AppliedMath

Article

Convection of Physical Quantities of Random Density

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Abstract: We study the random flow, through a thin cylindrical tube, of a physical quantity of random density, in the presence of random sinks and sources. We model convection in terms of the expectations of the flux and density and solve the initial value problem for the resulting convection equation. We propose a difference scheme for the convection equation, that is both stable and satisfies the Courant–Friedrichs–Lewy test, and estimate the difference between the exact and approximate solutions.

Keywords: convection equation; stochastic differential equations; finite-difference approximation schemes; Bochner integral reminder

1. Introduction and Statement of Main Results

Models involving the convection equation, were used to discuss the random flux of several physical quantities in a number of applied science problems (cf. G. C. Craig, B. G. Cohen, and R. S. Plant [1]; A. Hermoso, V. Holmar, and R. S. Plant [2]; D. D. Holm and W. Pan [3]), e.g., when dealing with atmosphere and ocean dynamics, where diffusive effects appear as negligible (cf. [3]). The stochastic models in [3] build on work by R.H. Kraichnan [4,5] and C. R. Doering, W. Horsthemke, and J. Riordan [6], on the effects of stochastic fluctuations on convection. The Holm–Pan models (cf. *op. cit.*) are essentially derived from a Hamiltonian principle for the deterministic case, by introducing noise. On the other hand, the very convection equation (without sources or drains)

$$u_t + c\, u_x = 0 \tag{1}$$

can be seen as a stochastic differential equation describing say random motion with bias c. Through the present paper we adopt a similar yet new approach, as follows. Let $\phi(x,\, t)$ be the one-dimensional flux of a physical quantity of density $u(x,\, t)$, and let $s(x,\, t)$ be a sink/source function. We assume that $\phi(x,\,\cdot\,)$, $u(x,\,\cdot\,)$ and $s(x,\,\cdot\,)$ are stochastic processes $\mathscr{T} \to L^2(\Omega)$ and derive, from elementary physical considerations, the conservation law

$$u_t(x,\, t) + \phi_x(x,\, t) = s(x,\, t), \quad 0 < x < \ell, \quad t \in \mathscr{T}. \tag{2}$$

Here, $\{\Omega,\, \mathscr{K},\, P\}$ is an infinite probability field, and partials are defined in the mean square. Equation (2) is derived from a conservation law in integral form, itself obtained by exhibiting Bochner integrals of vector-valued (i.e., $L^2(\Omega)$-valued) functions as mean square limits of "Riemann sums" (cf. e.g., [7] (p. 89)). When convection is manifest, one postulates, in the deterministic setting, that $\phi = c\, u$ for some constant $c > 0$ having the dimensions of a velocity. As ϕ and u have a random nature, we model convection in terms of the expectations of ϕ and u, i.e., we postulate that

$$M\, \phi(x,\, t) = c\, M\, u(x,\, t) \tag{3}$$

Citation: Barletta, E.; Dragomir, S.; Esposito, F. Convection of Physical Quantities of Random Density. *AppliedMath* **2024**, *4*, 225–249. https://doi.org/10.3390/appliedmath4010012

Academic Editor: Carlo Bianca

Received: 31 December 2023
Revised: 26 January 2024
Accepted: 2 February 2024
Published: 5 February 2024

for all $0 < x < \ell$ and $t \in \mathscr{T}$. Here, $M\xi = \int_\Omega \xi(\omega)\, d\,P(\omega)$ for any random variable ξ. This is one of the main novelties brought forth by the present paper, and under the additional requirement that

$$\left| r_{u(x,\,t)\,\phi(x,\,t)} \right| = 1, \tag{4}$$

we show that (3) yields

$$\phi - c\,u = \mathrm{v}\,\mathring{u} \tag{5}$$

for some function $\mathrm{v} : [0,\, \ell] \times \mathscr{T} \to \mathbb{R}$ (having the dimensions of a velocity), where $r_{\xi\,\eta}$ is the correlation coefficient of the random variables ξ and η, and $\mathring{\xi} = \xi - M\,\xi$. Consequently (2) leads to the *convection equation*

$$u_t + (c + \mathrm{v})\,u_x - \mathrm{v}\,M\big[u_x\big] + \mathrm{v}_x\,\mathring{u} = s, \tag{6}$$

and we consider the initial value problem

$$u(x,\,0) = f(x) \tag{7}$$

for (6). Taking expectations in (6) and (7) leads to the initial value problem for the ordinary convection, or transport, equation so that the mean value of the spatial derivative of the density can be computed and (6) becomes

$$u_t + (c + \mathrm{v})\,u_x + \mathrm{v}_x\,u = s + B, \tag{8}$$

$$B(x,\,t) = \frac{\partial}{\partial x}\Big\{ \mathrm{v}\,M\big[f(x - c\,t)\big] \Big\}.$$

