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Ghost Stars in General Relativity

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Abstract: We explore an idea put forward many years ago by Zeldovich and Novikov concerning the existence of compact objects endowed with arbitrarily small mass. The energy density of such objects, which we call "ghost stars", is negative in some regions of the fluid distribution, producing a vanishing total mass. Thus, the interior is matched on the boundary surface to Minkowski space–time. Some exact analytical solutions are exhibited and their properties are analyzed. Observational data that could confirm or dismiss the existence of this kind of stellar object are discussed.

Keywords: relativistic fluids; interior solutions; spherically symmetric sources

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1. Introduction

In their book on relativistic astrophysics, Zeldovich and Novikov (ZN) [1] (see also [2]), raise the question about the possibility of packaging the constituents of a self–gravitating fluid distribution in such a way that the total mass of the resulting compact object is arbitrarily small.

Specifically, they consider static spherically symmetric fluid distributions, for which the line element may be written as

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right), \tag{1}$$

where $\nu(r)$ and $\lambda(r)$ are functions of r, and c is the light velocity. In this section, we shall follow the notation of [1]; however, in the rest of the manuscript we shall use relativistic units, in which case we put $c = G = 1$.

The fluid distribution is bounded from the exterior by a surface, Σ, whose equation is $r = r_\Sigma =$ constant.

From (1) and the Einstein equations we may write

$$e^{-\lambda} = 1 - \frac{8\pi G}{rc^2} \int_0^r \mu r^2 dr, \tag{2}$$

and for the three-dimensional volume element we have

$$dV = 4\pi e^{\lambda/2} r^2 dr, \tag{3}$$

where μ denotes the energy density of the fluid.

Then, we have for the total mass (energy) the well-known expression

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$$E = Mc^2 = 4\pi c^2 \int_0^{r_\Sigma} \mu r^2 dr. \tag{4}$$

ZN also introduce the rest energy of the constituent particles, E_0, given by

$$E_0 = M_0 c^2 = N m_0 c^2, \tag{5}$$

where m_0 is the particle mass and N denotes the total number of particles that may be expressed through the particle density, n, as

$$N = \int_V n dV. \tag{6}$$

Also, denoting by E_1 the rest energy, E_0, plus the kinetic energy and the interaction energy of the constituents (excluding the gravitational interaction) we may write

$$E_1 = M_1 c^2 = c^2 \int_V \mu dV = 4\pi c^2 \int_0^{r_\Sigma} e^{\lambda/2} \mu r^2 dr. \tag{7}$$

Since $e^{\lambda/2} \geq 1$, then the mass defect $\Delta M = M_1 - M$ should be positive.

Thus, the original question posed by ZN may be rephrased as: can the constituents of a star be packaged in such a way that the mass defect equals M_1?

They answer affirmatively to the above question, and illustrate their point by analyzing the case of an ideal Fermi gas. Although their analysis is flawed, as we shall see below, the case for the existence of stars with arbitrarily small total mass should not be dismissed.

Let us first reproduce the analysis of ZN, following strictly their line of arguments (with only slight changes in notation).

Thus, let us consider an ultra-relativistic Fermi gas, characterized by an equation of state given by

$$\mu = \beta n^{4/3}, \qquad \beta \equiv \frac{3}{8}\hbar (3\pi^2)^{1/3}, \tag{8}$$

where $\hbar$ is the Planck constant over 2π.

Next, ZN assume for the distribution of energy density the form

$$\mu = \frac{a}{r^2}, \quad a = constant. \tag{9}$$

It is worth emphasizing that the above choice is justified by the fact that it coincides with the well-known Tolman VI solution [3], whose equation of state for large values of μ approaches that for a highly compressed Fermi gas.

