r>0,\tau\geq 0)$, $\vartheta=0$ and n = 0 in this operator, we obtain the operator $\mathcal{M}_{\tau,q}^{\sigma,r}$ defined by El-Deeb and Bulboacă [14] and Srivastava and El-Deeb [12];
4. Putting $a_j=\frac{\varsigma^{j-1}}{(j-1)!}\varrho^{-\varsigma}$ $(\varsigma>0)$ and n = 0 in this operator, we obtain the q-analogue of Poisson operator $I_q^{\vartheta,\varsigma}$ defined by El-Deeb et al. [15];
5. Putting $a_j=1,\vartheta=0$ in this operator, we obtain the operator $QTRcalB_{\vartheta,\sigma}^{\delta,q,n}$ defined as follows:

$$B_{\vartheta,\sigma}^{\delta,q,n}\mathrm{F}(\varepsilon)=\varepsilon+\sum_{j=2}^{+\infty}j^{2n}((\sigma-\vartheta)(j-1)+1)^{n}\frac{[j]_q!}{[\zeta]_{q,\epsilon-1}}h_j\,\varepsilon^{j}; \tag{10}$$

6. *Putting* $a_j = \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)}$ $(\rho > 0)$ *in this operator, we obtain the operator* $\mathcal{N}^{\sigma,n}_{\varsigma,p,q}$ *defined as follows:*

$$\begin{aligned}\mathcal{N}^{\sigma,n}_{\varsigma,p,q}\mathrm{F}(\varepsilon) &= \varepsilon + \sum_{j=2}^{+\infty} j^{2n}((\sigma-\vartheta)(j-1)+1)^n \frac{[j]_q!}{[\zeta+1]_{q,\varepsilon-1}} \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)} h_j\,\varepsilon^j \\ &= \varepsilon + \textstyle\sum_{j=2}^{+\infty} \varphi_j h_j\,\varepsilon^j,\end{aligned} \tag{11}$$

where

$$\varphi_j = j^{2n}((\sigma-\vartheta)(j-1)+1)^n \frac{[j]_q!}{[\zeta]_{q,\varepsilon-1}} \frac{(-1)^j\,\Gamma(\rho+1)}{4^{j-1}(j-1)!\Gamma(r+\rho)}, \tag{12}$$

7. *Putting* $a_j = \left(\frac{\tau+1}{\tau+j}\right)^r$ $(r > 0,\ \tau \geq\ 0)$ *in this operator, we obtain the operator* $\mathcal{M}^{\sigma,n,r}_{\tau,\theta,q}$ *defined as follows:*

$$\mathcal{M}^{\sigma,n,r}_{\tau,\theta,q}\mathrm{F}(\varepsilon) = \varepsilon + \sum_{j=2}^{+\infty} j^{2n}((\sigma-\vartheta)(j-1)+1)^n \left(\frac{\tau+1}{\tau+j}\right)^r \frac{[j]_q!}{[\zeta+1]_{q,\varepsilon-1}} h_j\,\varepsilon^j.$$