We solve the initial value problem (7) for (8) within the theory of characteristic manifolds (cf. [8]) (pp. 56–61). To write the equation of the characteristic curves Γ for (8), we consider generalized solutions to (8), i.e., solutions with jumps across Γ in their first derivatives. As a byproduct, we determine the law according to which the intensity of jumps propagates along Γ. The rather involved form (37) (in Section § 4) of the solution calls, for any practical purposes, for an approximation scheme. We consider two finite-difference equations, with either a forward- or backward-difference quotient for the x-partials, solve the two equations in the absence of sinks or sources ($s = 0$) and under the structural assumption $\mathrm{v} \in (0,\, +\infty)$, and investigate the stability of the two difference schemes. The problem remains open in more general cases, where v is no longer a constant. The difference scheme involving backward difference quotients for x-derivatives

$$\frac{1}{k}\,\{w(x,\,t+k) - w(x,\,t)\} + \frac{c+\mathrm{v}}{h}\,\{w(x+h,\,t) - w(x,\,t)\} = B_k(x,t), \tag{9}$$

$$B_k(x,\,t) := -\frac{\mathrm{v}}{c\,k}\,M\big[f\big(x - c\,(t+k)\big) - f(x - c\,t)\big],$$

turns out to be stable, and to satisfy the Courant–Friedrichs–Lewy test provided that

$$(c + \mathrm{v})\,\lambda \le 1 \tag{10}$$

where $\lambda = k/h$ is the mesh ratio of the grid $G = \big\{(m\,h,\, n\,k) \,:\, m \in \mathbb{Z},\quad n \in \mathbb{Z}_+\big\} \subset \mathbb{R} \times [0,\, +\infty)$ used to discretize (8). Provided that (10) holds, and under structural assumptions on the initial data, i.e., that $M\big[f'(x)\big] \ge 0$ and $f'' : \mathbb{R} \to L^2(\Omega)$ is bounded, we estimate in mean square the difference between the exact solution $u(x,\,t)$ and the approximate solution $w(x,\,t)$ (the solution to the initial value problem $w(x,\,0) = f(x)$ for (9))

$$\big\|u(x,\,t) - w(x,\,t)\big\| \le \frac{K\,h\,t}{\lambda}\,\sup_{x\in\mathbb{R}}\,\big\|f''(x)\big\| + \frac{\mathrm{v}}{c}\,M\big[f(x) - f(x - c\,t)\big] \tag{11}$$

for any $(x, t) \in G$. The main tool in deriving (11) is the truncated Taylor formula with Bochner integral rest, that we prepare in Section § 5 for functions $f : \mathbb{R} \to \mathfrak{X}$ with values in an arbitrary Fréchet space $\mathfrak{X}$. Our main references for vector-valued integration are W. Rudin [7] and W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander [9].

2. Conservation Laws

Let $(\Omega, \mathscr{K}, P)$ be a probability field, and let S_a be the set of all random signals, or stochastic processes $\xi : \mathscr{T} \to L^2(\Omega)$, with the time set $\mathscr{T} = [0, +\infty)$. Let $u = u(x, t)$, $\phi = \phi(x, t)$, and $s = s(x, t)$ be, respectively, the density of a given physical quantity, the flux of the quantity through a cylindrical tube $\mathscr{C}$ of cross-sectional area A, and the rate at which the quantity is created or destroyed within the section at x at time t, such that

$$u(x, \cdot\,),\ \phi(x, \cdot\,),\ s(x, \cdot\,) \in S_a\,, \quad 0 \le x \le \ell\,.$$

We adopt the following structural assumptions:

(i) The functions $u, \phi, s : [0, \ell] \times \mathscr{T} \to L^2(\Omega)$ are continuous.

