symmetry

MDPI

Article

The η-Anti-Hermitian Solution to a System of Constrained Matrix Equations over the Generalized Segre Quaternion Algebra

Bai-Ying Ren, Qing-Wen Wang * and Xue-Ying Chen

Department of Mathematics, Shanghai University, Shanghai 200444, China
* Correspondence: wqw@t.shu.edu.cn

Abstract: In this paper, we propose three real representations of a generalized Segre quaternion matrix. We establish necessary and sufficient conditions for the existence of the η-anti-Hermitian solution to a system of constrained matrix equations over the generalized Segre quaternion algebra. We also obtain the expression of the general η-anti-Hermitian solution to the system when it is solvable. Finally, we provide a numerical example to verify the main results of this paper.

Keywords: generalized Segre quaternion algebra; η-anti-Hermitian solution; real representation

Citation: Ren, B.-Y.; Wang, Q.-W.; Chen, X.-Y. The η-Anti-Hermitian Solution to a System of Constrained Matrix Equations over the Generalized Segre Quaternion Algebra. *Symmetry* **2023**, *15*, 592. https://doi.org/10.3390/sym15030592

Academic Editor: Alexei Kanel-Belov

Received: 31 January 2023
Revised: 17 February 2023
Accepted: 23 February 2023
Published: 24 February 2023

1. Introduction

In 1843, Hamilton [1] discovered the real quaternions

$$\mathbb{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k : \ i^2 = j^2 = k^2 = -1,\ ijk = -1,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$$

which is a four-dimensional non-commutative division algebra over the real number field $\mathbb{R}$.

The real quaternions have played an important role in many fields such as quantum physics, computer graphics and signal processing [2–4]. In these areas, the real quaternion algebra is more useful than the usual algebra. For example, Ling et al. [5] presented a new algorithm for solving the linear least squares problem over the quaternions. By means of direct quaternion arithmetics, the algorithm does not make the scale of the problem dilate exponentially, compared to the conventional real or complex representation methods. However, the multiplication of real quaternions is non-commutative; therefore, to avoid non-commutativity, the commutative quaternion algebra was introduced.

In 1892, Segre [6] defined the commutative quaternions

$$\mathbb{S} = \{a = a_0 + a_1 i + a_2 j + a_3 k : \ i^2 = -1,\ j^2 = 1, ij = ji = k,\ a_0, a_1, a_2, qa_3 \in \mathbb{R}\}$$

which is a four-dimensional commutative algebra that is not divisible over $\mathbb{R}$.

The commutative quaternions have been widely applied in various fields. For color image processing, Pei et al. [7] defined a simplified commutative quaternion polar form to represent color images, which is useful in the brightness–hue–saturation color space. After this, Pei et al. [8] developed the algorithms for calculating the eigenvalues, the eigenvectors and the singular value decompositions of commutative quaternion matrices. They employed the singular value decompositions of commutative quaternion matrices to implement a color image which reduces the computational complexity to one-forth of the conventional. Guo et al. [9] defined the reduced canonical transform of commutative quaternions which is the generalization of reduced Fourier transform of commutative quaternions. Lin et al. [10] established a commutative quaternion valued neural network (CQVNN) and studied the asymptotic stability of CQVNN.

The commutative quaternion matrix equations have been studied extensively. In [11], Kosal et al. studied some algebraic properties of commutative quaternion matrices using complex representations. Kosal et al. [12] gave the expression of the general solution to the Kalman–Yakubovich conjugate matrix equation over the commutative quaternions by means of real representations of a commutative quaternion matrix. Moreover, Kosal et al. [13] studied Sylvester-conjugate commutative quaternion matrix equations using real representations. In [14], Kosal et al. proposed a different kind of real representation of commutative quaternion matrices, and they also studied the general solution to matrix equation $AX = B$ over the commutative quaternions.

Segre [6] extended the commutative quaternion algebra to the generalized Segre quaternion algebra $\mathbb{S}_g$, which is defined as follows:

$$\mathbb{S}_g = \{a = a_0 + a_1 i + a_2 j + a_3 k : \ a_0, a_1, a_2, a_3 \in \mathbb{R} \ and \ i, j, k \notin \mathbb{R}\},$$

where i, j, k satisfy the following multiplication rules:

$$i^2 = k^2 = \alpha, \quad j^2 = 1, \quad ij = ji = k, \quad jk = kj = i, \quad ik = ki = \alpha j.$$

Here, we only consider the case in which $\alpha \neq 0$. In particular, $\mathbb{S}_g$ is the commutative quaternions $\mathbb{S}$ when $\alpha = -1$.

The generalized Segre quaternion algebra, which includes the commutative quaternions, shows the superiority of the proposed approach in signal processing over its counterpart in the real quaternions. Moreover, the Segre quaternions have potential applications in linear models, filtering and smoothing as well as signal detection [15].

The commutative quaternion algebra has many vital applications to areas of mathematics and physics. On this basis, the generalized Segre quaternion algebra is rarely studied. Out of this motivation, we focus on the generalized Segre quaternion algebra in this paper.

For $A \in \mathbb{S}_g^{m\times n}$, A can be uniquely expressed as $A = A_0 + A_1 i + A_2 j + A_3 k$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. We define η-conjugates [11], $\eta \in \{i, j, k\}$, as follows:

$$\begin{aligned} A^i &= A_0 - A_1 i + A_2 j - A_3 k, \\ A^j &= A_0 + A_1 i - A_2 j - A_3 k, \\ A^k &= A_0 - A_1 i - A_2 j + A_3 k, \end{aligned}$$

and η-conjugate transposes, $\eta \in \{i, j, k\}$, as follows:

$$\begin{aligned} A^{i*} &= {A_0}^T - A_1^T i + A_2^T j - A_3^T k, \\ A^{j*} &= A_0^T + A_1^T i - A_2^T j - A_3^T k, \\ A^{k*} &= A_0^T - A_1^T i - A_2^T j + A_3^T k. \end{aligned}$$

Definition 1. *For $\eta \in \{i, j, k\}$, $A \in \mathbb{S}_g^{n\times n}$ is called η-Hermitian matrix if $A = A^{\eta *}$, $A \in \mathbb{S}_g^{n\times n}$ is called η-anti-Hermitian matrix if $A = -A^{\eta *}$.*

The concept of η-(anti)-Hermitian matrix is a more generalized concept comparing with (skew-)Hermitian matrix. η-(anti)-Hermitian matrix has important applications in linear modeling and convergence analysis in statistical signal processing [16–18]. The η-(anti)-Hermitian solutions to matrix equations over real quaternions have been vastly investigated [19–22]. However, the η-anti-Hermitian solutions to matrix equations over the commutative quaternion algebra have received little attention. Yuan et al. [23] investigated the Hermitian solutions of commutative quaternion matrix equation $(AXB, CXD) = (E, G)$, which has wide applications in control and system theory, stability theory and neural network. Tian et al. [24] studied the anti-Hermitian solutions of matrix equations $AXA^H + BYB^H = C$ over the commutative quaternion algebra.

Motivated by significant application and research value of η-(anti)-Hermitian matrix and matrix equations, we consider the η-anti-Hermitian solution to the generalized Segre quaternion matrix equation

$$AX = B \tag{1}$$

and a system of constrained generalized Segre quaternion matrix equations

$$\begin{aligned} & E_1XE_2 + F_1YF_2 = H, \\ s.t. & \left\{ \begin{array}{rcl} A_1X & = & B_1, \\ XA_2 & = & B_2, \\ C_1Y & = & D_1, \\ YC_2 & = & D_2, \end{array} \right. \end{aligned} \tag{2}$$

where X and Y are unknown matrices and the other matrices are given with appropriate orders. The classic matrix equation $AX = B$ over the real quaternions is important in various applications, for example, Yuan et al. [25] discussed its application in color image restoration. However, to the best of our knowledge, currently known real representations of commutative matrices are not able to address the η-anti-Hermitian solutions of matrix equations. Our scientific innovation lies in not only establishing three different real representations, making up for the result of η-anti-Hermitian solutions over the commutative quaternions, but also generalizing it to the Segre quaternion algebra which is more extensive and of application value.

