mathematics

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Two-Dimensional System of Moment Equations and Macroscopic Boundary Conditions Depending on the Velocity of Movement and the Surface Temperature of a Body Moving in Fluid

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Abstract: This article is dedicated to the derivation of a two-dimensional system of moment equations depending on the velocity of movement and the surface temperature of a body submerged in fluid, and macroscopic boundary conditions for the system of moment equations approximating the Maxwell microscopic boundary condition for the particle distribution function. The initial-boundary value problem for the Boltzmann equation with the Maxwell microscopic boundary condition is approximated by a corresponding problem for the system of moment equations with macroscopic boundary conditions. The number of moment equations and the number of macroscopic boundary conditions are interconnected and depend on the parity of the approximation of the system of moment equations. The setting of the initial-boundary value problem for a non-stationary, nonlinear two-dimensional system of moment equations in the first approximation with macroscopic boundary conditions is presented, and the solvability of the above-mentioned problem in the space of functions continuous in time and square-integrable in spatial variables is proven.

Keywords: Boltzmann equation; two-dimensional system of moment equations; Maxwell microscopic boundary condition; macroscopic boundary conditions

MSC: 35Q20

Citation: Sakabekov, A.; Auzhani, Y.; Akimzhanova, S. Two-Dimensional System of Moment Equations and Macroscopic Boundary Conditions Depending on the Velocity of Movement and the Surface Temperature of a Body Moving in Fluid. *Mathematics* **2024**, *12*, 1491. https://doi.org/10.3390/math12101491

Academic Editor: James M. Buick

Received: 28 February 2024
Revised: 12 April 2024
Accepted: 23 April 2024
Published: 10 May 2024

1. Introduction

One of the most important tasks in hydrodynamics is studying the movement of solid bodies in fluids, particularly the forces that the medium exerts on the moving body. This problem has become increasingly significant due to the increasing speed of sea vessels and submarines. Determining the speed, surface temperature, density, flow velocity, and pressure of a body moving in a fluid flow is a crucial hydrodynamic task. Hydrodynamic characteristics serve as the foundation for predicting the navigability, seaworthiness, and maneuverability of marine objects. Unfortunately, there are no currently universally accepted calculation methods for assessing hydrodynamic characteristics. The primary method for determining them is still model experimentation, which can be costly. An increasingly popular alternative is numerical modeling methods. Utilizing computer modeling in the early stages of design enables the exploration of various design solutions and the assessment of control effectiveness and maneuvering capabilities. Therefore, the development and enhancement of reliable numerical methods for predicting hydrodynamic characteristics is a pressing and important issue in fluid mechanics.

The primary tool for describing the movement of gases and fluids is the single-particle distribution function, which satisfies the Boltzmann equation [1]. Applying the Boltzmann equation to calculate rarefied gas flows around aircraft or fluid flows around a solid body moving in a fluid flow involves solving this equation under appropriate boundary conditions. Defining boundary conditions on surfaces swept by rarefied gas is one of the most crucial issues in kinetic gas theory. In high-altitude aerodynamics, the interaction of gas with the surface of the swept body plays a significant role [2]. The aerothermodynamic characteristics of bodies in a gas flow are determined by the transfer of momentum and energy to the surface of the body, which involves the connection between the velocities and energies of molecules falling on the surface and molecules reflected from it. This connection is the essence of kinetic boundary conditions on the surface. In the case of gas or fluid flow around a moving solid body, boundary conditions are established in the form of a relationship between particles falling on the boundary and particles reflected from the boundary. The Maxwell boundary condition more accurately describes the interaction of gas molecules with the surface when solving specific tasks. It is worth nothing that in Ref. [3], two new models of boundary conditions were proposed: diffusive-moment and specular-moment, which generalize the well-known boundary conditions of Cercignani. Additionally, aerodynamic characteristics of spacecraft were studied using the direct statistical simulation method (Monte Carlo method), and various models of gas interaction with the surface and their impact on aerodynamic characteristics were analyzed in Ref. [4].

The conservation laws of mass, momentum, and energy are satisfied by the Boltzmann equation, resulting in five equations related to thirteen hydrodynamic parameters. To close the system of equations, the stress tensor and heat flux are expressed in terms of velocity components, density, temperature, and their derivatives. The heat flux is proportional to the temperature gradient, while the stress tensor is defined by the deformation velocity tensor. This approach leads to the derivation of Euler, Navier–Stokes, and Fourier equations. However, the linear relationship between the stress tensor, heat flux, and gradients of hydrodynamic quantities is only valid for flows with small Knudsen numbers. In more general cases, the flow cannot be fully described using only hydrodynamic quantities, and the system of five equations cannot be closed. Therefore, additional functions must be introduced to describe the flow and derive equations that meet the required conditions. One method that allows for obtaining a closed system of macroscopic equations at any Knudsen number is the moment method. The objective of this work is to derive a new two-dimensional nonstationary nonlinear system of moment equations that depend on the speed and temperature of a moving body's surface, along with macroscopic boundary conditions on the moving boundary, considering the temperature of the surface of the moving body. The new two-dimensional system of moment equations under appropriate macroscopic boundary conditions, will be used to determine the velocity, surface temperature of a body moving in a fluid flow, and fluid parameters. This process of determining fluid parameters such as density, fluid flow velocity, pressure, speed of movement, and surface temperature of a body moving in a fluid flow represents an inverse problem. To solve this inverse problem, a numerical experiment will be conducted.

Moment methods differ from each other based on the choice of different systems of basis functions. For example, Grad [5,6], in obtaining a moment system for the Boltzmann equation, decomposed the particle distribution function in terms of Hermite polynomials around the local Maxwell distribution. Grad used Cartesian velocity coordinates, and his moment system included unknown hydrodynamic characteristics of the gas such as density, temperature, and average velocity. These boundary conditions are dependent on the type and nature of the boundary conditions for the Boltzmann equation, making them a global problem in rarefied gas dynamics. For instance, the boundary conditions for Grad's 13 and 20 moment equations have not yet been definitively resolved. The properties of these systems depend on unknown parameters like gas density and average gas velocity, making formulating boundary conditions a complex task. In a different approach, in [7,8], we derived a moment system distinct from Grad's system of equations. We utilized spherical velocity coordinates and decomposed the distribution function into a series using the eigenfunctions of the linearized collision operator [1–9], a product of Sonine polynomials and spherical functions. The coefficients of decomposition, which are the moments of the distribution function, were defined differently than Grad's method. The resulting equation system, known as the Boltzmann moment equation system, corresponding to a partial sum of the series, is a nonlinear hyperbolic system concerning the moments of the particle distribution function. The differential part of this system is linear, with nonlinearity introduced as quadratic forms of the distribution function moments. The quadratic forms representing moments of the nonlinear collision integral were computed in Ref. [10] and are expressed through Talmi coefficients [11] and Clebsch–Gordan coefficients [12]. It is assumed in [5–7] that gas movement takes place within a bounded region with a stationary boundary.

In [13,14], moment systems for the spatially homogeneous Boltzmann equation and conditions for the representability of the solution of the spatially homogeneous Boltzmann equation in the form of a Poincaré series were obtained. It is worth noting that the method proposed in [13] (applying the Fourier transform to the velocity variable in the isotropic case) greatly simplified the collision integral and therefore the calculation of moments from the collision integral. Ref. [14] generalized the result of Ref. [13] for the case of anisotropic scattering. In the case of a spatially homogeneous Boltzmann equation, the moment equations are represented by a system of ordinary differential equations and the problem of boundary conditions does not arise.

In [15], the derivation of a systematic unperturbed hierarchy of a closed system of moment equations corresponding to classical theory is presented. The first member of the hierarchy is the Euler system of equations, which is based on Maxwellian distributions of velocities, while the second closure is based on non-isotropic Gaussian distributions of velocities. The closure procedure consists of two stages. The first stage ensures that each member of the hierarchy is hyperbolic, has entropy, and possesses realizability of its predicted moments. The second step involves modifying the collisional members, which is a nonlinear generalization of Grad's "diagonal approximation" and guarantees that members of the hierarchy beyond Gaussian closure recover the correct behavior of the Navier–Stokes equations. This article is a fundamental work describing a closed system of moment equations in the transitional regime.

Ref. [16] is dedicated to approximations of the Boltzmann equation using the moment method. The problem setting of moment closure with relative entropy is generalized to $\varphi\varphi$-divergence and the corresponding closure is based on minimizing $\varphi\varphi$-divergences. The proposed description includes, as particular cases, the classical Grad closure based on Hermite polynomial decomposition and Levermore's closure based on entropy. It has been established that generalizing closures based on divergence allows for the construction of extended thermodynamic theories that avoid significant limitations of standard moment closure formulations, such as the inadmissibility of the approximated distribution in phase space, potential loss of hyperbolicity, and singularity of flow functions in local equilibrium. Closure based on divergence leads to a hierarchy of treatable symmetric hyperbolic systems that preserve the fundamental structural properties of the Boltzmann equation.

In Ref. [17], a new and improved computational algorithm is developed, playing a crucial role in the moment method for solving the Boltzmann equation. This algorithm is founded on the principle of the collision integral's invariance concerning the selected base system of functions utilized for decomposing the distribution function. The relationships among matrix elements of the interaction matrix are methodically explored and analyzed. Additionally, recurrent relationships for matrix elements in the axially symmetric scenario are derived.

Ref. [18] aims to enhance the convergence of the moment method by introducing filtered moments of hyperbolic equations. These filtered moments do not result in additional computational costs while effectively reducing errors. The filter approach is developed through a thorough examination of averaged solutions from two neighboring moment systems. This averaging is reformulated using the artificial collision method, leading to the creation of a filter. The authors analyze the properties of the filter and provide numerical examples of one-dimensional tasks to demonstrate the improved quality of the new filtered moment method.

In Ref. [19], the development of continuum models for describing processes in gases where particle collisions cannot maintain thermal equilibrium is discussed. Such a situation typically occurs in rarefied or diluted gases for flows in microscopic conditions or whenever the Knudsen number becomes significant. Continuum models are based on a stochastic description of gas by the Boltzmann equation in kinetic gas theory. With moment approximations, extended hydrodynamics equations such as the regularized 13-moment equations can be obtained. The moment equations are detailed, and typical results for channel flow, cavity flow, and sphere flow under low Mach conditions, for which evolutionary equations and boundary conditions are well established, are considered. Conversely, nonlinear high-speed processes require special closures that are still under development. Modern approaches are considered, as well as the problem of calculating shock wave profiles based on continuum equations.

In Ref. [20], a globally hyperbolic regularization of the general Grad moment system in multidimensional spaces is proposed. Systems with moments up to an arbitrary order are studied. Characteristic velocities of the regularized moment system can be analytically determined and depend only on macroscopic velocity and temperature. The fully detailed structure of eigenvalues and eigenvectors of the coefficient matrix is elucidated. Furthermore, it is demonstrated that all characteristic waves are either nonlinear or linearly degenerate. The study also includes an analysis of the properties of rarefaction waves, contact discontinuities, and shock waves.

Grad proposed the Hermite series decomposition for approximating solutions to kinetic equations that have an unbounded velocity space [5]. However, for initial-boundary value problems, ill-posed boundary conditions lead to instability in the Hermite decomposition, resulting in non-convergent solutions. For linear kinetic equations, a method for establishing stable boundary conditions for (formally) Hermite approximations of any order was recently introduced. In [21], the L2-convergence of these stable Hermite approximations is examined, and explicit convergence rates are demonstrated under appropriate assumptions regarding the regularity of the exact solution. The convergence rates presented are validated through numerical experiments using the linearized BGK equation for rarefied gas dynamics.

Fluid dynamic modeling is often carried out using inexpensive macroscopic models such as the Euler equations. However, for rarefied gases in conditions close to equilibrium, macroscopic models are not accurate enough, and modeling using more precise microscopic models can be costly. In [22], a hierarchical micro–macro acceleration based on moment models is introduced, combining the speed of macroscopic models with the precision of microscopic models. The hierarchical micro–macro acceleration follows a flexible four-step procedure including a micro step, a restriction step, a macro step, and a matching stage. Several new micro–macro methods are developed from this and compared with existing methods. In test cases for 1D and 2D scenarios, the new methods demonstrate high accuracy and significant acceleration.

In [23], a foundation is presented for building interpretable and truly reliable reduced models for multiscale tasks in situations without scale separation. The hydrodynamic approximation of the kinetic equation is used as an example to illustrate the main stages and arising challenges. For this purpose, a set of generalized moments that optimally represent the main velocity distribution is first constructed. Then, the well-known closure problem is solved with the goal of best reflecting the dynamics associated with the kinetic equation. The problem of physical constraints, such as Galilean invariance, is addressed, and an active learning procedure is introduced to help ensure that the dataset used is sufficiently representative. The resulting system takes the form of a usual system of moments and operates independently of the numerical discretization used. Numerical results for the BGK (Bhatnagar–Gross–Krook) model and binary collision of Maxwell molecules are presented.

It is shown that the reduced model provides consistent accuracy across a wide range of Knudsen numbers from the hydrodynamic limit to free molecular flow.

Ref. [24] is dedicated to boundary conditions for the linearized hyperbolic system of moment equations of the one-dimensional Boltzmann equation. It is shown that even when the usual conditions of relaxation stability and the Kreiss condition are met, there exists an exponentially growing solution to the initial-boundary value problem for the moment system of equations. To clarify this issue, the generalized Kreiss condition for hyperbolic relaxation systems was checked. Using energy estimates, the stability of the moment system is proven when the generalized Kreiss condition is met. Furthermore, within the framework of the generalized Kreiss condition, reduced boundary conditions for the corresponding equilibrium system are derived. Numerical results confirm the convergence of the moment system solution to the solution of the equilibrium system with the obtained boundary conditions in the relaxation limit. Special attention should be paid to imposing boundary conditions for moment systems to observe the generalized Kreiss condition to ensure the relaxation limit of the initial-boundary value problem.

This paper presents the formulation of the initial-boundary value problem for the two-dimensional Boltzmann equation under microscopic Maxwell conditions. The application of the two-dimensional Boltzmann equation to calculate fluid flows around a solid body moving in a fluid flow under microscopic Maxwell conditions is studied for the first time. We will approximate the initial-boundary value problem for the two-dimensional nonstationary Boltzmann equation, considering the speed of the bodies moving in a fluid under Maxwell conditions on a moving boundary by the corresponding problem for a system of moment equations. We present the derivation of a new two-dimensional nonstationary nonlinear system of moment equations, dependent on the speed of movement, the temperature of the surface of a moving body, and the approximation of the microscopic Maxwell condition on the moving boundary. We show that the number of moment equations and macroscopic boundary conditions depend on the oddness and evenness of the approximation of the number of moment equations. We provide a method for closing the finite system of moment equations, resolving the closure of the moment system of equations individually for each specific approximation of the moment system of equations. In Section 2, we state the initial-boundary value problem for the system of moment equations in the first and second approximations under the macroscopic boundary conditions. We demonstrate the correctness of the initial-boundary value problem for a system of moment equations in the first approximation under the derived macroscopic boundary conditions.

The derived system depends on the speed of movement and the surface temperature of the moving body. Macroscopic boundary conditions for the moment system depend on the body's surface temperature. Thus, the system of moment equations under macroscopic boundary conditions allows us to determine the speed and surface temperature of a body moving in a fluid flow, as well as fluid parameters.

