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Some New Results on Stochastic Comparisons of Spacings of Generalized Order Statistics from One and Two Samples

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Abstract: Generalized order statistics (GOSs) are often adopted as a tool for providing a unified approach to several stochastic models dealing with ordered random variables. In this contribution, we first recall various useful results based on the notion of total positivity. Then, some stochastic comparisons between spacings of GOSs from one sample, as well as two samples, are developed under the more general assumptions on the parameters of the model. Specifically, the given results deal with the likelihood ratio order, the hazard rate order and the mean residual life order. Finally, an application is demonstrated for sequential systems.

Keywords: stochastic orders; logconvexity/logconcavity; total positivity; basic composition theorem; generalized order statistics

MSC: 62G30; 60E15

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1. Introduction

In the last three decades, a wide interest has arisen in studying stochastic orderings of order statistics, as well as of further kinds of ordered random variables and their spacings. The concept of generalized order statistics (GOSs) was first studied by Kamps [1,2], aiming to provide a unified approach to several models of ordered random variables. These notions are applied in many branches of statistical theory, with special attention to reliability and life testing. Stochastic comparisons of GOSs and their spacings have been discussed by several scholars (see, for instance, the investigations by Franco et al. [3], Belzunce et al. [4], Hu and Zhuang [5,6], Zhao and Balakrishnan [7], Balakrishnan et al. [8] and Alimohammadi [9]).

Let X and Y be two absolutely continuous nonnegative random variables. We denote, respectively, the cumulative distribution functions (cdf) by F and G, with survival functions (sf) $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, and probability density functions (pdf) f and g. We assume that $F^{-1}(0) = G^{-1}(0)$ (where F^{-1}, as customary, is the right-continuous inverse of F). Moreover, we denote the hazard rate (reversed hazard rate) functions of X and Y as $h_X = f/\bar{F}$ ($\kappa_X = f/F$) and $h_Y = g/\bar{G}$ ($\kappa_Y = g/G$), respectively. Also, let $\mathrm{M}^X(t) = E[X - t|X > t]$ and $\mathrm{M}^Y(t) = E[Y - t|Y > t]$ denote, respectively, the mean residual life functions of X and Y.

Given a set of independent and identically distributed random variables with pdf f and sf $\bar{F}$, we focus on the random variables

$$X_{(r,n,\widetilde{m}_n,k)}, \qquad r = 1, 2, \ldots, n, \tag{1}$$

where $n \in \mathbb{N}$, $k > 0$ and $\widetilde{m}_n = (m_1, \ldots, m_{n-1})$, if $n \geq 2$ ($\widetilde{m}_n \in \mathbb{R}$ is arbitrary, if $n = 1$), and where $m_1, \ldots, m_{n-1} \in \mathbb{R}$ are such that

$$\gamma_{(i,n,\widetilde{m}_n,k)} = k + n - i + \sum_{j=i}^{n-1} m_j \geq 1 \qquad \forall\, i \in \{1, \ldots, n-1\}.$$

We remark that the random variables given in (1) are referred to as GOSs if their joint density function is

$$\mathbf{f}(x_1, \ldots, x_n) = k\Big(\prod_{j=1}^{n-1} \gamma_{(j,n,\widetilde{m}_n,k)}\Big)\Big(\prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i)\Big)[\bar{F}(x_n)]^{k-1} f(x_n),$$

for any $F^{-1}(0) < x_1 \leq x_2 \leq \cdots \leq x_n < F^{-1}(1^-)$. We remark that special choices of parameters k and m_i refer to certain known submodels such as order statistics, record values, sequential order statistics and others. Specifically, for complete information on submodels of ordered random variables and their connection with the GOS model, we refer readers to Section 1 of Kamps [2].

In the rest of the paper, as customary, we use the term increasing (decreasing) meaning for nondecreasing (nonincreasing). Further, ratios and expectations are implicitly assumed to exist whenever used.

Let us now recall some useful notions of stochastic orders. The random variable X is said to be smaller than Y in the following:

- The likelihood ratio order ($X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x;
- The Hazard rate order ($X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x or, equivalently, $h_Y(x) \leq h_X(x)\ \forall x$;
- The reversed hazard rate order ($X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in x or, equivalently, $\kappa_X(x) \leq \kappa_Y(x)\ \forall x$;
- The mean residual life order ($X \leq_{mrl} Y$) if $\bar{G}(t)\int_t^\infty \bar{F}(x)\,dx \leq \bar{F}(t)\int_t^\infty \bar{G}(x)\,dx\ \forall t$ or, equivalently, $\mathrm{M}^X(t) \leq \mathrm{M}^Y(t)\ \forall t$;
- The usual stochastic order ($X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)\ \forall x$.

We recall the following implications (cf. Shaked and Shanthikumar [10]):

$$\begin{array}{ccccc} X \leq_{lr} Y & \Rightarrow & X \leq_{hr} Y & \Rightarrow & X \leq_{mrl} Y \\ \Downarrow & & \Downarrow & & \Downarrow \\ X \leq_{rh} Y & \Rightarrow & X \leq_{st} Y & \Rightarrow & E[X] \leq E[Y] \end{array}$$

A function $\lambda : \mathbb{R} \longmapsto \mathbb{R}_+$ is said logconcave (logconvex) if

$$\lambda(tx + (1-t)y) \geq (\leq)[\lambda(x)]^t[\lambda(y)]^{1-t},$$

for any $x, y \in \mathbb{R}$ and $t \in (0,1)$. An [11] showed that this definition is equivalent to

$$\lambda(y_1)\lambda(y_2) \leq (\geq)\lambda(y_1 + \epsilon)\lambda(y_2 - \epsilon), \tag{2}$$

for all $\epsilon \geq 0$ and $y_1 < y_2$.

Let us recall some aging notions. We say that X is the following:

- ILR (increasing likelihood ratio) if $f(x)$ is logconcave in $x \in \mathbb{R}_+$ or, equivalently, $f(x+\epsilon)/f(x)$ is an decreasing function of x for any $\epsilon > 0$, and, DLR (decreasing likelihood ratio) if $f(x)$ is logconvex in $x \in \mathbb{R}_+$;
- IFR (increasing failure rate) if $\bar{F}(x)$ is logconcave in $x \in \mathbb{R}_+$ or, equivalently, $\bar{F}(x+\epsilon)/\bar{F}(x)$ is an decreasing function of x for any $\epsilon > 0$, and, DFR (decreasing failure rate) if $\bar{F}(x)$ is logconvex in $x \in \mathbb{R}_+$;
- DMRL (decreasing mean residual life) if $\mathrm{M}^X(t)$ is a decreasing function of t, and, IMRL (increasing mean residual life) if $\mathrm{M}^X(t)$ is an increasing function of t.

It is well known that (cf. Barlow and Proschan [12])

$$\text{ILR (DLR)} \quad \Rightarrow \quad \text{IFR (DFR)} \quad \Rightarrow \quad \text{DMRL (IMRL)}.$$

We denote by $Y_{(r,n,\widetilde{m}_n,k)}$, $r = 1, \ldots, n$, the GOSs based on the cdf G. Moreover, the p-spacings of GOSs from F and G are expressed, respectively, for $2 \le r \le n - p + 1$ and $p \ge 1$, as

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} = X_{(r+p-1,n,\widetilde{m}_n,k)} - X_{(r-1,n,\widetilde{m}_n,k)}$$

and

$$W^{(p)}_{(r,n,\widetilde{m}_n,k)} = Y_{(r+p-1,n,\widetilde{m}_n,k)} - Y_{(r-1,n,\widetilde{m}_n,k)}.$$

For $p = 1$, 1-spacings are simple spacings in the literature, written as $V_{(r,n,\widetilde{m}_n,k)}$.

Wide attention has been paid in the literature to results such as stochastic orderings of GOSs and their spacings. However, most of the findings are based on restrictions on the parameters of the model. For instance, in some cases, the following conditions have been considered: $m_1 = m_2 = \cdots = m_{n-1}$ or $\gamma_{(i,n,\widetilde{m}_n,k)} \neq \gamma_{(j,n,\widetilde{m}_n,k)}$ for all $i \neq j$.

The main purpose of this article is not only to remove these restrictions but also to give the results for more general choices of different parameters m_i and m'_i. We remark that the choices of m_i and m'_i would simplify the comparison of submodels of GOSs. In this framework, the possible involved models deal with the following:

- Progressively Type II right censored order statistics with arbitrary censoring schemes;
- Order statistics under multivariate imperfect repair;
- Ordinary order statistics and sequential order statistics;
- Record values and Pfeifer's records, and so on.

For GOSs themselves, in the case of different m_i and m'_i, Franco et al. [3] obtained the results for the $\le_{st}$-order, $\le_{hr}$-order and $\le_{lr}$-order, but under the assumption $m_1 = m_2 = \cdots = m_{n-1}$. Then, Belzunce et al. [4] gave their results without the condition $m_1 = m_2 = \cdots = m_{n-1}$ in the multivariate case.

Now, we discuss about the spacings of GOSs. In the one-sample problem, Alimohammadi et al. [13] proved that if $k' \le k$, $m'_j \le m_i$ for all $i \le j$, $m'_i \ge -1$ $\forall i$ and m_i is decreasing in i, then

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \le_{lr} V^{(p)}_{(r',n',\widetilde{m}'_{n'},k')}, \qquad r \le r',\ n' - r' \le n - r, \tag{3}$$

provided that the following conditions are both satisfied:

(i) $m_i \ge 0$ $\forall i$, and X is DLR;
(ii) $-1 \le m_i < 0$ $\forall i$, X is DLR and h_X is logconvex.

Xie and Hu (2009) showed that if $m_i \ge -1$ $\forall i$ and m_i is decreasing in i, then

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \le_{hr} V^{(p)}_{(r',n',\widetilde{m}_n,k)}, \qquad r \le r',\ n' - r' \le n - r, \tag{4}$$

provided X is DFR.