Ma and Minda in [16] have given a unified treatment of various subclasses consisting of starlike and convex functions for either one of the quantities $\frac{\varepsilon\mathcal{T}'(\varepsilon)}{\mathcal{T}(\varepsilon)}$ or $1 + \frac{\varepsilon\mathcal{T}''(\varepsilon)}{\mathcal{T}(\varepsilon)}$ subordinate to a more general superordinate function. The $S^*(\phi)$ introduced by Ma and Minda [16] consists of function $\mathcal{T} \in \mathcal{A}$ satisfying $\frac{\varepsilon\mathcal{T}'(\varepsilon)}{\mathcal{T}(\varepsilon)} \prec \phi(z), z \in \mathfrak{U}$ and corresponding to class $k(\phi)$ of convex functions $\mathcal{T} \in \mathcal{A}$ satisfying $1 + \frac{\varepsilon\mathcal{T}''(\varepsilon)}{\mathcal{T}(\varepsilon)} \prec \phi(z), z \in \mathfrak{U}$, Ma and Minda [16], where ϕ is an analytic and univalent function with a positive real part in the unit disc $\mathfrak{U}$, satisfying $\phi(0) = 1$, $\phi'(0) > 0$, and $\phi(\mathfrak{U})$ is a starlike region with the respect to 1 and symmetric with the respect to the real axis. The functions in the classes $S^*(\phi)$ and $K(\phi)$, are called starlike functions of the Ma-Minda type or convex functions of the Ma-Minda type, respectively. By $S^*_{\sum_{\mathfrak{U}}}(\phi)$ and $K_{\sum_{\mathfrak{U}}}(\phi\,)$, we denote bi-starlike functions of Ma-Minda type and bi-convex functions of Ma-Minda type, respectively [16]. In this investigation, we assume that

$$\phi(\varepsilon) = 1 + B_1\varepsilon + B_2\varepsilon^2 + B_3\varepsilon^3 + \ldots,\quad B_1 > 0. \tag{13}$$

and

$$h(\varepsilon) = h_0 + h\varepsilon + h_2\varepsilon^2 + h_3\varepsilon^3 + \ldots. \tag{14}$$

The aim of this paper is to introduce new subclasses of the class $\sum_{\mathfrak{U}}$ and determine estimates of bounds on the coefficient $|h_2|$ and $|h_3|$ and for the functions in the above subclasses.

In [7] (see also [4,6,9,13,15–33]), certain subclasses of the bi-univalent analytic functions class B were introduced and non-sharp estimates on the first two coefficients $|h_2|$ and $|h_3|$ were found. The object of the present paper is to introduce two new subclasses as in Definitions 1 and 2 of the function class B using the linear q-convolution operator and determine estimates of the coefficients $|h_2|$ and $|h_3|$ for the functions in these new subclasses of the function class.

Lemma 1 ([9]). *Let* $p(\varepsilon) \in \mathcal{P}$*, then* $|p_i| \leq 2$ *for each* $i \in \mathbb{N}$*, where* $\mathcal{P}$ *is the family of all functions* p*, analytic in* $\mathfrak{U}$*, for which* $Re(p(\varepsilon)) > 0, (\varepsilon \in \mathfrak{U})$*, where*

$$p(z) = 1 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \ldots.$$

2. Coefficient Estimates for the Class $\mathfrak{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$

Definition 1. *A function* $\mathcal{T} \in \sum_{\mathfrak{U}}$ *defined by (1) is said to be in the class* $\mathfrak{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon\left(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right)+(1-\gamma)\left((1-\delta)\,\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)+\delta\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'\right.}\right]-1\prec_q(\varphi(\varepsilon)-1), \tag{15}$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\delta\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right)+(1-\gamma)\left((1-\delta)\;\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)+\delta\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'\right.}\right]-1\prec_q\;(\varphi(\varsigma)-1), \tag{16}$$

where $\gamma,\delta,\mu\in[0,1]$ *and* $\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)$ *is defined in Equation (7) and* $(\varepsilon,\varsigma\in\mathfrak{U})$.

For special values to parameters $\mu,\delta,\gamma,\zeta,n,\rho,\sigma,\vartheta$ and $\varphi(\varepsilon)$, leads to get known and new classes.

Remark 2. *For* $\delta=0$, *a function* $\mathcal{T}\in\sum_{\mathfrak{U}}$ *defined by Equation (7) is said to be in the class* $\mathfrak{f}_{q,\sum}^{\mu}\,(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'++(1-\gamma)\,\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)}\right]-1\prec_q(\varphi(\varepsilon)-1),$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+(1-\gamma)\,\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)}\right]-1\prec_q(\varphi(\varsigma)-1),$$

where ξ *is the inverse function of* $\mathcal{T}$ *and* $(\varepsilon,\varsigma\in\mathfrak{U})$.