2. Preliminaries

In this section, we specify the notations of the paper and propose three real representations of the matrix over the generalized Segre quaternion algebra. The algebraic properties of the real representations are also given.

2.1. Notations

Throughout this paper, we use the following notations:

- I_n denotes the $n \times n$ identity matrix;
- A^T, $rank(A)$ denote the transpose and rank of a matrix A, respectively;
- $A^\dagger$ denotes the Moore–Penrose inverse of A, which satisfies simultaneously $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$. Moreover, $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ are two projectors induced by A, respectively;
- $A \otimes B = (a_{pq}B) \in \mathbb{R}^{mt\times ns}$ denotes the Kronecker product of matrices $A = (a_{pq}) \in \mathbb{R}^{m\times n}$ and $B = (b_{pq}) \in \mathbb{R}^{t\times s}$;
- $Vec(A) = ({x_1}^T, {x_2}^T, \cdots, {x_n}^T)^T \in \mathbb{R}^{mn}$, where $x_i (i = 1, \cdots, n)$ is the i-th column vector of A, denotes the stretching function of a matrix A.

2.2. Real Representations

Theorem 1. *Let $A \in \mathbb{S}_g^{m\times n}$ and $B \in \mathbb{S}_g^{n\times s}$. Then,*

(1) $(A^\eta)^T = (A^T)^\eta = A^{\eta*}$,
(2) $(AB)^\eta = A^\eta B^\eta$,
(3) $(AB)^T = B^T A^T$,
(4) $(AB)^{\eta*} = B^{\eta*} A^{\eta*}$,
(5) $(A^{\eta*})^{\eta*} = A$.

Proof. For (2), it can be obtained by the proof of Theorem 3.1 in [11]. The others can be verified easily. □

Let $A \in \mathbb{S}_g^{m\times n}$, $A = A_0 + A_1 i + A_2 j + A_3 k$, where $A_0, A_1, A_2, A_3 \in \mathbb{R}^{m\times n}$. By generalizing the real representation in [14], we define a kind of real representations of A as follows:

$$A^{\sigma_i} = \begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ A_3 & A_2 & A_1 & A_0 \end{pmatrix}.$$

Similarly, we define two other kinds as follows:

$$A^{\sigma_j} = V_m A^{\sigma_i} = \begin{pmatrix} -A_0 & -\alpha A_1 & -A_2 & -\alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ A_2 & \alpha A_3 & A_0 & \alpha A_1 \\ -A_3 & -A_2 & -A_1 & -A_0 \end{pmatrix}, \quad A^{\sigma_k} = W_m A^{\sigma_i} = \begin{pmatrix} A_0 & \alpha A_1 & A_2 & \alpha A_3 \\ A_1 & A_0 & A_3 & A_2 \\ -A_2 & -\alpha A_3 & -A_0 & -\alpha A_1 \\ -A_3 & -A_2 & -A_1 & -A_0 \end{pmatrix},$$

where

$$V_m = \begin{pmatrix} -I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix} \quad and \quad W_m = \begin{pmatrix} I_m & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_m & 0 \\ 0 & 0 & 0 & -I_m \end{pmatrix}.$$

Let

$$G_n = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & -\alpha I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & -\alpha I_n \end{pmatrix}, \quad R_n = \begin{pmatrix} 0 & \alpha I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I_n \\ 0 & 0 & I_n & 0 \end{pmatrix},$$

$$S_n = \begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix} \quad and \quad T_n = \begin{pmatrix} 0 & 0 & 0 & \alpha I_n \\ 0 & 0 & I_n & 0 \\ 0 & \alpha I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}.$$

By direct calculation, the following properties of three real representations are obtained.

Proposition 1. *For $A, B \in \mathbb{S}_g^{m\times n}$, $C \in \mathbb{S}_g^{n\times s}$, $\lambda \in \mathbb{R}$, then*

(1) $A = B \Leftrightarrow A^{\sigma_\eta} = B^{\sigma_\eta}$, $\eta \in \{i, j, k\}$,

(2) $(A+B)^{\sigma_\eta} = A^{\sigma_\eta} + B^{\sigma_\eta}$, $\quad (\lambda A)^{\sigma_\eta} = \lambda A^{\sigma_\eta}$, $\eta \in \{i, j, k\}$,

(3) $(AC)^{\sigma_i} = A^{\sigma_i} C^{\sigma_i}$, $\quad (AC)^{\sigma_j} = A^{\sigma_j} V_n C^{\sigma_j}$, $\quad (AC)^{\sigma_k} = A^{\sigma_k} W_n C^{\sigma_k}$,

(4) *(a)* $R_m{}^{-1} A^{\sigma_i} R_n = A^{\sigma_i}$, $\quad S_m{}^{-1} A^{\sigma_i} S_n = A^{\sigma_i}$, $\quad T_m{}^{-1} A^{\sigma_i} T_n = A^{\sigma_i}$,

(b) $R_m{}^{-1} A^{\sigma_j} R_n = -A^{\sigma_j}$, $\quad S_m{}^{-1} A^{\sigma_j} S_n = -A^{\sigma_j}$, $\quad T_m{}^{-1} A^{\sigma_j} T_n = A^{\sigma_j}$,

(c) $R_m{}^{-1} A^{\sigma_k} R_n = A^{\sigma_k}$, $\quad S_m{}^{-1} A^{\sigma_k} S_n = -A^{\sigma_k}$, $\quad T_m{}^{-1} A^{\sigma_k} T_n = -A^{\sigma_k}$

(5) $(A^{i*})^{\sigma_i} = G_n{}^{-1} (A^{\sigma_i})^T G_m$, $\quad (A^{j*})^{\sigma_j} = G_n{}^{-1} (A^{\sigma_j})^T G_m$, $\quad (A^{k*})^{\sigma_k} = G_n{}^{-1} (A^{\sigma_k})^T G_m$,

(6) *(a)* $$A = \frac{1}{4}\begin{pmatrix} I_m & iI_m & jI_m & kI_m \end{pmatrix} A^{\sigma_i} \begin{pmatrix} I_n \\ \frac{1}{\alpha} iI_n \\ jI_n \\ \frac{1}{\alpha} kI_n \end{pmatrix},$$

(b) $$A = \frac{1}{4}\begin{pmatrix} -I_m & iI_m & jI_m & -kI_m \end{pmatrix} A^{\sigma_j} \begin{pmatrix} I_n \\ \frac{1}{\alpha} iI_n \\ jI_n \\ \frac{1}{\alpha} kI_n \end{pmatrix},$$

(c) $$A = \frac{1}{4}\begin{pmatrix} I_m & iI_m & -jI_m & -kI_m \end{pmatrix} A^{\sigma_k} \begin{pmatrix} I_n \\ \frac{1}{\alpha} iI_n \\ jI_n \\ \frac{1}{\alpha} kI_n \end{pmatrix}.$$

3. η-Anti-Hermitian solution to Equation (1) and the System (2)

In this section, by three real representations and related lemmas, we derive the necessary and sufficient conditions for Equation (1) and the system (2) to have η-anti-Hermitian solutions over $\mathbb{S}_g$ and obtain expressions of the general η-anti-Hermitian solution in Section 3.1 and Section 3.2, respectively. In Section 3.3, we give a numerical example.