2. Materials and Methods

In the first section of this work, the formulation of the initial-boundary value problem for the two-dimensional Boltzmann equation under Maxwell's microscopic conditions is presented. The Boltzmann equation includes a term dependent on the velocity of a body moving in a fluid, and the Maxwell condition incorporates a parameter dependent on the body's surface temperature. The moment method is used to obtain an approximate solution of the initial-boundary value problem for the two-dimensional Boltzmann equation under Maxwell's microscopic conditions. A new two-dimensional non-stationary nonlinear system of moment equations, dependent on the velocity and surface temperature of the moving body, and an approximation of Maxwell's microscopic boundary condition on the moving boundary, are derived. This approximation considers some molecules reflecting off the boundary specularly and others diffusely with a Maxwellian distribution. Macroscopic boundary conditions for the system of moment equations establish a connection between the moments of the particle distribution function falling on the boundary and those reflected from it. The number of moment equations and the quantity of macroscopic boundary conditions depend on the oddness and evenness of the approximation of the moment equations system. In the second section, the setting of the initial-boundary value problem for the system of moment equations in the first and second approximations under new macroscopic boundary conditions is provided. The difference in the boundary condition settings for the system of equations in the first and second approximations is demonstrated. The correctness of the initial-boundary value problem for the system of moment equations in the first approximation with the derived macroscopic boundary conditions is proven more precisely, demonstrating the existence of a unique solution to the aforementioned problem in the space of functions $C\big([0,T];L^2(G)\big)$, where G is a rectangle. The method of a priori estimation is used to prove the existence and uniqueness of the solution of the initial-boundary value problem for the system of moment equations in the first approximation.

3. Results

3.1. Derivation of the Non-Stationary Two-Dimensional System of Moment Equations and Approximation of Maxwell's Microscopic Boundary Condition

Accurate modeling and simulation of nonequilibrium processes in rarefied fluids or microflows represents one of the main challenges in modern fluid mechanics. Traditional models, developed centuries ago, lose their applicability in extreme physical situations. In these classical models, nonequilibrium variables such as the stress tensor and heat flux are combined with gradients of velocity and temperature as defined in the Navier–Stokes and Fourier (NSF) relations. These relations are valid and close to equilibrium; however, in rarefied fluids or microflows, particle collisions are insufficient to maintain equilibrium. Far from equilibrium, inertia and multiscale relaxation of higher orders in fluids becomes dominant and significant.

The main scaling parameter in kinetic theory is the Knudsen number, Kn, calculated as the ratio of the mean free path between collisions, λ, and a macroscopic length, L, such that $Kn = \lambda/L$. At $Kn = 0$, full equilibrium described by the non-dissipative Euler equations is assumed. In many processes, the Navier–Stokes and Fourier gas dynamics fail at $Kn \approx 0.01$ and sometimes at even smaller values [25].

At weak nonequilibrium, i.e., in the regime of gas flow as a continuum, it is possible to describe the flow using macroscopic equations that properly reflect the dynamic regime of the fluid when the mean free path is much smaller than the macroscopic length scales. These macroscopic equations are consequences of solving the Boltzmann equation outside thin Knudsen boundary layers and initial layers, i.e., as a result of solving the Boltzmann equation in the form of a normal Hilbert series (Chapman–Enskog method).

The Boltzmann equation describes gas flow at any value of the Knudsen number, particularly at values of the Knudsen number $10^{-3} < Kn < 10^{-1}$, where the gas flow can be considered a continuum, i.e., a flow of a continuous medium. When calculating the hydrodynamic characteristics of a body submerged in fluid, a term dependent on the velocity of the moving body must be introduced into the Boltzmann equation. Additionally, the condition on the moving boundary must contain a parameter dependent on the temperature of the moving body's surface. To calculate the flow of a continuous medium around a submerged body one must solve the Boltzmann equation under appropriate boundary conditions.

Hydrodynamic characteristics form the basis for predicting the navigability, seaworthiness, and maneuverability of marine vessels. Unfortunately, there are currently no universally accepted calculation methods for assessing these characteristics. The primary method for determining them is through model experimentation, which can be costly. An increasingly popular alternative is the use of numerical modeling methods. Utilizing computer modeling in the early stages of design enables the exploration of variable design solutions and the assessment of actors such as the effectiveness of control mechanisms and maneuvering capabilities. Therefore, the development and enhancement of reliable numerical methods for predicting hydrodynamic characteristics is a crucial and current challenge in fluid mechanics.

When calculating the hydrodynamic characteristics of a body moving in fluid or the aerodynamic characteristics of an aircraft in a high-speed flow of rarefied gas, a term dependent on the velocity of the body or aircraft movement must be introduced into the Boltzmann equation. Additionally, the condition on the moving boundary must include a parameter dependent on the surface temperature of the body or aircraft. To analyze the hydrodynamic characteristics of the fluid and the velocity of movement of the body submerged in it, one can use the complete integro-differential Boltzmann equation:

$$\frac{\partial f}{\partial t} + \left(c, \frac{\partial f}{\partial x}\right) + \left(U, \frac{\partial f}{\partial x}\right) = J(f, f) \tag{1}$$

where $f = f(t, x, c)$ is the particle distribution function in space over time and velocities, $c = v - U$ is the relative speed, $U = (U_1, U_2, U_3)$ is the velocity of the body moving in the fluid, and $J(f, f)$ is the collision integral. Assuming the particle distribution function is even in c_2, i.e., $f(t, x, c_1, -c_2, c_3) = f(t, x, c_1, c_2, c_3)$, Equation (1) becomes the following equation:

$$\frac{\partial f}{\partial t} + c_1 \frac{\partial f}{\partial x_1} + c_3 \frac{\partial f}{\partial x_3} + U_1 \frac{\partial f}{\partial x_1} + U_3 \frac{\partial f}{\partial x_3} = J(f, f), c = (c_1, c_2, c_3) \in R_3^c$$

Problem 1. *Find a solution to the following initial-boundary value problem for the Boltzmann equation [1,9]:*

$$\begin{gathered} \frac{\partial f}{\partial t} + c_1 \frac{\partial f}{\partial x_1} + c_3 \frac{\partial f}{\partial x_3} + U_1 \frac{\partial f}{\partial x_1} + U_3 \frac{\partial f}{\partial x_3} = J(f, f), t \in (0, T], \\ x = (x_1, x_3) \in G, c \in R_3^c, \end{gathered} \tag{2}$$

$$f|_{t=0} = f^0(x, c), (x, c) \in G \times R_3^c, \tag{3}$$

$$\begin{gathered} f(t, x_{\partial G}, c) = \beta f(t, x_{\partial G}, c - 2(c, n_{\partial G}) n_{\partial G}) + (1 - \beta) exp\left(-c^2/(2R)\right), \\ (n_{\partial G}, c) > 0, x_{\partial G} \in \partial G, \end{gathered} \tag{4}$$

where $f \equiv f(t, x, c)$ *is the particle distribution function in space by velocity and time,* $f^0 \equiv f^0(x, c)$ *is the particle distribution at the initial moment in time (a given function),* $J(f, f) \equiv \int \left[f(c')f(c_1') - f(c)f(c_1)\right]\sigma(cos\chi)dc_1 dn$ *is a nonlinear collision operator written for Maxwellian molecules, G isa bounded area in two-dimensional space,* $n_{\partial G}$ *is the external unit normal vector of the boundary* ∂G *of area G, and* $c = v - U$.

Condition (4) is the microscopic boundary condition of Maxwell for the Boltzmann equation. According to Condition (4), a certain part of the incoming particles is specularly reflected, while the other particles are absorbed by the wall and then emitted with a Maxwellian distribution corresponding to the wall temperature Θ. $\alpha^2 = \frac{1}{R\Theta}$ is also a function of time and coordinates, and R is the Boltzmann constant. The parameter β belongs to the interval (0, 1), where $\beta = 1$ corresponds to purely specular reflection from the boundary.

Formula (4) is written under the assumption that the boundary (wall or surface) moves at a speed $U = (U_1, U_3)$. The Equations (2)–(4) are written in the coordinate system associated with the moving wall, where the speed of movement is a function of time and coordinates, i.e., $U = U(t, x)$. The Boltzmann equation is a complex nonlinear integro-differential equation for which it is impossible to construct an exact analytical solution. Various methods are therefore applied to construct an approximate solution to the Boltzmann equation. By expanding the particle distribution function in a Fourier series around the Maxwellian distribution over a complete orthogonal system of functions in the space R_3^c and substituting it into the Boltzmann equation and then integrating over the velocity space, the Boltzmann equation can be transformed into an infinite system of partial differential equations with respect to the expansion coefficients. In practice, the study is limited to a finite system of differential equations, leading to the following interrelated questions: (1) How should the infinite system of differential equations be truncated? (2) How can the microscopic boundary condition for the particle distribution function be approximated? (3) Is the obtained problem for the finite system of equations correct, i.e., does a solution exist for the new problem, and if so, to what space does it belong? This work aims to provide answers these questions by proposing one method of truncating the infinite system of differential equations, approximating Maxwell's microscopic boundary condition, and proving the correctness of the initial-boundary value problem for the finite system of partial differential equations.

The approximate solution to Equations (2)–(4) can be found using the moment method.

$$f_k(t,x,c) = f_0(\alpha|c|)\sum_{2n+l=0}^{k}\left(\sum_{m=0}^{l} f_{nl}^{(m)}(t,x)\Phi_{nlm}^{(c)}(\alpha c)\right), \tag{5}$$

$$\begin{gathered}\int_{R_3^c}\left[\frac{\partial f_k}{\partial t}+U_1\frac{\partial f_k}{\partial x_1}+U_3\frac{\partial f_k}{\partial x_3}+c_1\frac{\partial f_k}{\partial x_1}+c_3\frac{\partial f_k}{\partial x_3}-J(f_k,f_k)\right]\Phi_{nlm}^{(c)}(\alpha c)dc=0,\\ 2n+l=0,1,\ldots,k, m=0,1,\ldots,1,\end{gathered} \tag{6}$$

$$\begin{gathered}\int_{R_3^c}\left(f_k(0,x,c)-f^0(x,c)\right)\Phi_{nlm}^{(c)}(\alpha c)dc=0,\\ 2n+l=0,1,\ldots,k, m=0,1,\ldots,1,\end{gathered} \tag{7}$$

$$\begin{gathered}\int_{(n_{\partial G},c)>0}(n_{\partial G},c)f_{2N+1}^{+(t,x_{\partial G},c)}\Phi_{n,2l,m}^{(c)}(\alpha c)dc-\\ \beta\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)f_{2N+1}^{-(t,x_{\partial G},c)}\Phi_{n,2l,m}^{(c)}(\alpha c)dc-\\ (1-\beta)\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)exp\left(\frac{-|c|^2}{2R\Theta}\right)\Phi_{n,2l,m}^{(c)}(\alpha c)dc=0,\\ 2(n+l)=0,2,\ldots,2N, m=0,1,\ldots,2l,\end{gathered} \tag{8}$$

when $k = 2N+1$,

$$\begin{gathered}\int_{(n_{\partial G},c)>0}(n_{\partial G},c)f_{2N}^{+(t,x_{\partial G},c)}\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc-\\ \beta\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)f_{2N}^{-(t,x_{\partial G},c)}\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc-\\ (1-\beta)\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)exp\left(\frac{-|c|^2}{2R\Theta}\right)\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc=0,\\ 2(n+l)+1=1,3,\ldots,2N-1, m=0,1,\ldots,2l+1,\end{gathered} \tag{9}$$

when $k = 2N$, where

$$f_{nl}^{(m)}(t,x) = \int_{R_3^c} f_k(t,x,c)\Phi_{nlm}^{(c)}(\alpha c)dc, \tag{10}$$

$$f_k^0(x,c) = f_0(\alpha|c|)\sum_{2n+l=0}^{k}\left(\sum_{m=0}^{l} f_{nl}^{0(m)}(t,x)\Phi_{nlm}^{(c)}(\alpha c)\right), \tag{11}$$

$$f_{nl}^{0(m)}(t,x) = \int_{R_3^c} f_k^0(x,c)\Phi_{nlm}^{(c)}(\alpha c)dc, \tag{12}$$

$\Phi_{nlm}^{(c)}(\alpha c) = \gamma_{nl}^{(m)}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^l S_n^{l+1/2}\left(\frac{\alpha^2|c|^2}{2}\right)P_l^{(m)}(cos\theta)cosm\psi$ are the eigenfunctions of the linearized collision operator [1,8], $S_n^{l+1/2}\left(\frac{\alpha^2|c|^2}{2}\right)$ are the Sonine polynomials, $P_l^{(m)}(cos\theta)$ are the associated Legendre functions, $\gamma_{nl}^{(m)} = \sqrt{\frac{\sqrt{\pi}n!(2l+1)(l-m)!}{2\Gamma(n+l+3/2)(l+m)!}}$ —is the normalization coefficient, Γ is the gamma function, $|c|, \theta, \psi$ are the polar coordinates in velocity space, and $f_0(\alpha|c|) = \left(\alpha^2/2\pi\right)^{3/2}exp\left(-\alpha^2c^2/2\right)$ is the local Maxwellian distribution.

Representing f as in Equation (5) can be viewed as an attempt to approximate the solution of the Boltzmann equation, comprising a partial sum of the particle distribution function's Fourier series expansion in eigenfunctions of the linearized collision operator. The system of Equation (6), corresponding to the partial sum of Equation (5), will be referred to as the system of moment equations in the k-th approximation. The finite system of moment equations for a specific problem replaces the Boltzmann equation with a certain degree of accuracy [1,9]. It also presents the task of setting boundary conditions for the finite system of equations approximating the microscopic boundary conditions for the Boltzmann equation, meaning it is necessary to approximately replace the boundary conditions for the particle distribution function with a certain number of macroscopic conditions for the moments. Choosing boundary conditions that the solutions of the moment equations must satisfy is a significant issue for the system of moment equations.

Equations (8) and (9) serve as approximations of the microscopic Maxwell condition depending on the oddness and evenness of the approximation of the system of moment equations, providing macroscopic boundary conditions for the system of moment equations at $k = 2N + 1$ and $k = 2N$, respectively. At $k = 2N + 1$ ($k = 2N$), even (odd) moments of the second index of the Maxwell condition's discrepancy are set to zero, i.e., for $k = 2N + 1$ ($k = 2N$), both sides of the Maxwell condition's discrepancy are multiplied by $\Phi^{(c)}_{n,2l,m}(\alpha c)$ ($\Phi^{(c)}_{n,2l+1,m}(\alpha c)$) and integrated over the velocity half-space. The number of macroscopic boundary conditions also depends on the oddness and evenness of the approximation of the system of moment equations. The number of equations in Equation (6) depends on the chosen approximation and moments. Through calculation, it is found that the number of moment equations at $k = 2N + 1$ equals to $\frac{1}{24}(k+1)(k+3)(2k+7)$, and the number of macroscopic boundary conditions equals to $\frac{1}{6}(N+1)(N+2)(2N+3)$, while the number of moment equations at $k = 2N$ equals to $\frac{1}{24}(k+2)(k+4)(2k+3)$, and the number of macroscopic boundary conditions equals to $\frac{1}{3}N(N+1)(N+2)$. This shows that the number of moment equations and macroscopic boundary conditions depends on the oddness and evenness of the approximation of the number of moment equations. At $k = 1(k = 2)$, the number of moment equations equals 3 in Equation (7), and the number of macroscopic boundary conditions equals 1 in Equation (2). For the one-dimensional system of Boltzmann moment equations, the approximation of the microscopic Maxwell boundary condition at a constant α is given in [26]. In [27], a new one-dimensional system of moment equations depending on the flight speed and surface temperature of the aircraft was derived, and the microscopic Maxwell boundary condition was approximated at $\alpha = \alpha(t, x)$. This work presents for the first time the derivation of a two-dimensional system of moment equations and the approximation of the microscopic Maxwell Condition (4) for the particle distribution function in the case of a two-dimensional Boltzmann equation.