Remark 3. *For* $\delta=1$, *a function* $\mathcal{T}\in\sum_{\mathfrak{U}}$ *defined by Equation (7) is said to be in the class* $\mathfrak{f}_{q,\sum}^{\mu}\,(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'+\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''+(1-\gamma)\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right]-1\prec_q(\varphi(\varepsilon)-1),$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'+\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''+(1-\gamma)\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right]-1\prec_q(\varphi(\varsigma)-1),$$

where ξ *is the inverse function of* $\mathcal{T}$ *and* $\varepsilon,\varsigma\in\mathfrak{U}$,

Theorem 1. *If the function* $\mathcal{T}$ *belongs to the class* $\mathfrak{f}_{q,\sum}^{\mu}\,(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$, *then we have*

$$|h_2|\le\frac{|A_0|\;B_1\sqrt{B_1}}{\sqrt{(1+2\gamma)(3\mu-2\delta-1)A_0B_1^2\psi_3-(1+\gamma)^2\left[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2\right]}}, \tag{17}$$

and

$$|h_3| \leq \frac{B_1(|A_0|+|A_1|)}{(1+2\gamma)(3\mu-2\delta-1)\psi_3} + \frac{A_0^2 B_1^2}{4(1+\gamma)^2(2\mu-\delta-1)^2\psi_2^2}, \quad B_1>1, \tag{18}$$

Proof. Let $\mathcal{T} \in \mathscr{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$. There exist two analytic functions u, v and $u, v : \mathfrak{U} \longrightarrow \mathfrak{U}$ with $u(0) = v(0) = 0, |u(\varepsilon)| < 1$ and $|v(\varsigma)| \leq 1$ for all $\varepsilon, \varsigma \in \mathfrak{U}$, satisfying the following conditions.

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \gamma\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon\left(\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \delta\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right) + (1-\gamma)\left((1-\delta)\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon) + \delta\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'\right.}\right] - 1 \prec_q h(\varepsilon)(\varphi(u(\varepsilon)-1)), \ \ \varepsilon \in \mathfrak{U} \tag{19}$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \gamma\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \delta\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right) + (1-\gamma)\left((1-\delta)\ \mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma) + \delta\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'\right.}\right] - 1 \prec_q h(\varsigma)(\varphi(v(\varsigma)-1)), \ \varsigma \in \mathfrak{U}, \tag{20}$$

where ξ is the inverse function of $\mathcal{T}$ and $(\varepsilon, \varsigma \in \mathfrak{U})$. Determine the definition of the functions $p(\varepsilon)$ and $q(\varsigma)$ by

$$p(\varepsilon) = \frac{1+u(\varepsilon)}{1-u(\varepsilon)} = 1 + c_1\varepsilon^2 + c_2\varepsilon^2 + \cdots \tag{21}$$

and

$$q(\varsigma) = \frac{1+v(\varsigma)}{1-v(\varsigma)} = 1 + d_1\varsigma^2 + d_2\varsigma^2 + \cdots. \tag{22}$$

Equivalently,

$$u(\varepsilon) := \frac{p(\varepsilon)-1}{p(\varepsilon)+1} = \frac{1}{2}\left\{c_1\varepsilon + \left(c_2 - \frac{c_1^2}{2}\right)\varepsilon^2 + \cdots\right\}, \tag{23}$$

and

$$v(\varsigma) := \frac{q(\varsigma)-1}{q(\varsigma)+1} = \frac{1}{2}\ \left\{b_1\varsigma + \left(b_2 - \frac{b_1^2}{2}\right)\varsigma^2 + \cdots\right\}. \tag{24}$$