3.1. η-Anti-Hermitian Solution to Equation (1)

Lemma 1. *[26] Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$. Then, the matrix equation $AX = B$ has a skew symmetric solution $X = -X^T \in \mathbb{R}^{n\times n}$ if and only if*

$$R_A B = 0,\ BA^T = -AB^T.$$

In which case, the general skew symmetric solution to $AX = B$ is

$$X = A^{\dagger}B - (A^{\dagger}B)^T + A^{\dagger}AB^T(A^{\dagger})^T + L_A U (L_A)^T,$$

where $U = -U^T \in \mathbb{R}^{n\times n}$ is an arbitrary matrix.

Theorem 2. *Let $A \in {\mathbb{S}_g}^{m\times n}$, $B \in {\mathbb{S}_g}^{m\times n}$, $\eta \in \{i, j, k\}$,*

$$A_\eta = \begin{cases} A^{\sigma_i}{G_n}^{-1}, & \eta = i \\ A^{\sigma_j}V_n{G_n}^{-1}, & \eta = j \\ A^{\sigma_k}W_n{G_n}^{-1}, & \eta = k \end{cases} \text{ and } B_\eta = B^{\sigma_\eta}.$$

Then, the Equation (1) has an η-anti-Hermitian solution $X = -X^{\eta^*} \in {\mathbb{S}_g}^{n\times n}$ if and only if

(1) The corresponding real matrix equation $A_\eta Y_\eta = B_\eta$ has a skew symmetric solution $Y_\eta \in \mathbb{R}^{4n\times 4n}$.

(2) The following conditions hold:

$$R_{A_\eta}B_\eta = 0,\ B_\eta {A_\eta}^T = -A_\eta {B_\eta}^T. \tag{3}$$

The above two statements are equivalent to each other. In which case, the general η-anti-Hermitian solution to Equation (1) can be expressed as follows:

(a) in the case of $\eta = i$,

$$X = \frac{1}{16}\begin{pmatrix} I_n & -\frac{1}{\alpha}iI_n & jI_n & -\frac{1}{\alpha}kI_n \end{pmatrix}(Y_i - {R_n}^TY_i{R_n}^{-1} + {S_n}^TY_i{S_n}^{-1} - {T_n}^TY_i{T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix},$$

(b) in the case of $\eta = j$,

$$X = \frac{1}{16}\begin{pmatrix} -I_n & -\frac{1}{\alpha}iI_n & jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(Y_j + {R_n}^TY_j{R_n}^{-1} - {S_n}^TY_j{S_n}^{-1} - {T_n}^TY_j{T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix}, \tag{4}$$

(c) in the case of $\eta = k$,

$$X = \frac{1}{16}\begin{pmatrix} I_n & -\frac{1}{\alpha}iI_n & -jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(Y_k - {R_n}^TY_k{R_n}^{-1} - {S_n}^TY_k{S_n}^{-1} + {T_n}^TY_k{T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix}.$$

In above (a)–(c),

$$Y_\eta = {A_\eta}^\dagger B_\eta - ({A_\eta}^\dagger B_\eta)^T + {A_\eta}^\dagger A_\eta {B_\eta}^T ({A_\eta}^\dagger)^T + L_{A_\eta} U {L_{A_\eta}}^T, \tag{5}$$

where $U = -U^T \in \mathbb{R}^{4n\times 4n}$ is an arbitrary matrix.

Proof. We only prove the case of $\eta = j$ and the other cases can be conducted in similar ways.

First, we show that any skew symmetric solution to real matrix equation

$$A_j Y_j = B_j \tag{6}$$

can generate a j-anti-Hermitian solution to Equation (1) over $\mathbb{S}_g$.

Let us suppose that Equation (6) has a skew symmetric solution Y. Applying (4) of Proposition 1 to Equation (6), we obtain the following three equations:

$$A^{\sigma_j} R_n V_n {G_n}^{-1} Y {R_n}^{-1} = B^{\sigma_j}, \quad A^{\sigma_j} S_n V_n {G_n}^{-1} Y {S_n}^{-1} = B^{\sigma_j}, \quad A^{\sigma_j} T_n V_n {G_n}^{-1} Y {T_n}^{-1} = B^{\sigma_j}.$$

By direct computation, we have

$$R_n V_n = -V_n R_n, \quad R_n {G_n}^{-1} = -{G_n}^{-1} {R_n}^T, \tag{7}$$

$$S_n V_n = -V_n S_n, \quad S_n {G_n}^{-1} = {G_n}^{-1} {S_n}^T, \tag{8}$$

$$T_n V_n = V_n T_n, \quad T_n {G_n}^{-1} = -{G_n}^{-1} {T_n}^T, \tag{9}$$

$${R_n}^{-1} = \frac{1}{\alpha} R_n, \quad {S_n}^{-1} = S_n, \quad {T_n}^{-1} = \frac{1}{\alpha} T_n.$$

It is easy to show that ${R_n}^T Y {R_n}^{-1}$, $-{S_n}^T Y {S_n}^{-1}$ and $-{T_n}^T Y {T_n}^{-1}$ are skew symmetric solutions to Equation (6). Let

$$\mathcal{Y} = \frac{1}{4}(Y + {R_n}^T Y {R_n}^{-1} - {S_n}^T Y {S_n}^{-1} - {T_n}^T Y {T_n}^{-1}),$$

then, $\mathcal{Y}$ is also a skew symmetric solution to Equation (6).

Assume that Y has the form:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{pmatrix}, \quad Y_{ij} \in \mathbb{R}^{n\times n}, \quad i,j = 1,2,3,4,$$

then, we have

$${G_n}^{-1}\mathcal{Y} = \begin{pmatrix} -Z_0 & -\alpha Z_1 & -Z_2 & -\alpha Z_3 \\ Z_1 & Z_0 & Z_3 & Z_2 \\ Z_2 & \alpha Z_3 & Z_0 & \alpha Z_1 \\ -Z_3 & -Z_2 & -Z_1 & -Z_0 \end{pmatrix},$$

where

$$Z_0 = \frac{1}{4}(-Y_{11} - \frac{1}{\alpha}Y_{22} + Y_{33} + \frac{1}{\alpha}Y_{44}), \quad Z_1 = \frac{1}{4\alpha}(-Y_{12} - Y_{21} + Y_{34} + Y_{43}),$$
$$Z_2 = \frac{1}{4}(-\frac{1}{\alpha}Y_{24} + \frac{1}{\alpha}Y_{42} + Y_{31} - Y_{13}), \quad Z_3 = \frac{1}{4\alpha}(-Y_{14} + Y_{41} - Y_{23} + Y_{32}).$$

It is easy to find that ${G_n}^{-1}\mathcal{Y}$ is the j-real representation of the matrix over $\mathbb{S}_g$, so by (6) of Proposition 1, we can construct a new matrix X over $\mathbb{S}_g$:

$$\begin{aligned} X &= \frac{1}{4}\begin{pmatrix} -I_n & iI_n & jI_n & -kI_n \end{pmatrix} {G_n}^{-1}\mathcal{Y}\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} -I_n & -\frac{1}{\alpha}iI_n & jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(Y + {R_n}^T Y {R_n}^{-1} - {S_n}^T Y {S_n}^{-1} - {T_n}^T Y {T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix}. \end{aligned}$$

Evidently, ${G_n}^{-1}\mathcal{Y}$ is the j-real representation of X. Note that $\mathcal{Y}$ is a skew symmetric solution to Equation (6). By Proposition 1, we obtain

$$(AX)^{\sigma_j} = A_j\mathcal{Y} = B_j = B^{\sigma_j}, \quad (X^{j*})^{\sigma_j} = {G_n}^{-1}\mathcal{Y}^T = -{G_n}^{-1}\mathcal{Y} = -X^{\sigma_j},$$

clearly, X is a j-anti-Hermitian solution to Equation (1) over $\mathbb{S}_g$. Therefore, any skew symmetric solution to Equation (6) can generate a j-anti-Hermitian solution to Equation (1).