Let us introduce the notation $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U_1\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}$ and rewrite Equality (6) as

$$\begin{gathered}\int_{R_3^c}\left[\frac{\partial f_k}{\partial t} + U_1\frac{\partial f_k}{\partial x_1} + U_3\frac{\partial f_k}{\partial x_3} + c_1\frac{\partial f_k}{\partial x_1} + c_3\frac{\partial f_k}{\partial x_3} - J(f_k, f_k)\right]\Phi^{(c)}_{nlm}(\alpha c)dc = \\ \int_{R_3^c}\{\tfrac{d}{dt}\left(f_k\Phi^{(c)}_{nlm}\right) + \tfrac{\partial}{\partial x_1}\left(c_1 f_k\Phi^{(c)}_{nlm}\right) + \tfrac{\partial}{\partial x_3}\left(c_3 f_k\Phi^{(c)}_{nlm}\right) - f_k\left[\tfrac{d}{dt}\left(\Phi^{(c)}_{nlm}\right) + \tfrac{\partial}{\partial x_1}\left(c_1\Phi^{(c)}_{nlm}\right) + \tfrac{\partial}{\partial x_3}\left(c_3\Phi^{(c)}_{nlm}\right)\right] \\ -J(f_k, f_k)\Phi^{(c)}_{nlm}\}dc = 0.\end{gathered} \tag{13}$$

By definition of the coefficients $f^{(m)}_{nl}$ we have

$$\int_{R_3^c}\frac{d}{dt}\left(f_k\Phi^{(c)}_{nlm}\right)dc = \frac{d}{dt}\int_{R_3^c}\left(f_k\Phi^{(c)}_{nlm}\right)dc = \frac{df^{(m)}_{nl}}{dt} \equiv \frac{\partial f^{(m)}_{nl}}{\partial t} + U_1\frac{\partial f^{(m)}_{nl}}{\partial x_1} + U_3\frac{\partial f^{(m)}_{nl}}{\partial x_3} \tag{14}$$

Let us calculate the following integral:

$$\int_{R_3^c}\left[\frac{\partial}{\partial x_1}\left(c_1 f_k\Phi^{(c)}_{nlm}\right) + \frac{\partial}{\partial x_3}\left(c_3 f_k\Phi^{(c)}_{nlm}\right)\right]dc = \frac{\partial}{\partial x_1}\int_{R_3^c} c_1 f_k\Phi^{(c)}_{nlm}dc + \frac{\partial}{\partial x_3}\int_{R_3^c} c_3 f_k\Phi^{(c)}_{nlm}dc. \tag{15}$$

On the right-hand side of the last equation, we will express c_1 and c_3 in spherical coordinates and replace $\Phi^{(c)}_{nlm}$ with its value

$$\begin{gathered}\int_{R_3^c} c_1 f_k \Phi_{nlm}^{(c)} dc + \frac{\partial}{\partial x_3}\int_{R_3^c} c_3 f_k \Phi_{nlm}^{(c)} dc = \gamma_{nl}^{(m)} \frac{\partial}{\partial x_1}\int_{R_3^c} |c| sin(\Theta) cos(\psi) f_k cosm\psi \left(\frac{\alpha|c|}{\sqrt{2}}\right)^l \\ \times S_n^{l+\frac{1}{2}}\left(\frac{\alpha^2|c|^2}{2}\right) P_l^{(m)} cos\Theta |c|^2 d|c| sin\Theta d\Theta d\psi \\ +\gamma_{nl}^{(m)} \frac{\partial}{\partial x_3}\int_{R_3^c} |c| cos(\Theta) f_k cosm\psi \left(\frac{\alpha|c|}{\sqrt{2}}\right)^l S_n^{l+\frac{1}{2}}\left(\frac{\alpha^2|c|^2}{2}\right) P_l^{(m)} cos\Theta |c|^2 d|c| sin\Theta d\Theta d\psi = \\ \gamma_{nl}^{(m)} \frac{\partial}{\partial x_1}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2 c^2}{2}\right)|c|^2 d|c| \int_{-1}^1\int_0^{2\pi}\sqrt{1-\mu^2} P_l^{(m)}(\mu) cos\psi cosm\psi d\mu d\psi \\ +\gamma_{nl}^{(m)} \frac{\partial}{\partial x_3}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2 c^2}{2}\right)|c|^2 d|c| \int_{-1}^1\int_0^{2\pi}\mu P_l^{(m)}(\mu) cosm\psi d\mu d\psi = \\ \gamma_{nl}^{(m)} \frac{\partial}{\partial x_1}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2 c^2}{2}\right)|c|^2 d|c| \int_{-1}^1\int_0^{2\pi}\sqrt{1-\mu^2} P_l^{(m)}(\mu) \\ \times[cos(m+1)\psi cos(m-1)\psi] d\mu d\psi \\ +\gamma_{nl}^{(m)} \frac{\partial}{\partial x_3}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2 c^2}{2}\right)|c|^2 d|c| \int_{-1}^1\int_0^{2\pi}\mu P_l^{(m)}(\mu) cosm\psi d\mu d\psi\end{gathered} \tag{16}$$

Hence, using the relations [28]

$$\sqrt{1-\mu^2}P_l^{(m)}(\mu) = \begin{cases} \left(P_{l+1}^{(m+1)}(\mu) - P_{l-1}^{(m+1)}(\mu)\right)/(2l+1) \\ \left((l+m)(l+m-1)P_{l-1}^{(m-1)}(\mu) - (l-m+1)(l-m+2)P_{l+1}^{(m-1)}(\mu)\right)/(2l+1) \end{cases} \tag{17}$$
$$(2l+1)\mu P_l^{(m)}(\mu) = (l-m+1)P_{l+1}^{(m)}(\mu) + (l+m)P_{l-1}^{(m)}(\mu)$$

rewrite Equality (16) as

$$\begin{gathered}\frac{\partial}{\partial x_1}\int_{R_3^c} c_1 f_k \Phi_{nlm}^{(c)} dc + \frac{\partial}{\partial x_3}\int_{R_3^c} c_3 f_k \Phi_{nlm}^{(c)} dc = \frac{\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_1}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2c^2}{2}\right)|c|^2 d|c| * \\ \int_{-1}^1\int_0^{2\pi}\left\{\left(P_{l+1}^{(m+1)}(\mu) - P_{l-1}^{(m+1)}(\mu)\right) cos(m+1)\psi + \left((l+m)(l+m-1)P_{l-1}^{(m-1)}(\mu)\right.\right. \\ \left.\left. -(l-m+1)(l-m+2)P_{l+1}^{(m-1)}(\mu)\right) cos(m-1)\psi\right\} d\mu d\psi + \\ \frac{+\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_3}\int_0^\infty f_k \frac{\sqrt{2}}{\alpha}\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1} S_n^{l+1/2}\left(\frac{\alpha^2c^2}{2}\right)|c|^2 d|c| \\ \times\int_{-1}^1\int_0^{2\pi}\left[(l-m+1)P_{l+1}^{(m)}(\mu) + (l+m)P_{l-1}^{(m)}(\mu)\right] cosm\psi d\mu d\psi\end{gathered} \tag{18}$$

Furthermore, the following relationships for Sonine polynomials [28] are known:

$$\begin{gathered} yS_n^{\beta+1}(y) = (n+\beta+1)S_n^\beta(y) - (n+1)S_{n+1}^\beta, \\ S_n^{\beta-1}(y) = S_n^\beta(y) - S_{n-1}^\beta(y) \end{gathered} \tag{19}$$

Using the Equation (19), we transform Equality (18) into the following:

$$\begin{gathered}\frac{\partial}{\partial x_1}\int_{R_3^c} c_1 f_k \Phi_{nlm}^{(c)} dc + \frac{\partial}{\partial x_3}\int_{R_3^c} c_3 f_k \Phi_{nlm}^{(c)} dc = \frac{\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_1}\int_{R_3^c} f_k \frac{\sqrt{2}}{\alpha} \\ *\left\{\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1}\left[S_n^{l+3/2} - S_{n-1}^{l+3/2}\right][P_{l+1}^{(m+1)}(\mu) cos(m+1)\psi\right. \\ -(l-m+1)(l-m+2)P_{l+1}^{(m-1)}(\mu) cos(m-1)\psi] \\ -\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\left[(n+l+1/2)S_n^{l-1/2} - (n+1)S_{n+1}^{l-1/2}\right]\left[P_{l-1}^{(m+1)} cos(m+1)\psi\right. \\ \left.\left. -(l+m)(l+m-1)P_{l-1}^{(m-1)} cos(m-1)\psi\right]\right\} dc \\ +\frac{\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_3}\int_{R_3^c} f_k \frac{\sqrt{2}}{\alpha}\{\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+1}\left[S_n^{l+3/2} - S_{n-1}^{l+3/2}\right](l-m \\ +1)P_{l+1}^{(m)} cosm\psi + \left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\left[(n+l+1/2)S_n^{l-1/2} - (n+1)S_{n+1}^{l-1/2}\right](l \\ +m)P_{l-1}^{(m)} cosm\psi\} dc.\end{gathered}$$

From here, considering the notation in Equation (10), we write Equation (16) as

$$
\begin{gathered}
\int_{R_3^c}\left[\frac{\partial}{\partial x_1}\left(c_1 f_k \Phi_{nlm}^{(c)}\right)+\frac{\partial}{\partial x_3}\left(c_3 f_k \Phi_{nlm}^{(c)}\right)\right]dc=\frac{\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_1}\frac{\sqrt{2}}{\alpha} \\
\cdot\left\{\frac{f_{n,l+1}^{(m+1)}}{\gamma_{n,l+1}^{(m+1)}}-\frac{f_{n-1,l+1}^{(m+1)}}{\gamma_{n-1,l+1}^{(m+1)}}-(l-m+1)(l-m+2)\left(\frac{f_{n,l+1}^{(m-1)}}{\gamma_{n,l+1}^{(m-1)}}-\frac{f_{n-1,l+1}^{(m-1)}}{\gamma_{n-1,l+1}^{(m-1)}}\right)-\frac{n+l+1/2}{\gamma_{n,l-1}^{(m+1)}}f_{n,l-1}^{(m+1)}+\frac{n+1}{\gamma_{n+1,l-1}^{(m+1)}}f_{n+1,l-1}^{(m+1)}\right. \\
\left.+(l+m)(l+m-1)\left(\frac{n+l+1/2}{\gamma_{n,l-1}^{(m-1)}}f_{n,l-1}^{(m-1)}-\frac{n+1}{\gamma_{n+1,l-1}^{(m-1)}}f_{n+1,l-1}^{(m-1)}\right)\right\} \\
+\frac{\gamma_{nl}^{(m)}}{2l+1}\frac{\partial}{\partial x_3}\frac{\sqrt{2}}{\alpha}\left[(l-m+1)\left(\frac{f_{n,l+1}^{(m)}}{\gamma_{n,l+1}^{(m)}}-\frac{f_{n-1,l+1}^{(m)}}{\gamma_{n-1,l+1}^{(m)}}\right)\right. \\
\left.+(l+m)\left(\frac{n+l+1/2}{\gamma_{n,l-1}^{(m)}}f_{n,l-1}^{(m)}-\frac{n+1}{\gamma_{n+1,l-1}^{(m)}}f_{n+1,l-1}^{(m)}\right)\right].
\end{gathered}
\tag{20}
$$

On the right-hand side of Equality (20), instead of $\gamma_{nl}^{(m)},\gamma_{n,l+1}^{(m+1)},\ldots$ we substitute the values of the normalization coefficients:

$$
\begin{gathered}
\int_{R_3^c}\left[\frac{\partial}{\partial x_1}\left(c_1 f_k \Phi_{nlm}^{(c)}\right)+\frac{\partial}{\partial x_3}\left(c_3 f_k \Phi_{nlm}^{(c)}\right)\right]dc=\frac{\partial}{\partial x_1}\frac{\sqrt{2}}{\alpha}\times \\
\left[\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m+1)}-\sqrt{n}f_{n-1,l+1}^{(m+1)}\right)\right. \\
-\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m-1)}-\sqrt{n}f_{n-1,l+1}^{(m-1)}\right) \\
-\sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m+1)}-\sqrt{n+1}f_{n+1,l-1}^{(m+1)}\right) \\
\left.+\sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m-1)}-\sqrt{n+1}f_{n+1,l-1}^{(m-1)}\right)\right]+ \\
\frac{+\partial}{\partial x_3}\frac{\sqrt{2}}{\alpha}\left[\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m)}-\sqrt{n}f_{n-1,l+1}^{(m)}\right)\right. \\
\left.+\sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m)}-\sqrt{n+1}f_{n+1,l-1}^{(m)}\right)\right]
\end{gathered}
\tag{21}
$$

We write the result of calculating the integral from the fourth term in Equation (13) as (due to the complexity, we omitted the calculations of the integrals)

$$
\begin{gathered}
-\int_{R_3^c} f_k\left[\frac{d}{dt}\left(\Phi_{nlm}^{(c)}\right)+\frac{\partial}{\partial x_1}\left(c_1\Phi_{nlm}^{(c)}\right)+\frac{\partial}{\partial x_3}\left(c_3\Phi_{nlm}^{(c)}\right)\right]dc \\
=\frac{dln\alpha}{dt}a_1\left(f_{nl}^{(m)}\right)+\alpha\frac{dU_1}{dt}a_2\left(f_{nl}^{(m)}\right)+\alpha\frac{dU_3}{dt}a_3\left(f_{nl}^{(m)}\right)+\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}b_1\left(f_{nl}^{(m)}\right)+\frac{\partial U_1}{\partial x_1}b_2\left(f_{nl}^{(m)}\right) \\
+\frac{\partial U_3}{\partial x_1}b_3\left(f_{nl}^{(m)}\right)+\frac{\sqrt{2}}{\alpha}\frac{\partial ln\alpha}{\partial x_3}c_1\left(f_{nl}^{(m)}\right)+\frac{\partial U_1}{\partial x_3}c_2\left(f_{nl}^{(m)}\right)+\frac{\partial U_3}{\partial x_3}c_3\left(f_{nl}^{(m)}\right),
\end{gathered}
\tag{22}
$$

where the coefficient values are $a_1\left(f_{nl}^{(m)}\right), a_2\left(f_{nl}^{(m)}\right), a_3\left(f_{nl}^{(m)}\right), b_1\left(f_{nl}^{(m)}\right), b_2\left(f_{nl}^{(m)}\right), b_3\left(f_{nl}^{(m)}\right)$, $c_1\left(f_{nl}^{(m)}\right)$, $c_2\left(f_{nl}^{(m)}\right)$, and $c_3\left(f_{nl}^{(m)}\right)$ (references are provided at the end of the article (see Appendix A)).