Xie and Zhuang [14] proved that if $m_i \ge -1$ $\forall i$ and m_i is decreasing in i, then

$$V_{(r,n,\widetilde{m}_n,k)} \le_{mrl} V_{(r',n',\widetilde{m}_n,k)}, \qquad r \le r',\ n' - r' \le n - r, \tag{5}$$

provided $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL.

In the two-sample problem, Hu and Zhuang [6] proved that if $m_1 = \cdots = m_{n-1} = m$, then

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \le_{lr} W^{(p)}_{(r,n,\widetilde{m}_n,k)}, \tag{6}$$

provided that the following conditions are both satisfied:

(i) $m \ge 0$, $X \le_{lr} Y$ and X or Y is DLR;
(ii) $-1 \le m < 0$, $X \le_{hr} Y$, $h_Y(x)/h_X(x)$ is increasing in x and, either X is DLR and h_X is logconvex or Y is DLR and h_Y is logconvex.

Finally, Alimohammadi [9] showed that

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{hr} W^{(p)}_{(r,n,\widetilde{m}_n,k)}, \tag{7}$$

provided that the following conditions are both satisfied:

(i) $m_i \geq 0\ \forall i$, $X \leq_{lr} Y$ and, X or Y is DFR;
(ii) $-1 \leq m_i < 0\ \forall i$, $X \leq_{hr} Y$, $h_Y(x)/h_X(x)$ is increasing in x and, X or Y is DFR.

Paper Organization

This article is organized as follows.

In Section 2, we give preliminaries, including several useful lemmas, as well as recalling the marginal and joint density functions of GOSs and providing certain useful recursive formulas for the involved functions. Then, we list some well-known concepts of total positivity, i.e., the TP_2 and RR_2 notions, as well as the extended basic composition theorem. In Section 3, we give the hr and mrl ordering results in (4) and (5) for different m_i and m_i'. These are concerning the case of a single sample. In Section 4, we establish the lr ordering result in (6) without the restriction $m_1 = \cdots = m_{n-1}$, and dealing with double samples. Each of the Sections 3 and 4 contain new further results. In Section 5, an application of these results is demonstrated for the hazard rate ordering of p-spacings between failures in sequential $(n-r+1)$-out-of-n systems, and involves the Pareto distribution. Finally, in Section 6, some concluding remarks are provided, drawing the motivation, contribution and findings of this study.

2. Preliminaries

The marginal density functions of GOSs have been represented in several ways (see, for instance, Kamps [1,2], Kamps and Cramer [15], and Cramer and Kamps [16]). In particular, Cramer et al. [17] provided the following representation:

$$f_{X_{(r,n,\widetilde{m}_n,k)}}(x) = c_{r-1}[\bar{F}(x)]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1}\xi_{r,\widetilde{m}_r}(F(x))f(x), \qquad x \in \mathbb{R}, \tag{8}$$

where

$$c_{r-1} = \prod_{i=1}^{r}\gamma_{(i,n,\widetilde{m}_n,k)}, \quad r = 1,\ldots,n, \qquad \gamma_{(n,n,\widetilde{m}_n,k)} = k,$$

and $\xi_{r,\widetilde{m}_r}$ is a particular Meijer's G-function. Since we aim to obtain our results for different parameters $\widetilde{m}$ and $\widetilde{m}'$, we define $\tilde{\eta}_{r+i} = (m_{r+1},\ldots,m_{r+i})$ for each $i \geq 1$ ($\tilde{\eta}_{r+i}$ is arbitrary, if $i = 0$) for showing the dependency of the function ψ (below) on these parameters. Concerning the joint pdf of $X_{(r,n,\widetilde{m}_n,k)}$ and $X_{(s,n,\widetilde{m}_n,k)}$, $1 \leq r < s \leq n$, we recall that Tavangar and Asadi [18] gave the following density:

$$\begin{aligned} f_{X_{(r,n,\widetilde{m}_n,k)},X_{(s,n,\widetilde{m}_n,k)}}(x_1,x_2) &= c_{s-1}[\bar{F}(x_1)]^{\gamma_{(r,n,\widetilde{m}_n,k)}-\gamma_{(s,n,\widetilde{m}_n,k)}-1}\xi_{r,\widetilde{m}_r}(F(x_1)) \\ &\times[\bar{F}(x_2)]^{\gamma_{(s,n,\widetilde{m}_n,k)}-1}\psi_{s-r-1,\tilde{\eta}_{s-1}}\left(\frac{\bar{F}(x_2)}{\bar{F}(x_1)}\right)f(x_1)f(x_2), \qquad x_1 < x_2, \end{aligned} \tag{9}$$

(zero elsewhere), where $\psi_{0,\tilde{\eta}_i}(t) = 1$ for any i, $\psi_{1,\tilde{\eta}_{r+1}}(t) = \delta_{m_{r+1}}(1-t)$,

$$\psi_{\alpha,\tilde{\eta}_{r+\alpha}}(t) = \int_t^1\int_{u_{\alpha-1}}^1\cdots\int_{u_2}^1\delta_{m_{r+1}}(1-u_1)\prod_{i=1}^{\alpha-1}u_i^{m_{r+i+1}}du_1\ldots du_{\alpha-2}du_{\alpha-1}, \qquad 0 \leq t \leq 1,$$

with $\alpha = 2,3,\ldots$, and, for $t \in (0,1)$,

$$\delta_m(t) = \begin{cases} \dfrac{1}{m+1}(1-(1-t)^{m+1}), & m \neq -1 \\ -\ln(1-t), & m = -1. \end{cases}$$

It is worth mentioning that the following recursive formulas hold (see Lemmas 2.1 and 3.1 of Alimohammadi and Alamatsaz [19]):

$$\xi_{r,\widetilde{m}_r}(t) = \int_0^t \xi_{r-1,\widetilde{m}_{r-1}}(u)[1-u]^{m_{r-1}}du, \qquad 0 \le t \le 1,\ r = 2,\dots,n, \tag{10}$$

with $\xi_{1,\widetilde{m}_i}(t) = 1$ for any i, and,

$$\psi_{\alpha,\widetilde{\eta}_{r+\alpha}}(t) = \int_t^1 \psi_{\alpha-1,\widetilde{\eta}_{r+\alpha-1}}(u)u^{m_{r+\alpha}}du, \qquad 0 \le t \le 1,\ \alpha = 1,2,\dots \tag{11}$$

with $\psi_{0,\widetilde{\eta}_i}(t) = 1$ for any i. Moreover, Cramer et al. [17] investigated some convexity properties of $\xi_{r,\widetilde{m}_r}$ and GOSs.

For any $u \in \mathbb{R}_+$ such that $\bar{F}(u) > 0$, we write $\bar{F}_u(x) = \bar{F}(x+u)/\bar{F}(u)$, $x \in \mathbb{R}_+$, for the survival function of the residual lifetime. Then, substituting r with $r-1$ and s with $r+p-1$ in (9), after few calculations, for $2 \le r \le n-p+1$, we obtain

$$\begin{aligned} f_{V^{(p)}_{(r,n,\widetilde{m}_n,k)}}(x) &= c_{r+p-2}\int_0^{+\infty}[\bar{F}(x+u)]^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\psi_{p-1,\widetilde{\mu}_{r+p-1}}(\bar{F}_u(x))f(x+u) \\ &\times [\bar{F}(u)]^{\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)du, \qquad x \ge 0, \end{aligned} \tag{12}$$

where $\widetilde{\mu}$ is $\widetilde{\eta}$ for $r-1$ (i.e., $\widetilde{\mu}_{r+i} = (m_r,\dots,m_{r+i-1})$ for any $i \ge 1$ and if $i = 0$, then $\widetilde{\mu}_{r+i}$ is arbitrary), and according to (11) for $r-1$,

$$\psi_{p-1,\widetilde{\mu}_{r+p-1}}(\bar{F}_u(x)) = \int_{\bar{F}_u(x)}^1 \psi_{p-2,\widetilde{\mu}_{r+p-2}}(u)u^{m_{r+p-2}}du, \qquad 2 \le p \le n-r+1, \tag{13}$$

with $\psi_{0,\widetilde{\mu}_i}(t) = 1$ for any i. Also, for $2 \le r \le n-p+1$, from (12) we arrive at

$$\bar{F}_{V^{(p)}_{(r,n,\widetilde{m}_n,k)}}(x) = c_{r+p-2}\int_0^{+\infty}[\bar{F}(x+u)]^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}} \tag{14}$$

$$\begin{aligned} &\times \left[\int_0^1 z^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\psi_{p-1,\widetilde{\mu}_{r+p-1}}(z\bar{F}_u(x))dz\right] \\ &\times [\bar{F}(u)]^{\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)du, \qquad x \ge 0. \end{aligned} \tag{15}$$

Several studies in the past have been based on the following notion of total positivity.

Definition 1 (Karlin [20])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of the real line $\mathbb{R}$. A function $\lambda : \mathcal{A}\times\mathcal{B} \to \mathbb{R}$ such that*

$$\lambda(x_1,y_1)\lambda(x_2,y_2) - \lambda(x_1,y_2)\lambda(x_2,y_1) \ge (\le) 0 \quad \textit{for all } x_1 \le x_2 \textit{ in } \mathcal{A} \textit{ and } y_1 \le y_2 \textit{ in } \mathcal{B}$$

is said to be totally positive of order 2 (TP_2) (reverse regular of order 2 (RR_2)).

Note that the TP_2 (RR_2) property can be equivalently expressed as

$$\frac{\lambda(x_2,y)}{\lambda(x_1,y)} \text{ is increasing (decreasing) in } y \text{ when } x_1 \le x_2\text{, whenever this ratio exists.}$$

Also note that the product of two TP_2 (RR_2) functions is TP_2 (RR_2). Moreover, if $\lambda(x,y)$ is TP_2 (RR_2) in (x,y), then $\lambda_1(x)\lambda(x,y)\lambda_2(y)$ is TP_2 (RR_2) in (x,y) when the functions λ_1 and λ_2 are nonnegative (cf. Karlin [20]).