(ii) The partials $\dfrac{\partial u}{\partial t}, \dfrac{\partial \phi}{\partial x} : [0, \ell] \times \mathscr{T} \to L^2(\Omega)$ are well defined and continuous.

Let $Q = [a, b] \subset [0, \ell]$ be an arbitrary subinterval, and let $\{E_j : 1 \le j \le N\}$ be a partition of Q, i.e., a mutually disjoint family of Borel subsets $E_j \subset Q$ covering Q, and let $x_j \in E_j$, $1 \le j \le N$. The amount of the quantity in the tube between the sections at $x = a$ and $x = b$ at time t is approximated by $A \sum_{j=0}^{N-1} \mathfrak{m}(E_j)\, u(x_j\,, t)$, where $\mathfrak{m}$ is the Lebesgue measure. By the structural assumption (i), for every $t \in \mathscr{T}$, the subset $u(Q, t) \subset L^2(\Omega)$ is compact, and hence $\overline{\mathrm{co}}\big[u(Q, t)\big]$ is compact, too, by Milman's theorem (cf. e.g., Theorem 3.25 in [7] (p. 76)). Here, $\overline{\mathrm{co}}(A)$ denotes the closed convex hull of the set $A \subset L^2(\Omega)$. Indeed, Milman's theorem applies to compact subsets of locally convex spaces and, in particular, Banach spaces such as $L^2(\Omega)$. Consequently, by (Theorem 3.27 in [7] (p. 78)), there is a unique $v_t \in \overline{\mathrm{co}}\big[u(Q, t)\big]$ (the Bochner integral $v_t = \int_Q u(\,\cdot\,, t)\; d\,\mathfrak{m}_1$) such that $\Lambda(v_t) = \int_Q \Lambda\big(u(\,\cdot\,, t)\big)\; d\,\mathfrak{m}_1$ for every $\Lambda \in L^2(\Omega)^*$. Here, $d\,\mathfrak{m}_1(x) = |Q|^{-1}\; d\,\mathfrak{m}(x)$ with $|Q| = b - a$. For every $\nu \ge 1$, there is a partition $\{E_j^\nu : 1 \le j \le N_\nu\}$ of Q such that for any $x_j^\nu \in E_j^\nu$

$$\int_Q u(x, t)\; d\,\mathfrak{m}_1(x) - \sum_{j=1}^{N_\nu} \mathfrak{m}_1\big(E_j^\nu\big)\, u\big(x_j^\nu\,, t\big) \in B_{1/\nu}(0) \subset L^2(\Omega)$$

Thus, $\lim_{\nu\to\infty} \sum_{j=0}^{N_\nu - 1} \mathfrak{m}_1\big(E_j^\nu\big)\, u\big(x_j^\nu\,, t\big)$ exists in the topology of $L^2(\Omega)$, and $A\,|Q|\, \int_Q u(\,\cdot\,, t)\; d\,\mathfrak{m}_1$ is the amount of quantity in the cylinder $\mathscr{C}_Q$ of generator Q.

Lemma 1. *Let $F \in S_a$ be given by $F(t) = \int_Q u(x, t)\; d\,\mathfrak{m}_1(x)$. Then, F is differentiable (in the mean square) and*

$$F'(t) = \int_Q \frac{\partial u}{\partial t}(x, t)\; d\,\mathfrak{m}_1(x)\,. \tag{12}$$

Proof. The proof is straightforward. Indeed, for any $\Lambda \in L^2(\Omega)^*$, the function $\Lambda(u) : Q \times \mathscr{T} \to \mathbb{R}$ is continuous, and its partial derivative $\partial\Lambda(u)/\partial t : Q \times \mathscr{T} \to \mathbb{R}$ is well defined and continuous. Then,

$$\frac{d}{dt} \int_a^b \Lambda\big[u(x, t)\big]\; d\,\mathfrak{m}_1(x) = \int_a^b \frac{\partial \Lambda(u)}{\partial t}(x, t)\; d\,\mathfrak{m}_1(x) =$$

$$= \Lambda\Big(\int_Q \frac{\partial u}{\partial t}(x, t)\; d\,\mathfrak{m}_1(x)\Big)$$