By substituting the values of the integrals from Equations (15), (21), and (22) into Equation (13), we obtain the following two-dimensional system of moment equations:

$$
\begin{gathered}
\frac{df_{nl}^{(m)}}{dt}+\frac{\partial}{\partial x_1} \\
-\sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m-1)}-\sqrt{n}f_{n-1,l+1}^{(m-1)}\right) \\
-\sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m+1)}-\sqrt{n+1}f_{n+1,l-1}^{(m+1)}\right) \\
+\sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m-1)}-\sqrt{n+1}f_{n+1,l-1}^{(m-1)}\right)\Bigg\}+ \\
\frac{+\partial}{\partial x_3}\left\{\frac{\sqrt{2}}{\alpha}\left[\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m)}-\sqrt{n}f_{n-1,l+1}^{(m)}\right)\right.\right. \\
\left.\left.+\sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m)}-\sqrt{n+1}f_{n+1,l-1}^{(m)}\right)\right]\right\}+ \\
\frac{dln\alpha}{dt}a_1\left(f_{nl}^{(m)}\right)+\alpha\frac{dU_1}{dt}a_2\left(f_{nl}^{(m)}\right)+\alpha\frac{dU_3}{dt}a_3\left(f_{nl}^{(m)}\right)+\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}b_1\left(f_{nl}^{(m)}\right)+\frac{\partial U_1}{\partial x_1}b_2\left(f_{nl}^{(m)}\right)+\frac{\partial U_3}{\partial x_1}b_3\left(f_{nl}^{(m)}\right) \\
+\frac{\sqrt{2}}{\alpha}\frac{\partial ln\alpha}{\partial x_3}c_1\left(f_{nl}^{(m)}\right)+\frac{\partial U_1}{\partial x_3}c_2\left(f_{nl}^{(m)}\right)+\frac{\partial U_3}{\partial x_3}c_3\left(f_{nl}^{(m)}\right)=J_{nl}^{(m)}, \\
2n+l=0,1,\ldots,k;\, m=0,1,\ldots,l,\, t\in(0,T], x=(x_1,x_3)\in G,
\end{gathered}
\tag{23}
$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x_1} + U_3 \frac{\partial}{\partial x_3}$,

$$J_{nl}^{(m)} = \sum \begin{pmatrix} N_3 L_3 M_3 \downarrow nlm \\ n_3 l_3 m_3 \uparrow 000 \end{pmatrix} \begin{pmatrix} N_3 L_3 M_3 \downarrow n_1 l_1 m_1 \\ n_3 l_3 m_3 \uparrow n_2 l_2 m_2 \end{pmatrix} V^{(l_3)} f_{n_1 l_1}^{(m_1)} f_{n_2 l_2}^{(m_2)}, \tag{24}$$

$$V^{(l_3)} = \sigma_{l_3} - \sigma_0, \begin{pmatrix} N_3 L_3 M_3 \downarrow n_1 l_1 m_1 \\ n_3 l_3 m_3 \uparrow n_2 l_2 m_2 \end{pmatrix} - \text{Talmic oefficients.}$$

Equation (23) represents a nonlinear system of partial differential equations concerning the moments of the particle distribution function. The differential part of this system includes coefficients such as the velocity of movement and the surface temperature of a body moving in a fluid. Additionally, the lower terms of the Equation (23) also include derivatives of the velocity of movement and surface temperature as coefficients. For the Equation (23), it is necessary to formulate an initial-boundary value problem and demonstrate the correctness of the new problem concerning the moments of the distribution function. Establishing boundary conditions for the finite system of moment equations remains a complex and unresolved issue, comparable in difficulty to the boundary condition problem for Grad's equation system, which is a global issue in the dynamics of rarefied gas. From the microscopic boundary conditions for the Boltzmann equation, an infinite number of boundary conditions can be derived for any type of decomposition. However, the number of boundary conditions is not determined by the number of moment equations, and the number of moment equations does influence the amount of boundary conditions. Moreover, the boundary conditions must be consistent with the moment equations. The differential part of the Equation (23) contains three unknown parameters, U_1, U_3 and α, which depend on time and spatial variables (x_1, x_2). Therefore, the characteristics of the finite equation system derived from Equation (23) depend on the velocity of movement and the surface temperature of a body moving in a fluid, which are functions of time and spatial variables. The microscopic Maxwell condition is approximated using Equations (8) and (9). The next section will present the derivation of macroscopic boundary conditions for the system of moment equations in the first and second approximations and demonstrate the correctness of the initial-boundary value problem for the two-dimensional system of moment equations in the first approximation. The Equation (23) system differs from Grad's equation system, as the moments of the distribution function are defined differently than by Grad and the Boltzmann moment equation systems introduced in Refs. [7,8] by one of the authors of this work.

Grad expanded the particle distribution function in Hermite polynomials around the local Maxwell function, with the expansion coefficients determined by the formula:

$$a^N = \frac{1}{n} \int f \, H^{(N)}(v) d\xi,$$

where

$$v = \sqrt{\frac{m}{\mathrm{kT}}} c = \sqrt{\frac{m}{\mathrm{kT}}} (\xi - u)$$

is the relative velocity (relative to the average velocity). The coefficients of the particle distribution function's expansion in Hermite polynomials depend on an unknown parameter $n = \int f d\xi$—the zeroth-order moment. In case of the Equation (23) system, the moments $f_{nl}^{(m)}$ are determined using Equation (10) with $c = v - U$, where $U = (U_1, U_3)$ is the velocity of the body moving in the fluid. Therefore, the coefficients of the particle distribution function's expansion in Hermite polynomials and the moments $f_{nl}^{(m)}$ (coefficients of the particle distribution function's expansion around the local Maxwellian distribution using the eigenfunctions of the linearized collision operator) differ. Additionally, the structures of the Grad equation system and Equation (23) are different. Indeed, let us write down the differential equation for $a^{(2)}$ from the Grad equation system [5]:

$$\begin{gathered}\frac{\partial a_{ij}^{(2)}}{\partial t}+u_r\frac{\partial a_{ij}^{(2)}}{\partial x_r}+a_{ir}^{(2)}\frac{\partial u_j}{\partial x_r}+(a_{ij}^{(2)}+\delta_{ij})\frac{1}{\mathrm{RT}}\frac{\mathrm{dRT}}{\mathrm{dt}}+\\ \sqrt{\mathrm{RT}}\frac{\partial a_{ijr}^{(3)}}{\partial x_r}+\frac{\sqrt{\mathrm{RT}}}{\rho}a_{ijr}^{(3)}\frac{\partial\rho}{\partial x_r}+\frac{3}{2\mathrm{RT}}a_{ijr}^{(3)}\frac{\partial\mathrm{RT}}{\partial x_r}+\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}=J_{ij}^{(2)}.\end{gathered}\tag{25}$$

In Equation (25), u_i, u_j, u_r are the components representing the average velocity of the gas, and T is the gas temperature, i.e., the coefficients at $a_{ij}^{(2)}$, $a_{ijr}^{(3)}$ and their derivatives depend on the macroscopic characteristics of the gas. Now, let us write down the differential equation for $f_{02}^{(0)}$ from the system of Equation (23), corresponding to the values $2n+l=2, n=0,$ and $l=2$ (the moments $f_{nl}^{(m)}$ at $2n+l>3$ are set to zero)

$$\begin{gathered}\frac{\partial f_{02}^{(0)}}{\partial t}+U_1\frac{\partial f_{02}^{(0)}}{\partial x_1}+U_3\frac{\partial f_{02}^{(0)}}{\partial x_3}+\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\big(-\sqrt{\tfrac{2}{3}}f_{01}^{(1)}\big)\right)+\frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\big(\tfrac{2}{\sqrt{3}}f_{01}^{(0)}\big)\right)+\frac{dln\alpha}{dt}(-2f_{02}^{(0)})\\ +\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\big(\tfrac{2}{\sqrt{3}}f_{01}^{(1)}\big)\\ +\frac{\partial U_1}{\partial x_1}\left(-\tfrac{5}{7}f_{02}^{(0)}\right)+\frac{\sqrt{2}}{\alpha}\frac{\partial ln\alpha}{\partial x_3}\left(2\sqrt{\tfrac{2}{3}}f_{01}^{(0)}\right)+\frac{\partial U_3}{\partial x_3}\left(-\tfrac{11}{7}f_{02}^{(0)}\right)=J_{02}^{(0)},\end{gathered}\tag{26}$$

In Equation (26), (U_1, U_3) represents the velocity of the body's movement, and $\alpha=\sqrt{\frac{1}{R\Theta}}$, where Θ is the surface temperature of the body. The derivatives of $f_{02}^{(0)}$ with respect to x_1 and x_3 include coefficients $U_1\wedge U_3$, and the derivatives $\frac{\partial}{\partial x_1}$ contain $\frac{1}{\alpha}$ as a coefficient. Additionally, derivatives with respect to time and the spatial variable from the movement velocity) and α also enter Equation (26) as coefficients for the lower order terms. Thus, the Grad equation system and the moment system of Equation (23) differ. Using the corresponding problem for the Grad equation system, one can determine the macroscopic characteristics of the gas, provided boundary conditions can be set, and with the initial-boundary value problem for the moment system of Equation (23), one can determine the macroscopic characteristics of the fluid, as well as the movement velocity and surface temperature of the body. Determining the movement velocity and surface temperature of the body using Equations (23)–(26) is an inverse problem for the nonlinear hyperbolic equation system.

In deriving the Boltzmann moment equation system, the particle distribution function was decomposed using the eigenfunctions of the linearized collision operator around the global Maxwellian distribution, meaning it relied on a constant value $\alpha=\sqrt{\frac{1}{R\Theta}}$ and the Boltzmann moment equation system depended only on one parameter. However, in the case of the Equation (23) system, both quantities (U_1, U_3) and α are functions of time and coordinates, leading to different structures for the Equation (23) system and the Boltzmann moment equation system. If, in Equation (23), the parameter α is constant and $U_1=0\wedge U_3=0$, then the Boltzmann moment equation system is obtained [7,8].

$$\begin{gathered}\frac{\partial f_{nl}^{(m)}}{\partial t}+\frac{\sqrt{2}}{\alpha}\frac{\partial}{\partial x_1}\Big\{\Big[\sqrt{\tfrac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m+1)}-\sqrt{n}f_{n-1,l+1}^{(m+1)}\right)-\\ -\sqrt{\tfrac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m-1)}-\sqrt{n}f_{n-1,l+1}^{(m-1)}\right)-\\ -\sqrt{\tfrac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m+1)}-\sqrt{n+1}f_{n+1,l-1}^{(m+1)}\right)+\\ +\sqrt{\tfrac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m-1)}-\sqrt{n+1}f_{n+1,l-1}^{(m-1)}\right)\Big\}+\\ +\frac{\sqrt{2}}{\alpha}\frac{\partial}{\partial x_3}\Big\{\Big[\sqrt{\tfrac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}f_{n,l+1}^{(m)}-\sqrt{n}f_{n-1,l+1}^{(m)}\right)\\ +\sqrt{\tfrac{(l-m)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}f_{n,l-1}^{(m)}-\sqrt{n+1}f_{n+1,l-1}^{(m)}\right)\Big]\Big\}=J_{nl}^{(m)},\\ 2n+l=0,1,\ldots,k;\ m=0,1,\ldots,l.\end{gathered}$$

The Boltzmann moment equation system is a special case of the Equation (23) system. The moment Equation (23) system represents a finite system of equations. This raises the issue of closing the finite system of equations, as for values of $2n + l = 0, 1, \ldots, k \wedge m = 0, 1, \ldots, l(n = 0)$ (with $n = 0$) moments of the distribution function with negative indices appear. Furthermore, the left side of the moment system of Equation (23) contains moments not included in the sum in Equation (5), i.e., the number of unknowns exceeds the number of equations. The partial sum in Equation (5) can be considered a method for closing the finite system of moment equations. The closure of the Equation (23) system is resolved individually for each specific approximation of the moment equation system.

3.2. Correctness of the Initial-Boundary Value Problem for a Two-Dimensional Moment System of Equations in the First Approximation

Let us present the formulation of the initial-boundary value problem for a two-dimensional moment system of equations in the first approximation ($k = 1$) under macroscopic boundary conditions. If in the system of Equation (23), the index $2n + l$ takes values 0 and 1, and m also takes values 0 and 1, so we obtain the following equations:

$$\begin{gathered}
\frac{df_{00}^{(0)}}{dt} + \frac{\partial}{\partial x_1}\left(\frac{\sqrt{2}}{\alpha} f_{01}^{(1)}\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha} f_{01}^{(0)}\right) + \frac{\partial U_1}{\partial x_1}\left(-\frac{1}{3} f_{00}^{(0)}\right) + \frac{\partial U_3}{\partial x_3}\left(-\frac{1}{3} f_{00}^{(0)}\right) = 0, \\
\frac{df_{01}^{(0)}}{dt} + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha} f_{00}^{(0)}\right) + \frac{dln\alpha}{dt}\left(-f_{01}^{(0)}\right) + \frac{\partial U_1}{\partial x_1}\left(-\frac{1}{5} f_{01}^{(0)}\right) + \frac{\partial U_3}{\partial x_3}\left(-\frac{6}{5} f_{01}^{(0)}\right) = 0, \\
\frac{df_{01}^{(1)}}{dt} + \frac{\partial}{\partial x_1}\left(\frac{\sqrt{2}}{\alpha} f_{00}^{(0)}\right) + \frac{dln\alpha}{dt}\left(-f_{\ 01}^{(1)}\right) + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(-f_{00}^{(0)}\right) + \frac{\partial U_1}{\partial x_1}\left(-\frac{2}{5} f_{01}^{(1)}\right) + \frac{\partial U_3}{\partial x_3}\left(-\frac{2}{5} f_{01}^{(1)}\right) = 0,\ t \\
\in (0, T],\ (x_1, x_3) \in G.
\end{gathered} \tag{27}$$

The two-dimensional moment system of equations in the first approximation in Equation (27) includes three equations. Now, let us derive the macroscopic boundary conditions for $k = 1$.

In Equation (8), we set $N = 0$, $2(n + l) = 0$, and $m = 0$, which gives us the following boundary conditions for the first approximation of the two-dimensional moment system of equations:

$$\begin{gathered}
\int_{(n_{\partial G}, c) > 0} (n_{\partial G}, c) f_1^{+(t, x_{\partial G}, c)} \Phi_{000}^{(c)}(\alpha c) dc - \\
\beta \int_{(n_{\partial G}, c) < 0} (n_{\partial G}, -c) f_1^{-(t, x_{\partial G}, c)} \Phi_{000}^{(c)}(\alpha c) dc - \\
(1 - \beta) \int_{(n_{\partial G}, c) < 0} (n_{\partial G}, -c) exp\left(\frac{-|c|^2}{2R\Theta}\right) \Phi_{000}^{(c)}(\alpha c) dc = 0,
\end{gathered} \tag{28}$$

where $\Phi_{000}^{(c)}(\alpha c) = 1$,

$$f_1(t, x, c) = f_0(\alpha|c|) \sum_{2n+l=0}^{1} \left(\sum_{m=0}^{l} f_{nl}^{(m)}(t, x) \Phi_{nlm}^{(c)}(\alpha c) \right) \tag{29}$$

Let the domain $G = \{-a_j < x_j < a_j, j = 1, 3\}$ be a rectangle. In Equation (28), we substitute the value of $f_1(t, x, c)$ with Formula (29) and perform integration over half-spaces.