Then, using (9) in (4), it follows at once

$$M = 4\pi a r_\Sigma. \tag{10}$$

On the other hand, using (2), (3), (6), (8) and (9), we obtain for N

$$N = \frac{\alpha r_\Sigma^{3/2}}{\sqrt{1 - \frac{8\pi G a}{c^2}}}, \quad \alpha \equiv \frac{8\pi}{3}\left(\frac{a}{\beta}\right)^{3/4}, \tag{11}$$

implying

$$r_\Sigma = \alpha^{-2/3} N^{2/3} \left(1 - \frac{8\pi G a}{c^2}\right)^{1/3}. \tag{12}$$

Feeding back (12) into (10) produces

$$M \sim N^{2/3} \left(1 - \frac{8\pi G a}{c^2}\right)^{1/3}. \tag{13}$$

From (13), ZN conclude that, in the limit when $a \rightarrow \frac{c^2}{8\pi G}$, the total mass, M, tends to zero.

Such a conclusion is incorrect, as (11) and (12) imply that, in the limit $a \rightarrow \frac{c^2}{8\pi G}$, N diverges as $N^{2/3} \sim \frac{1}{(1-\frac{8\pi Ga}{c^2})^{1/3}}$, thereby canceling the term $(1-\frac{8\pi Ga}{c^2})^{1/3}$ in (13). This is also evident from (10), which shows that M does not tend to zero for any value of a (different from zero).

In general, it should be clear from its very definition (4), that M cannot be zero for any positive defined energy-density function, μ. Thus, vanishing total mass is only possible if we accept the existence of fluid distributions allowing negative energy density, or in the trivial case $\mu = 0$.

The appearance of negative energy density (mass) in general relativity has been considered in the past by several researchers, starting with a paper by Bondi [4]. This issue also appears in relation to the Reissner–Nordstrom solution and classical electron models (see [5–8] and references therein). More recently, negative masses have been invoked in the construction of some cosmological models (see [9,10] and references therein). Also, it is worth mentioning that negative energy density appears in hyperbolically symmetric fluids (see [11,12] and references therein). In all the cases above, quantum effects were not taken into account. However, in spite of these examples, we believe that it is fair to say that the assumption of positive energy density is well justified, at the classic level, for any realistic fluid.

Notwithstanding, the situation is quite different in the quantum regime. Indeed, as it has been argued in the recent past (see [13–17] and references therein), the appearance of negative energy density is possible, whenever quantum effects are expected to be relevant.

Thus, the idea of compact objects with arbitrarily small total mass is still feasible, if we accept the possibility of negative energy density. We call such objects "ghost stars", in analogy with a somehow similar situation observed in some Einstein–Dirac neutrinos (named ghost neutrinos), which do not produce a gravitational field but still are characterized by non-vanishing current density [18–20].

In this work, we shall explore such a possibility by presenting explicit analytical models of ghost stars.

2. The Einstein Equations for Static Locally Anisotropic Fluids

In what follows, we shall briefly summarize the definitions and main equations required for describing spherically symmetric static anisotropic fluids. We shall heavily rely on [21], and therefore we shall omit many steps in the calculations, details of which the reader may find in that reference.

We consider a spherically symmetric distribution of static fluid, bounded by a spherical surface, Σ. The fluid is assumed to be locally anisotropic (principal stresses unequal).

The justification to consider anisotropic fluids, instead of isotropic ones, is provided by the fact that pressure anisotropy is produced by many different physical phenomena of the kind expected in a gravitational collapse scenario (see [22] and references therein). In particular, we expect that the final stages of stellar evolution should be accompanied by intense dissipative processes, which, as shown in [23], should produce pressure anisotropy.

In curvature coordinates (using relativistic units), the line element reads (please notice that we are using signature -2, instead $+2$, as in [21])

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + r^2 sin^2\theta d\phi^2), \tag{14}$$

which has to satisfy the Einstein equations. For a locally anisotropic fluid they are

$$8\pi\mu = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{15}$$

$$8\pi P_r = -\frac{1}{r^2} + e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right), \tag{16}$$

$$8\pi P_\perp = \frac{e^{-\lambda}}{4}\left(2\nu'' + \nu'^2 - \lambda'\nu' + 2\frac{\nu' - \lambda'}{r}\right), \tag{17}$$

where primes denote derivatives with respect to r, and μ, P_r and $P_\perp$ are proper energy density, radial pressure and tangential pressure, respectively.