Conversely, let us suppose that Equation (1) has a j-anti-Hermitian solution X over $\mathbb{S}_g$, by Proposition 1, we obtain

$$A_j(G_n X^{\sigma_j}) = (AX)^{\sigma_j} = B^{\sigma_j} = B_j, \quad G_n X^{\sigma_j} = G_n(-X^{j*})^{\sigma_j} = -(G_n X^{\sigma_j})^T.$$

Thus, $G_n X^{\sigma_j}$ is a skew symmetric solution to Equation (6). Hence, any j-anti Hermitian solution to Equation (1) can generate a skew symmetric solution to Equation (6).

It is easy to see that Equation (1) has a j-anti-Hermitian solution if and only if the Equation (6) has a skew symmetric solution. By Lemma 1, Equation (6) has a skew symmetric solution Y_j as shown in Formula (5) if and only if condition (3) holds. In this case, by substituting Y_j into Formula (4), we can obtain a j-anti-Hermitian solution to Equation (1). □

3.2. *η-Anti-Hermitian Solution to the System (2)*

Lemma 2. *[27] Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. Then, the equation $Ax = b$ is consistent if and only if*

$$AA^{\dagger}b = b.$$

In which case, the general solution is $x = A^{\dagger}b + L_A u$, where $u \in \mathbb{R}^n$ is an arbitrary vector.

Lemma 3. *Let us suppose that $A \in \mathbb{R}^{k\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{k\times s}$, $D \in \mathbb{R}^{s\times l}$, $E \in \mathbb{R}^{k\times l}$,*

$$Z_{ij} = \frac{\sqrt{2}}{2}({e_i}^{(n)}({e_j}^{(n)})^T - {e_j}^{(n)}({e_i}^{(n)})^T), \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \tag{10}$$

$$W_{pq} = \frac{\sqrt{2}}{2}({e_p}^{(s)}({e_q}^{(s)})^T - {e_q}^{(s)}({e_p}^{(s)})^T), \quad p = 1,2,\ldots,s-1;\ q = p+1,\ldots,s, \tag{11}$$

where ${e_i}^{(n)}$ is the i-th column vector of I_n.

Set

$$\begin{aligned} S_1 &= \begin{pmatrix} Z_{12} & \cdots & Z_{1n} & Z_{23} & \cdots & Z_{2n} & \cdots & Z_{n-1,n} \end{pmatrix} \in \mathbb{R}^{n\times \frac{1}{2}n^2(n-1)}, \\ S_2 &= \begin{pmatrix} W_{12} & \cdots & W_{1s} & W_{23} & \cdots & W_{2s} & \cdots & W_{s-1,s} \end{pmatrix} \in \mathbb{R}^{s\times \frac{1}{2}s^2(s-1)}, \end{aligned}$$

$$G = \begin{pmatrix} Vec(Z_{12}) & \cdots & Vec(Z_{1n}) & Vec(Z_{23}) & \cdots & Vec(Z_{2n}) & \cdots & Vec(Z_{n-1,n}) \end{pmatrix} \in \mathbb{R}^{n^2\times \frac{1}{2}n(n-1)}, \tag{12}$$

$$H = \begin{pmatrix} Vec(W_{12}) & \cdots & Vec(W_{1s}) & Vec(W_{23}) & \cdots & Vec(W_{2s}) & \cdots & Vec(W_{s-1,s}) \end{pmatrix} \in \mathbb{R}^{s^2 \times \frac{1}{2}s(s-1)}, \quad (13)$$

$$M = (B^T \otimes A)G, \quad N = (D^T \otimes C)H, \quad d = Vec(E), \quad Q = \begin{pmatrix} M & N \end{pmatrix}. \quad (14)$$

Then, the real matrix equation

$$AXB + CYD = E \quad (15)$$

has a skew symmetric solution (X, Y) *if and only if*

$$QQ^{\dagger}d = d. \quad (16)$$

In this case, the general skew symmetric solution to Equation (15) can be expressed as

$$X = S_1(a \otimes I_n), \quad Y = S_2(b \otimes I_s), \quad (17)$$

where

$$a = \begin{pmatrix} I_{\frac{1}{2}n(n-1)} & 0 \end{pmatrix}\sigma, \quad b = \begin{pmatrix} 0 & I_{\frac{1}{2}s(s-1)} \end{pmatrix}\sigma, \quad \sigma = Q^{\dagger}d + L_Q u, \quad (18)$$

and $u \in \mathbb{R}^{\frac{1}{2}n(n-1)+\frac{1}{2}s(s-1)}$ *is an arbitrary vector.*

Proof. From the definition of $\{Z_{ij}\}$ and $\{W_{pq}\}$ in (10) and (11), it is easy to verify that $\{Z_{ij}\}$ and $\{W_{pq}\}$ form the orthonormal bases of the set of all skew symmetric matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{s\times s}$, respectively. That is,

$$(Z_{ij}, Z_{pq}) = \begin{cases} 0, & i \neq p \text{ or } j \neq q \\ 1, & i = p \text{ and } j = q \end{cases}, \quad (W_{ij}, W_{pq}) = \begin{cases} 0, & i \neq p \text{ or } j \neq q \\ 1, & i = p \text{ and } j = q \end{cases}.$$

Now, if X and Y are skew symmetric matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{R}^{s\times s}$, respectively, they can be expressed as

$$X = \sum_{i,j} a_{ij} Z_{ij}, \quad Y = \sum_{p,q} b_{pq} W_{pq}, \quad (19)$$

where the real numbers a_{ij} ($i = 1, \ldots, n-1$, $j = i+1, \ldots, n$) and b_{pq} ($p = 1, \ldots, s-1$, $q = p+1, \ldots, s$) are yet to be determined. Substitute Formula (19) into Equation (15), we obtain

$$\sum_{i,j} a_{ij} A Z_{ij} B + \sum_{p,q} b_{pq} C W_{pq} D = E. \quad (20)$$

Set

$$a = \begin{pmatrix} a_{12} & \cdots & a_{1n} & a_{23} & \cdots & a_{2n} & \cdots & a_{n-1,n} \end{pmatrix}^T,$$

$$b = \begin{pmatrix} b_{12} & \cdots & b_{1s} & b_{23} & \cdots & b_{2s} & \cdots & b_{s-1,s} \end{pmatrix}^T, \quad \sigma = \begin{pmatrix} a^T & b^T \end{pmatrix}^T,$$

and let G, H, M, N, d and Q be defined in (12)–(14). Applying the stretching function to both sides of Equation (20), by direct computation, we have

$$Ma + Nb = d. \quad (21)$$

From the definition of the stretching function, it is evident that Equation (15) is equivalent to Equation (21), which is then equivalent to the following equation:

$$Q\sigma = d. \quad (22)$$

It follows from Lemma 2 that Equation (22) is consistent if and only if condition (16) holds, in which case, the general solution σ to Equation (22) can be shown in Formula (18). Therefore, the general skew symmetric solution to Equation (15) can be shown in Formulas (17) and (18). □

Theorem 3. *Suppose $A_1 \in \mathbb{S}_g^{m_1\times n}$, $B_1 \in \mathbb{S}_g^{m_1\times n}$, $A_2 \in \mathbb{S}_g^{n\times m_2}$, $B_2 \in \mathbb{S}_g^{n\times m_2}$, $C_1 \in \mathbb{S}_g^{t_1\times s}$, $D_1 \in \mathbb{S}_g^{t_1\times s}$, $C_2 \in \mathbb{S}_g^{s\times t_2}$, $D_2 \in \mathbb{S}_g^{s\times t_2}$, $E_1 \in \mathbb{S}_g^{k\times n}$, $E_2 \in \mathbb{S}_g^{n\times l}$, $F_1 \in \mathbb{S}_g^{k\times s}$, $F_2 \in \mathbb{S}_g^{s\times l}$, $H \in \mathbb{S}_g^{k\times l}$, $\eta \in \{i,j,k\}$. Let Z_{ij}, W_{pq}, S_1, S_2, G, H be given as in Lemma 3 where n and s change to $4n$ and $4s$, respectively.*