Thus, for the system of Equation (27), we obtain the boundary conditions (the calculation of integrals is omitted due to its complexity):

$$\begin{gathered}\left(\frac{1}{\sqrt{2}\alpha}\left(f_{01}^{(1)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\left(f_{00}^{(0)}\right)^{+}\right)\right)(t, \pm a_1) \\ = \beta\left(\frac{1}{\sqrt{2}\alpha}\left(f_{01}^{(1)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\left(f_{00}^{(0)}\right)^{-}\right)\right)(t, \pm a_1) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_1, \\ \left(\frac{1}{\sqrt{2}\alpha}\left(f_{01}^{(0)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\left(f_{00}^{(0)}\right)^{+}\right)\right)(t, \pm a_3) \\ = \beta\left(\frac{1}{\sqrt{2}\alpha}\left(f_{01}^{(0)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\left(f_{00}^{(0)}\right)^{-}\right)\right)(t, \pm a_3) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_3, \\ F_1 = \frac{1}{8\sqrt{2}},\ F_2 = \frac{1}{4\sqrt{2}},\ t \in (0, T].\end{gathered} \tag{30}$$

$\left(\mathrm{f}_{01}^{(1)}\right)^{+}, \ldots \left(\left(\mathrm{f}_{01}^{(1)}\right)^{-}, \ldots\right)$ corresponds to the moments of particles impacting the boundary (reflected from the boundary) of the distribution function, indicating that the positive sign (+) (negative sign ($-$)) is associated with the moments of particles impacting the boundary (reflecting from the boundary) of the distribution function. The macroscopic boundary conditions for the moment system of equations create a link between the moments of particles impacting and reflecting off the boundary of the distribution function.

The expressions $\frac{1-\beta}{\alpha\sqrt{\pi}}F_1$ and $\frac{1-\beta}{\alpha\sqrt{\pi}}F_3$ are moments of the second term on the right-hand side of Condition (4). From Equation (30), it is evident that one boundary condition is applied at the ends of the interval $\left(-a_j, a_j\right), j = 1,\ 3$. We introduce the following vectors and matrices:

$$\begin{gathered}u = f_{00}^{(0)}, w = \left(f_{01}^{(0)}, f_{01}^{(1)}\right)', \overline{A_1} = \left(0, \sqrt{2}\right), \overline{A_3} = (1, 0), B_1 = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\right), B_3 = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\right), \\ D_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, C_1 = \begin{pmatrix} -1/5 & 0 \\ 0 & -2/5 \end{pmatrix}, C_3 = \begin{pmatrix} -5/6 & 0 \\ 0 & -2/5 \end{pmatrix}.\end{gathered}$$

Then, the initial-boundary value problem for the system of Equation (27) under the boundary in Condition (30) is formulated as follows:

$$\begin{gathered}\frac{du}{dt} + \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 w\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 w\right) = 0, \\ \frac{dw}{dt} + \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1{}' u\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3{}' u\right) + \frac{dln\alpha}{dt}D_1 w + \frac{\partial U_1}{\partial x_1}C_1 w \frac{+\partial U_3}{\partial x_3}C_3 w + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\begin{pmatrix} 0 \\ -1 \end{pmatrix}u = 0, t \in (0, T], x = (x_1, x_3) \in G,\end{gathered} \tag{31}$$

$$u(0, x) = u_0(x), w(0, x) = w_0(x), x = (x_1, x_3) \in G, \tag{32}$$

$$\begin{gathered}\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_j w^{+} \pm B_j u^{+}\right)(t, \pm a_j) = \beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_j w^{-} \mp B_j u^{-}\right)(t, \pm a_j) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_j, \\ t \in (0, T],\ j = 1, 3,\end{gathered} \tag{33}$$

where $\overline{A}_j'$ are the transposed matrices, u_0 is a given function, and $w_0(x)$—is a given vector function

$$A_1 = \begin{pmatrix} 0 & \overline{A}_1 \\ \overline{A}_1{}' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & \overline{A}_3 \\ \overline{A}_3{}' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The goal is to find a solution to the system of Equation (31) that satisfies the initial Condition (32) and boundary Condition (33). When $U_1 = U_3 = 0$ and $\alpha - const$, the system of Equation (31) is strictly hyperbolic [29]. Indeed, the matrices A_1 and A_3 are symmetric, and the roots of the equation det $\left(\tau I - \frac{1}{\alpha}A_1\xi_1 - \frac{1}{\alpha}A_2\xi_3\right) = 0$ are real and distinct $\left(\tau_1 = 0, \tau_2 = \frac{1}{\alpha}\sqrt{2\xi_1^2 + \xi_3^2}, \tau_3 = \frac{-1}{\alpha}\sqrt{2\xi_1^2 + \xi_3^2}\right)$. The matrix $A = A_1\xi_1 + A_3\xi_3$ can be diagonalized for any $\xi = (\xi_1, \xi_3)\epsilon R^2$. The eigenvalues of matrix A_1 are denoted by

$\lambda_1 = 0, \lambda_{2/3} = \pm\sqrt{2}$, while the eigenvalues of matrix A_3 are such that $\lambda_1 = 0, \lambda_{2/3} = \pm 1$. The matrices $A_j, j = 1, 3$ have one positive, one negative, and one zero eigenvalue; therefore, it is sufficient to set one boundary condition at the ends of the interval $(-a_j, a_j), j = 1, 3$.

For Equations (31)–(33), the following theorem holds true:

Theorem 1. *If $W_0(x) = (u_0(x), w_0(x))'$ belongs to the space $L_2(G)$ and $\alpha = \alpha(t,x) > 0, U_j = U_j(t,x), j = 1, 3$ are continuously differentiable functions in the domain $[0,T] \times G$, then Equations (31)–(33) have a unique solution, $W(t,x) = (u(t,x), w(t,x))'$, in the space $[0,T] \times G$, and the following a priori estimate applies:*

$$\|W\|_{C([0,T];L_2(G))} \leq C_1 \|W_0\|_{L_2(G)} \tag{34}$$

where C_1 is a constant independent of $W(t,x), 0 < t < T$, and T is any positive, finite number.

Proof. Let $W_0(x) \in L_2(G)$. We prove the validity of Estimate (34) by multiplying the first equation of the system in Equation (31) by u and the second equation scalarly by w, then integrating over the domain G:

$$\begin{gathered}\iint_G \left[\frac{1}{2}\frac{d}{dt}\left(u^2 + (w,w)\right) + \left(\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 w\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 w\right)\right)u\right. \\ \left. + \left(\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1{}' u\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3{}' u\right) + \frac{d ln\alpha}{dt} D_1 w + \frac{\partial U_1}{\partial x_1} C_1 w \frac{+\partial U_3}{\partial x_3} C_3 w + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} u\right), w\right)\right] dx = 0.\end{gathered} \tag{35}$$

Please provide the expressions you would like to transform.

$$\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 w\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 w\right)\right)u = \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 wu\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 wu\right) - \frac{1}{\alpha}\overline{A}_1 w\frac{\partial u}{\partial x_1} - \frac{1}{\alpha}\overline{A}_3 w\frac{\partial u}{\partial x_3}$$

$$\begin{gathered}\left(\left(\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1{}' u\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3{}' u\right) + \frac{d ln\alpha}{dt} D_1 w + \frac{\partial U_1}{\partial x_1} C_1 w \frac{+\partial U_3}{\partial x_3} C_3 w + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} u\right), w\right) \\ = \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1' u, w\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3' u, w\right) - \left(\frac{1}{\alpha}\overline{A}_1' u, \frac{\partial w}{\partial x_1}\right) - \left(\frac{1}{\alpha}\overline{A}_3' u, \frac{\partial w}{\partial x_3}\right) + \\ \frac{d ln\alpha}{dt}(D_1 w, w) + \frac{\partial U_1}{\partial x_1}(C_1 w, w) + \frac{\partial U_3}{\partial x_3}(C_3 w, w) + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} u, w\right).\end{gathered} \tag{36}$$

Considering Relation (36), we rewrite Equation (35) as:

$$\begin{gathered}\iint_G \frac{1}{2}\frac{d}{dt}\left(u^2 + (w,w)\right) dx \\ + \iint_G \left[\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 wu\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 wu\right) - \frac{1}{\alpha}\overline{A}_1 w\frac{\partial u}{\partial x_1} - \frac{1}{\alpha}\overline{A}_3 w\frac{\partial u}{\partial x_3} + \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1' u, w\right)\right. \\ + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3' u, w\right) - \left(\frac{1}{\alpha}\overline{A}_1' u, \frac{\partial w}{\partial x_1}\right) - \left(\frac{1}{\alpha}\overline{A}_3' u, \frac{\partial w}{\partial x_3}\right) + \frac{d ln\alpha}{dt}(D_1 w, w) \\ \left. + \frac{\partial U_1}{\partial x_1}(C_1 w, w) + \frac{\partial U_3}{\partial x_3}(C_3 w, w) + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix} u, w\right)\right] dx = 0\end{gathered} \tag{37}$$

Using the boundary conditions in Equation (33), we transform the following expression:

$$\begin{gathered}\iint_G \left[\frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1 wu\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3 wu\right) + \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}\overline{A}_1' u, w\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}\overline{A}_3' u, w\right)\right] dx = \\ 2\sqrt{2}\iint_G \left[\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_1 wu\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_3 wu\right)\right] = 2\sqrt{2}\int_{a_3}^{a_3}\left[\frac{1}{\sqrt{2}\alpha}\overline{A}_1 wu\right]_{x_1=-a_1}^{x_1=a_1} dx_3 + \\ 2\sqrt{2}\int_{-a_1}^{a_1}\left[\frac{1}{\sqrt{2}\alpha}\overline{A}_3 wu\right]_{x_3=-a_3}^{x_3=a_3} dx_1 = 2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_1 w^- - B_1 u^-\right)u^+ + \frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ - B_1 (u^+)^2\right]_{x_1=a_1} + \right. \\ \left. \left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_1 w^- + B_1 u^-\right)u^+ - \frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ + B_1 (u^+)^2\right]_{x_1=-a_1}\right\} dx_3 \\ +2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_3 w^- - B_3 u^-\right)u^+ + \frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ - B_3 (u^+)^2\right]_{x_3=a_3} + \left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_3 w^- + B_3 u^-\right)u^+ - \frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ + \right. \right. \\ \left. B_3 (u^+)^2\right] x_3 = -a_3\} dx_1.\end{gathered} \tag{38}$$

We substitute the value of the integral from Relation (38) into Equation (37):

$$\iint_G \frac{1}{2}\frac{d}{dt}\left(u^2+(w,w)\right)dx+\iint_G\left[-\frac{1}{\alpha}\overline{A}_1 w\frac{\partial u}{\partial x_1}-\frac{1}{\alpha}\overline{A}_3 w\frac{\partial u}{\partial x_3}-\left(\frac{1}{\alpha}\overline{A}_1' u,\frac{\partial w}{\partial x_1}\right)-\left(\frac{1}{\alpha}\overline{A}_3' u,\frac{\partial w}{\partial x_3}\right)+\frac{dln\alpha}{dt}(D_1 w,w)+\frac{\partial U_1}{\partial x_1}(C_1 w,w)+\frac{\partial U_3}{\partial x_3}(C_3 w,w)+\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix}0\\-1\end{pmatrix}u,w\right)\right]dx+2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_1 w^- - B_1 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ - B_1(u^+)^2\right]x_1=a_1+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_1 w^- + B_1 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ + B_1(u^+)^2\right]_{x_1=-a_1}\right\}dx_3+2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_3 w^- - B_3 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ - B_3(u^+)^2\right]x_3=a_3+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\overline{A}_3 w^- + B_3 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ + B_3(u^+)^2\right]_{x_3=-a_3}\right\}dx_1=0 \tag{39}$$

Consider the derivative:

$$\frac{1}{2}\frac{d}{dt}\left(u^2+(w,w)\right)=\frac{1}{2}\left(\frac{\partial}{\partial t}+U_1\frac{\partial}{\partial x_1}+U_3\frac{\partial}{\partial x_3}\right)\left(u^2+(w,w)\right)=\frac{1}{2}\frac{\partial}{\partial t}\left(u^2+(w,w)\right)+\frac{1}{2}\frac{\partial}{\partial x_1}\left[U_1\left(u^2+(w,w)\right)\right]+\frac{1}{2}\frac{\partial}{\partial x_3}\left[U_3\left(u^2+(w,w)\right)\right]-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_1}{\partial x_1}-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_3}{\partial x_3} \tag{40}$$

The value of the derivative from Equation (40) is substituted into Relation (39):

$$\iint_G \left\{\frac{1}{2}\frac{\partial}{\partial t}\left(u^2+(w,w)\right)+\frac{1}{2}\frac{\partial}{\partial x_1}\left[U_1\left(u^2+(w,w)\right)\right]+\frac{1}{2}\frac{\partial}{\partial x_3}\left[U_3\left(u^2+(w,w)\right)\right]-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_1}{\partial x_1}-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_3}{\partial x_3}\right\}dx+\iint_G\left[-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_1}{\partial x_1}-\frac{1}{2}\left(u^2+(w,w)\right)\frac{\partial U_3}{\partial x_3}-\frac{1}{\alpha}\bar{A}_1 w\frac{\partial u}{\partial x_1}-\frac{1}{\alpha}\bar{A}_3 w\frac{\partial u}{\partial x_3}-\left(\frac{1}{\alpha}\bar{A}_1' u,\frac{\partial w}{\partial x_1}\right)-\left(\frac{1}{\alpha}\bar{A}_3' u,\frac{\partial w}{\partial x_3}\right)+\frac{dln\alpha}{dt}(D_1 w,w)+\frac{\partial U_1}{\partial x_1}(C_1 w,w)+\frac{\partial U_3}{\partial x_3}(C_3 w,w)+\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix}0\\-1\end{pmatrix}u,w\right)\right]dx+2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1 w^- - B_1 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ - B_1(u^+)^2\right]_{x_1=a_1}+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1 w^- + B_1 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ + B_1(u^+)^2\right]_{x_1=-a_1}\right\}dx_3+2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3 w^- - B_3 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ - B_3(u^+)^2\right]_{x_3=a_3}+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3 w^- + B_3 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ + B_3(u^+)^2\right]_{x_3=-a_3}\right\}dx_1=0.$$

Hence,

$$\frac{1}{2}\frac{\partial}{\partial t}\iint_G \left(u^2+(w,w)\right)dx+\iint_G\left[-\frac{1}{\alpha}\bar{A}_1 w\frac{\partial u}{\partial x_1}-\frac{1}{\alpha}\bar{A}_3 w\frac{\partial u}{\partial x_3}-\left(\frac{1}{\alpha}\bar{A}_1' u,\frac{\partial w}{\partial x_1}\right)-\left(\frac{1}{\alpha}\bar{A}_3' u,\frac{\partial w}{\partial x_3}\right)+\frac{dln\alpha}{dt}(D_1 w,w)+\frac{\partial U_1}{\partial x_1}(C_1 w,w)+\frac{\partial U_3}{\partial x_3}(C_3 w,w)+\frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix}0\\-1\end{pmatrix}u,w\right)\right]dx+2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1 w^- - B_1 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ - B_1(u^+)^2\right]_{x_1=a_1}+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1 w^- + B_1 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_1 u^+ + B_1(u^+)^2\right]_{x_1=-a_1}\right\}dx_3+2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3 w^- - B_3 u^-\right)u^+ +\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ - B_3(u^+)^2\right]_{x_3=a_3}+\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3 w^- + B_3 u^-\right)u^+ -\frac{1-\beta}{\alpha\sqrt{\pi}}F_3 u^+ + B_3(u^+)^2\right]_{x_3=-a_3}\right\}dx_1+\int_{-a_3}^{a_3}\left\{\frac{1}{2}\left[U_1\left(u^2+(w,w)\right)\right]_{x_1=a_1}+\frac{1}{2}\left[U_1\left(u^2+(w,w)\right)\right]_{x_1=-a_1}\right\}dx_3+\int_{-a_1}^{a_1}\left\{\frac{1}{2}\left[U_3\left(u^2+(w,w)\right)\right]_{x_3=a_3}+\frac{1}{2}\left[U_3\left(u^2+(w,w)\right)\right]_{x_3=-a_3}\right\}dx_1=0. \tag{41}$$