In the following lemma, we recall the well-known extended basic composition theorem. For part $i.(a)$, see Karlin [20]; for the other parts, see Esna-Ashari et al. [21].

Lemma 1 (Extended basic composition theorem). *Let $\lambda_1 : \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathbb{R}_+$, $\lambda_2 : \mathcal{A} \times \mathcal{B} \times \mathcal{C} \to \mathbb{R}_+$ and $\lambda : \mathcal{A} \times \mathcal{B} \to \mathbb{R}_+$ be Borel-measurable functions satisfying*

$$\lambda(x,y) = \int_{\mathcal{Z}} \lambda_1(x,y,z)\lambda_2(x,y,z)d\mu(z),$$

where μ denotes a sigma-finite measure defined on $\mathcal{C}$.

i. (a) If λ_1 and λ_2 are TP_2 in each pair of variables,
or,
(b) if λ_1 and λ_2 are RR_2 in (y,z) and (x,z), and if λ_1 and λ_2 are TP_2 in (x,y), then λ is TP_2 in (x,y);

ii. (a) If λ_1 and λ_2 are RR_2 in (y,z) and (x,y), and if λ_1 and λ_2 are TP_2 in (x,z),
or,
(b) if λ_1 and λ_2 are RR_2 in (x,y) and (x,z), and if λ_1 and λ_2 are TP_2 in (y,z), then λ is RR_2 in (x,y).

Hereafter, we provide a result that is often adopted for establishing the monotonicity of fractions when the numerator and the denominator are expressed by integrals or by summations (see Misra and van der Meulen [22]).

Lemma 2. *Let Θ be a subset of the real line $\mathbb{R}$, and let Y be a nonnegative random variable whose cdf belongs to the family $\mathcal{P} = \{\Xi(\cdot|\theta), \theta \in \Theta\}$, and assume that*

$$\Xi(\cdot|\theta_1) \leq_{st} (\geq_{st})\Xi(\cdot|\theta_2), \quad \text{whenever } \theta_1 \leq \theta_2, \quad \theta_1, \theta_2 \in \Theta.$$

If $\phi(y,\theta)$ is a real valued function defined on $\mathbb{R} \times \Theta$, which is measurable in y for each θ such that $E_\theta[\phi(Y,\theta)]$ exists, then $E_\theta[\phi(Y,\theta)]$ is

(i) increasing in θ, if $\phi(y,\theta)$ is increasing in θ and increasing (decreasing) in y;
(ii) decreasing in θ, if $\phi(y,\theta)$ is decreasing in θ and decreasing (increasing) in y.

Let us now provide two lemmas that play a crucial role in the following. They are also useful on their own.

Lemma 3.

(i) If m_i is decreasing (increasing) in i, then
(a) $\psi_{p-1,\tilde{\mu}_{r+p-1}}(t)$ is RR_2 (TP_2) in $(r,t) \in \{2,\ldots,n-p+1\} \times (0,1)$ for any $p \geq 2$;
(b) $\xi_{r,\widetilde{m}_r}(t)$ is TP_2 (RR_2) in $(r,t) \in \{2,\ldots,n-p+1\} \times (0,1)$.

(ii) If $m'_i \leq (\geq) m_i$ for all i, then
(a) $\psi_{p-1,\tilde{\mu}'_{r+p}}(t)/\psi_{p-1,\tilde{\mu}_{r+p}}(t)$ is decreasing (increasing) in t for any $p \geq 2$;
(b) $\xi_{r,\widetilde{m}'_r}(t)/\xi_{r,\widetilde{m}_r}(t)$ is increasing (decreasing) in t for any $r \geq 2$,
where $\tilde{\mu}'_{r+p}$ and $\widetilde{m}'_r$ are vectors with elements m'_i.

Proof. We prove the results for the function ψ, while for the function ξ the proof can proceed in an analogous manner.
(i) From (13), we have

$$\psi_{p-1,\tilde{\mu}_{r+p-1}}(t) = \int_{\mathbb{R}} I_{\{0 \leq u \leq t\}} \psi_{p-2,\tilde{\mu}_{r+p-2}}(u) u^{m_{r+p-2}} du,$$

where I_A is the indicator function. Let $p = 2$. If m_i is decreasing (increasing) in i, then $u^{m_{r+p-2}}$ is RR_2 (TP_2) in (r,u). Then, noting that $I_{\{0 \leq u \leq t\}}$ is TP_2 in (u,t), we obtain the desired result by induction and using Lemma 1 (ii.a) (resp. (i.a)).
(ii) For the purpose of this proof, we denote m_i by $\mathfrak{m}_1$ and m'_i by $\mathfrak{m}_2$ for any i. Let $p = 2$. By a similar argument used in part (i), one can see that $\int_{\mathbb{R}} I_{\{0 \leq u \leq t\}} u^{\mathfrak{m}_j} du$ is RR_2 (TP_2) in

$(j,t) \in \{1,2\} \times (0,1)$ provided that $m_i' \leq (\geq) m_i$ for all i and, thus, part (ii) follows by induction. □

Lemma 4. *The function $\psi_{p-1,\tilde{\mu}_{r+p-1}}(zt)$ is RR_2 in $(z,t) \in (0,1) \times (0,1)$ for any $p \geq 2$.*

Proof. From (13), we have

$$\psi_{p-1,\tilde{\mu}_{r+p-1}}(zt) = \int_{\mathbb{R}} I_{\{z \leq u \leq \frac{1}{t}\}} \psi_{p-2,\tilde{\mu}_{r+p-2}}(ut) u^{m_{r+p-2}} t^{m_{r+p-2}+1} du.$$

Let $p = 2$. Since $I_{\{z \leq u \leq \frac{1}{t}\}}$ is TP_2 in (u,z) and RR_2 in (u,t) and (t,z), we have the desired result using induction and Lemma 1 (ii). □

Regarding the conditional GOSs, we now obtain an applicable result.

Lemma 5. *Given a baseline absolutely continuous cdf F, let $X_{(r,n,\widetilde{m}_n,k)}$, $r = 1,\ldots,n$, be the corresponding GOS. The conditional distribution of $X_{(r,n,\widetilde{m}_n,k)}$, given that $X_{(r-1,n,\widetilde{m}_n,k)} = t$, is identical to the distribution of the first GOS obtained from a sample of size $n-r+1$ taken from a population whose distribution is F truncated on the left at t, i.e., $1 - \frac{\bar{F}(x)}{\bar{F}(t)}$, $x \geq t$.*

Proof. From (8) and (9), we have, for $x \geq t$,

$$\begin{aligned} f_{X_{(r,n,\widetilde{m}_n,k)}}(x | X_{(r-1,n,\widetilde{m}_n,k)} = t) &= \frac{f_{X_{(r-1,n,\widetilde{m}_n,k)},X_{(r,n,\widetilde{m}_n,k)}}(t,x)}{f_{X_{(r-1,n,\widetilde{m}_n,k)}}(t)} \\ &= \gamma_{(r,n,\widetilde{m}_n,k)} \left[\frac{\bar{F}(x)}{\bar{F}(t)}\right]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1} \frac{f(x)}{\bar{F}(t)}. \end{aligned} \tag{16}$$

The result follows from (16) by realizing that $\frac{\bar{F}(x)}{\bar{F}(t)}$ and $\frac{f(x)}{\bar{F}(t)}$ are the sf and pdf of the population whose distribution is obtained by truncating the distribution F on the left at t and taking into account the following identity:

$$\gamma_{(r,n,\widetilde{m}_n,k)} = \gamma^*_{(1,n-r+1,\widetilde{m}^*_{n-r+1},k)},$$

where

$$\gamma^*_{(1,n-r+1,\widetilde{m}^*_{n-r+1},k)} = k + (n-r+1) - 1 + \sum_{j=1}^{n-r} m_j{}^*,$$

in which $m_j^* = m_{r+j-1}$, $j = 1,\ldots,n-r$. □

We also need the following technical lemma from Alimohammadi and Alamatsaz [19].

Lemma 6. *The function $\xi_{r,\widetilde{m}_r}(x)$ is logconcave for any $r \geq 2$ provided $m_i \geq 0\ \forall i$.*

The proof of the following lemma is similar to that of Lemma 2.5 in Hu and Wei [23], and hence omitted.

Lemma 7. *Let $\mathrm{M}_{(1,n)}$ be the mean residual life function of the first GOS from n independent observations on F. If F is IMRL (DMRL), then $\mathrm{M}_{(1,n)}(t) \leq (\geq) \mathrm{M}^X(t)/\gamma_{(1,n,\widetilde{m}_n,k)}\ \forall t$.*

At the end of this section, we recall the following two lemmas from Brown and Proschan [24] and Xie and Zhuang [14], respectively.

Lemma 8. *If F is IMRL (DMRL), then $1 - \bar{F}^p$, for $0 < p < 1$, is also IMRL (DMRL).*

Lemma 9. *Let $\mathrm{M}_{(1,n-r+1)}$ be the mean residual life function of the first GOS from $n-r+1$ independent observations on F. If $m_i \geq -1\ \forall i$, then*

(i) $\mathrm{M}_{(1,n-(r-1))}(t) \leq \mathrm{M}_{(1,n-r)}(t)$, $t \in \mathbb{R}_+$;
(ii) $\mathrm{M}_{(1,(n+1)-(r-1))}(t) \leq \mathrm{M}_{(1,n-(r-1))}(t)$, $t \in \mathbb{R}_+$;
(iii) $\mathrm{M}_{(1,n-(r-1))}(t) \leq \mathrm{M}_{(1,(n+1)-r)}(t)$, $t \in \mathbb{R}_+$, *if* $m_n \leq \min\{m_1, \ldots, m_{n-1}\}$.