Applying Equations (23) and (24) in Equations (19) and (20), respectively, we have

$$\left[\frac{\varepsilon\left[\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \gamma\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right]^{\mu}}{\gamma\varepsilon\left(\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)' + \delta\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)''\right) + (1-\gamma)\left((1-\delta)\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon) + \delta\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'\right.}\right] - 1 = h(\varepsilon)\left(\varphi\left(\frac{p(\varepsilon)-1}{p(\varepsilon)+1}\right) - 1\right), \tag{25}$$

and

$$\left[\frac{\varsigma\left[\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \gamma\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right]^{\mu}}{\gamma\varsigma\left(\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)' + \delta\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)''\right) + (1-\gamma)\left((1-\delta)\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma) + \delta\varsigma\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\xi(\varsigma)\right)'\right.}\right] - 1 = h(\varsigma)\left(\varphi\left(\frac{q(\varsigma)-1}{q(\varsigma)+1}\right) - 1\right). \tag{26}$$

Utilizing Equations (22) and (23) in the right-hands (RH) of the relations Equations (25) and (26), we obtain

$$h(\varepsilon)\left(\varphi\left(\frac{p(\varepsilon)-1}{p(\varepsilon)+1}\right) - 1\right) = \frac{1}{2}A_0B_1c_1\varepsilon + \left\{\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2\right\}\varepsilon^2 + \cdots. \tag{27}$$

and

$$h(\varsigma)\left(\varphi\left(\frac{q(\varsigma)-1}{q(\varsigma)+1}\right)-1\right)=\frac{1}{2}A_0B_1d_1\varsigma+\left\{\frac{1}{2}A_1B_1d_1+\frac{1}{2}A_0B_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{A_0B_2}{4}d_1^2\right\}\varsigma^2+\cdots. \tag{28}$$

By equalizing Equations (25)–(28), respectively, we obtain

$$(1+\gamma)(2\mu-\delta-1)h_2\psi_2=\frac{1}{2}A_0B_1c_1, \tag{29}$$

$$\begin{gathered}\left[(1+2\gamma)(3\mu-2\delta-1)h_3\psi_3+(1+\gamma)^2[2\mu(\mu-1)-(1+\delta)(2\mu-\delta-1)]h_2^2\psi_2^2\right]\\=\tfrac{1}{2}A_1B_1c_1+\tfrac{1}{2}A_0B_1\left(c_2-\tfrac{c_1^2}{2}\right)+\tfrac{A_0B_2}{4}c_1^2.\end{gathered} \tag{30}$$

and

$$-(1+\gamma)(2\mu-\delta-1)h_2\psi_2=\frac{1}{2}A_0B_1b_1 \tag{31}$$

$$\begin{gathered}\left[\begin{gathered}\left[(1+\gamma)^2[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]+2(1+2\gamma)(3\mu-2\delta-1)\right]h_2^2\psi_2^2\\-(1+2\gamma)(3\mu-2\delta-1)h_3\psi_3\end{gathered}\right]\\=\tfrac{1}{2}A_1B_1d_1+\tfrac{1}{2}A_0B_1+\left(d_2-\tfrac{d_1^2}{2}\right)+\tfrac{A_0B_2}{4}d_1^2.\end{gathered} \tag{32}$$

From Equations (29) and (31), we have

$$h_2=\frac{A_0B_1c_1}{2(1+\gamma)(2\mu-\delta-1)\psi_2}=-\frac{A_0B_1d_1}{2(1+\gamma)(2\mu-\delta-1)\psi_2} \tag{33}$$

It follows that

$$c_1=-d_1\,, \tag{34}$$

and

$$8(1+\gamma)^2(2\mu-\delta-1)^2\,h_2^2\psi_2^2=A_0^2B_1^2\left(d_1^2+c_1^2\right). \tag{35}$$