yields (12). □

Therefore, $A|Q| \int_Q (\partial u/\partial t)(x,\, t)\; d\,\mathfrak{m}_1(x)$ is the rate of change of the total amount of quantity in $\mathscr{C}_Q$. Similarly, $A|Q| \int_Q s(x,\, t)\; d\,\mathfrak{m}_1(x)$ represents the amount of the quantity that is created (destroyed) in $\mathscr{C}_Q$. By convention, the flux is positive if the flow (parallel to the cylinder's axis) is to the right and negative if the flow is to the left; hence, the flux contribution (to the rate of change of the quantity in $\mathscr{C}_Q$) at the moment $t \in \mathscr{T}$ is $-A\,\phi(\,\cdot\,,\, t)\big|_{\partial Q}$. Then,

$$|Q| \int_Q \frac{\partial u}{\partial t}(x,\, t)\; d\,\mathfrak{m}_1(x) = -\phi(\,\cdot\,,\, t)\big|_{\partial Q} + |Q| \int_Q s(x,\, t)\; d\,\mathfrak{m}_1(x) \tag{13}$$

i.e., the rate of change of the amount of quantity in $\mathscr{C}_Q$ equals the rate at which it flows in at $x = a$, minus the rate at which it flows out at $x = b$ plus the rate at which it is created (destroyed) within $\mathscr{C}_Q$. By the Leibniz–Newton formula (applied to $\Lambda(\phi(\,\cdot\,,\, t))$ with $t \in \mathscr{T}$ fixed and arbitrary $\Lambda \in L^2(\Omega)^*$), the conservation law (13) becomes

$$\int_Q \left\{ \frac{\partial u}{\partial t}(x,\, t) + \frac{\partial \phi}{\partial x}(x,\, t) - s(x,\, t) \right\} d\,\mathfrak{m}_1(x) = 0,$$

yielding

$$\frac{\partial u}{\partial t}(x,\, t) + \frac{\partial \phi}{\partial x}(x,\, t) = s(x,\, t), \quad 0 < x < \ell, \;\; t > 0 \tag{14}$$

by the arbitrariness of the interval $Q \subset [0,\, \ell]$. Developments to other geometries and other-than-circular cross-section shapes (as suggested by the reviewer) may be considered. For instance, if the area of the cross-section at x is a smooth function $A(x)$, then an approximation by Riemann sums argument similar to the above leads to the conservation law

$$A(x) \left\{ \frac{\partial u}{\partial t}(x,\, t) + \frac{\partial \phi}{\partial x}(x,\, t) - s(x,\, t) \right\} + A'(x)\,\phi(x,\, t) = 0. \tag{15}$$

The conservation law (15) was adopted by A. M. Berezhoskii, M. A. Pustovoit, and S. M. Bezrukov (cf. (1.5) in [10] (p. 134706-2)) to study the reduction of the three-dimensional diffusion with a position-dependent diffusion coefficient, in tubes of varying cross-sections, to a one-dimensional description.

3. Convection Equation

Let $r_{\xi\,\eta} = \operatorname{cov}(\xi,\, \eta)/(\sigma_\xi\, \sigma_\eta)$ denote the correlation coefficient of the random variables $\xi,\, \eta$, where

$$\operatorname{cov}(\xi,\, \eta) = \langle \mathring{\xi},\, \mathring{\eta} \rangle_{L^2(\Omega)}\,, \quad \sigma_\xi = \|\mathring{\xi}\|_{L^2(\Omega)}\,, \quad \mathring{\xi} = \xi - M\,\xi\,,$$

hence $|r_{\xi\,\eta}| \le 1$. Also, $M\xi$ is the expectation, or mean value, of the random variable ξ

$$M\xi = \int_\Omega \xi(\omega)\; d\,P\,(\omega).$$

When convection is manifest, the state equation in the deterministic setting is

$$\phi = c\,u \tag{16}$$

for some constant $c > 0$, yielding

$$\left| r_{u(x,t)\,\phi(x,t)} \right| = 1, \quad 0 \le x \le \ell, \;\; t \ge 0.$$

For random flows, a reasonable assumption (modeling convection) should involve the mean values of the variables in (16) i.e.,