3. Stochastic Orders from One Sample

We start the section with a completely new result.

Theorem 1. *If X is IFR, then*

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \geq_{hr} V^{(p)}_{(r+1,n+1,\widetilde{m}'_{n+1},k')}$$

provided $k' \geq k$, $m'_j \geq m_i$ *for all* $i \leq j$ *and* m_i *is increasing in* i.

Proof. Let us define

$$\begin{aligned}\phi_2(z,x,u) &= z^{\gamma_{(r+p,n+1,\widetilde{m}'_{n+1},k')}-\gamma_{(r+p-1,n,\widetilde{m}_n,k)}} \frac{\psi_{p-1,\tilde{\mu}'_{r+p}}(z\bar{F}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x))} \\ &= \left(z^{k'-k+\sum_{j=r+p}^{n}(m'_j-m_{j-1})}\right)\left(\frac{\psi_{p-1,\tilde{\mu}_{r+p}}(z\bar{F}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x))}\right)\left(\frac{\psi_{p-1,\tilde{\mu}'_{r+p}}(z\bar{F}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p}}(z\bar{F}_u(x))}\right).\end{aligned}$$

From (14), we have

$$\frac{\bar{F}_{V^{(p)}_{(r+1,n+1,\widetilde{m}'_{n+1},k)}}(x)}{\bar{F}_{V^{(p)}_{(r,n,\widetilde{m}_n,k)}}(x)} = E[\phi_1(U_1,x)],$$

where

$$\begin{aligned}\phi_1(u,x) &\propto [\bar{F}_u(x)]^{\gamma_{(r+p,n+1,\widetilde{m}'_{n+1},k')}-\gamma_{(r+p-1,n,\widetilde{m}_n,k)}}[\bar{F}(u)]^{\gamma_{(r,n+1,\widetilde{m}'_{n+1},k)}-\gamma_{(r-1,n,\widetilde{m}_n,k)}} \\ &\times \frac{\xi_{r,\widetilde{m}'_r}(F(u))}{\xi_{r-1,\widetilde{m}_{r-1}}(F(u))} E[\phi_2(Z,x,u)] \\ &= [\bar{F}_u(x)]^{k'-k+\sum_{j=r+p}^{n}(m'_j-m_{j-1})}[\bar{F}(u)]^{\sum_{j=r}^{n}(m'_j-m_{j-1})} \\ &\times \left(\frac{\xi_{r,\widetilde{m}_r}(F(u))}{\xi_{r-1,\widetilde{m}_{r-1}}(F(u))}\right)\left(\frac{\xi_{r,\widetilde{m}'_r}(F(u))}{\xi_{r,\widetilde{m}_r}(F(u))}\right) E[\phi_2(Z,x,u)], \end{aligned} \tag{17}$$

U_1 and Z are nonnegative random variables whose respective cdfs belong to the families $\mathcal{P}_1 = \{\mathcal{U}_1(\cdot|x), x \in \mathbb{R}_+\}$ and $\mathcal{P}_2 = \{\mathcal{Z}(\cdot|x,u), x,u \in \mathbb{R}_+\}$. The corresponding pdfs are given by

$$\begin{aligned} l_1(u|x) &= c_1(x) I_{\{0\leq u\}}[\bar{F}(x+u)]^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}}\left[\int_0^1 z^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x))dz\right] \\ &\times [\bar{F}(u)]^{\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)\end{aligned}$$

and

$$l_2(z|x,u) = c_2(x,u) I_{\{0\leq z\leq 1\}} z^{\gamma_{(r+p-1,n,\widetilde{m}_n,k)}-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x)),$$

in which

$$c_1(x) = \left[\int_0^{\infty}[\bar{F}(x+u)]^{\gamma(r+p-1,n,\tilde{m}_n,k)}\left[\int_0^1 z^{\gamma(r+p-1,n,\tilde{m}_n,k)-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x))dz\right]\right.$$
$$\left.\times\ [\bar{F}(u)]^{\gamma(r-1,n,\tilde{m}_n,k)-\gamma(r+p-1,n,\tilde{m}_n,k)-1}\xi_{r-1,\tilde{m}_{r-1}}(F(u))f(u)du\right]^{-1}$$

and

$$c_2(x,u) = \left[\int_0^1 z^{\gamma(r+p-1,n,\tilde{m}_n,k)-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x))dz\right]^{-1}$$

are the normalizing constants.

All the terms in parentheses in (17) are decreasing in x and u according to the conditions of the theorem and parts (ib) and (iib) of Lemma 3. Now, we show that $E[\phi_2(Z,x,u)]$ is decreasing in x and u. When X is IFR, $\bar{F}_u(x)$ is decreasing in u. Obviously, it is decreasing in x. So, according to the conditions of theorem and Lemma 3, $\phi_2(z,x,u)$ is increasing in z and decreasing in x and u. For any $x_1 \le x_2$ and $u_1 \le u_2$,

$$\frac{l_2(z|x_2,u)}{l_2(z|x_1,u)} \propto \frac{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x_2))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_u(x_1))}$$

and

$$\frac{l_2(z|x,u_2)}{l_2(z|x,u_1)} \propto \frac{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_{u_2}(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(z\bar{F}_{u_1}(x))}$$

are increasing in z according to Lemma 4 or, equivalently, $\mathcal{Z}(\cdot|x_1,u) \le_{lr} \mathcal{Z}(\cdot|x_2,u)$ and $\mathcal{Z}(\cdot|x,u_1) \le_{lr} \mathcal{Z}(\cdot|x,u_2)$. Thus, $E[\phi_2(Z,x,u)]$ is decreasing in both x and u by part (ii) of Lemma 2. By a similar approach, one can see that $\mathcal{U}_1(\cdot|x_1) \le_{lr} \mathcal{U}_1(\cdot|x_2)$ for any $x_1 \le x_2$. Therefore, by part (ii) of Lemma 2, we have that $E[\phi_1(U,x)]$ is decreasing in x. The proof is thus completed. □

From now on, we extend some existing results to the very flexible cases.

Theorem 2. *If X is DFR, then*

(i) $V^{(p)}_{(r,n,\tilde{m}_n,k)} \le_{hr} V^{(p)}_{(r+1,n,\tilde{m}'_n,k')}$,

(ii) $V^{(p)}_{(r,n+1,\tilde{m}_{n+1},k)} \le_{hr} V^{(p)}_{(r,n,\tilde{m}'_n,k')}$,

(ii) $V^{(p)}_{(r,n,\tilde{m}_n,k)} \le_{hr} V^{(p)}_{(r+1,n+1,\tilde{m}'_{n+1},k')}$,

if $k' \le k$, $m'_j \le m_i$ *for all* $i \le j$, $m'_i \ge -1$ $\forall i$ *and* m_i *is decreasing in* i.

Proof. Let us show the proof of Part (i). The proof of the other two parts can be given in a similar way. First, we set $z = \bar{F}(t)/\bar{F}(x+u)$ in (14). Then, one has

$$\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x) = c_{r+p-2}\int_0^{+\infty}\left[\int_{x+u}^{+\infty}[\bar{F}(t)]^{\gamma(r+p-1,n,\tilde{m}_n,k)-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)f(t)dt\right]$$
$$\times\ [\bar{F}(u)]^{\gamma(r-1,n,\tilde{m}_n,k)-\gamma(r+p-1,n,\tilde{m}_n,k)-1}\xi_{r-1,\tilde{m}_{r-1}}(F(u))f(u)du. \tag{18}$$

Let us define

$$\phi_4(t,x,u) = [\bar{F}(t)]^{\gamma(r+p,n,\tilde{m}'_n,k')-\gamma(r+p-1,n,\tilde{m}_n,k)}\frac{\psi_{p-1,\tilde{\mu}'_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}.$$

Equation (18) gives

$$\frac{\bar{F}_{V^{(p)}_{(r+1,n,\tilde{m}'_n,k)}}(x)}{\bar{F}_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)} = E[\phi_3(U_3,x)],$$

for

$$\begin{aligned}\phi_3(u,x) &= E[\phi_4(T,x,u)][\bar{F}(u)]^{(\gamma_{(r,n,\tilde{m}'_n,k')}-\gamma_{(r+p,n,\tilde{m}'_n,k')})-(\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(r+p-1,n,\tilde{m}_n,k)})} \\ &\quad\times \frac{\xi_{r,\tilde{m}'_r}(F(u))}{\xi_{r-1,\tilde{m}_{r-1}}(F(u))} \end{aligned} \tag{19}$$

$$\begin{aligned} &= \left([\bar{F}(u)]^{(m'_r-m_{r-1})}\right)\left(\frac{\xi_{r,\tilde{m}_r}(F(u))}{\xi_{r-1,\tilde{m}_{r-1}}(F(u))}\right)\left(\frac{\xi_{r,\tilde{m}'_r}(F(u))}{\xi_{r,\tilde{m}_r}(F(u))}\right) \\ &\quad\times \left(E[\phi_4(T,x,u)]\cdot[\bar{F}(u)]^{\sum_{j=r}^{r+p-2}(m'_{j+1}-m_j)}\right). \end{aligned} \tag{20}$$

Moreover, the nonnegative random variables U_3 and T have cdfs belonging, respectively, to the families $\mathcal{P}_3 = \{\mathcal{U}_3(\cdot|x), x \in \mathbb{R}_+\}$ and $\mathcal{P}_4 = \{\mathcal{T}(\cdot|x,u), x,u \in \mathbb{R}_+\}$. The respective pdfs are given by

$$\begin{aligned} l_3(u|x) &= c_3(x)I_{\{0\le u\}}\left[\int_{x+u}^{+\infty}[\bar{F}(t)]^{\gamma_{r+p-1}-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)f(t)dt\right] \\ &\quad\times [\bar{F}(u)]^{\gamma_{r-1}-\gamma_{r+p-1}-1}\xi_{r-1,\tilde{m}_{r-1}}(F(u))f(u) \end{aligned}$$

and

$$l_4(t|x,u) = c_4(x,u)I_{\{x+u\le t\}}[\bar{F}(t)]^{\gamma_{r+p-1}-1}\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)f(t),$$

in which $c_3(x)$ and $c_4(x,u)$ are the normalizing constants. We show that $\phi_1(u,x)$ is increasing in u and x in (20).