Now, by summing Equations (33) and (35), in light of Equations (33) and (34), we obtain

$$\begin{gathered}8\left[(1+\gamma)^2[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]A_0B_1^2\psi_2^2+(1+2\gamma)(3\mu-2\delta-1)\psi_3A_0B_1^2\right]h_2^2\\=2A_0^2B_1^3(c_2+d_2)+\left(8(1+\gamma)^2(2\mu-\delta-1)^2\,(B_2-B_1)h_2^2\psi_2^2\right),\end{gathered} \tag{36}$$

which implies

$$h_2^2=\frac{2A_0^2B_1^3(c_2+d_2)}{8\left\{(1+2\gamma)(3\mu-2\delta-1)A_0B_1^2\psi_3-(1+\gamma)^2\left[(2\mu-\delta-1)^2(B_2-B_1)-[2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]A_0B_1^2\right]\right\}}. \tag{37}$$

Applying Lemma 1 $|c_i|\le 2, |d_i|\le 2$ to Equation (37), we obtain the desired result Equation (17).

Next, for the bound on $|a_3|$, by subtracting Equation (32) from Equation (30), we obtain

$$4\left\{(1+2\gamma)(3\mu-2\delta-1)\psi_3h_3-(1+2\gamma)(3\mu-2\delta-1)\psi_3\,h_2^2\right\}=2A_1B_1c_1+A_0B_1(c_2-d_2) \tag{38}$$

By substituting Equation (32) from Equation (30), and with further computation using Equations (34) and (35), we obtain

$$h_3=\frac{2A_1B_1c_1}{4(1+2\gamma)(3\mu-2\delta-1)\psi_3}+\frac{A_0B_1(c_2-d_2)}{4(1+2\gamma)(3\mu-2\delta-1)\psi_3}+\frac{A_0^2B_1^2\left(c_1^2+d_1^2\right)}{8(1+\gamma)^2(2\mu-\delta-1)^2\psi_2^2}. \tag{39}$$

Applying Lemma 1. $|c_i|\le 2, |d_i|\le 2$, in Equation (38), we obtain Equation (18). This completes the proof of Theorem 1. □

By putting $\delta = 0$ in Theorem 1, we obtain the following Corollary:

Corollary 1. *If the function $\mathcal{T}(\varepsilon)$ given by (1) belongs to the class $\not{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, \gamma, 0, \varphi)$, then*

$$|h_2| \leq \frac{|A_0|\, B_1\sqrt{B_1}}{\sqrt{(1+2\gamma)(3\mu-1)A_0B_1^2\psi_3 - (1+\gamma)^2\Big[(2\mu-1)^2(B_2-B_1) - [2\mu(\mu-1)-(2\mu-1)]\psi_2^2A_0B_1^2\Big]}},$$

and

$$|h_3| \leq \frac{B_1(|A_0|+|A_1|)}{(1+2\gamma)(3\mu-1)\psi_3} + \frac{A_0^2B_1^2}{4(1+\gamma)^2(2\mu-1)^2\psi_2^2}.$$

By putting $\delta = 1$ in Theorem 1, we obtain the following Corollary:

Corollary 2. *Let $\mathcal{T}(\varepsilon)$ given by (1) belong to the class $\not{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, \gamma, 1, \varphi)$. Then,*

$$|h_2| \leq \frac{|A_0|\, B_1\sqrt{B_1}}{\sqrt{3(1+2\gamma)(\mu-1)A_0B_1^2\psi_3 - (1+\gamma)^2\Big[(2\mu-2)^2(B_2-B_1) - 2[\mu(\mu-1)-(2\mu-2)]\psi_2^2A_0B_1^2\Big]}},$$

and

$$|h_3| \leq \frac{B_1(|A_0|+|A_1|)}{3(1+2\gamma)(\mu-1)\psi_3} + \frac{A_0^2B_1^2}{8(1+\gamma)^2(\mu-1)^2\psi_2^2}.$$

By putting $\gamma = 1$ in Theorem 1, we have the following Corollary:

Corollary 3. *Let $\mathcal{T}(\varepsilon)$ given by (1) belong to the class $\not{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, 1, \delta, \varphi)$. Then,*

$$|h_2| \leq \frac{|A_0|\, B_1\sqrt{B_1}}{\sqrt{3(3\mu-2\delta-1)A_0B_1^2\psi_3 - 4\Big[(2\mu-\delta-1)^2(B_2-B_1) - [2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2\Big]}},$$

and

$$|h_3| \leq \frac{B_1(|A_0|+|A_1|)}{3(3\mu-2\delta-1)\psi_3} + \frac{A_0^2B_1^2}{16(2\mu-\delta-1)^2\psi_2^2}, \quad B_1 > 1.$$

By putting $\gamma = 0$ in Theorem 1, we have the following Corollary:

Corollary 4. *Let $\mathcal{T}(\varepsilon)$ given by (1) belong to the class $\not{f}^{\mu}_{q,\sum}(\zeta, n, \rho, \sigma, \vartheta, 0, \delta, \varphi)$.*

Then, $|h_2| \leq \frac{|A_0|B_1\sqrt{B_1}}{\sqrt{(3\mu-2\delta-1)A_0B_1^2\psi_3 - \left[(2\mu-\delta-1)^2(B_2-B_1) - [2\mu(\mu-1)-(2\mu-\delta-1)(1+\delta)]\psi_2^2A_0B_1^2\right]}}$,

and

$$|h_3| \leq \frac{B_1(|A_0|+|A_1|)}{(3\mu-2\delta-1)\psi_3} + \frac{A_0^2B_1^2}{4(2\mu-\delta-1)^2\psi_2^2}, \quad B_1 > 1.$$

3. Coefficients Estimates for the Subclass $\aleph^{q,\delta}_{\sum}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$

Definition 2. *A function $\mathcal{T} \in \sum_{\mathfrak{U}}$ defined by (1) is said to be in the class $\aleph^{q,\delta}_{\sum}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$ if the following quasi-subordination conditions are satisfied:*

$$1 + \frac{1}{\gamma}\left[(1-\delta)\frac{\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}{(1-\lambda)\varepsilon + \lambda\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)} + \delta\left(\frac{\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'' + \left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}{\lambda\varepsilon\left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'' + \left(\mathcal{Q}^{\zeta,q,n}_{\rho,\sigma,\vartheta}\mathcal{T}(\varepsilon)\right)'}\right) - 1\right] \prec_q (\varphi(\varepsilon) - 1) \tag{40}$$

and

$$1+\frac{1}{\gamma}\left[(1-\delta)\frac{\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{(1-\lambda)\varsigma+\lambda\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)}+\delta\left(\frac{\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''+\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{\lambda\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''+\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right)-1\right]\prec_q(\varphi(\varsigma)-1), \tag{41}$$

where $(0\leq\lambda<1, 0\leq\delta\leq 1, \gamma\in\mathbb{C}\setminus\{0\}, \varepsilon,\in\mathfrak{U})$.

For special values of parameters λ and δ, we obtain new and well-known classes.

Remark 4. *For* $\lambda=0$, *a function* $\mathcal{T}\in\sum_{\mathfrak{U}}$ *defined by Equation (1) is said to be in the class* $\aleph_{\Sigma}^{q,\delta}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ *if the following quasi-subordination conditions are satisfied:*

$$1+\frac{1}{\gamma}\left[(1-\delta)\frac{\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}{\varepsilon}+\delta\left(\frac{\varepsilon\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)''+\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}{\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\mathcal{T}(\varepsilon)\right)'}\right)-1\right]\prec_q(\varphi(\mathrm{z})-1)$$

and

$$1+\frac{1}{\gamma}\left[(1-\delta)\frac{\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{\varsigma}+\delta\left(\frac{\varsigma\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)''+\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}{\left(\mathcal{Q}_{\rho,\sigma,\vartheta}^{\zeta,q,n}\xi(\varsigma)\right)'}\right)-1\right]\prec_q(\varphi(\mathrm{w})-1)$$