(iii)

$$M\,\phi = c\,M\,u \tag{17}$$

for some constant $c > 0$. Let $\mathring{u} : [0, \ell] \times \mathscr{T} \to L^2(\Omega)$ be given by $\mathring{u}(x, t) = u(x, t) - M\big[u(x, t)\big]$. Through the following calculation, we omit (x, t) for simplicity. Then, (by (17)),

$$\mathring{\phi} = \phi - c\, M\, u, \tag{18}$$

$$\operatorname{cov}(u, \phi) = \big\langle \mathring{u}\,, \mathring{\phi} \big\rangle = c\, \|\mathring{u}\|^2 + \big\langle \mathring{u},\, \phi - c\, u \big\rangle\,, \tag{19}$$

$$\sigma_\phi^2 = \|\mathring{\phi}\|^2 = \|\phi - cu\|^2 + c^2\, \sigma_u^2 + 2\, c\, \big\langle \mathring{u},\, \phi - c\, u \big\rangle\,. \tag{20}$$

Moreover, we adopt the structural assumption

(iv) $\big|r_{u\,\phi}\big| = 1$

i.e., intuitively, the random variables $u(x,t)$ and $\phi(x,t)$ are as far from being uncorrelated as possible. This is equivalent to (by (18)–(20))

$$\big|\big\langle \mathring{u}\,, \phi - c\, u \big\rangle\big| = \sigma_u\, \|\phi - c\, u\|$$

yielding

$$\phi - c\, u = \mathrm{v}\, \mathring{u} \tag{21}$$

for some function $\mathrm{v} : [0, \ell] \times \mathscr{T} \to \mathbb{R}$. An elementary dimensional analysis of (21) shows that v has the dimensions of a velocity (i.e., $[\mathrm{v}]_{\mathrm{SI}} = L \cdot T^{-1}$). We adopt the structural assumptions

(v) The partial $\dfrac{\partial u}{\partial x} : [0, \ell] \times \mathscr{T} \to L^2(\Omega)$ exists and is continuous.

(vi) $\mathrm{v} \in C^1\big([0, \ell] \times \mathscr{T}\big)$.

Therefore, we replace the deterministic convection model (16) by the probabilistic (17), yet (by (21)) our assumptions imply that the mean square error $\|\phi - c\, u\|$ is proportional to the mean square deviation $\sigma_u = \sqrt{Du}$ by a proportionality factor $|\mathrm{v}(x, t)|$. That is, the mean square approximation to which ϕ and u satisfy (16) at (x, t) depends on the size of the dispersion of the random variable $u(x, t)$. Substitution from (21) into the conservation law (14) leads to

$$\frac{\partial u}{\partial t} + (c + \mathrm{v})\, \frac{\partial u}{\partial x} - \mathrm{v}\, M\Big(\frac{\partial u}{\partial x}\Big) + \frac{\partial \mathrm{v}}{\partial x}\, \mathring{u} = s. \tag{22}$$

Let us consider the initial value problem

$$u(x, 0) = f(x) \tag{23}$$

for Equation (22) in the absence of sinks or sources, i.e.,

(vii) $M\, s = 0$,

where

(viii) $f \in C^1\big[[0, \ell],\, L^2(\Omega)\big]$.

Passing to expectations in (14) and (23) leads (by (17)) to the initial value problem

$$\frac{\partial}{\partial t}(Mu) + c\, \frac{\partial}{\partial x}(Mu) = 0, \quad M\big[u(x, 0)\big] = M\big[f(x)\big], \tag{24}$$

with the solution

$$Mu(x, t) = M\big[f(x - ct)\big]. \tag{25}$$

Finally, substitution from (25) into (22) leads to

$$Lu \equiv \frac{\partial u}{\partial t} + (c + \mathrm{v})\, \frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial x}\, u = s + \frac{\partial}{\partial x}\Big\{\mathrm{v}\, M\big[f(x - ct)\big]\Big\}. \tag{26}$$

The solution to the initial value problem (23) for Equation (26) relies on the classical theory of characteristic manifolds and the Cauchy problem (cf., e.g., [8] (pp. 56–61)).