The terms in the first three parentheses are increasing in u according to the conditions of theorem and Lemma 3. Now, we show that the term in the last parentheses is increasing in u. We first prove that

$$\frac{\psi_{p-1,\tilde{\mu}'_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}\cdot[\bar{F}(u)]^{\sum_{j=r}^{r+p-2}(m'_{j+1}-m_j)} \tag{21}$$

is increasing in u. This is carried out proceeding by induction on p. For $p=2$, due to (13) we have

$$\frac{\psi_{1,\tilde{\mu}'_{r+2}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{1,\tilde{\mu}_{r+1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}\cdot[\bar{F}(u)]^{(m'_{r+1}-m_r)} = \frac{\int_u^t(\bar{F}(z))^{m'_{r+1}}f(z)dz}{\int_u^t(\bar{F}(z))^{m_r}f(z)dz} = E[\phi_5(Z,t,u)],$$

where

$$\phi_5(z,t,u) = [\bar{F}(z)]^{m'_{r+1}-m_r}.$$

Moreover, the cdf of the nonnegative random variable Z belongs to the family $\mathcal{P}_5 = \{\mathcal{Z}(\cdot|t,u), t,u \in \mathbb{R}_+\}$. The pdf of Z is

$$l_5(z|t,u) = c_5(t,u)I_{\{u\le z\le t\}}[\bar{F}(z)]^{m_r}f(z),$$

with normalizing constant $c_5(t,u)$. From the conditions of the theorem, we have that $\phi_5(z,t,u)$ is increasing in z, and it is constant in u. Noting that $I_{\{u\le z\le t\}}$ is TP_2 in (z,u), we obtain $\mathcal{Z}(\cdot|t,u_1) \le_{lr} \mathcal{Z}(\cdot|t,u_2)$ for $u_1 \le u_2$. Furthermore, part (i) of Lemma 2 yields that $E[\phi_5(Z,t,u)]$ is increasing in u. Applying a similar reasoning, since

$$\frac{\psi_{p-1,\tilde{\mu}'_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}[\bar{F}(u)]^{\sum_{j=r}^{r+p-2}(m'_{j+1}-m_j)} = \frac{\int_u^t \psi_{p-2,\tilde{\mu}'_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)[\bar{F}(u)]^{\sum_{j=r}^{r+p-3} m'_{j+1}}[\bar{F}(z)]^{m'_{r+p-1}}f(z)dz}{\int_u^t \psi_{p-2,\tilde{\mu}'_{r+p-2}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)[\bar{F}(u)]^{\sum_{j=r}^{r+p-3} m_j}[\bar{F}(z)]^{m_{r+p-2}}f(z)dz},$$

the term in (21) is increasing in u by induction.

Noting that X is DFR and that $I_{\{x+u\le t\}}$ is TP_2 in (t,u), we obtain, in a similar way, that $\mathcal{T}(\cdot|x,u_1) \le_{lr} \mathcal{T}(\cdot|x,u_2)$ for $u_1 \le u_2$. Moreover, part (i) of Lemma 2 yields that the term in the third parentheses is increasing in u. Also, one can see that $E[\phi_4(T,x,u)]$ is increasing in x because of

$$\phi_4(t,x,u) = [\bar{F}(t)]^{k'-k-m'_{r+p-1}-1+\sum_{j=r+p-1}^{n-1}(m'_j-m_j)} \cdot \frac{\psi_{p-1,\tilde{\mu}_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{p-1,\tilde{\mu}_{r+p-1}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)} \cdot \frac{\psi_{p-1,\tilde{\mu}'_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}{\psi_{p-1,\tilde{\mu}_{r+p}}\left(\frac{\bar{F}(t)}{\bar{F}(u)}\right)}$$

and $\mathcal{T}(\cdot|x_1,u) \le_{lr} \mathcal{T}(\cdot|x_2,u)$ for $x_1 \le x_2$. Since $\mathcal{U}(\cdot|x_1) \le_{lr} \mathcal{U}(\cdot|x_2)$ for $x_1 \le x_2$, from the part (i) of Lemma 2, we finally obtain that $E[\phi_1(U,x)]$ increases in x. □

Corollary 1. *For a baseline absolutely continuous cdf F, let $X_{(r,n,\widetilde{m}_n,k)}$ and $X_{(r',n',\widetilde{m}'_{n'},k')}$ be the corresponding GOSs. Suppose that $k' \le k$, $m'_j \le m_i$ for all $i \le j$, $m'_i \ge -1$ $\forall i$ and m_i is decreasing in i. If X is DFR, then*

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \le_{hr} V^{(p)}_{(r',n',\widetilde{m}'_{n'},k')}, \qquad r \le r',\ n'-r' \le n-r.$$

Remark 1. *Xie and Hu [25] proved the results in Theorem 2 and Corollary 1 for the special case when $k = k'$ and $m_i = m'_i$.*

Theorem 3. *If $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL, then*

(i) $V_{(r,n,\widetilde{m}_n,k)} \le_{mrl} V_{(r+1,n,\widetilde{m}'_n,k')}$;
(ii) $V_{(r,n+1,\widetilde{m}_{n+1},k)} \le_{mrl} V_{(r,n,\widetilde{m}'_n,k')}$;
(ii) $V_{(r,n,\widetilde{m}_n,k)} \le_{mrl} V_{(r+1,n+1,\widetilde{m}'_{n+1},k')}$;

provided $k' \le k$, $m'_j \le m_i$ for all $i \le j$, $m'_i \ge -1$ $\forall i$ and m_i is decreasing in i.

Proof. Let us prove Part (i). The proof of the other two parts follows a similar line. For convenience, let $\mathbb{M}_{(r,n)}$ denote the mean residual life function of $V_{(r,n,\widetilde{m}_n,k)}$. From (14), we have

$$\begin{aligned}
\mathbb{M}_{(r,n)}(t) &= \frac{\int_t^\infty \bar{F}_{V_{(r,n,\widetilde{m}_n,k)}}(x)dx}{\bar{F}_{V_{(r,n,\widetilde{m}_n,k)}}(t)} \\
&= \frac{\int_t^\infty \int_0^{+\infty}[\bar{F}_u(x)]^{\gamma(r,n,\widetilde{m}_n,k)}[\bar{F}(u)]^{\gamma(r-1,n,\widetilde{m}_n,k)-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)dudx}{\int_0^{+\infty}[\bar{F}_u(t)]^{\gamma(r,n,\widetilde{m}_n,k)}[\bar{F}(u)]^{\gamma(r-1,n,\widetilde{m}_n,k)-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)du} \\
&= \frac{\int_0^{+\infty}\left[\int_t^\infty[\bar{F}_u(x)]^{\gamma(r,n,\widetilde{m}_n,k)}dx\right][\bar{F}(u)]^{\gamma(r-1,n,\widetilde{m}_n,k)-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)du}{\int_0^{+\infty}[\bar{F}_u(t)]^{\gamma(r,n,\widetilde{m}_n,k)}[\bar{F}(u)]^{\gamma(r-1,n,\widetilde{m}_n,k)-1}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u)du} \\
&= E[\phi_5(U_5,t)],
\end{aligned}$$

where

$$\phi_6(u,t) = \frac{\int_t^\infty [\bar{F}_u(x)]^{\gamma(r,n,\widetilde{m}_n,k)} dx}{[\bar{F}_u(t)]^{\gamma(r,n,\widetilde{m}_n,k)}}. \tag{22}$$

Let U_6 be a nonnegative random variable whose cdf belongs to the family $\mathcal{P}_6 = \{\mathcal{U}_6(\cdot|t), t \in \mathbb{R}_+\}$. Its pdf is:

$$l_6(u|t) = c_6(t) I_{\{0\le u\}} [\bar{F}_u(t)]^{\gamma(r,n,\widetilde{m}_n,k)} [\bar{F}(u)]^{\gamma(r-1,n,\widetilde{m}_n,k)-1} \xi_{r-1,\widetilde{m}_{r-1}}(F(u)) f(u),$$

where $c_6(t)$ is the normalizing constant. According to Lemma 5, we have $\phi_6(u,t) = E[\mathrm{M}_{(1,n-r+1)}(t+u)]$ where $\mathrm{M}_{(1,n-r+1)}$ denotes the mean residual life function of the first GOS taken from $n-r+1$ independent observations on F. So, we arrive at

$$\mathbb{M}_{(r,n)}(t) = E[\mathrm{M}_{(1,n-(r-1))}(t+U_6)]$$

and

$$\mathbb{M}_{(r+1,n)}(t) = E[\mathrm{M}_{(1,n-r)}(t+U_7)].$$

Here, U_7 is a nonnegative random variable such that its cdf is a member of the family $\mathcal{P}_7 = \{\mathcal{U}_7(\cdot|t), t \in \mathbb{R}_+\}$. Its pdf is

$$l_7(u|t) = c_7(t) I_{\{0\le u\}} [\bar{F}_u(t)]^{\gamma(r+1,n,\widetilde{m}'_n,k')} [\bar{F}(u)]^{\gamma(r,n,\widetilde{m}'_n,k')-1} \xi_{r,\widetilde{m}'_r}(F(u)) f(u),$$

in which $c_7(t)$ is the normalizing constant. Observe that

$$\frac{l_7(u|t)}{l_6(u|t)} \propto [\bar{F}(u+t)]^{k'-k-m'_r-1+\sum_{j=r}^{n-1}(m'_j-m_j)} [\bar{F}(u)]^{m'_r-m_{r-1}} \cdot \frac{\xi_{r,\widetilde{m}_r}(F(u))}{\xi_{r-1,\widetilde{m}_{r-1}}(F(u))} \cdot \frac{\xi_{r,\widetilde{m}'_r}(F(u))}{\xi_{r,\widetilde{m}_r}(F(u))}$$

is increasing in u according to the conditions of the theorem and Lemma 3. Thus, $\mathcal{U}_6(\cdot|t) \le_{lr} \mathcal{U}_7(\cdot|t)$, which implies that $\mathcal{U}_6(\cdot|t) \le_{st} \mathcal{U}_7(\cdot|t)$. Furthermore, it is easy to see that according to Lemma 8, if $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL then $X_{(1,n',\widetilde{m}_n,k)}$ is IMRL for $n' < n$ provided that $m_i \ge -1$. Therefore,

$$\begin{aligned}
\mathbb{M}_{(r+1,n)}(t) &= E[\mathrm{M}_{(1,n-r)}(t+U_7)] \\
&\ge E[\mathrm{M}_{(1,n-r)}(t+U_6)] \\
&\ge E[\mathrm{M}_{(1,n-(r-1))}(t+U_6)] \\
&= \mathbb{M}_{(r,n)}(t),
\end{aligned}$$

where the last inequality follows from Lemma 9(i). The proof is thus completed. □