Theorem 2. *If the function* $\mathcal{T}$ *belongs to the class* $\aleph_{\Sigma}^{q,\delta}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$, *then we have*

$$|h_2|\leq\frac{\gamma|A_0|\,B_1\sqrt{B_1}}{\sqrt{2(1-\lambda)(1+2\delta)A_0B_1^2\psi_3-(1-\lambda)^2\left[(1+3\delta)A_0B_1^2-(1+\delta)^2(B_2-B_1)\right]\psi_2^2}} \tag{42}$$

and

$$|h_3|\leq\frac{\gamma B_1(|A_0|+|A_1|)}{(1-\lambda)(1+2\delta)\psi_3}+\frac{A_0^2B_1^2\gamma^2}{(1+\delta)^2(1-\lambda)^2\psi_2^2}\ ,B_1>1, \tag{43}$$

where $0\leq\delta\leq 1, 0\leq\lambda\leq 1,\ \gamma\in\mathfrak{U}-\{0\}$.

Proof. Proceeding as in the proof of Theorem 1, we can obtain the relations as follows:

$$\frac{1}{\gamma}(1+\delta)(1-\lambda)h_2\psi_2=\frac{1}{2}A_0B_1c_1, \tag{44}$$

$$\begin{aligned}&\tfrac{1}{\gamma}\left[2(1-\lambda)(1+2\delta)h_3\psi_3-(1-\lambda)(1+\lambda)((1+3\delta))h_2^2\psi_2^2\ \right]\\&\qquad=\tfrac{1}{2}A_1B_1c_1+\tfrac{1}{2}A_0B_1\left(c_2-\tfrac{c_1^2}{2}\right)+\ \tfrac{A_0B_2}{4}c_1^2\Big\}\end{aligned} \tag{45}$$

and

$$-\frac{1}{\gamma}(1+\delta)(1-\lambda)h_2\psi_2=\frac{1}{2}A_0B_1b_1, \tag{46}$$

$$\begin{aligned}&\tfrac{1}{\gamma}\left[[4(1-\lambda)(1+2\delta)\psi_3-(1-\lambda)(1+\lambda)(1+3\delta)]h_2^2\psi_2^2-2(1-\lambda)(1+2\delta)\psi_3h_3\right]\\&\qquad=\tfrac{1}{2}A_1B_1b_1+\tfrac{1}{2}A_0B_1\left(b_2-\tfrac{b_1^2}{2}\right)+\tfrac{A_0B_2}{4}b_1^2\ \Big\}.\end{aligned} \tag{47}$$

From Equations (44) and (46), we obtain

$$c_1=-d_1 \tag{48}$$

and

$$h_2=\frac{\gamma A_0B_1c_1}{2(1+\delta)(1-\lambda)\psi_2}=-\frac{\gamma A_0B_1b_1}{2(1+\delta)(1-\lambda)\psi_2} \tag{49}$$

and

$$8(1+\delta)^2(1-\lambda)^2\, h_2^2\psi_2^2 = A_0^2 B_1^2 \gamma^2 \left(d_1^{\,2} + c_1^2\right). \tag{50}$$

Now, by summing Equations (45) and (47) and using Equation (50), we obtain

$$\begin{aligned} &\tfrac{8}{\gamma}\{(2(1-\lambda)(1+2\delta)\psi_3 - (1-\lambda)(1+\lambda)(1+3\delta)\psi_2^2\}h_2^{\,2} \\ &= 2A_0B_1(c_2+d_2) + A_0(B_2-B_1)(c_1^2+d_1^2), \end{aligned} \tag{51}$$

which implies

$$h_2^2 = \frac{2A_0^2B_1^3(c_2+d_2)}{8\left\{2(1-\lambda)(1+2\delta)A_0B_1^2\psi_3 - (1-\lambda)^2\left[\left[(1+3\delta)A_0B_1^2 - (1+\delta)^2(B_2-B_1)\right]\psi_2^2\right]\right\}}. \tag{52}$$