Set

$$\mathcal{A}_\eta = \begin{pmatrix} A_1 \\ -A_2^{\eta*} \end{pmatrix}, \quad \mathcal{B}_\eta = \begin{pmatrix} B_1 \\ B_2^{\eta*} \end{pmatrix}, \quad \mathcal{C}_\eta = \begin{pmatrix} C_1 \\ -C_2^{\eta*} \end{pmatrix}, \quad \mathcal{D}_\eta = \begin{pmatrix} D_1 \\ D_2^{\eta*} \end{pmatrix},$$

$$A_\eta = \begin{cases} \mathcal{A}_i^{\sigma_i} G_n^{-1}, & \eta = i \\ \mathcal{A}_j^{\sigma_j} V_n G_n^{-1}, & \eta = j \\ \mathcal{A}_k^{\sigma_k} W_n G_n^{-1}, & \eta = k \end{cases}, \quad B_\eta = \mathcal{B}_\eta{}^{\sigma_\eta}, \quad C_\eta = \begin{cases} \mathcal{C}_i^{\sigma_i} G_s^{-1}, & \eta = i \\ \mathcal{C}_j^{\sigma_j} V_s G_s^{-1}, & \eta = j \\ \mathcal{C}_k^{\sigma_k} W_s G_s^{-1}, & \eta = k \end{cases}, \quad D_\eta = \mathcal{D}_\eta{}^{\sigma_\eta},$$

$$E_{1\eta} = \begin{cases} E_1{}^{\sigma_i} G_n^{-1}, & \eta = i \\ E_1{}^{\sigma_j} V_n G_n^{-1}, & \eta = j \\ E_1{}^{\sigma_k} W_n G_n^{-1}, & \eta = k \end{cases}, \quad E_{2\eta} = \begin{cases} E_2{}^{\sigma_i}, & \eta = i \\ V_n E_2{}^{\sigma_j}, & \eta = j \\ W_n E_2{}^{\sigma_k}, & \eta = k \end{cases},$$

$$F_{1\eta} = \begin{cases} F_1{}^{\sigma_i} G_s^{-1}, & \eta = i \\ F_1{}^{\sigma_j} V_s G_s^{-1}, & \eta = j \\ F_1{}^{\sigma_k} W_s G_s^{-1}, & \eta = k \end{cases}, \quad F_{2\eta} = \begin{cases} F_2{}^{\sigma_i}, & \eta = i \\ V_s F_2{}^{\sigma_j}, & \eta = j \\ W_s F_2{}^{\sigma_k}, & \eta = k \end{cases}, \quad H_\eta = H^{\sigma_\eta},$$

$$\widehat{E}_{1_\eta} = E_{1\eta} L_{A_\eta}, \quad \widehat{E}_{2_\eta} = L_{A_\eta}^T E_{2\eta}, \quad \widehat{F}_{1_\eta} = F_{1\eta} L_{C_\eta}, \quad \widehat{F}_{2_\eta} = L_{C_\eta}^T F_{2\eta}, \tag{23}$$

$$\begin{aligned} \widehat{H}_\eta = H_\eta &- E_{1\eta}[A_\eta{}^\dagger B_\eta - (A_\eta{}^\dagger B_\eta)^T + A_\eta{}^\dagger A_\eta B_\eta{}^T (A_\eta{}^\dagger)^T] E_{2\eta} \\ &- F_{1\eta}[C_\eta{}^\dagger D_\eta - (C_\eta{}^\dagger D_\eta)^T + C_\eta{}^\dagger C_\eta D_\eta{}^T (C_\eta{}^\dagger)^T] F_{2\eta} \end{aligned}, \tag{24}$$

$$\widehat{M}_\eta = ((\widehat{E}_{2_\eta})^T \otimes \widehat{E}_{1_\eta}) G, \quad \widehat{N}_\eta = ((\widehat{F}_{2_\eta})^T \otimes \widehat{F}_{1_\eta}) H, \quad d_\eta = Vec(\widehat{H}_\eta), \quad Q_\eta = \begin{pmatrix} \widehat{M}_\eta & \widehat{N}_\eta \end{pmatrix}. \tag{25}$$

Then, the system (2) has an η-anti-Hermitian solution (X, Y) over $\mathbb{S}_g$ if and only if

(1) The corresponding system of real matrix equations

$$\begin{cases} A_\eta M_\eta &= B_\eta \\ C_\eta N_\eta &= D_\eta \\ E_{1\eta} M_\eta E_{2\eta} + F_{1\eta} N_\eta F_{2\eta} &= H_\eta \end{cases} \tag{26}$$

has a skew symmetric solution (M_η, N_η).

(2) The following conditions hold:

$$R_{A_\eta} B_\eta = 0,\ B_\eta A_\eta{}^T = -A_\eta B_\eta{}^T,\ R_{C_\eta} D_\eta = 0,\ D_\eta C_\eta{}^T = -C_\eta D_\eta{}^T, \tag{27}$$

$$Q_\eta Q_\eta^\dagger d_\eta = d_\eta. \tag{28}$$

The above two statements are equivalent to each other. In which case, the general η-anti-Hermitian solution to the system (2) can be expressed as follows:

(a) in the case of $\eta = i$,

$$X = \frac{1}{16}\begin{pmatrix} I_n & -\frac{1}{\alpha}iI_n & jI_n & -\frac{1}{\alpha}kI_n \end{pmatrix}(M_i - {R_n}^T M_i {R_n}^{-1} + {S_n}^T M_i {S_n}^{-1} - {T_n}^T M_i {T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix},$$

$$Y = \frac{1}{16}\begin{pmatrix} I_s & -\frac{1}{\alpha}iI_s & jI_s & -\frac{1}{\alpha}kI_s \end{pmatrix}(N_i - {R_s}^T N_i {R_s}^{-1} + {S_s}^T N_i {S_s}^{-1} - {T_s}^T N_i {T_s}^{-1})\begin{pmatrix} I_s \\ \frac{1}{\alpha}iI_s \\ jI_s \\ \frac{1}{\alpha}kI_s \end{pmatrix},$$

(b) in the case of $\eta = j$,

$$X = \frac{1}{16}\begin{pmatrix} -I_n & -\frac{1}{\alpha}iI_n & jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(M_j + {R_n}^T M_j {R_n}^{-1} - {S_n}^T M_j {S_n}^{-1} - {T_n}^T M_j {T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix}, \tag{29}$$

$$Y = \frac{1}{16}\begin{pmatrix} -I_s & -\frac{1}{\alpha}iI_s & jI_s & \frac{1}{\alpha}kI_s \end{pmatrix}(N_j + {R_s}^T N_j {R_s}^{-1} - {S_s}^T N_j {S_s}^{-1} - {T_s}^T N_j {T_s}^{-1})\begin{pmatrix} I_s \\ \frac{1}{\alpha}iI_s \\ jI_s \\ \frac{1}{\alpha}kI_s \end{pmatrix}, \tag{30}$$

(c) in the case of $\eta = k$,

$$X = \frac{1}{16}\begin{pmatrix} I_n & -\frac{1}{\alpha}iI_n & -jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(M_k - {R_n}^T M_k {R_n}^{-1} - {S_n}^T M_k {S_n}^{-1} + {T_n}^T M_k {T_n}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix},$$

$$Y = \frac{1}{16}\begin{pmatrix} I_s & -\frac{1}{\alpha}iI_s & -jI_s & \frac{1}{\alpha}kI_s \end{pmatrix}(N_k - {R_s}^T N_k {R_s}^{-1} - {S_s}^T N_k {S_s}^{-1} + {T_s}^T N_k {T_s}^{-1})\begin{pmatrix} I_s \\ \frac{1}{\alpha}iI_s \\ jI_s \\ \frac{1}{\alpha}kI_s \end{pmatrix}.$$