The above is a system of three ordinary differential equations for the five unknown functions ν, λ, μ, P_r and $P_\perp$, and accordingly their solutions would depend on two arbitrary functions.

From the above field equations, the Tolman–Oppenheimer–Volkof equation follows

$$P_r' = -\frac{(m + 4\pi P_r r^3)}{r(r - 2m)}(\mu + P_r) + \frac{2(P_\perp - P_r)}{r}, \tag{18}$$

where we have introduced the mass function, m [24], defined by

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}. \tag{19}$$

In [21], a general algorithm to express any solution for anisotropic fluids in terms of two generating functions was proposed (see also [25]). It generalizes a previous work by Lake for isotropic fluids [26].

Specifically, it was shown that the general line element corresponding to any solution to the system (15)–(17) may be written as

$$\begin{aligned} ds^2 &= e^{\int (2z(r) - 2/r)dr} dt^2 - \frac{z^2(r) e^{\int (\frac{4}{r^2 z(r)} + 2z(r))dr}}{r^6 \left(-2\int \frac{z(r)(1 + \Pi(r) r^2) e^{\int (\frac{4}{r^2 z(r)} + 2z(r))dr}}{r^8} dr + C\right)} dr^2 \\ &- r^2 d\theta^2 - r^2 sin^2\theta d\phi^2. \end{aligned} \tag{20}$$

with $\Pi(r) = 8\pi(P_r - P_\perp)$ and

$$e^{\nu(r)} = e^{\int (2z(r) - 2/r)dr} \tag{21}$$

where C is a constant of integration.

The expression (20) follows from (21) and the formal integration of $\Pi(r) = 8\pi(P_r - P_\perp)$, after replacing P_r and $P_\perp$ by their expressions in (16) and (17) (see [21] for details).

In the next sections, z will be obtained from specific restrictions on the fluid distribution (e.g., conformal flatness, vanishing complexity factor).

The physical variables may be written as

$$4\pi P_r = \frac{z(r - 2m) + m/r - 1}{r^2}, \tag{22}$$

$$4\pi\mu = \frac{m'}{r^2}, \tag{23}$$

and

$$4\pi P_\perp = \left(1 - \frac{2m}{r}\right)\left(z' + z^2 - \frac{z}{r} + \frac{1}{r^2}\right) + z\left(\frac{m}{r^2} - \frac{m'}{r}\right). \tag{24}$$

In order to match smoothly the metric (14) with the Schwarzschild metric on the boundary surface $r = r_\Sigma = constant$, we require the continuity of the first and the second fundamental forms across that surface, producing

$$e^{\nu_\Sigma} = 1 - \frac{2M}{r_\Sigma}, \tag{25}$$

$$e^{-\lambda_\Sigma} = 1 - \frac{2M}{r_\Sigma}, \tag{26}$$

$$[P_r]_\Sigma = 0, \tag{27}$$

where subscript Σ indicates that the quantity is evaluated on the boundary surface, Σ.

The above conditions hold for any value of M, including $M = 0$.

For configurations with $M = 0$, we obtain from (22) and (27)

$$z_\Sigma = \frac{1}{r_\Sigma}. \tag{28}$$

We shall next present solutions describing fluid spheres with vanishing total mass. To do that, we shall resort to a variety of assumptions, some of which are usually invoked in the modeling of relativistic stars.

3. Conformally Flat Ghost Stars

The Weyl tensor is known to play a very important role in the structure and evolution of compact objects (see [27] and references therein), which explains why the vanishing Weyl tensor condition (conformal flatness) has been used so frequently in the study of self-gravitating objects.