To obtain an a priori estimate, we use the spherical representation of u and w [30]: $u = y(t)\omega_1(t,x)$ and $w = y(t)\omega_2(t,x)$, where $\|\omega\|_{L^2[-a,a]} = 1, \omega(t,x) = (\omega_1(t,x), \omega_2(t,x))'$ is a unit vector. We substitute the values $u = y(t)\omega_1(t,x)$ and $w = y(t)\omega_2(t,x)$ into Equation (41). Then we will have an ordinary differential equation:

$$\frac{dy}{dt} + yp(t) = f(t), \tag{42}$$

where

$$p(t) = \iint_G \left[-\frac{1}{\alpha}\bar{A}_1\omega_2\frac{\partial\omega_1}{\partial x_1} - \frac{1}{\alpha}\bar{A}_3\omega_2\frac{\partial\omega_1}{\partial x_3} - \left(\frac{1}{\alpha}\bar{A}_1'\omega_1, \frac{\partial\omega_2}{\partial x_1}\right) - \left(\frac{1}{\alpha}\bar{A}_3'\omega_1, \frac{\partial\omega_2}{\partial x_3}\right) + \right.$$
$$\left. \frac{dln\alpha}{dt}(D\omega_{2_1}, \omega_2) + \frac{\partial U_1}{\partial x_1}(C_1\omega_2, \omega_2) + \frac{\partial U_3}{\partial x_3}(C_3\omega_2, \omega_2) + \frac{\sqrt{2}}{\alpha^2}\frac{\partial ln\alpha}{\partial x_1}\left(\begin{pmatrix} 0 \\ -1 \end{pmatrix}\omega_1, \omega_2\right)\right] dx +$$
$$2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1\omega_2^- - B_1\omega_1^-\right)\omega_1^+ + \frac{1-\beta}{\alpha\sqrt{\pi}}F_1\omega_1^+ - B_1(\omega_1^+)^2\right]_{x_1=a_1} + \left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_1\omega_2^- + \right.\right.\right.$$
$$\left.\left.\left. B_1\omega_1^-\right)\omega_1^+ - \frac{1-\beta}{\alpha\sqrt{\pi}}F_1\omega_1^+ + B_1(\omega_1^+)^2\right]_{x_1=-a_1}\right\} dx_3 + 2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3\omega_2^- - \right.\right.\right.$$
$$\left. B_3\omega_1^-\right)\omega_1^+ + \frac{1-\beta}{\alpha\sqrt{\pi}}F_3\omega_1^+ - B_3(\omega_1^+)^2\right]_{x_3=a_3} + \left[\beta\left(\frac{1}{\sqrt{2}\alpha}\bar{A}_3\omega_2^- + B_3\omega_1^-\right)\omega_1^+ - \right.$$
$$\left.\left. \frac{1-\beta}{\alpha\sqrt{\pi}}F_3\omega_1^+ + B_3(\omega_1^+)^2\right]_{x_3=-a_3}\right\} dx_1 +$$
$$\int_{-a_3}^{a_3}\left\{\frac{1}{2}\left[U_1\left(\omega_1^2 + (\omega_2,\omega_2)\right)\right]_{x_1=a_1} + \frac{1}{2}\left[U_1\left(\omega_1^2 + (\omega_2,\omega_2)\right)\right]_{x_1=-a_1}\right\} dx_3$$
$$+ \int_{-a_1}^{a_1}\left\{\frac{1}{2}\left[U_3\left(\omega_1^2 + (\omega_2,\omega_2)\right)\right]_{x_3=a_3} + \frac{1}{2}\left[U_3\left(\omega_1^2 + (\omega_2,\omega_2)\right)\right]_{x_3=-a_3}\right\} dx_1,$$
$$f(t) = 2\sqrt{2}\int_{-a_3}^{a_3}\left\{\left[\frac{1-\beta}{\alpha\sqrt{\pi}}F_1\omega_1^+\right]_{x_1=a_1} + \left[\frac{1-\beta}{\alpha\sqrt{\pi}}F_1\omega_1^+\right]_{x_1=-a_1}\right\} dx_3 +$$
$$2\sqrt{2}\int_{-a_1}^{a_1}\left\{\left[\frac{1-\beta}{\alpha\sqrt{\pi}}F_3\omega_1^+\right]_{x_3=a_3} + \left[\frac{1-\beta}{\alpha\sqrt{\pi}}F_3\omega_1^+\right]_{x_3=-a_3}\right\} dx_1.$$

Equation (42) is studied under the initial condition:

$$y(0) = \|W_0\| = \|W_0\|_{L^2[-a,a]}. \tag{43}$$

The solution to Equations (42) and (43) takes the form:

$$y(t) = exp\left(-\int_0^t p(\tau)d\tau\right)\left[\|W_0\| + \int_0^t f(\tau)exp\left(\int_0^\tau p(\xi)d\xi\right)\right]. \tag{44}$$

From Equation (44) it follows that $y(t)$ is bounded by $\forall t: 0 < t < T$, where T is any arbitrary positive finite number. Hence, for all $\forall t \in [0,T]$, the a priori estimate in Equation (34) holds. The existence of a solution to Equations (31)–(33) can be demonstrated using the Galerkin method. The uniqueness of the solution to Equations (31)–(33) follows from the a priori estimate of Equation (34). □

For comparison, let us consider the formulation of the initial-boundary value problem for the two-dimensional moment equation system in the second approximation under macroscopic boundary conditions. If in the system of Equation (24), the index $2n + l$ takes values 0, 1, and 2, and m also takes values from 0 to l, then we obtain the following equations:

$$
\begin{aligned}
&\tfrac{df_{01}^{(0)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\sqrt{2}f_{02}^{(1)}\right)+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}\left[\tfrac{2}{\sqrt{3}}f_{02}^{(0)}+f_{00}^{(0)}-\sqrt{\tfrac{2}{3}}f_{10}^{(0)}\right]\right)+\tfrac{dln\alpha}{dt}\left(-f_{01}^{(0)}\right)+\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(-f_{02}^{(1)}\right)\\
&+\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{1}{5}f_{01}^{(0)}\right)+\tfrac{\sqrt{2}}{\alpha}\tfrac{\partial ln\alpha}{\partial x_3}\left[\sqrt{\tfrac{2}{3}}f_{02}^{(0)}+\tfrac{1}{\sqrt{2}}\left(f_{00}^{(0)}-f_{10}^{(0)}\right)\right]+\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{6}{5}f_{01}^{(0)}\right)=0,\\
&\tfrac{df_{01}^{(1)}}{dt}+\tfrac{\partial}{\partial x_1}\left\{\tfrac{1}{\alpha}\left[2f_{02}^{(2)}-\sqrt{\tfrac{2}{3}}f_{02}^{(0)}+\sqrt{2}f_{00}^{(0)}-\tfrac{2}{\sqrt{3}}f_{10}^{(0)}\right]\right\}+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}f_{02}^{(1)}\right)+\tfrac{dln\alpha}{dt}\left(-f_{01}^{(1)}\right)+\\
&\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(-\sqrt{2}f_{02}^{(2)}+\tfrac{1}{\sqrt{3}}f_{02}^{(0)}-f_{00}^{(0)}+\sqrt{\tfrac{2}{3}}f_{10}^{(0)}\right)+\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{4}{5}f_{01}^{(1)}\right)+\tfrac{\sqrt{2}}{\alpha}\tfrac{\partial ln\alpha}{\partial x_3}\left(\tfrac{1}{\sqrt{2}}f_{02}^{(1)}\right)+\\
&\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{2}{5}f_{01}^{(1)}\right)=0,\\
&\tfrac{df_{00}^{(0)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\sqrt{2}f_{01}^{(1)}\right)+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}f_{01}^{(0)}\right)+\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{1}{3}f_{00}^{(0)}\right)+\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{1}{3}f_{00}^{(0)}\right)=0,\\
&\tfrac{df_{02}^{(0)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\left(-\sqrt{\tfrac{2}{3}}f_{01}^{(1)}\right)\right)+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}\left(\tfrac{2}{\sqrt{3}}f_{01}^{(0)}\right)\right)+\tfrac{dln\alpha}{dt}\left(-2f_{02}^{(0)}\right)+\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(\tfrac{2}{\sqrt{3}}f_{01}^{(1)}\right)\\
&+\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{5}{7}f_{02}^{(0)}\right)+\tfrac{\sqrt{2}}{\alpha}\tfrac{\partial ln\alpha}{\partial x_3}\left(2\sqrt{\tfrac{2}{3}}f_{01}^{(0)}\right)+\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{11}{7}f_{02}^{(0)}\right)=J_{02}^{(0)},\\
&\tfrac{df_{02}^{(1)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\sqrt{2}f_{01}^{(0)}\right)+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}f_{01}^{(1)}\right)+\tfrac{dln\alpha}{dt}\left(-2f_{02}^{(1)}\right)+\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(-2f_{01}^{(0)}\right)+\\
&\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{6}{7}f_{02}^{(1)}\right)+\tfrac{\sqrt{2}}{\alpha}\tfrac{\partial ln\alpha}{\partial x_3}\left(\sqrt{2}f_{01}^{(1)}\right)+\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{9}{7}f_{02}^{(1)}\right)=J_{02}^{(1)},\\
&\tfrac{df_{02}^{(2)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\left(2f_{01}^{(1)}\right)\right)+\tfrac{dln\alpha}{dt}\left(-2f_{02}^{(2)}\right)+\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(-2\sqrt{2}f_{01}^{(1)}\right)+\\
&\tfrac{\partial U_1}{\partial x_1}\left(-\tfrac{9}{7}f_{02}^{(2)}\right)+\tfrac{\partial U_3}{\partial x_3}\left(-\tfrac{3}{7}f_{02}^{(2)}\right)=J_{02}^{(2)},\\
&\tfrac{df_{10}^{(0)}}{dt}+\tfrac{\partial}{\partial x_1}\left(\tfrac{1}{\alpha}\left(-\tfrac{2}{\sqrt{3}}f_{01}^{(1)}\right)\right)+\tfrac{\partial}{\partial x_3}\left(\tfrac{1}{\alpha}\left(-\sqrt{\tfrac{2}{3}}f_{01}^{(0)}\right)\right)+\tfrac{dln\alpha}{dt}\left(-2f_{10}^{(0)}+\sqrt{6}f_{00}^{(0)}\right)+\\
&\alpha\tfrac{dU_1}{dt}\left(\tfrac{1}{\sqrt{3}}f_{01}^{(1)}\right)+\alpha\tfrac{dU_3}{dt}\left(\sqrt{\tfrac{2}{3}}f_{01}^{(0)}\right)+\tfrac{\sqrt{2}}{\alpha^2}\tfrac{\partial ln\alpha}{\partial x_1}\left(5\sqrt{\tfrac{2}{3}}f_{01}^{(1)}\right)+\tfrac{\partial U_1}{\partial x_1}\left(-f_{10}^{(0)}-\tfrac{\sqrt{6}}{3}f_{00}^{(0)}\right)+\\
&\tfrac{\partial U_3}{\partial x_1}\left(\sqrt{3}f_{02}^{(1)}\right)+\tfrac{\sqrt{2}}{\alpha}\tfrac{\partial ln\alpha}{\partial x_3}\left(-\tfrac{5}{\sqrt{3}}f_{01}^{(0)}\right)+\tfrac{\partial U_1}{\partial x_3}\left(\tfrac{1}{\sqrt{3}}f_{02}^{(1)}\right)+\tfrac{\partial U_3}{\partial x_3}\left[-\left(\tfrac{4}{3\sqrt{2}}f_{02}^{(0)}+f_{10}^{(0)}\right)\right]=0.
\end{aligned}
\tag{45}
$$

The system of Equation (45) contains seven equations regarding the moments of the particle distribution function, of which four equations correspond to the laws of conservation of mass, momentum, and energy. $J_{02}^{(0)}, J_{02}^{(1)}$, and $J_{02}^{(2)}$ are moments of the collision integral.

To derive a boundary condition for the system of Equation (45), we use Relation (10):

$$
\begin{gathered}
\int_{(n_{\partial G},c)>0}(n_{\partial G},c)f_2^{+(t,x_{\partial G},c)}\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc-\\
\beta\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)f_2^{-(t,x_{\partial G},c)}\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc-\\
(1-\beta)\int_{(n_{\partial G},c)<0}(n_{\partial G},-c)exp\left(\frac{-|c|^2}{2R\Theta}\right)\Phi_{n,2l+1,m}^{(c)}(\alpha c)dc=0,\\
2(n+l)+1=1, m=0,1,\ldots,2l+1,
\end{gathered}
\tag{46}
$$

where

$$
f_2(t,x,c)=f_0(\alpha|c|)\sum_{2n+l=0}^{2}\left(\sum_{m=0}^{l}f_{nl}^{(m)}(t,x)\Phi_{nlm}^{(c)}(\alpha c)\right).
\tag{47}
$$

From the equation $2(n+l)+1=1$, it follows that $2(n+l)=0\rightarrow n=l=0$, and thus $m=0,1$. Note that $\Phi_{010}^{(c)}(\alpha c)=\alpha|c|\mu, \Phi_{011}^{(c)}(\alpha c)=\left(\frac{\alpha|c|}{\sqrt{2}}\right)\sqrt{1-\mu^2}cos\psi$.