In above (a)–(c),

$$M_\eta = {A_\eta}^\dagger B_\eta - ({A_\eta}^\dagger B_\eta)^T + {A_\eta}^\dagger A_\eta {B_\eta}^T ({A_\eta}^\dagger)^T + L_{A_\eta} U_\eta {L_{A_\eta}}^T, \tag{31}$$

$$N_\eta = {C_\eta}^\dagger D_\eta - ({C_\eta}^\dagger D_\eta)^T + {C_\eta}^\dagger C_\eta {D_\eta}^T ({C_\eta}^\dagger)^T + L_{C_\eta} V_\eta {L_{C_\eta}}^T, \tag{32}$$

where

$$U_\eta = S_1(a_\eta \otimes I_{4n}), \quad V_\eta = S_2(b_\eta \otimes I_{4s}), \tag{33}$$

$$a_\eta = \begin{pmatrix} I_{2n(4n-1)} & 0 \end{pmatrix}\sigma_\eta, \quad b_\eta = \begin{pmatrix} 0 & I_{2s(4s-1)} \end{pmatrix}\sigma_\eta, \quad \sigma_\eta = Q_\eta^\dagger d_\eta + L_{Q_\eta} u, \tag{34}$$

and $u \in \mathbb{R}^{2n(4n-1)+2s(4s-1)}$ is an arbitrary vector.

Proof. We only prove the case of $\eta = j$ and the other cases can be conducted in similar ways.

At first, it is clear that the following system of matrix equations

$$\begin{cases} A_1X &= B_1 \\ XA_2 &= B_2 \end{cases}$$

has a j-anti-Hermitian solution X over $\mathbb{S}_g$ if and only if

$$\begin{cases} A_1X &= B_1 \\ -A_2^{j*}X &= B_2^{j*} \end{cases} \quad i.e., \ \mathcal{A}_jX = \mathcal{B}_j$$

has a j-anti-Hermitian solution X over $\mathbb{S}_g$, by Theorem 2, if and only if

$$A_jM_j = B_j$$

has a skew symmetric solution M over $\mathbb{R}$. Similarly,

$$\begin{cases} A_1X &= B_1 \\ XA_2 &= B_2 \\ C_1Y &= D_1 \\ YC_2 &= D_2 \end{cases}$$

has a j-anti-Hermitian solution (X, Y) over $\mathbb{S}_g$ if and only if

$$\begin{cases} A_jM_j &= B_j \\ C_jN_j &= D_j \end{cases} \tag{35}$$

has a skew symmetric solution (M, N) over $\mathbb{R}$.

Then, we show that any skew symmetric solution (M, N) to the system

$$\begin{cases} A_jM_j &= B_j \\ C_jN_j &= D_j \\ E_{1j}M_jE_{2j} + F_{1j}N_jF_{2j} &= H_j \end{cases} \tag{36}$$

can generate a j-anti-Hermitian solution (X, Y) to the system (2).

Let us suppose that the system (36) has a skew symmetric solution (M, N). By the proof of Theorem 2,

$$\mathcal{M} = \frac{1}{4}(M + {R_n}^TM{R_n}^{-1} - {S_n}^TM{S_n}^{-1} - {T_n}^TM{T_n}^{-1})$$

and

$$\mathcal{N} = \frac{1}{4}(N + {R_s}^TN{R_s}^{-1} - {S_s}^TN{S_s}^{-1} - {T_s}^TN{T_s}^{-1})$$

are also the skew symmetric solutions to the system (35). Apply (4) of Proposition 1 to

$$E_{1j}M_jE_{2j} + F_{1j}N_jF_{2j} = H_j. \tag{37}$$

By (7)–(9), we obtain the following three equations

$$\begin{aligned} E_{1j}({R_n}^TM{R_n}^{-1})E_{2j} + F_{1j}({R_s}^TN{R_s}^{-1})F_{2j} = H_j, \\ E_{1j}(-{S_n}^TM{S_n}^{-1})E_{2j} + F_{1j}(-{S_s}^TN{S_s}^{-1})F_{2j} = H_j, \\ E_{1j}(-{T_n}^TM{T_n}^{-1})E_{2j} + F_{1j}(-{T_s}^TN{T_s}^{-1})F_{2j} = H_j. \end{aligned}$$

Hence, $\mathcal{M}$ and $\mathcal{N}$ are also the skew symmetric solutions to Equation (37) and therefore to the system (36).

From the proof of Theorem 2, we know that $G_n{}^{-1}\mathcal{M}$ and $G_s{}^{-1}\mathcal{N}$ are the j-real representations of matrices over $\mathbb{S}_g$; moreover, we can construct new matrices X and Y over $\mathbb{S}_g$:

$$X = \frac{1}{4}\begin{pmatrix} -I_n & iI_n & jI_n & -kI_n \end{pmatrix} G_n{}^{-1}\mathcal{M}\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} -I_n & -\frac{1}{\alpha}iI_n & jI_n & \frac{1}{\alpha}kI_n \end{pmatrix}(M + R_n{}^T M R_n{}^{-1} - S_n{}^T M S_n{}^{-1} - T_n{}^T M T_n{}^{-1})\begin{pmatrix} I_n \\ \frac{1}{\alpha}iI_n \\ jI_n \\ \frac{1}{\alpha}kI_n \end{pmatrix},$$

$$Y = \frac{1}{4}\begin{pmatrix} -I_s & iI_s & jI_s & -kI_s \end{pmatrix} G_s{}^{-1}\mathcal{N}\begin{pmatrix} I_s \\ \frac{1}{\alpha}iI_s \\ jI_s \\ \frac{1}{\alpha}kI_s \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} -I_s & -\frac{1}{\alpha}iI_s & jI_s & \frac{1}{\alpha}kI_s \end{pmatrix}(N + R_s{}^T N R_s{}^{-1} - S_s{}^T N S_s{}^{-1} - T_s{}^T N T_s{}^{-1})\begin{pmatrix} I_s \\ \frac{1}{\alpha}iI_s \\ jI_s \\ \frac{1}{\alpha}kI_s \end{pmatrix}.$$

Clearly, $G_n{}^{-1}\mathcal{M}$ and $G_s{}^{-1}\mathcal{N}$ are the j-real representations of X and Y, respectively. According to Theorem 2, (X, Y) is a j-anti-Hermitian solution to the system (35) over $\mathbb{S}_g$.

Note that $(\mathcal{M}, \mathcal{N})$ is a skew symmetric solution to Equation (37), we obtain

$$(E_1XE_2)^{\sigma_j} + (F_1YF_2)^{\sigma_j} = E_{1j}\mathcal{M}E_{2j} + F_{1j}\mathcal{N}F_{2j} = H_j = H^{\sigma_j},$$

i.e.,

$$E_1XE_2 + F_1YF_2 = H. \tag{38}$$

Hence, (X, Y) is a j-anti-Hermitian solution to Equation (38) and therefore to the system (2). Consequently, any skew symmetric solution (M, N) to the system (36) can generate a j-anti-Hermitian solution to the system (2).

Conversely, let us suppose that the system (2) has a j-anti-Hermitian solution (X, Y) over $\mathbb{S}_g$, then we obtain

$$A_j(G_nX^{\sigma_j}) = (\mathcal{A}_jX)^{\sigma_j} = \mathcal{B}_j^{\sigma_j} = B_j, \quad C_j(G_sY^{\sigma_j}) = (\mathcal{C}_jY)^{\sigma_j} = \mathcal{D}_j^{\sigma_j} = D_j,$$

$$E_{1j}(G_nX^{\sigma_j})E_{2j} + F_{1j}(G_sY^{\sigma_j})F_{2j} = (E_1XE_2)^{\sigma_j} + (F_1YF_2)^{\sigma_j} = H^{\sigma_j} = H_j.$$

Thus, $(G_nX^{\sigma_j}, G_sY^{\sigma_j})$ is a skew symmetric solution to the system (36) over $\mathbb{R}$, then any j-anti-Hermitian solution to the system (2) can generate a skew symmetric solution to the system (36).