4. Generalized Solutions, Characteristic Curves, and the Initial Value Problem

Let $\Gamma = \{\Phi = 0\} \subset \mathbb{R}^2$ be a curve with $\Phi(x, t) = x - \varphi(t)$ and $\varphi \in C^2(0, +\infty)$. Let us assume that
$$\mathbb{R}^2 \setminus \Gamma = \Omega^+ \cup \Omega^-, \quad \Omega^+ \cap \Omega^- = \varnothing .$$

Let u be a *generalized solution* to (26), i.e.,
$$u \in C(\mathbb{R}^2), \quad u^\pm \in C^1(\Omega^\pm \cup \Gamma), \quad u^\pm := u\big|_{\Omega^\pm \cup \Gamma},$$
$$Lu^\pm(x, t) = s(x, t) + \frac{\partial}{\partial x}\Big\{\mathrm{v}(x, t)\, M\big[f(x - ct)\big]\Big\}, \quad (x, t) \in \Omega^\pm \cup \Gamma .$$

Given a function $w(x,t)$ defined for $(x,t) \in \mathbb{R}^2$, we denote by
$$\big[w\big] = w^+\big(\varphi(t), t\big) - w^-\big(\varphi(t), t\big), \quad w^\pm := w\big|_{\Omega^\pm \cup \Gamma},$$
the jump of w at a point $(\varphi(t), t)$ of Γ. Then, $[u] = 0$ so that $d\,[u]/d\,t = 0$ yields
$$\left[\frac{\partial u}{\partial x}\right] \varphi'(t) + \left[\frac{\partial u}{\partial t}\right] = 0. \tag{27}$$

Let us subtract the two equations
$$(Lu^\pm)(\varphi(t), t) = s\big(\varphi(t), t\big) + \frac{\partial}{\partial x}\Big\{\mathrm{v}\, M\big[f(x - ct)\big]\Big\}_{x=\varphi(t)}$$
and form jumps. We obtain
$$(c + \mathrm{v}) \left[\frac{\partial u}{\partial x}\right] + \left[\frac{\partial u}{\partial t}\right] = 0. \tag{28}$$

If we set $\mu(t) := \big[\partial u/\partial x\big]$, then
$$\left[\frac{\partial u}{\partial x}\right] = \mu, \quad \left[\frac{\partial u}{\partial t}\right] = -\varphi'\,\mu, \tag{29}$$
and substitution from (29) into (28) yields (unless $\mu = 0$, i.e., unless $u \in C^1$)
$$\varphi'(t) = \mathrm{v}\big(\varphi(t), t\big) + c, \tag{30}$$
i.e., Γ is a characteristic curve. According to (29), the quantity μ measures the "intensity" of the jumps in the first derivatives. As $t \in \mathscr{T}$ is time, one interprets φ as the point $x = \varphi(t)$ moving along the x-axis. For every t one has in $u(x, t)$, a function of x whose first derivative is discontinuous at the moving point $\varphi(t)$. The speed dx/dt of "propagation of the discontinuity" is determined by (30).

The jumps in different points of Γ are therefore related to each other. Let us determine the law according to which the *intensity* μ of the jump propagates along Γ. By (29),
$$\mu' = \frac{d\mu}{dt} = \left[\frac{\partial^2 u}{\partial x^2}\right] \varphi' + \left[\frac{\partial^2 u}{\partial x\,\partial t}\right], \tag{31}$$
$$-(\varphi'\,\mu)' = \left[\frac{\partial^2 u}{\partial t\,\partial x}\right] \varphi' + \left[\frac{\partial^2 u}{\partial t^2}\right]. \tag{32}$$

From now on, we assume that $u^\pm \in C^2(\Omega^\pm \cup \Gamma)$. Let us differentiate in $Lu^\pm = s + \partial\big\{\mathrm{v}\, M\big[f(x - ct)\big]\big\}/\partial x$ with respect to x
$$\frac{\partial^2 u^\pm}{\partial x\,\partial t} + \Big\{c + \mathrm{v}(x, t)\Big\}\, \frac{\partial^2 u^\pm}{\partial x^2} + 2\,\frac{\partial \mathrm{v}}{\partial x}\,\frac{\partial u^\pm}{\partial x} + \frac{\partial^2 \mathrm{v}}{\partial x^2}\, u^\pm =$$

$$= \frac{\partial s}{\partial x} + \frac{\partial^2}{\partial x^2}\Big\{ \mathrm{v}\, M\big[f(x-ct)\big]\Big\}$$

evaluate along Γ, subtract the two resulting equations, and form jumps. We obtain

$$(c+\mathrm{v})\left[\frac{\partial^2 u}{\partial x^2}\right] + \left[\frac{\partial^2 u}{\partial x\,\partial t}\right] + 2\,\frac{\partial \mathrm{v}}{\partial x}\left[\frac{\partial u}{\partial x}\right] = 0. \tag{33}$$