Let the area $G=\{-a_j<x_j<a_j, j=1,3\}$ be a rectangle. In Equation (46), instead of $f_2(t,x,c)$, we substitute its value according to Formula (47) and perform integration over half-spaces. Then, for the system of Equation (45), we have a boundary condition (the calculation of integrals is omitted due to their complexity):

$$\left(\frac{2}{\sqrt{2}\alpha}\left(f_{02}^{(1)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\left(f_{01}^{(0)}\right)^{+}\right)\right)(t,\pm a_1) = \beta\left(\frac{2}{\sqrt{2}\alpha}\left(f_{02}^{(1)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\left(f_{01}^{(0)}\right)^{-}\right)\right)(t,\pm a_1) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_1,$$

$$\left(\frac{1}{\alpha}\left(\sqrt{2}f_{00}^{(0)} - \sqrt{\frac{2}{3}}f_{02}^{(0)} + 2f_{02}^{(2)} - \frac{2}{\sqrt{3}}f_{10}^{(0)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\left(f_{01}^{(1)}\right)^{+}\right)\right)(t,\pm a_1) = \beta\left(\frac{1}{\alpha}\left(\sqrt{2}f_{00}^{(0)} - \sqrt{\frac{2}{3}}f_{02}^{(0)} + 2f_{02}^{(2)} - \frac{2}{\sqrt{3}}f_{10}^{(0)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\left(f_{01}^{(1)}\right)^{-}\right)\right)(t,\pm a_1) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_1$$

$$\left(\frac{1}{\alpha}\left(f_{00}^{(0)} + \frac{2}{\sqrt{3}}f_{02}^{(0)} - \sqrt{\frac{2}{3}}f_{10}^{(0)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\sqrt{2}\left(f_{01}^{(0)}\right)^{+}\right)\right)(t,\pm a_3) = \beta\left(\frac{1}{\alpha}\left(f_{00}^{(0)} + \frac{2}{\sqrt{3}}f_{02}^{(0)} - \sqrt{\frac{2}{3}}f_{10}^{(0)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\sqrt{2}\left(f_{01}^{(0)}\right)^{-}\right)\right)(t,\pm a_3) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_3,$$

$$\left(\frac{1}{\alpha}\left(f_{02}^{(1)}\right)^{+} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\sqrt{2}\left(f_{01}^{(1)}\right)^{+}\right)\right)(t,\pm a_3) = \beta\left(\frac{1}{\alpha}\left(f_{02}^{(1)}\right)^{-} \pm \frac{1}{\alpha\sqrt{\pi}}\left(\sqrt{2}\left(f_{01}^{(1)}\right)^{-}\right)\right)(t,\pm a_3) \pm \frac{1-\beta}{\alpha\sqrt{\pi}}F_3,$$

$$F_1 = \frac{1}{8\sqrt{2}}, F_2 = \frac{1}{4\sqrt{2}}, t \in (0,T]. \tag{48}$$

The upper sign +(−) corresponds to the moments of particle distribution function falling on the boundary (reflected from the boundary). From Equation (48), it is evident that at the ends of the interval $\left(-a_j, a_j\right), j = 1,3$, two boundary conditions are set. Introducing the following vectors and matrices:

$$w = \left(f_{01}^{(0)}, f_{01}^{(1)}\right)', \ u = \left(f_{00}^{(0)}, f_{02}^{(0)}, f_{02}^{(1)}, f_{02}^{(2)}, f_{10}^{(0)}\right)', \ W = (w,u)', \ A_1 = \begin{pmatrix} 0 & \mathcal{A}_1 \\ \mathcal{A}_1' & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \mathcal{A}_3 \\ \mathcal{A}_3' & 0 \end{pmatrix},$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & -\sqrt{2/3} & 0 & 2 & -2/\sqrt{3} \end{pmatrix}, \ \mathcal{A}_3 = \begin{pmatrix} 1 & 2/\sqrt{3} & 0 & 0 & -\sqrt{2/3} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_1 = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{2\sqrt{2}}\right), \ B_3 = \frac{1}{\alpha\sqrt{\pi}}\left(\frac{1}{\sqrt{2}}\right), \ J(W,W) = \left(0,0,0,J_{02}^{(0)}, J_{02}^{(1)}, J_{02}^{(2)}, 0\right)'.$$

A_1 and A_3—are symmetric square matrices of the seventh order.

$$A_1 = \begin{pmatrix} 0 & \mathcal{A}_1 \\ \mathcal{A}_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2/3} & 0 & 2 & -\sqrt{2/3} \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2/3} & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2/3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \mathcal{A}_3 \\ \mathcal{A}_3' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 2/\sqrt{3} & 0 & 0 & -\sqrt{2/3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2/\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{2/3} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote by $\mathcal{F} = \mathcal{F}\left(\mathrm{W}, \alpha, \frac{\mathrm{d}\ln\alpha}{\mathrm{dt}}, \alpha\frac{\mathrm{dU_1}}{\mathrm{dt}}, \alpha\frac{\mathrm{dU_3}}{\mathrm{dt}}, \frac{1}{\alpha^2}\frac{\partial\ln\alpha}{\partial x_1}, \frac{1}{\alpha}\frac{\partial\ln\alpha}{\partial x_3}, \frac{\partial U_1}{\partial x_1}, \frac{\partial U_1}{\partial x_3}, \frac{\partial U_3}{\partial x_1}, \frac{\partial U_3}{\partial x_3}\right)$ a vector consisting of the lower order terms of the system of equations (2.1). The initial-boundary value problem for the system of Equation (45) under boundary conditions in Equation (48) is written in vector–matrix form:

$$\frac{dW}{dt} + \frac{\partial}{\partial x_1}\left(\frac{1}{\alpha}A_1 W\right) + \frac{\partial}{\partial x_3}\left(\frac{1}{\alpha}A_3 W\right) + \mathcal{F} = J(W,W),\ t \in (0,T],\ x = (x_1, x_3) \in G, \quad (49)$$

$$W(0,x) = W_0(x), x \in G, \quad (50)$$

$$\left(\tfrac{1}{\sqrt{2\alpha}}\mathcal{A}_j u^+ \pm B_j w^+\right)(t, \pm a_j) = \beta\left(\tfrac{1}{\sqrt{2\alpha}}\mathcal{A}_j u^- \mp B_j w^-\right)(t, \pm a_j) \pm \tfrac{1-\beta}{\alpha\sqrt{\pi}}F_j,\ j = 1,3, \quad t \in (0,T], \quad (51)$$

The eigenvalues of matrix A_1:

$$\lambda_1 = -2.8284, \lambda_2 = -1.4142, \lambda_3 = -0.0000, \lambda_4 = 0.0000, \lambda_5 = 0.0000, \lambda_6 = 1.4142, \lambda_7 = 2.8284.$$

The eigenvalues of matrix A_3: $\lambda_1 = -1.7321$, $\lambda_2 = -1.0000$, $\lambda_3 = -0.0000$, $\lambda_4 = -0.0000$, $\lambda_5 = 0.0000$, $\lambda_6 = 1.0000$, $\lambda_7 = 1.7321$.

Each matrix has two positive, two negative, and three zero eigenvalues. The moment equation system in the second approximation contains seven equations regarding the moments of the particle distribution function, and the number of macroscopic boundary conditions at the ends of the interval $(-a_j, a_j)$, $j = 1,\ 3$ equals two.

4. Discussion

The initial-boundary value problem for the two-dimensional Boltzmann equation under microscopic Maxwellian conditions is approximated by the corresponding problem for a two-dimensional nonstationary system of moment equations under macroscopic boundary conditions. In this case, the Boltzmann equation contains a term depending on the speed of a body moving in a liquid, and the Maxwell condition contains a parameter that depends on the temperature of the body's surface. Macroscopic boundary conditions for the moment system of equations depend on the body's surface temperature. The theorem in the second section proves the existence of a global solution in time to the initial-boundary value problem for a two-dimensional system of moment equations in the first approximation in the function space $C([0,T]; L_2(G))$. The questions of existence for a two-dimensional system of moment equations in the second approximation and higher approximations, as well as determining the speed of a body moving in a liquid and the temperature of the body's surface, will be investigated in subsequent works.

5. Conclusions

The study of the motion of solid bodies in fluids, particularly the investigation of the forces that the medium exerts on a moving body, is an important and relevant problem in hydrodynamics. The increase in the speed of movement of ships and submarines requires numerical experiments to determine the hydrodynamic characteristics of bodies moving in a fluid. The velocity of movement and surface temperature of the body, as well as the hydrodynamic characteristics of the fluid, can be determined using the initial-boundary value problem for the moment system of Equation (23) under the boundary conditions in Equations (8) and (9).

In the case of body motion in fluids, the moment system of Equation (23) depends on the velocity of the moving body and the surface temperature of the body, while the boundary conditions in Equations (8) and (9) depend on the surface temperature of the moving body. The system of Equation (23) represents a complex nonlinear hyperbolic system of equations regarding the moments of the particle distribution function. The left side of system in Equation (23) depends on unknown parameters α and (U_1, U_3), while the right side is a quadratic form—moments of the collision integral. Additionally, the boundary conditions in Equations (8) and (9) also depend on an unknown parameter. The characteristics of the moment equation system depend on three unknown parameters: the movement velocity and surface temperatures of the body moving in the fluid stream. Generally, formulating boundary conditions for such systems of equations is very difficult. However, we have managed to formulate boundary conditions for the moment equation system in the form of Equations (8) and (9).

It is crucial to provide proof of the correctness of initial-boundary problems for the system of Equation (23) in different approximations under the boundary conditions in Equations (8) or (9) from a mathematical standpoint. Understanding the hydrodynamic characteristics of a body in motion through fluids is a significant problem in continuum mechanics.

Determining the velocity of movement and the surface temperature of a body moving in a fluid stream, as well as fluid parameters, is an important current task in hydrodynamics. To determine the unknown coefficients of the system of moment equations, additional information about solving the direct problem is provided. The next step is solving the inverse problem. The plan is to develop an iterative numerical method for solving both direct and inverse problems for a nonstationary nonlinear system of moment equations, implement software algorithms, and apply them to solve the described inverse problems.

Author Contributions: Conceptualization, A.S. and Y.A.; methodology, A.S.; validation, A.S., formal analysis, Y.A.; investigation, A.S.; writing—original draft preparation, A.S.; writing—review and editing, Y.A; visualization, S.A.; supervision, S.A.; project administration, A.S.; funding acquisition, Y.A. and S.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan, grant no. AP23486726.

Data Availability Statement: The data used to support the findings of this study are included within the article.

Conflicts of Interest: The author declares no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Appendix A

$$a_1\left(f_{nl}^{(m)}\right) = -(l+2n)f_{nl}^{(m)} + 2\sqrt{n\left(n+l+{}^1\!/_2\right)}f_{n-1,l}^{(m)},$$

$$a_2\left(f_{nl}^{(m)}\right) = \frac{1}{\sqrt{2}}\left\{\frac{l+2n}{l+2}\sqrt{\frac{n(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}f_{n-1,l+1}^{(m+1)}\right.$$
$$-\sqrt{\frac{n(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}f_{n-1,l+1}^{(m-1)} - \frac{2\sqrt{n(n+l+1/2)}}{l+2}$$
$$\times\left[\sqrt{\frac{(n-1)(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}f_{n-2,l+1}^{(m+1)} - \sqrt{\frac{(n-1)(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}f_{n-2,l+1}^{(m-1)}\right]$$
$$-\frac{(l+2n)\gamma_{nl}^{(m)}}{\sqrt{2}(2l+1)(l+2)}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\left[\left(n+l+{}^1\!/_2\right)S_{n-1}^{l+{}^1\!/_2} - nS_n^{l+{}^1\!/_2}\right]$$
$$\times\left[\left((l+m)(l+m-1)-1\right)P_{l-1}^{(m-1)}\,cos(m-1)\,\psi\right]f_k dc$$
$$-\frac{\sqrt{2}}{(2l+1)(l+2)\alpha^3}\left((l+2n)\gamma_{nl}^{(m)} + \left(n+l+{}^1\!/_2\right)\gamma_{n-1,l}^{(m)}\right)D_{nl}^{(m)}(d_{|c|}f_k)$$
$$+\frac{2\left(n+l+{}^1\!/_2\right)\gamma_{nl}^{(m)}}{l+2}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\left[(n+l-1/2)S_{n-2}^{l+1/2} - (n-1)S_{n-1}^{l+1/2}\right]$$
$$\times\left\{\frac{1}{\sqrt{2}(2l+1)}[(l+m)(l+m-1)P_{l-1}^{(m-1)}\,cos(m-1)\,\psi - P_{l-1}^{(m+1)}\,cos(m+1)\,\psi]\right\}f_k dc$$
$$-\frac{2\sqrt{2}\left(n+l+{}^1\!/_2\right)\gamma_{nl}^{(m)}}{(2l+1)(l+2)\alpha^2}D_{n-1,l}^{(m)}(d_{|c|}f_k),$$

$$a_3\left(f_{nl}^{(m)}\right) = \frac{1}{\sqrt{2}}\left\{\frac{2(l+2n)}{l+2}\sqrt{\frac{n(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}f_{n-!,l+1}^{(m)}\right.$$
$$\left.-\frac{4(n+l+1/2)}{l+2}\sqrt{\frac{(n-1)(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}f_{n-2,l+1}^{(m)}\right\}$$
$$-\frac{(l+2n)\gamma_{nl}^{(m)}}{\sqrt{2}(2l+1)(l+2)}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\left[\left(n+l+{}^1\!/_2\right)S_{n-1}^{l+{}^1\!/_2} - nS_n^{l+{}^1\!/_2}\right]\left[2lP_{l-1}^{(m-1)}cosm\psi\right]f_k dc$$
$$-\frac{2}{(2l+1)(l+2)\alpha^3}\left((l+2n)\gamma_{nl}^{(m)} + \left(n+l+{}^1\!/_2\right)\gamma_{n-1,l}^{(m)}\right)\widetilde{D}_{nl}^{(m)}(d_{|c|}f_k)$$
$$+\frac{4\left(n+l+{}^1\!/_2\right)\gamma_{nl}^{(m)}}{l+2}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l-1}\times$$
$$\left[(n+l-1/2)S_{n-2}^{l+1/2} - (n-1)S_{n-1}^{l+1/2}\right]\left\{\sqrt{2}(l+m)P_{l-1}^{(m)}cosm\psi\right\}f_k dc - \frac{2\sqrt{2}\left(n+l+{}^1\!/_2\right)\gamma_{nl}^{(m)}}{(2l+1)(l+2)\alpha^2}\widetilde{D}_{n-1,l}^{(m)}(d_{|c|}f_k);$$

$$b_1\left(f_{nl}^{(m)}\right) = -(l+2n)\left[\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}\,f_{n,l+1}^{(m+1)} - \sqrt{n}f_{n-1,l+1}^{(m+1)}\right)\right.$$
$$- \sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}\,f_{n,l+1}^{(m-1)} - \sqrt{n}f_{n-1,l+1}^{(m-1)}\right)$$
$$- \sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}\,f_{n,l-1}^{(m+1)} - \sqrt{n+1}f_{n+1,l-1}^{(m+1)}\right)$$
$$\left.+ \sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}\,f_{n,l-1}^{(m-1)} - \sqrt{n+1}f_{n+1,l-1}^{(m-1)}\right)\right]$$
$$+ 2\sqrt{n(n+l+1/2)}\left[\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+1/2}\,f_{n-1,l+1}^{(m+1)}\right.\right.$$
$$\left. - \sqrt{n-1}f_{n-2,l+1}^{(m+1)}\right)$$
$$- \sqrt{\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+1/2}\,f_{n-1,l+1}^{(m-1)} - \sqrt{n-1}f_{n-2,l+1}^{(m-1)}\right)$$
$$- \sqrt{\frac{(l-m-1)(l-m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l-1/2}\,f_{n-1,l-1}^{(m+1)} - \sqrt{n}f_{n,l-1}^{(m+1)}\right)$$
$$\left.+ \sqrt{\frac{(l+m-1)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l-1/2}\,f_{n-1,l-1}^{(m-1)} - \sqrt{n}f_{n,l-1}^{(m-1)}\right)\right],$$