Corollary 2. *Given an absolutely continuous cdf F, let $X_{(r,n,\widetilde{m}_n,k)}$ and $X_{(r',n',\widetilde{m}'_{n'},k')}$ be the corresponding GOSs. Suppose $k' \le k$, $m'_j \le m_i$ for all $i \le j$, $m'_i \ge -1$ $\forall i$ and m_i is decreasing in i. If $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL, then*

$$V_{(r,n,\widetilde{m}_n,k)} \le_{mrl} V_{(r',n',\widetilde{m}'_{n'},k')}, \qquad r \le r',\ n'-r' \le n-r.$$

Remark 2. *Xie and Zhuang [14] proved the results in Theorem 3 for the special case when $k = k'$ and $m_i = m'_i$, and also under the assumption that $\gamma_{(i,n,\widetilde{m}_n,k)} \ne \gamma_{(j,n,\widetilde{m}_n,k)}$ for all $i \ne j$.*

4. Stochastic Orders from Two Samples

This section deals with comparisons of GOSs' spacings from two samples. Again, we start the section with a completely new result.

Theorem 4. *Given the absolutely continuous cdfs F and G, let $X_{(r,n,\widetilde{m}_n,k)}$ and $Y_{(r,n,\widetilde{m}_n,k)}$, $r = 1,\ldots,n$, be corresponding GOSs, respectively. Then,*

$$V_{(r,n,\widetilde{m}_n,k)} \geq_{lr} W_{(r,n,\widetilde{m}_n,k)}$$

if $m_i \geq 0\ \forall i$, $X \geq_{lr} Y$, and if X or Y is DLR.

Proof. From (12), we have

$$\frac{f_{W_{(r,n,\widetilde{m}_n,k)}}(x)}{f_{V_{(r,n,\widetilde{m}_n,k)}}(x)} = E[\phi_8(U_8,x)],$$

where

$$\begin{aligned}\phi_8(u,x) &= \left[\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1}\frac{g(x+u)}{f(x+u)}\left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\gamma_{(r-1,n,\widetilde{m}_n,k)}-\gamma_{(r,n,\widetilde{m}_n,k)}-1} \\ &\quad\times\frac{\xi_{r-1,\widetilde{m}_{r-1}}(G(u))}{\xi_{r-1,\widetilde{m}_{r-1}}(F(u))}\frac{g(u)}{f(u)} \\ &= \left(\left[\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1}\frac{g(x+u)}{f(x+u)}\right)\left(\left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m_{r-1}}\frac{g(u)}{f(u)}\right)\left(\frac{\xi_{r-1,\widetilde{m}_{r-1}}(G(u))}{\xi_{r-1,\widetilde{m}_{r-1}}(F(u))}\right).\end{aligned} \tag{23}$$

Here, the nonnegative random variable U_8 possesses the cdf, which is a member of the family $\mathcal{P}_8 = \{\mathcal{U}_8(\cdot|x), x \in \mathbb{R}_+\}$. Moreover, its pdf is

$$l_8(u|x) = c_8(x)I_{\{0\leq u\}}[\bar{F}(x+u)]^{\gamma_{(r,n,\widetilde{m}_n,k)}-1}f(x+u)[\bar{F}(u)]^{m_{r-1}}\xi_{r-1,\widetilde{m}_{r-1}}(F(u))f(u),$$

with normalizing constant $c_8(x)$. The terms in the first two sets of parentheses in (23) are decreasing in u and x according to the conditions of the theorem. For the third set of parentheses, we note that because of $X \geq_{st} Y$ and $X \geq_{rh} Y$, for $x_1 \leq x_2$, we have

$$F(x_2) \leq G(x_2), \qquad \frac{G(x_1)}{F(x_1)} \geq \frac{G(x_2)}{F(x_2)}.$$

So,

$$F(x_1) - F(x_2) \leq \frac{F(x_2)}{G(x_2)}G(x_1) - F(x_2) = \frac{F(x_2)}{G(x_2)}(G(x_1) - G(x_2)) \leq G(x_1) - G(x_2),$$

or, equivalently,

$$F(x_1) + G(x_2) - G(x_1) \leq F(x_2). \tag{24}$$

Now, we prove that $\xi_{r-1,\widetilde{m}_{r-1}}(G(u))/\xi_{r-1,\widetilde{m}_{r-1}}(F(u))$ is decreasing in u or, equivalently, for $x_1 \leq x_2$,

$$\xi_{r-1,\widetilde{m}_{r-1}}(F(x_1))\xi_{r-1,\widetilde{m}_{r-1}}(G(x_2)) \leq \xi_{r-1,\widetilde{m}_{r-1}}(F(x_2))\xi_{r-1,\widetilde{m}_{r-1}}(G(x_1)).$$

According to (2) with $\epsilon = G(x_2) - G(x_1) \geq 0$, $F(x_1) = y_1 < y_2 = G(x_2)$, and Lemma 6, we have

$$\begin{aligned}\xi_{r-1,\widetilde{m}_{r-1}}(F(x_1))\xi_{r-1,\widetilde{m}_{r-1}}(G(x_2)) &\leq \check{\xi}_{r-1,\widetilde{m}_{r-1}}(F(x_1) + G(x_2) - F(x_1))\xi_{r-1,\widetilde{m}_{r-1}}(G(x_1)) \\ &\leq \check{\xi}_{r-1,\widetilde{m}_{r-1}}(F(x_2))\xi_{r-1,\widetilde{m}_{r-1}}(G(x_1)),\end{aligned}$$

where the last inequality follows from (24) because $\xi_{r-1,\tilde{m}_{r-1}}(x)$ is increasing in x. Finally, since X is DLR we have $\mathcal{U}_8(\cdot|x_1) \leq_{lr} \mathcal{U}_8(\cdot|x_2)$ for $x_1 \leq x_2$ and, thus, part (ii) of Lemma 2 implies that $E[\phi_8(U_8, x)]$ is decreasing in x.

Analogously, when Y is DLR, one has that the ratio $f_{V_{(r,n,\tilde{m}_n,k)}}(x)/f_{W_{(r,n,\tilde{m}_n,k)}}(x)$ is increasing in x, due to part (i) of Lemma 2. This completes the proof. □

Theorem 5. *Given the absolutely continuous cdfs F and G, denote by $X_{(r,n,\tilde{m}_n,k)}$ and $Y_{(r,n,\tilde{m}_n,k)}$, $r = 1, \dots, n$, the corresponding GOSs. Then,*

$$V^{(p)}_{(r,n,\tilde{m}_n,k)} \leq_{lr} W^{(p)}_{(r,n,\tilde{m}_n,k)}$$

if the following conditions are both satisfied:

(i) $m_i \geq 0\ \forall i$, *$X \leq_{lr} Y$, and X or Y is DLR;*
(ii) $-1 \leq m_i < 0\ \forall i$, *$X \leq_{hr} Y$, $h_Y(x)/h_X(x)$ is increasing in x and, either X is DLR and h_X is logconvex or Y is DLR and h_Y is logconvex.*

Proof. First assume that X is DLR in condition (i) or X is DLR and h_X is logconvex in condition (ii). From (12), we obtain

$$\frac{f_{W^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)}{f_{V^{(p)}_{(r,n,\tilde{m}_n,k)}}(x)} = E[\phi_9(U_9, x)],$$

for

$$\begin{aligned}
\phi_9(u,x) &= \left[\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right]^{\gamma_{(r+p-1,n,\tilde{m}_n,k)}-1} \frac{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{G}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{F}_u(x))} \frac{g(x+u)}{f(x+u)} \\
&\times \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(r+p-1,n,\tilde{m}_n,k)}-1} \frac{\xi_{r-1,\tilde{m}_{r-1}}(G(u))}{\xi_{r-1,\tilde{m}_{r-1}}(F(u))} \frac{g(u)}{f(u)} \\
&= \left[\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right]^{\gamma_{(r+p-1,n,\tilde{m}_n,k)}-1} \frac{g(x+u)}{f(x+u)} \left(\left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m_{r-1}} \frac{g(u)}{f(u)}\right) \left(\frac{\xi_{r-1,\tilde{m}_{r-1}}(G(u))}{\xi_{r-1,\tilde{m}_{r-1}}(F(u))}\right) \\
&\times \left(\frac{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{G}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{F}_u(x))} \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\sum_{j=r}^{r+p-2}(m_j+1)}\right) \\
&= \left(\left[\frac{\bar{G}(x+u)}{\bar{F}(x+u)}\right]^{\gamma_{(r+p-1,n,\tilde{m}_n,k)}} \frac{h_Y(x+u)}{h_X(x+u)}\right) \left(\left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{m_{r-1}+1} \frac{h_Y(u)}{h_X(u)}\right) \frac{\xi_{r-1,\tilde{m}_{r-1}}(G(u))}{\xi_{r-1,\tilde{m}_{r-1}}(F(u))} \\
&\times \left(\frac{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{G}_u(x))}{\psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{F}_u(x))} \cdot \left[\frac{\bar{G}(u)}{\bar{F}(u)}\right]^{\sum_{j=r}^{r+p-2}(m_j+1)}\right).
\end{aligned}$$

Moreover, U_9 is a nonnegative random variable whose cdf belongs to the family $\mathcal{P}_9 = \{\mathcal{U}_9(\cdot|x), x \in \mathbb{R}_+\}$. Its pdf is

$$\begin{aligned}
l_9(u|x) = c_9(x) I_{\{0 \leq u\}} [\bar{F}(x+u)]^{\gamma_{(r+p-1,n,\tilde{m}_n,k)}-1} \psi_{p-1,\tilde{\mu}_{r+p-1}}(\bar{F}_u(x)) f(x+u) \\
\times [\bar{F}(u)]^{\gamma_{(r-1,n,\tilde{m}_n,k)}-\gamma_{(r+p-1,n,\tilde{m}_n,k)}-1} \xi_{r-1,\tilde{m}_{r-1}}(F(u)) f(u)
\end{aligned}$$

in which $c_9(x)$ is the normalizing constant.