Similarly, differentiation with respect to t

$$\frac{\partial^2 u^\pm}{\partial t^2} + \frac{\partial \mathrm{v}}{\partial t}\,\frac{\partial u^\pm}{\partial x} + (c+\mathrm{v})\,\frac{\partial^2 u^\pm}{\partial x\,\partial t} + \frac{\partial^2 \mathrm{v}}{\partial x\,\partial t}\,u^\pm + \frac{\partial \mathrm{v}}{\partial x}\,\frac{\partial u^\pm}{\partial t} =$$

$$= \frac{\partial s}{\partial t} + \frac{\partial^2}{\partial x\,\partial t}\Big\{ \mathrm{v}\, M\big[f(x-ct)\big]\Big\}$$

leads to

$$\left[\frac{\partial^2 u}{\partial t^2}\right] + \frac{\partial \mathrm{v}}{\partial t}\left[\frac{\partial u}{\partial x}\right] + (c+\mathrm{v})\left[\frac{\partial^2 u}{\partial x\,\partial t}\right] + \frac{\partial \mathrm{v}}{\partial x}\left[\frac{\partial u}{\partial t}\right] = 0. \tag{34}$$

Let us eliminate the jumps between Equations (29) and (31)–(34) to obtain

$$\mu' + 2\,\frac{\partial \mathrm{v}}{\partial x}(\varphi(t),\, t)\,\mu = 0, \tag{35}$$

$$(\varphi'\,\mu)' + \left\{\varphi'\,\frac{\partial \mathrm{v}}{\partial x}(\varphi(t),\, t) - \frac{\partial \mathrm{v}}{\partial t}(\varphi(t),\, t)\right\}\mu = 0. \tag{36}$$

The characteristic Equation (30) yields

$$\varphi'' = \frac{\partial \mathrm{v}}{\partial x}(\varphi(t),\, t)\,\varphi' + \frac{\partial \mathrm{v}}{\partial t}(\varphi(t),\, t),$$

and a straightforward calculation shows that (35) and (36) are equivalent. Therefore, the ODE (35) for the jump intensity μ governs its growth during the propagation of singularities. The general solution to (35) is

$$\mu(t) = C\,\exp\left\{-2\int_0^t \frac{\partial \mathrm{v}}{\partial x}(\varphi(\tau),\, \tau)\,d\tau\right\},$$

so if $\mu = 0$ at some point of Γ, then $\mu = 0$ along Γ, and no jump occurs.

Let Γ_ξ be the characteristic curve of equation $x = \varphi(t, \xi)$, that is, $\varphi(\cdot\,, \xi) : (-r_0\,, r_0) \to \mathbb{R}$ is the solution to the Cauchy problem $\varphi'(t) = \mathrm{v}\big(\varphi(t),\, t\big) + c$ and $\varphi(0) = \xi$. An inspection of the (proof of the) classical existence and uniqueness of the solution to the Cauchy problem for a first-order ODE shows that to form the domain of the solution, one chooses $0 < r_0 < \min\{r,\ R/M,\ 1/L\}$ with

$$r > 0,\quad R > 0,\quad M = \sup_{(x,t)\in K}\big|\mathrm{v}(x,\, t)\big|,\quad L = \sup_{(x,t)\in K}\left|\frac{\partial \mathrm{v}}{\partial x}(x,\, t)\right|,$$

where $K = \big[\xi - R,\, \xi + R\big] \times \big[-r,\, r\big]$. Let u be a C^1 solution to the initial value problem (23) and (26), and let us set

$$U(\xi,\, t) := u\big(\varphi(t\,, \xi),\ t\big).$$

Differentiation with respect to t gives

$$U_t + \frac{\partial \mathrm{v}}{\partial x}(\varphi(t,\, \xi),\, t)\,U = s\big(\varphi(t,\, \xi),\, t\big) +$$

$$+ \frac{\partial \mathrm{v}}{\partial x}(\varphi(t,\, \xi),\, t)\,M\big[f(\varphi(t,\, \xi) - ct)\big] + \mathrm{v}\big(\varphi(t,\, \xi),\, t\big)\,M\big[f'(\varphi(t,\, \xi) - ct)\big].$$

Let us set

$$H(\xi,\, t) = \int_0^t \frac{\partial \mathrm{v}}{\partial x}\big(\varphi(\tau,\, \xi),\, \tau\big)\, d\,\tau.$$