$$b_2\left(f_{nl}^{(m)}\right) = \left\{\frac{-1}{4(2l+1)}\left[\frac{(l+m+2)(l+m+1)}{2l+3} + \frac{(l-m-1)(l-m)}{2l-1} + \frac{(l+m)(l+m-1)}{2l-1} + \frac{(l-m+1)(l-m+2)}{2l+3}\right]\left((l+2n+\right.\right.$$
$$\left.\left. 1)f_{nl}^{(m)} + 2\sqrt{n\left(n+l+\frac{1}{2}\right)}\,f_{n-1,l}^{(m)}\right) - \left(\left((l+2n+1)\right)D1_{nl}^{(m)} + 2\sqrt{n\left(n+l+\frac{1}{2}\right)}\,D1_{n-1,l}^{(m)}\right)\right\},$$

$$b_3\left(f_{nl}^{(m)}\right) = (l+2n+1)c_2^{(1)}\left(f_{nl}^{(m)}\right) + 2\sqrt{n\left(n+l+\frac{1}{2}\right)}\,c_2^{(1)}\left(f_{n-1,l}^{(m)}\right)$$

$$c_1\left(f_{nl}^{(m)}\right) = (l+2n)\left[\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+3/2}\,f_{n,l+1}^{(m)} - \sqrt{n}f_{n-1,l+1}^{(m)}\right)\right.$$
$$\left.+\sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l+1/2}\,f_{n,l-1}^{(m)} - \sqrt{n+1}f_{n+1,l-1}^{(m)}\right)\right]$$
$$-2\sqrt{n\left(n+l+\frac{1}{2}\right)}\left[\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}\left(\sqrt{n+l+1/2}\,f_{n-1,l+1}^{(m)} - \sqrt{n-1}f_{n-2,l+1}^{(m)}\right)\right.$$
$$\left.+\sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}}\left(\sqrt{n+l-1/2}\,f_{n-1,l-1}^{(m)} - \sqrt{n}f_{n,l-1}^{(m)}\right)\right];$$

$$c_2\left(f_{nl}^{(m)}\right) = c_2^{(1)}\left(f_{nl}^{(m)}\right) + c_2^{(1)}\left(f_{n-1,l}^{(m)}\right),$$

$$c_2^{(1)}\left(f_{nl}^{(m)}\right) = \frac{l+2n+1}{(l+3)}\left\{\frac{l-m+1}{2l+3}\right.$$
$$\times\left[\sqrt{\frac{(l+m+1)(l+m+2)(l+m+3)n(n+l+3/2)}{(2l+1)(2l+5)(l-m+1)}}f_{n-1,l+2}^{(m+1)}\right.$$
$$-\sqrt{\frac{(l+m+1)(l-m+2)(l-m+3)n(n+l+3/2)}{(2l+1)(2l+5)(l-m+1)}}f_{n-1,l+2}^{(m-1)}$$
$$+\left(\frac{1}{2l+3}\left[(l+m+1)\sqrt{n(n+l+1/2)(l+m)(l-m-1)}f_{n-1,l}^{(m-1)}\right.\right.$$
$$\left.-(l-m+1)\sqrt{n(n+l+1/2)(l-m)(l+m+1)}f_{n-1,l}^{(m+1)}\right]$$
$$+\frac{(l+m)}{2l-1}\left[\sqrt{n(n+l+1/2)(l-m)(l+m+1)}f_{n-1,l}^{(m+1)}\right.$$
$$\left.-(l-m)\sqrt{\frac{n(n+l+1/2)(l-m+1)}{l+m}}f_{n-1,l}^{(m-1)}\right]$$
$$-\frac{n}{2l+3}\left[(l+m+1)\sqrt{(l-m+1)(l+m)}f_{n,l}^{(m-1)}\right.$$
$$\left.-(l-m+1)\sqrt{(l+m+1)(l-m)}f_{n,l}^{(m+1)}\right]$$
$$+\frac{n(l+m)}{2l-1}\sqrt{(l+m+1)(l-m)}f_{n,l}^{(m+1)} - \frac{n(l-m)}{2l-1}\sqrt{\frac{l-m+1}{l+m}}f_{n,l}^{(m-1)}$$
$$+\frac{l+m}{2l-1}\left(\sqrt{\frac{n(n+l+1/2)(n+l-1/2)(2l+1)(l-m)}{(n+l-3/2)(2l-3)(l+m)}}f_{n-1,l-2}^{(m-1)}\right.$$
$$\left.-\sqrt{\frac{n(n+l+1/2)(n+l-1/2)(2l+1)(l-m)(l-m-1)(l-m-2)}{(n+l-3/2)(2l-3)(l+m)}}f_{n-1,l-2}^{(m+1)}\right)$$
$$-2n\sqrt{\frac{n+l+1/2}{n+l-1/2}}\left(\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)}}f_{n,l-2}^{(m-1)}\right.$$
$$\left.-\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)(l-m-1)(l-m-2)}}f_{n,l-2}^{(m+1)}\right)$$
$$+n\sqrt{\frac{(n+1)}{n+l+1/2}}\left(\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)}}f_{n+1,l-2}^{(m-1)}\right.$$
$$\left.\left.\left.\left.-\sqrt{\frac{(2l+1)(l-m)(l-m+1)(l-m+2)}{(2l-3)(l+m)}}f_{n+1,l-2}^{(m+1)}\right)\right]\right)\right)\right\} - \gamma_{nl}^{(m)}D_{nl}^{(m)}\left(d_{|c|}f_k\right);$$

$$c_2^{(1)}\left(f_{n-1,l}^{(m)}\right)=\frac{2(n+l+1/2)}{(l+3)}\left\{\frac{l-m+1}{2l+3}\right.$$

$$\times\left[\sqrt{\frac{(l+m+1)(l+m+2)(l+m+3)(n-1)(n+l+1/2)}{(2l+1)(2l+5)(l-m+1)}}f_{n-2,l+2}^{(m+1)}\right.$$

$$-\sqrt{\frac{(l+m+1)(l-m+2)(l-m+3)(n-1)(n+l+1/2)}{(2l+1)(2l+5)(l-m+1)}}f_{n-2,l+2}^{(m-1)}$$

$$+\left(\frac{1}{2l+3}\left[(l+m+1)\sqrt{(n-1)(n+l-1/2)(l+m)(l-m-1)}f_{n-2,l}^{(m-1)}\right.\right.$$

$$\left.-(l-m+1)\sqrt{(n-1)(n+l-1/2)(l-m)(l+m+1)}f_{n-2,l}^{(m+1)}\right]$$

$$+\frac{(l+m)}{2l-1}\left[\sqrt{(n-1)(n+l-1/2)(l-m)(l+m+1)}f_{n-2,l}^{(m+1)}\right.$$

$$\left.-(l-m)\sqrt{\frac{(n-1)(n+l-1/2)(l-m+1)}{l+m}}f_{n-2,l}^{(m-1)}\right]$$

$$-\frac{n-1}{2l+3}\left[(l+m+1)\sqrt{(l-m+1)(l+m)}f_{n-1,l}^{(m-1)}\right.$$

$$\left.-(l-m+1)\sqrt{(l+m+1)(l-m)}f_{n-1,l}^{(m+1)}\right]$$

$$+\frac{(n-1)(l+m)}{2l-1}\sqrt{(l+m+1)(l-m)}f_{n-1,l}^{(m+1)}-\frac{(n-1)(l-m)}{2l-1}\sqrt{\frac{l-m+1}{l+m}}f_{n-1,l}^{(m-1)}$$

$$+\frac{l+m}{2l-1}\left(\sqrt{\frac{(n-1)(n+l-1/2)(n+l-3/2)(2l+1)(l-m)}{(n+l-5/2)(2l-3)(l+m)}}f_{n-2,l-2}^{(m-1)}\right.$$

$$\left.-\sqrt{\frac{(n-1)(n+l-1/2)(n+l-3/2)(2l+1)(l-m)(l-m-1)(l-m-2)}{(n+l-5/2)(2l-3)(l+m)}}f_{n-2,l-2}^{(m+1)}\right)$$

$$-2(n-1)\sqrt{\frac{n+l-1/2}{n+l-3/2}}\left(\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)}}f_{n-1,l-2}^{(m-1)}\right.$$

$$\left.-\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)(l-m-1)(l-m-2)}}f_{n-1,l-2}^{(m+1)}\right)$$

$$+(n-1)\sqrt{\frac{n}{n+l-1/2}}\left(\sqrt{\frac{(2l+1)(l-m)}{(2l-3)(l+m)}}f_{n,l-2}^{(m-1)}\right.$$

$$\left.\left.\left.\left.-\sqrt{\frac{(2l+1)(l-m)(l-m+1)(l-m+2)}{(2l-3)(l+m)}}f_{n,l-2}^{(m+1)}\right)\right]\right)\right\}-\gamma_{n-1,l}^{(m)}D_{n-1,l}^{(m)}\left(d_{|c|}f_k\right)$$

$$c_3\left(f_{nl}^{(m)}\right)=-\left(\frac{2(l+2n+1)}{(2l+3)(l+3)}\sqrt{\frac{(l-m+1)(l-m+2)(l+m+1)(l+m+2)}{(2l+1)(2l+5)}}\left[\sqrt{n(n+l+3/2)}f_{n-1,l+2}^{(m)}\right.\right.$$

$$\left.-\sqrt{n(n-1)}f_{n-2,l+2}^{(m)}\right]+\frac{(l+2n+1)}{2l+1}\left(\frac{(l-m+1)(l+m+1)}{2l+3}+\frac{(l-m)(l+m)}{2l-1}\right)f_{nl}^{(m)}$$

$$+(l+2n+1)\gamma_{nl}^{(m)}\left\{\frac{2(l+m)(l+m-1)}{(2l+1)(2l-1)(l+3)}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^l\left[\left(n+l+\frac{1}{2}\right)S_{n-1}^{l+\frac{1}{2}}-nS_n^{l+\frac{1}{2}}\right]P_{l-2}^{(m)}cosm\psi f_k dc\right.$$

$$\left.\left.-\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\partial_{n,l-2}^{(m)}\left(d_{|c|}f_k\right)-\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\partial_{n,l+2}^{(m)}\left(d_{|c|}f_k\right)\right\}\right)$$

$$
-\left(\frac{4\sqrt{n\left(n+l+\frac{1}{2}\right)}}{(2l+3)(l+3)}\sqrt{\frac{(l-m+1)(l-m+2)(l+m+1)(l+m+2)}{(2l+1)(2l+5)}}\right.
$$

$$
\times\left[\sqrt{(n-1)(n+l+1/2)}\,f_{n-2,l+2}^{(m)}-\sqrt{(n-1)(n-2)}f_{n-3,l+2}^{(m)}\right]
$$

$$
+\frac{2\sqrt{n\left(n+l+\frac{1}{2}\right)}\left(\frac{(l-m+1)(l+m+1)}{2l+3}+\frac{(l-m)(l+m)}{2l-1}\right)f_{n-1,l}^{(m)}}{2l+1}
$$

$$
+2\sqrt{n\left(n+l+\frac{1}{2}\right)}\gamma_{n-1,l}^{(m)}
$$

$$
\times\left\{\frac{2(l+m)(l+m-1)}{(2l+1)(2l-1)(l+3)}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l}\left[\left(n+l-\frac{1}{2}\right)S_{n-2}^{l+\frac{1}{2}}\right.\right.
$$

$$
\left.-(n-1)S_{n-1}^{l+\frac{1}{2}}\right]P_{l-2}^{(m)}cosm\psi f_k dc-\frac{(l+m)(l+m-1)}{(2l+1)(2l-1)}\partial_{n-1,l-2}^{(m)}(d_{|c|}f_k)
$$

$$
\left.\left.-\frac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}\partial_{n-1,l+2}^{(m)}(d_{|c|}f_k)\right\}\right),
$$

$$
D_{nl}^{(m)}(d_{|c|}f_k)=\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+2}S_n^{l+1/2}\left\{\left[P_{l+1}^{(m+1)}-P_{l-1}^{(m+1)}\right]cos(m+1)\psi\right.
$$

$$
+\left[(l+m)(l+m-1)P_{l-1}^{(m-1)}-(l-m+1)(l-m+2)P_{l+1}^{(m-1)}\right]cos(m
$$

$$
\left.-1)\psi\right\}(d_{|c|}f_k)d\mu d\,\psi;
$$

$$
\widetilde{D}_{nl}^{(m)}(d_{|c|}f_k)=\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+2}S_n^{l+1/2}\left[(l-m+1)P_{l+1}^{(m)}(\mu)+(l\right.
$$

$$
\left.+m)P_{l-1}^{(m)}(\mu)\quad\right]cosm\psi(d_{|c|}f_k)d\mu d\psi.
$$

$$
\partial_{n,l\pm2}^{(m)}(df_k)=\frac{2\sqrt{2}}{(l+3)\alpha^3}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+3}S_n^{l+\frac{1}{2}}P_{l\pm2}^{(m)}cosm\psi df_k d\mu d\psi
$$

$$
\overline{D_{nl}^{(m)}}(df_k)=\frac{\gamma_{nl}^{(m)}}{(2l+1)(l+3)}\frac{\sqrt{2}}{\alpha^3}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l+3}S_n^{l+\frac{1}{2}}\left\{\frac{l-m+1}{2l+3}\left[\left(P_{l+2}^{(m+1)}-P_l^{(m+1)}\right)cos(m+1)\psi+\right.\right.
$$

$$
\left.\left((l+m+1)(l+m)P_l^{(m-1)}-(l-m+2)(l-m+3)P_{l+2}^{(m-1)}\right)cos(m-1)\psi\right]+
$$

$$
\frac{l+m}{2l-1}\left[\left(P_l^{(m+1)}-P_{l-2}^{(m+1)}\right)cos(m+1)\psi+\left((l+m-1)(l+m-2)P_{l-2}^{(m-1)}-\right.\right.
$$

$$
\left.\left.\left.(l-m)(l-m+1)P_l^{(m-1)}\right)cos(m-1)\psi\right]\right\}df_k d\mu d\psi.
$$

$$
D1_{nl}^{(m)}=\frac{\gamma_{nl}^{(m)}}{4(2l+1)}\int\left(\frac{\alpha|c|}{\sqrt{2}}\right)^{l}S_n^{l+\frac{1}{2}}\left\{\frac{1}{2l+3}\left[\left(P_{l+2}^{(m+2)}-P_l^{(m+2)}\right)cos(m+2)\psi\right.\right.
$$

$$
\left.-(l-m+1)(l-m+2)P_{l+2}^{(m)}cosm\psi\right]
$$

$$
-\frac{1}{2l-1}\left[\left(P_l^{(m+2)}-P_{l-2}^{(m+2)}\right)cos(m+2)\,\psi\right.
$$

$$
\left.+(l+m)(l+m-1)P_{l-2}^{(m)}cosm\psi\right]
$$

$$
+\frac{1}{2l-1}\left[-(l+m)(l+m-1)P_{l-2}^{(m)}cosm\psi\right.
$$

$$
+\left((l+m-2)(l+m-3)P_{l-2}^{(m-2)}\right.
$$

$$
\left.\left.-(l-m+1)(l-m+2)P_l^{(m-2)}\right)cos\,(m-2)\psi\right]
$$

$$
+\frac{1}{2l+3}\left[-(l-m+1)(l-m+2)P_{l+2}^{(m)}cosm\psi\right.
$$

$$
+\left((l+m)(l+m-1)P_l^{(m-2)}-(l-m+3)(l-m+4)P_{l+2}^{(m-2)}\right)cos\,(m
$$

$$
\left.\left.-2)\psi\right]\right\}f_k dc.
$$

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