The rest of the proof is omitted for brevity, since it follows the same lines of Theorems 2 and 4. □

Remark 3. *Hu and Zhuang [6] proved the result in Theorem 5 imposing the restrictive condition $m_1 = m_2 = \cdots = m_{n-1}$ into the model.*

Theorem 6. *For the absolutely continuous cdfs F and G, let $X_{(r,n,\widetilde{m}_n,k)}$ and $Y_{(r,n,\widetilde{m}_n,k)}$, $r = 1,\dots,n$, be the respective GOSs. Then,*

$$V_{(r,n,\widetilde{m}_n,k)} \leq_{mrl} W_{(r,n,\widetilde{m}_n,k)}$$

if $X \leq_{mrl} Y$, X is IMRL and Y is DMRL.

Proof. Let $\mathrm{M}^X(\mathrm{M}^Y)$, $\mathrm{M}^X_{(1,n)}(\mathrm{M}^Y_{(1,n)})$ and $\mathbb{M}^X_{(r,n)}(\mathbb{M}^Y_{(r,n)})$ denote the mean residual life functions of $X(Y)$, $X_{(1,n,\widetilde{m}_n,k)}(Y_{(1,n,\widetilde{m}_n,k)})$ and $V_{(r,n,\widetilde{m}_n,k)}(W_{(r,n,\widetilde{m}_n,k)})$, respectively. As shown in Theorem 3, we have

$$\mathbb{M}^X_{(r,n)}(t) = E[\mathrm{M}_{(1,n-r+1)}(t+U_{10})]$$

and

$$\mathbb{M}^Y_{(r,n)}(t) = E[\mathrm{M}_{(1,n-r+1)}(t+U_{11})],$$

where U_{10} and U_{11} are some random variables. By the assumption, there exists a constant M_0 such that $\mathrm{M}^X(u) \leq \mathrm{M}_0 \leq \mathrm{M}^Y(v)$ for all $u,v \geq 0$. Thus, by Lemma 7, we have $\gamma_{(1,n,\widetilde{m}_n,k)}\mathrm{M}^X_{(1,n)}(u) \leq \mathrm{M}_0 \leq \gamma_{(1,n,\widetilde{m}_n,k)}\mathrm{M}^Y_{(1,n)}(v)$ for all $u,v \geq 0$. This implies $\mathbb{M}^X_{(r,n)}(t) \leq \mathbb{M}^Y_{(r,n)}(t)$ for any $t \geq 0$, and hence the result. □

Remark 4. *Hu and Wei [23] and Zhao et al. [26] obtained the result in Theorem 6 for ordinary order statistics and record values, respectively.*

The forthcoming corollary can be obtained by combining Theorem 5 and Equation (3).

Corollary 3. *Given the absolutely continuous cdfs F and G, let $X_{(r,n,\widetilde{m}_n,k)}$ and $Y_{(r',n',\widetilde{m}'_{n'},k')}$ be the corresponding GOSs, respectively. Assume that $k' \leq k$, $m'_j \leq m_i$ for all $1 \leq i \leq j \leq \min\{n-1, n'-1\}$, $m'_i \geq -1\ \forall i$ and m_i is decreasing in i, $r \leq r'$ and $n'-r' \leq n-r$. Then,*

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{lr} W^{(p)}_{(r',n',\widetilde{m}'_{n'},k')}$$

if the following conditions are both satisfied:

(i) $m_i \geq 0\ \forall i$, *$X \leq_{lr} Y$ and X or Y is DLR;*
(ii) $-1 \leq m_i < 0\ \forall i$, *$X \leq_{hr} Y$, $h_Y(x)/h_X(x)$ is increasing in x and, either X is DLR and h_X is logconvex or Y is DLR and h_Y is logconvex.*

In Corollary 3, the DFR instead, then the result established can be weakened from the likelihood ratio order to hazard rate order by combining Corollary 1 and (7).

Corollary 4. *Given the absolutely continuous cdfs F and G, assume that $X_{(r,n,\widetilde{m}_n,k)}$ and $Y_{(r',n',\widetilde{m}'_{n'},k')}$ are the corresponding GOSs, respectively. Suppose $k' \leq k$, $m'_j \leq m_i$ for all $1 \leq i \leq j \leq \min\{n-1, n'-1\}$, $m'_i \geq -1\ \forall i$ and m_i is decreasing in i, $r \leq r'$ and $n'-r' \leq n-r$. Then,*

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{hr} W^{(p)}_{(r',n',\widetilde{m}'_{n'},k')}$$

if the following conditions are both satisfied:

i. $m_i \geq 0\ \forall i$, *$X \leq_{lr} Y$ and, X or Y is DFR;*
ii. $-1 \leq m_i < 0\ \forall i$, *$X \leq_{hr} Y$, $h_Y(x)/h_X(x)$ is increasing in x and, X or Y is DFR.*

The following corollary is an immediate consequence of Theorems 3 and 6.

Corollary 5. *Given the absolutely continuous cdfs F and G, denote by $X_{(r,n,\widetilde{m}_n,k)}$ and $Y_{(r',n',\widetilde{m}'_{n'},k')}$ the corresponding GOSs, respectively. Suppose $k' \leq k$, $m'_j \leq m_i$ for all $1 \leq i \leq j \leq \min\{n-1, n'-1\}$, $m'_i \geq -1$ $\forall i$ and m_i is decreasing in i, $r \leq r'$ and $n'-r' \leq n-r$. Then,*

$$V^{(p)}_{(r,n,\widetilde{m}_n,k)} \leq_{mrl} W^{(p)}_{(r',n',\widetilde{m}'_{n'},k')}$$

provided that $X_{(1,n,\widetilde{m}_n,k)}$ is IMRL, $X \leq_{mrl} Y$, X is IMRL and Y is DMRL.

We conclude this section by pointing out that the condition that $h_Y(x)/h_X(x)$ is increasing in x, given in Corollaries 3 and 4, does define a suitable stochastic order that expresses a concept of relative aging of two probability distributions (see, for instance, Sengupta and Deshpande [27], and Wu and Westling [28], and references therein).

5. An Application in Submodels

As stated before, different choices of m_i and m'_i would enhance the comparisons of submodels of GOSs between themselves, and more generally, between different submodels. In this section, we present the application of Corollary 4 in sequential $(n-r+1)$-out-of-n systems. The applications of other results in further useful submodels can be treated in a similar way.

In a sequential $(n-r+1)$-out-of-n system, which includes the classical $(n-r+1)$-out-of-n system, the sequence of observed components' failure times are named sequential order statistics (SOS). In this model, the system fails after the r-th failure. Hence, the system lifetime is described by the r-th SOS. Right after the failure of the i-th system's component, the lifetimes of the remaining components possess a distribution which is adjusted by a parameter α_i (cf. Cramer and Kamps [29]). This fact is symptomatic of two events: (i) a damage caused by the previous failures, and (ii) a higher load superimposed on the remaining components, which leads to a possible shorter residual life. Furthermore, we note that SOSs under the proportional hazard rates model are included in GOSs (cf. Kamps [1,2]). In this case, given the baseline cdf F and the positive real numbers $\alpha_1, \ldots, \alpha_n$, the family of distribution functions

$$F_i(x) = 1 - (1 - F(x))^{\alpha_i}, \qquad x \in \mathbb{R},$$

leads to the model of GOSs characterized by parameters $k = \alpha_n$, $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$, $1 \leq i \leq n-1$, so that $\gamma_i = (n-i+1)\alpha_i$, $1 \leq i \leq n$. Note that the positions $\alpha_1 = \cdots = \alpha_n = 1$ yield the classical $(n-r+1)$-out-of-n systems.

Now, we discuss the hazard rate ordering of p-spacings between failures in sequential $(n-r+1)$-out-of-n systems. Let $V^{SOS}_{(r,n,\widetilde{\alpha})}$ and $W^{SOS}_{(r,n,\widetilde{\alpha})}$ represent the p-spacings of two sequential systems when the components have the lifetime distributions F and G, respectively. Let us consider the following Pareto distributions for X and Y, respectively:

$$\bar{F}(x) = \begin{cases} 1, & x < 0 \\ \left(1 + \dfrac{x}{\sigma_1}\right)^{-\tau_1}, & x \geq 0, \end{cases} \qquad \bar{G}(x) = \begin{cases} 1, & x < 0 \\ \left(1 + \dfrac{x}{\sigma_2}\right)^{-\tau_2}, & x \geq 0, \end{cases}$$

with $\tau_1 > \tau_2 > 0$ and $\sigma_2 > \sigma_1 > 0$, and the systems parameters $\widetilde{\alpha} = \{1, 1.3, 2.2\}$ and $\widetilde{\alpha}' = \{1, 1.25, 1.7, 1.9\}$.