If we assume the space–time within the fluid distribution to be conformally flat, then the two generating functions read

$$z = \frac{2}{r} \pm \frac{e^{\frac{\lambda}{2}}}{r}\, tanh\left(\int \frac{e^{\frac{\lambda}{2}}}{r} dr\right). \tag{29}$$

and

$$\Pi = r\left(\frac{1 - e^{-\lambda}}{r^2}\right)'. \tag{30}$$

In (29), we shall choose the minus sign, since the plus sign leads (in this case) to a model not satisfying the boundary condition (28).

We shall present two conformally flat models of ghost star. For that purpose, we shall complement the conformal flatness condition with some additional restrictions.

3.1. Ghost Star with a Given Density Profile

Let us assume a density profile of the form

$$4\pi\mu = \sum_{i=0}^{n} a_i r^{i-2}, \tag{31}$$

which using (23) produces

$$m = \sum_{i=0}^{n} \frac{a_i}{i+1} r^{i+1}. \tag{32}$$

Since the total mass is assumed to vanish, then the following condition has to be satisfied

$$\sum_{i=0}^{n} \frac{\bar{a}_i}{i+1} = 0, \tag{33}$$

with $\bar{a}_i = a_i r_\Sigma^i$.

In order to describe a specific model, let us restrict the expression (31) to $n = 2$.

Thus, we obtain for the energy density and the mass function

$$4\pi\mu = -\frac{3}{2r^2} + \frac{a_1}{r} + a_2, \tag{34}$$

and

$$m = -\frac{3}{2}r + \frac{a_1}{2}r^2 + \frac{a_2}{3}r^3, \tag{35}$$

where we have chosen $a_0 = -\frac{3}{2}$ to simplify the calculation of the second term on the right of (29).

Then the condition (33) reads

$$a_2 = \frac{9}{2r_\Sigma^2} - \frac{3a_1}{2r_\Sigma}, \tag{36}$$

and using (35) and (36) in (19) we obtain

$$e^{-\lambda} = 4 - \frac{3r^2}{r_\Sigma^2} - a_1 r\left(1 - \frac{r}{r_\Sigma}\right). \tag{37}$$

With the expression for λ given by (37), the two generating functions for this case become

$$z = \frac{5}{2r} - \sqrt{\frac{a_2}{-24 + 6a_1 r + 4a_2 r^2}}, \tag{38}$$

and

$$\Pi = \frac{6}{r^2} - \frac{a_1}{r}. \tag{39}$$

The constant a_1 may be easily obtained from (38), and, using condition (28), it reads

$$a_1 = \frac{12}{r_\Sigma}, \tag{40}$$

which combined with (36) produces

$$a_2 = -\frac{27}{2r_\Sigma^2}. \tag{41}$$

With the two expressions above, we finally obtain for z and m

$$z = \frac{6r - 5r_\Sigma}{r(3r - 2r_\Sigma)}, \tag{42}$$

$$m = -\frac{3}{2}r + \frac{6r^2}{r_\Sigma} - \frac{9r^3}{2r_\Sigma^2}, \tag{43}$$

and using using (22), (34) and (39) we obtain for the energy density, the radial pressure and Π

$$4\pi\mu = -\frac{3}{2r^2} + \frac{12}{r_\Sigma r} - \frac{27}{2r_\Sigma^2}, \tag{44}$$

$$4\pi P_r = \frac{27}{2r_\Sigma^2} - \frac{21}{r r_\Sigma} + \frac{15}{2r^2}, \tag{45}$$

$$\Pi = \frac{6}{r^2} - \frac{12}{r r_\Sigma}. \tag{46}$$

Using (44) and (46), the reader can easily check that the condition of conformal flatness (see Equation (29) in [28])

$$P_r - P_\perp = \frac{1}{r^3}\int_0^r r^3 \mu' dr, \tag{47}$$

is satisfied.