It is clear that the system (2) has a j-anti-Hermitian solution if and only if the system (36) has a skew symmetric solution. By Lemma 1, the system (35) has a skew symmetric solution if and only if condition (27) holds, the general skew symmetric solutions (M_j, N_j) to the system (35) are as shown in Formulas (31) and (32), where $U_j = -U_j{}^T \in \mathbb{R}^{4n\times 4n}$ and $V_j = -V_j{}^T \in \mathbb{R}^{4s\times 4s}$ are arbitrary matrices.

By substituting Formulas (31) and (32) into Equation (37), we have

$$\widehat{E}_{1_j}U_j\widehat{E}_{2_j} + \widehat{F}_{1_j}V_j\widehat{F}_{2_j} = \widehat{H}_j, \tag{39}$$

where $\widehat{E}_{1_j}$, $\widehat{E}_{2_j}$, $\widehat{F}_{1_j}$, $\widehat{F}_{2_j}$ and $\widehat{H}_j$ are defined as in (23)–(24).

According to Lemma 3, Equation (39) has a skew symmetric solution (U_j, V_j) if and only if condition (28) holds, where $\widehat{M}_j$, $\widehat{N}_j$, d_j and Q_j are defined as in Formula (25); then, the general expression of the skew symmetric solution to the Equation (39) as shown in Formulas (33) and (34).

In conclusion, the system (2) has a j-anti-Hermitian solution over $\mathbb{S}_g$ if and only if conditions (27) and (28) hold. In this case, the expression of the general j-anti-Hermitian solution to the system (2) can be given by (29)–(34). $\square$

Corollary 1. *Under the same definitions in Theorem 3, if conditions (27) and (28) hold, and*

$$rank(Q_\eta) = 2n(4n-1) + 2s(4s-1), \tag{40}$$

then the system of real matrix Equations (26) has a unique skew symmetric solution

$$M_\eta = \begin{pmatrix} M_{\eta_{11}} & M_{\eta_{12}} & M_{\eta_{13}} & M_{\eta_{14}} \\ M_{\eta_{21}} & M_{\eta_{22}} & M_{\eta_{23}} & M_{\eta_{24}} \\ M_{\eta_{31}} & M_{\eta_{32}} & M_{\eta_{33}} & M_{\eta_{34}} \\ M_{\eta_{41}} & M_{\eta_{42}} & M_{\eta_{43}} & M_{\eta_{44}} \end{pmatrix}, \quad N_\eta = \begin{pmatrix} N_{\eta_{11}} & N_{\eta_{12}} & N_{\eta_{13}} & N_{\eta_{14}} \\ N_{\eta_{21}} & N_{\eta_{22}} & N_{\eta_{23}} & N_{\eta_{24}} \\ N_{\eta_{31}} & N_{\eta_{32}} & N_{\eta_{33}} & N_{\eta_{34}} \\ N_{\eta_{41}} & N_{\eta_{42}} & N_{\eta_{43}} & N_{\eta_{44}} \end{pmatrix},$$

where $M_{\eta_{pq}} \in \mathbb{R}^{n\times n}$ and $N_{\eta_{pq}} \in \mathbb{R}^{s\times s}$. In this case, the system (2) also has a unique η-anti-Hermitian solution $X = X_0 + X_1 i + X_2 j + X_3 k$, $Y = Y_0 + Y_1 i + Y_2 j + Y_3 k$, which can be given as follows:

(1) in the case of $\eta = \{i, k\}$,

$$X_0 = M_{\eta_{11}}, \quad X_1 = \frac{1}{\alpha} M_{\eta_{12}}, \quad X_2 = M_{\eta_{13}}, \quad X_3 = \frac{1}{\alpha} M_{\eta_{14}},$$
$$Y_0 = N_{\eta_{11}}, \quad Y_1 = \frac{1}{\alpha} N_{\eta_{12}}, \quad Y_2 = N_{\eta_{13}}, \quad Y_3 = \frac{1}{\alpha} N_{\eta_{14}},$$

(2) in the case of $\eta = j$,

$$X_0 = -M_{j_{11}}, \quad X_1 = -\frac{1}{\alpha} M_{j_{12}}, \quad X_2 = -M_{j_{13}}, \quad X_3 = -\frac{1}{\alpha} M_{j_{14}},$$
$$Y_0 = -N_{j_{11}}, \quad Y_1 = -\frac{1}{\alpha} N_{j_{12}}, \quad Y_2 = -N_{j_{13}}, \quad Y_3 = -\frac{1}{\alpha} N_{j_{14}}.$$

Proof. Since rank equality (40) holds, the column vectors of Q_j are linearly independent. By Lemma 2 in Chapter 1 of [27], we obtain

$$Q_j^{\dagger} Q_j = I_{2n(4n-1)+2s(4s-1)}.$$

In this case, $L_{Q_j} = 0$. According to Theorem 3, the system (26) has a unique skew symmetric solution (M, N) and the system (2) has at least one η-anti-Hermitian solution. Let us assume that the system (2) has two different solutions (X, Y) and $(\widehat{X}, \widehat{Y})$. By the proof of Theorem 3, $(G_n X^{\sigma_\eta}, G_s Y^{\sigma_\eta})$ and $(G_n \widehat{X}^{\sigma_\eta}, G_s \widehat{Y}^{\sigma_\eta})$ are two different skew symmetric solutions to the system (26) which conflicts with our assumption. Therefore, the system (2) has a unique η-anti-Hermitian solution (X, Y). Let us assume (M_η, N_η) is a unique symmetric solution to the system (26); we can obtain X_t and Y_t $(t = 0, 1, 2, 3)$ from X^{σ_η} and Y^{σ_η}, where $X^{\sigma_\eta} = G_n^{-1} M_\eta$ and $Y^{\sigma_\eta} = G_s^{-1} N_\eta$. $\square$

3.3. Numerical Examples

Consider the general j-anti-Hermitian solution to the system (2) over H_g when $\alpha = -1$. Let

$$A_1 = \begin{pmatrix} 2 - i + 4j + k & -i + 3j + 2k \end{pmatrix}, \quad B_1 = \begin{pmatrix} 12 + 6i & 6 + 6j - 6k \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1-2i+j+k \\ 2-i+j+k \end{pmatrix}, \quad B_2 = \begin{pmatrix} 6+6j-6k \\ 6i \end{pmatrix}, \quad C_1 = \begin{pmatrix} 2+i-j & 1+j+k \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -k & 1+2i+j \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1+i+j+2k \\ -1-i+j \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1-2i-j \\ 2j-k \end{pmatrix},$$

$$E_1 = \begin{pmatrix} j+k & 1+i \\ 2+2j & 2i+j+k \\ 1+2j+k & 3k \end{pmatrix}, \quad E_2 = \begin{pmatrix} i-5j & 3+2k & 1+2i+5k \\ -2-k & 6j & 3i+j+3k \end{pmatrix},$$

$$F_1 = \begin{pmatrix} 6-j & 3i-6k \\ i+2k & -5 \\ 3+k & -2i+3j \end{pmatrix}, \quad F_2 = \begin{pmatrix} i-j+2k & 3+k & 2j \\ -5 & -2i+3j & 6+3i+j-6k \end{pmatrix},$$

$$H = \begin{pmatrix} -53-69i-48j+3k & 62-19i+15j+45k & -35+80i+152j-37k \\ -41-42i-67j-37k & 41+53i+27j+42k & 21+84i+54j+153k \\ -15+5i-83j-43k & 86+30i-44j+18k & 42+34i-3j+81k \end{pmatrix}.$$