Now, let us consider the following systems:

- System 1: a sequential 2-out-of-3 system (i.e., $r = 2$ and $n = 3$) with parameters $\widetilde{\alpha}$, in which the initial components have reliability $\bar{F}$;
- System 2: a sequential 2-out-of-4 system (i.e., $r = 3$ and $n = 4$) with parameters $\widetilde{\alpha}'$, in which the initial components have reliability $\bar{F}$;
- System 3: a sequential 2-out-of-4 system (i.e., $r = 3$ and $n = 4$) with parameters $\widetilde{\alpha}'$, in which the initial components have reliability $\bar{G}$.

Then, we have

$$V^{SOS}_{(2,3,\tilde{\alpha})} \leq_{hr} V^{SOS}_{(3,4,\tilde{\alpha}')} \leq_{hr} W^{SOS}_{(3,4,\tilde{\alpha}')},$$

since, for the first inequality, we have $1.9 = k' \leq k = 2.2$, $2 = r \leq r' = 3$, $4 - 3 = n' - r' \leq n - r = 3 - 2$,

$$m_1 = -0.6,\ m_2 = -0.6,$$
$$m'_1 = -0.75,\ m'_2 = -0.65,\ m'_3 = 0.5,$$

and thus $m'_j \leq m_i$ for all $1 \leq i \leq j \leq \min\{3-1, 4-1\}$, $m'_i \geq -1\ \forall i$, and m_i is decreasing in i. Also, for the second inequality, the conditions (i) and (ii) in Corollary 4 are satisfied because both X and Y are DFR, the ratio of pdfs $g(x)/f(x)$ is increasing in x for $\tau_1 > \tau_2$ and $\sigma_2 > \sigma_1$, so that $X \leq_{lr,hr} Y$, and $h_Y(x)/h_X(x)$ is increasing in x for $\sigma_2 > \sigma_1$.

Finally, it is worth mentioning that computing the hazard rate functions of p-spacings between failures in such systems is quite hard in general.

6. Concluding Remarks

This paper has been devoted to ascertain various stochastic comparison involving GOSs and their spacings, based on the main stochastic orders, i.e., the likelihood ratio order, the hazard rate order, the reversed hazard rate order, the mean residual life order, the usual stochastic order. We recall that such stochastic orders are typically adopted to compare two random quantities, such as two random variables or random vectors, in order to establish which one is 'larger', according to eligible criteria (i.e., dispersion, hazard rate, residual lifetimes, concentration and so on). The usefulness of the given results arises also in statistical decision making, where the selection of the 'best' random quantity deserves high interest especially for fulfilling suitable optimization criteria. The ordering relations have been obtained herewith by resorting to typical tools in this framework, such as aging notions for random lifetimes (ILR, IFR, IMRL, as well as their dual notions) and useful concepts of total positivity (TP_2 and RR_2 properties, and the extended basic composition theorem).

It is worth noting the stochastic comparison of GOSs and spacings attracted many investigations, due to their great relevance in statistics and life testing. For instance, a large number of goodness-of-fit tests are based on sample spacings and their transformations. Useful examples of models that attract interest in reliability theory are the spacings of the form $X_{i+1,n} - X_{i,n}$, that may be considered as the additional lifetime to be gained on using a $(n-i)$-out-of-n (ordinary or sequential) system (viewed as two submodels of GOSs) rather than a $(n-i+1)$-out-of-n (ordinary or sequential) system. Furthermore, the literature in this area deals with order statistics with non-integral sample size, record values, censoring schemes and so on (see, e.g., Kamps [2]).

Other examples that illustrate the role of the mentioned stochastic comparisons are connected with the nonhomogeneous Poisson process. Wide attention is given to this counting process, since it arises quite naturally in various applications of probability. For instance, in reliability theory, the repair times of items that are being continuously minimally repaired indeed are the epoch times of a nonhomogeneous Poisson process. Hence, the analysis of records (as a submodel of GOSs) includes the times of the consecutive record values of a sequence of independent and identically distributed nonnegative random variables, viewed as the epoch times of a nonhomogeneous Poisson process. Similarly, the analysis of spacing involves the interepoch times of different nonhomogeneous Poisson processes.

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References

1. Kamps, U. A concept of generalized order statistics. *J. Stat. Plann. Infer.* **1995**, *48*, 1–23. [CrossRef]
2. Kamps, U. *A Concept of Generalized Order Statistics*; Teubner: Stuttgart, Germany, 1995; p. 210.
3. Franco, M.; Ruiz, J.M.; Ruiz, M.C. Stochastic orderings between spacings of generalized order statistics. *Prob. Engin. Inform. Sci.* **2002**, *16*, 471–484. [CrossRef]
4. Belzunce, F.; Mercader, J.A.; Ruiz, J.M. Stochastic comparisons of generalized order statistics. *Prob. Engin. Inform. Sci.* **2005**, *19*, 99–120. [CrossRef]
5. Hu, T.; Zhuang, W. Stochastic properties of p-spacings of generalized order statistics. *Prob. Engin. Inform. Sci.* **2005**, *19*, 257–276. [CrossRef]
6. Hu, T.; Zhuang, W. Stochastic orderings between p-spacings of generalized order statistics from two samples. *Prob. Engin. Inform. Sci.* **2006**, *20*, 465–479. [CrossRef]
7. Zhao, P.; Balakrishnan, N. Stochastic comparisons and properties of conditional generalized order statistics. *J. Stat. Plann. Infer.* **2009**, *139*, 2920–2932. [CrossRef]
8. Balakrishnan, N.; Belzunce, F.; Sordo, M.A.; Suárez-Llorens, A. Increasing directionally convex orderings of random vectors having the same copula, and their use in comparing ordered data. *J. Multivar. Anal.* **2012**, *105*, 45–54. [CrossRef]
9. Alimohammadi, M. Resolving an open problem on the hazard rate ordering of p-spacings. *Prob. Engin. Inform. Sci.* **2023**, *37*, 1020–1028. [CrossRef]
10. Shaked, M.; Shanthikumar, J.G. *Stochastic Orders*; Springer: New York, NY, USA, 2007; p. 473.
11. An, M.H. Logconcavity versus logconvexity: A complete characterization. *J. Econom. Theory* **1998**, *80*, 350–369. [CrossRef]
12. Barlow, R.E.; Proschan, F. *Statistical Theory of Reliability and Life Testing*; Holt, Rinehart and Winston: New York, NY, USA, 1975; p. 290.
13. Alimohammadi, M.; Esna-Ashari, M.; Navarro, J. Likelihood ratio comparisons and logconvexity properties of p-spacings from generalized order statistics. *Prob. Engin. Inform. Sci.* **2023**, *37*, 86–105. [CrossRef]
14. Xie, H.; Zhuang, W. Some new results on ordering of simple spacings of generalized order statistics. *Prob. Engin. Inform. Sci.* **2011**, *25*, 71–81. [CrossRef]
15. Kamps, U.; Cramer, E. On distributions of generalized order statistics. *Statistics* **2001**, *35*, 269–280. [CrossRef]
16. Cramer, E.; Kamps, U. Marginal distributions of sequential and generalized order statistics. *Metrika* **2003**, *58*, 293–310. [CrossRef]
17. Cramer, E.; Kamps, U.; Rychlik, T. Unimodality of uniform generalized order statistics, with applications to mean bounds. *Ann. Inst. Stat. Math.* **2004**, *56*, 183–192. [CrossRef]
18. Tavangar, M.; Asadi, M. Some unified characterization results on the generalized Pareto distributions based on generalized order statistics. *Metrika* **2012**, *75*, 997–1007. [CrossRef]
19. Alimohammadi, M.; Alamatsaz, M.H. Some new results on unimodality of generalized order statistics and their spacings. *Stat. Prob. Lett.* **2011**, *81*, 1677–1682. [CrossRef]
20. Karlin, S. *Total Positivity*; Stanford University Press: Stanford, CA, USA, 1968; p. 576.
21. Esna-Ashari, M.; Alimohammadi, M.; Cramer, E. Some new results on likelihood ratio ordering and aging properties of generalized order statistics. *Comm. Stat. Theory Meth.* **2022**, *51*, 4667–4691. [CrossRef]
22. Misra, N.; van der Meulen, E.C. On stochastic properties of m-spacings. *J. Stat. Plann. Infer.* **2003**, *115*, 683–697. [CrossRef]
23. Hu, T.; Wei, Y. Stochastic comparisons of spacings from restricted families of distributions. *Stat. Prob. Lett.* **2001**, *53*, 91–99. [CrossRef]
24. Brown, M.; Proschan, F. Imperfect repair. *J. Appl. Prob.* **1983**, *20*, 851–859. [CrossRef]
25. Xie, H.; Hu, T. Ordering p-spacings of generalized order statistics revisited. *Prob. Engin. Inform. Sci.* **2009**, *23*, 1–16. [CrossRef]
26. Zhao, P.; Li, X.; Li, Z.; Xu, M. Stochastic comparisons of spacings of record values from one or two sample sequences. *Statistics* **2008**, *42*, 167–177. [CrossRef]
27. Sengupta, D.; Deshpande, J.V. Some results on the relative ageing of two life distributions. *J. Appl. Prob.* **1994**, *31*, 991–1003. [CrossRef]

28. Wu, Y.; Westling, T. Nonparametric inference under a monotone hazard ratio order. *Electron. J. Stat.* **2023**, *17*, 3181–3225. [CrossRef]
29. Cramer, E.; Kamps, U. Sequential order statistics and k-out-of-n systems with sequentially adjusted failure rates. *Ann. Inst. Stat. Math.* **1996**, *48*, 535–549. [CrossRef]