From (44), we see that μ is negative in the intervals $0 < r \lessapprox 0.15 r_\Sigma$ and $r \gtrapprox 0.73 r_\Sigma$. As it is apparent from the expressions of the physical variables, the fluid distribution has a singularity at the origin ($r = 0$), and therefore the center should be excluded from the discussion. The best way to handle this drawback consists in assuming that a vacuum cavity surrounds the center. Denoting the equation of the boundary of the cavity by $r = r_i = constant$, we obtain from (43) $r_i = \frac{r_\Sigma}{3}$, which ensures the continuity of the mass function on that surface. However, the radial pressure is discontinuous on that surface, and therefore it is a thin shell, endowed with a singular matter distribution satisfying the Israel conditions [29].

3.2. Ghost Star with the Gokhroo and Mehra Ansatz

We shall now complement the conformal flatness condition with an ansatz proposed by Gokhroo and Mehra [30]. Its virtue consists in providing physically satisfactory models for compact objects.

Thus, we shall assume for λ the condition

$$e^{-\lambda} = 1 - \alpha r^2 + \frac{3K\alpha r^4}{5r_\Sigma^2}, \tag{48}$$

producing, because of (15) and (19),

$$\mu = \mu_0 \left(1 - \frac{Kr^2}{r_\Sigma^2}\right), \tag{49}$$

and

$$m(r) = \frac{4\pi\mu_0 r^3}{3}\left(1 - \frac{3Kr^2}{5r_\Sigma^2}\right), \tag{50}$$

where K is a constant, μ_0 is the central density and

$$\alpha \equiv \frac{8\pi\mu_0}{3}. \tag{51}$$

Since we must impose $m(r_\Sigma) = 0$, then $K = \frac{5}{3}$.

Feeding back this value of K into (48)–(50), we obtain

$$4\pi\mu = \frac{6}{r_\Sigma^2}\left(1 - \frac{5r^2}{3r_\Sigma^2}\right), \tag{52}$$

$$m = \frac{2r^3}{r_\Sigma^2}\left(1 - \frac{r^2}{r_\Sigma^2}\right), \tag{53}$$

and

$$e^{-\lambda} = 1 - \frac{4r^2}{r_\Sigma^2} + \frac{4r^4}{r_\Sigma^4}, \tag{54}$$

where we have chosen $\alpha = \frac{4}{r_\Sigma^2}$, in order to facilitate the calculation of the second term on the right of (29). Thus, we obtain for z

$$z = \frac{3}{r} - \frac{2r}{2r^2 - r_\Sigma^2}, \tag{55}$$

whereas for Π we obtain from (30)

$$\Pi = -\frac{8r^2}{r_\Sigma^4}, \tag{56}$$

and from (22) we obtain the expression for P_r

$$8\pi P_r = \frac{4}{r^2}\left(1 - \frac{4r^2}{r_\Sigma^2} + \frac{3r^4}{r_\Sigma^4}\right). \tag{57}$$

As it follows from (52), the energy density becomes negative for $r \gtrapprox 0.77 r_\Sigma$.

As in the precedent model, there appears a singularity at the center, which could be embedded in a vacuum cavity bounded by a thin shell.

4. Ghost Stars with Vanishing Complexity Factor

The complexity factor, usually denoted by Y_{TF}, is a scalar function intended to measure the degree of complexity of a given fluid distribution, and was introduced in [28] for static spherically symmetric configurations. A rigorous definition of complexity has been the goal of many scientists in different branches of sciences, with such interest being motivated by the intuitive idea that complexity should, somehow, measure a basic property describing the structures existing within a system.

Mathematically, the complexity factor describes the trace-free part of the electric Riemann tensor and may be written as (see [28] for details)

$$Y_{TF} = \Pi - \frac{4\pi}{r^3}\int_0^r \tilde{r}^3 \mu' d\tilde{r}, \tag{58}$$

and, accordingly, the vanishing complexity factor condition reads

$$\Pi = \frac{4\pi}{r^3}\int_0^r \tilde{r}^3 \mu' d\tilde{r}, \tag{59}$$

and please notice that the symbol Π here differs from the one in [28] by a factor 8π.