Applying Lemma 1. in Equation (52), we obtain the desired result Equation (42).
Next, for the bound on $|h_3|$, by subtracting Equation (45) from (47), we obtain

$$\frac{8}{\gamma}\left\{(1-\lambda)(1+2\delta)\psi_3 h_3 - (1-\lambda)(1+2\delta)\psi_3\, h_2^2\right\} = 2A_{_1}B_1c_1 + A_0B_1(c_2-d_2)$$

By substituting Equation (47) from Equation (45), and with further computation using Equations (48) and (49), we obtain

$$h_3 = \frac{2\gamma A_1B_1c_1}{4(1-\lambda)(1+2\delta)\psi_3} + \frac{\gamma A_0B_1(c_2-d_2)}{4(1-\lambda)(1+2\delta)} + \frac{A_0^2B_1^2\gamma^2(c_1^{\,2}+d_1^2)}{8(1+\delta)^2(1-\lambda)^2\psi_2^{\,2}} \tag{53}$$

From Equations (53) and (52), we obtain the desired result Equation (43). The proof is complete. □

Corollary 5. *If $\mathcal{T}(\varepsilon) \in \aleph_{\Sigma}^{q,\delta}(1,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ defined in (1), then we have*

$$|h_2| \le \frac{\gamma|A_0|\, B_1\sqrt{B_1}}{\sqrt{2(1+2\delta)A_0B_1^2\psi_3 - \left[(1+3\delta)A_0B_1^2 - (1+\delta)^2(B_2-B_1)\right]\psi_2^2}}$$

and

$$|h_3| \le \frac{\gamma B_1(|A_0|+|A_1|)}{(1+2\delta)\psi_3} + \frac{A_0^2B_1^2\gamma^2}{(1+\delta)^2\psi_2^{\,2}}\ \ , B_1 > 1.$$

Corollary 6. *If $\mathcal{T}(\varepsilon) \in \aleph_{\Sigma}^{q,1}(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ defined in (1), then we have*

$$|h_2| \le \frac{\gamma|A_0|\, B_1\sqrt{B_1}}{\sqrt{6(1-\lambda)A_0B_1^2\psi_3 - (1-\lambda)^2\left[4A_0B_1^2 - 4(B_2-B_1)\right]\psi_2^2}}$$

and

$$|h_3| \le \frac{\gamma B_1(|A_0|+|A_1|)}{3(1-\lambda)\psi_3} + \frac{A_0^2B_1^2\gamma^2}{4(1-\lambda)^2\psi_2^{\,2}}\ \ , B_1 > 1.$$

4. Conclusions

We introduce and investigate new subclasses $\mathfrak{f}_{q,\Sigma}^{\mu}(\zeta,n,\rho,\sigma,\vartheta,\gamma,\delta,\varphi)$ and $\aleph_{\Sigma}^{q,\delta}$ $(\lambda,\zeta,n,\rho,\sigma,\vartheta,\varphi)$ of the analytic function class of bi-univalent functions defined in open unit disk connected with a linear q-convolution operator, which are associated with quasi-subordination. We find coefficient estimates $|h_2|$, $|h_3|$ for functions in these subclasses. Several known and new consequences of these results are also pointed out. The results contained in the paper could inspire ideas for continuing the study, and we opened some windows for authors to generalize our new subclasses to obtain some new results in bi-univalent function theory. There is symmetry between the results of the subclass $\mathfrak{b}^{\mu}_{q,\Sigma}(\zeta, n, \rho, \sigma, \vartheta, \gamma, \delta, \varphi)$ and the results of the subclass $\aleph^{q,\delta}_{\Sigma}(\lambda, \zeta, n, \rho, \sigma, \vartheta, \varphi)$.

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