By direct computation of A_j, B_j, C_j, D_j, Q_j and d_j, it can be verified that

$$R_{A_j}B_j = 0, \quad B_j{A_j}^T = -A_j{B_j}^T, \quad R_{C_j}D_j = 0, \quad D_j{C_j}^T = -C_j{D_j}^T, \quad Q_jQ_j^{\dagger}d_j = d_j,$$

and

$$rank(Q_j) = 56.$$

According to Corollary 1, the system (26) has a unique skew symmetric solution. By computation, we obtain the unique j-anti-Hermitian solution to the system (2) over H_g is

$$X = \begin{pmatrix} 3j+3k & 2-j-3k \\ -2-j-3k & 3j+3k \end{pmatrix}, \quad Y = \begin{pmatrix} k & 2+i+2j \\ -2-i+2j & -j-k \end{pmatrix}.$$

4. Conclusions

In this paper, we discuss the existence and general expression of the η-anti-Hermitian solution to the generalized Segre quaternion matrix Equation (1) and the system (2) by using the real representations of matrices over the generalized Segre quaternion algebra, Moore–Penrose generalized inverse, Kronecker product and the stretching function. In the end, a numerical example is given to verify the main results. The η-Hermitian solution to the generalized Segre quaternion matrix Equation (1) and the system (2) and other systems may be considered in the future.

Author Contributions: Methodology, B.-Y.R. and Q.-W.W.; software, B.-Y.R. and X.-Y.C.; writing—original draft preparation, Q.-W.W. and B.-Y.R.; writing—review and editing, Q.-W.W., B.-Y.R. and X.-Y.C.; supervision, Q.-W.W.; project administration, Q.-W.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by National Natural Science Foundation of China (11971294) and (12171369).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors have declared that there is no conflict of interest.

References

1. Hamilton, W.R. *Lectures on Quaternions*; Hodges and Smith: Dublin, Ireland, 1853.
2. Li, Y.; Zhang, Y.Z.; Wei, M.S.; Zhao, H. Real structure-preserving algorithm for quaternion equality constrained least squares problem. *Math. Meth. Appl. Sci.* **2020**, *43*, 4558–4566. [CrossRef]

3. Le Bihan, N.; Mars, J. Singular value decomposition of quaternion matrices: A new tool for vector-sensor signal processing. *Signal Process.* **2004**, *84*, 1177–1199. [CrossRef]
4. Adler, S.L. Scattering and decay theory for quaternionic quantum mechanics, and the structure of induced T nonconservation. *Phys. Rev. D* **1988**, *37*, 3654–3662. [CrossRef] [PubMed]
5. Ling, S.T.; Wang, M.H.; Cheng, X.H. A new implementation of LSMR algorithm for the quaternionic least squares problem . *J. Math. Phys.* **2018**, *59*, 073510-1–073510-10. [CrossRef]
6. Segre, C. The real representations of complex elements and extensions to bicomplex systems. *Math. Ann.* **1892**, *40*, 413–467. (In Italian) [CrossRef]
7. Pei, S.C.; Chang, J.H.; Ding, J.J. Commutative reduced biquaternions and their Fourier transform for signal and image processing applications. *IEEE Trans. Signal Process.* **2004**, *52*, 2012–2031. [CrossRef]
8. Pei, S.C.; Chang, J.H.; Ding, J.J.; Chen, M.Y. Eigenvalues and singular value decompositions of reduced biquaternion matrices. *IEEE Trans. Circuits Syst. I. Regul. Pap* **2008**, *55*, 2673–2685.
9. Guo, L.; Zhu, M.; Ge, X. Reduced biquaternion canonical transform, convolution and correlation. *Signal Process.* **2011**, *91*, 2147–2543. [CrossRef]
10. Lin, D.Y.; Chen, X.F.; Sun, W.T.; Li, W.K.; Xia, Y.N. Stability analysis of commutative quaternion valued neural network with time varying delays. *Chin. J. Intell. Sci. Technol.* **2020**, *2*, 80–87. (In Chinese)
11. Kosal, H.H.; Tosun, M. Commutative quaternion matrices. *Adv. Appl. Clifford Algebr.* **2014**, *24*, 769–799. [CrossRef]
12. Kosal, H.H.; Akyigit, M.; Tosun, M. Consimilarity of commutative quaternion matrices. *Miskolc Math. Notes* **2015**, *16*, 965–977. [CrossRef]
13. Kosal, H.H.; Tosun, M. Some equivalence relations and results over the commutative quaternions and their matrices. *An. Stiint. Univ. "Ovidius" Constanta Ser. Mat.* **2017**, *25*, 125–142. [CrossRef]
14. Kosal, H.H.; Tosun, M. Universal similarity factorization equalities for commutative quaternions and their matrices. *Linear Multilinear A* **2019**, *67*, 926–938. [CrossRef]
15. Navarro-Moreno, J.; Fernández-Alcalá, R.M.; Ruiz-Molina, J.C. Proper ARMA modeling and forecasting in the generalized Segre's quaternions domain. *Mathematics* **2022**, *10*, 1083. [CrossRef]
16. Took, C.C.; Mandic, D.P. Augmented second-order statistics of quaternion random signals. *Signal Process.* **2011**, *91*, 214–224. [CrossRef]
17. Took, C.C.; Mandic, D.P. Quaternion-valued stochastic gradient-based adaptive IIR filtering. *IEEE Trans. Signal Process.* **2010**, *58*, 3895–3901. [CrossRef]
18. Took, C.C.; Mandic, D.P. The quaternion LMS algorithm for adaptive filtering of hypercomplex real world processes. *IEEE Trans. Signal Process.* **2009**, *57*, 1316–1327. [CrossRef]
19. Liu, X. The η-anti-Hermitian solution to some classic matrix equations. *Appl. Math. Comput.* **2018**, *320*, 264–270. [CrossRef]
20. Rehman, A.; Wang, Q.W.; He, Z.H. Solution to a system of real quaternion matrix equations encompassing η-Hermicity. *Appl. Math. Comput.* **2015**, *265*, 945–957. [CrossRef]
21. Yuan, S.F. Least squares η-hermitian Solution for Quaternion Matrix Equation $AXB = C$. *Inform. Comput. Appl.* **2012**, *307*, 300–305.
22. Zhang, Y.; Wang, R.H. The exact solution of a system of quaternion matrix equations involving η-Hermicity. *Appl. Math. Comput.* **2013**, *222*, 201–209. [CrossRef]
23. Yuan, S.F.; Tian, Y.; Li, M.Z. On Hermitian solutions of the reduced biquaternion matrix equation $(AXB, CXD) = (E, G)$. *Linear Multilinear A* **2020**, *68*, 1355–1373. [CrossRef]
24. Tian, Y.; Yuan, S.F.; Li, M.Z. On anti-Hermitian solutions of the reduced biquaternion matrix equation $AXA^H + BYB^H = C$. *Pure Math.* **2018**, *8*, 624–631. (In Chinese) [CrossRef]
25. Yuan, S.F.; Wang, Q.W.; Duan, X.F. On solutions of the quaternion matrix equation $AX = B$ and their applications in color image restoration. *Appl. Math. Comput.* **2013**, *221*, 10–20. [CrossRef]
26. Khatri, C.G.; Mitra, S.K. Hermitian and nonnegative definite solutions of linear matrix equations. *SIAM J. Appl. Math* **1976**, *31*, 579–585. [CrossRef]
27. Ben-Israel A.; Greville T.N.E. *Generalized Inverses: Theory and Applications*; John Wiley and Sons: New York, NY, USA, 1974.