Using (22)–(24) and (59), we are led to the following differential equation for z

$$\begin{aligned} &2\left(1-\frac{2m}{r}\right)\left(z'+z^2\right)-\left(\frac{2}{r}-\frac{5m}{r^2}+\frac{m'}{r}\right)\left(2z-\frac{1}{r}\right)\\ &+\frac{2}{r}-\frac{4m}{r^3}=0. \end{aligned} \tag{60}$$

The first integral of the above equation (for m) reads

$$1-\frac{2m}{r} = e^{\int_r^{r_\Sigma} \frac{4(r^2z'+r^2z^2-2rz+2)}{2r^2z-r}dr}, \tag{61}$$

from which we see that, for any z satisfying (28), we have a model with a vanishing complexity factor. However, we shall follow here a different strategy, and we shall present two models of ghost stars satisfying the vanishing complexity factor condition, by imposing two different additional restrictions.

4.1. A Model with a Given Energy-Density Profile

In order to specify this model, we shall propose the following energy-density profile,

$$8\pi\mu = \frac{1-9(\frac{r}{r_\Sigma})^8}{r^2}, \tag{62}$$

producing for m

$$m = \frac{r}{2}\left[1-\left(\frac{r}{r_\Sigma}\right)^8\right], \tag{63}$$

the reason behind this choice being simply that it allows the integration of (60).

Indeed, feeding back (63) into (60), we may easily integrate this equation for z, obtaining

$$z = \frac{1}{c_1 r^2 - r}, \tag{64}$$

where c_1 is a constant of integration, which according to (28) reads

$$c_1 = \frac{2}{r_\Sigma}. \tag{65}$$

Having obtained the two generators of the solution, we may write for P_r and $P_\perp$

$$8\pi P_r = -\frac{1}{r^2} + \frac{r^6}{r_\Sigma^8}\left(\frac{3 - \frac{2r}{r_\Sigma}}{\frac{2r}{r_\Sigma} - 1}\right), \tag{66}$$

$$8\pi P_\perp = \frac{4r^7}{r_\Sigma^9\left(\frac{2r}{r_\Sigma} - 1\right)}. \tag{67}$$

In this model, the energy density becomes negative for values of r in the interval $[0.76r_\Sigma < r, r = r_\Sigma]$. As in the previous model, the fluid presents a singularity at the origin, which could be surrounded by a cavity bounded by a thin shell.

4.2. Ghost Star with Vanishing Active Gravitational Mass

For this model, we shall additionally assume that the active gravitational (Tolman) mass [31] vanishes.

This last condition implies (see Equation (7.30) in [22])

$$m + 4\pi P_r r^3 = 0. \tag{68}$$

Feeding the above condition into (18) and using (59), we obtain

$$P_r' + \frac{\Pi}{4\pi r} = 0, \tag{69}$$

which can be easily transformed into

$$P_r'' + \frac{4P_r'}{r} + \frac{\mu'}{r} = 0. \tag{70}$$

In order to find a solution to the above equation, we shall split it in two equations, as

$$P_r'' + \frac{3P_r'}{r} = 0, \tag{71}$$

and

$$\frac{P_r'}{r} + \frac{\mu'}{r} = 0, \tag{72}$$

whose solutions reads

$$P_r = b\left(\frac{1}{r^2} - \frac{1}{r_\Sigma^2}\right), \tag{73}$$

and

$$\mu = b\left(\frac{3}{r_\Sigma^2} - \frac{1}{r^2}\right), \tag{74}$$

where boundary condition (27) has been used and b is a constant of integration.

Using (73) in (68), we obtain for the mass

$$m = 4\pi r^3 b\left(\frac{1}{r_\Sigma^2} - \frac{1}{r^2}\right), \tag{75}$$

while using (74) in (59), we obtain for Π

$$\Pi = \frac{8\pi b}{r^2}. \tag{76}$$

In this model, the energy density becomes negative in the interval $0 < r \lessapprox 0.58 r_\Sigma$ (if we assume $b > 0$). As in the previous models, the physical variables exhibit a singular behavior at the center, and any surface delimiting a vacuum cavity surrounding the center would be a thin shell.

5. Discussion

Exploring the possibility of the existence of compact objects endowed with vanishing total mass (energy), we have presented four exact solutions to Einstein equations for static spherical distribution of anisotropic fluids, sharing this property. Such solutions must, within some regions of the distribution, be endowed with negative energy density. Negative energy-density values appear indistinctly in outer or in inner regions, depending on the model, not a universal pattern of distribution having been detected.

Although some of the assumptions adopted to obtain the presented solutions (e.g., the vanishing complexity factor or the conformal flatness) are physically meaningful, the obtained solutions are intended only to illustrate the above-mentioned possibility but not to describe any specific astrophysical scenario. A pending problem regarding this issue consists in finding exact solutions for ghost stars, directly related to relevant astrophysical data.

In the same order of ideas, an important open question concerning ghost stars is related to possible astrophysical observations that could confirm (or dismiss) the existence of this kind of object. We have in mind, for example, a new trend of investigations based on the recent observations of shadow images of the gravitationally collapsed objects at the center of the elliptical galaxy $M87$ and at the center of the Milky Way galaxy by the Event Horizon Telescope (EHT) Collaboration (see [32–35] and references therein). More specifically, we wonder if it could be possible to establish the existence of a ghost star by its shadow.

The solutions we have presented should be considered as the final state of collapsing stars, where quantum effects become relevant during the evolution process. Accordingly, it is of utmost interest to describe the process leading to the final stage with vanishing total mass. To do that, we should find non-static exact solutions describing such a process. Additionally, a detailed description of the mechanism by means of which quantum effects allow negative energy density should be provided. These two problems are out of the scope of this manuscript, but remain among the most relevant questions to solve concerning the physical viability of ghost stars.

Regarding the formation of a ghost star, it should be clear from elementary physical considerations that, as a final product of gravitational collapse, the formation of such configurations must be preceded by an intense radiative process. The problem regarding the efficiency of energy release in gravitational collapse has been discussed by several authors (see [36–38] and references therein). Some of these authors conclude that a 100% efficiency (all the mass is radiated away) is possible under rather mild restrictions [36,38], while others [37] claim that 100% efficiency is forbidden under physically meaningful conditions, among which positive energy density plays a relevant role. Thus, the violation of such a condition, as it happens in our models, is a strong argument to believe that 100% efficiency could be a likely possibility. In such a case, the detection of a strong emission of radiation might indicate the location of a ghost star.

We would like to conclude with five remarks oriented to encourage future research on this issue

- We have explored the possibility of ghost stars within the context of general relativity. It would be interesting to explore such a possibility under some of the extended theories of gravity [39].
- For reasons exposed before, we have considered anisotropic fluids. However, it seems clear that ghost star models described by isotropic fluids should also exist. It could be interesting to find some models of this kind.

- All the models presented here exhibit a singularity at the origin. In order to exclude such a region, we have proposed to surround the center with a vacuum cavity. However, in all examples analyzed, the boundary surface of such a cavity appears to be a thin shell. It would be interesting to find singularity-free solutions and/or singular solutions whose center could be embedded in a vacuum cavity delimited by a regular boundary.
- We would like to insist on the importance of finding exact (analytical or numerical) solutions describing the evolution leading to a ghost star.
- Alternatively, it could be also of interest to find solutions describing the evolution of an initial ghost star leading to a $M > 0$ object, by absorbing radiation. As strange as this scenario might look like (compact object absorbing radiation), it is worth noticing that it has been invoked in the past to explain the origin of gas in quasars [40]. A semi-numerical example for such a model is described in [41].

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