Then,

$$U(\xi,\, t) = \exp\big(-H(\xi,\, t)\big)\,\Big\{\int_0^t \exp\big(H(\xi,\, \tau)\big)\, s\big(\varphi(\tau,\, \xi),\, \tau\big)\, d\,\tau +$$

$$+\int_0^t \exp\big(H(\xi,\, \tau)\big)\,\Big[\frac{\partial \mathrm{v}}{\partial x}\big(\varphi(\tau,\, \xi),\, \tau\big)\, M\big[f\big(\varphi(\tau,\, \xi) - c\tau\big)\big] +$$

$$+\mathrm{v}\big(\varphi(\tau,\, \xi),\, \tau\big)\, M\big[f'\big(\varphi(\tau\,,\, \xi) - c\tau\big)\big]\Big]\, d\,\tau\Big\} + f(\xi).$$

Let $\xi_0 \in \mathbb{R}$, and let $\Gamma_{\xi_0} = \{x = \varphi(t\,,\, \xi_0)\}$ be a characteristic curve with $\varphi_\xi(0, \xi_0) \neq 0$. Let $F \in C^1(\mathbb{R}^3)$ be given by $F(x,\, t,\, \xi) := x - \varphi(t,\, \xi)$ so that $F(\xi_0\,,\, 0,\, \xi_0) = 0$ and $F_\xi(\xi_0\,,\, 0,\, \xi_0) \neq 0$. By the implicit function theorem, there exist $\delta > 0$ and $\sigma > 0$, and there is a unique C^1 function $\psi : B_\delta(\xi_0\,,\, 0) \to (\xi_0 - \sigma\,,\, \xi_0 + \sigma)$ such that $\psi(\xi_0\,,\, 0) = \xi_0$ and $F\big(x,\, t,\, \psi(x,\, t)\big) = 0$. Then,

$$u(x,\, t) = \exp\big(-H(\psi(x, t),\, t)\big) \times \tag{37}$$

$$\times\Big(\int_0^t \exp\big(H(\psi(x, t),\, \tau)\big)\, s\big(g(x,\, t,\, \tau),\, \tau\big)\, d\,\tau +$$

$$+\int_0^t \exp\big(H(\psi(x, t),\, \tau)\big)\,\Big\{\frac{\partial \mathrm{v}}{\partial x}\big(g(x,\, t,\, \tau),\, \tau\big)\, M\big[f\big(g(x,\, t,\, \tau) - c\tau\big)\big] +$$

$$+\mathrm{v}\big(g(x,\, t,\, \tau),\, \tau\big)\, M\big[f'\big(g(x,\, t,\, \tau) - c\tau\big)\big]\Big\}\, d\,\tau\Big) + f\big(\psi(x, t)\big)$$

where $g(x,\, t,\, \tau) := \varphi\big(\tau,\, \psi(x,\, t)\big)$. The next section is devoted to a recast of the convection Equation (26) as either the finite-difference Equation (38) or (53), where x-derivatives are approximated by a forward, respectively, a backward, difference quotient. For the sake of simplicity, we assume that $s = 0$. Also, our analysis is confined to the case where $\|\phi - c\, u\|/\sigma_u =$ constant, i.e., precisely

(ix) $\mathrm{v} \in (0, +\infty)$.

We solve both (38) and (53) and examine the stability of the two difference schemes. The scheme provided by (53) is shown to be stable, and then we parallel the exact solution (to the initial value problem for (26) with $s = 0$ and $\mathrm{v} \in (0, +\infty)$) and the approximate solution (to the initial value problem for (53)).

5. Two Finite-Difference Equations

Let $G = \big\{(mh,\, nk) \in \mathbb{R} \times [0, +\infty) \;:\; m \in \mathbb{Z},\;\; n \in \mathbb{Z}_+\big\}$ be a grid in $\mathbb{R}^2$ of mesh size $h > 0$ in the x-direction and $k > 0$ in the t-direction. Let us consider the finite-differences equation

$$\frac{1}{k}\big\{w(x,\, t+k) - w(x,\, t)\big\} + \tag{38}$$

$$+(c + \mathrm{v})\,\frac{1}{h}\big\{w(x+h,\, t) - w(x,\, t)\big\} = B(x,\, t)$$

where

$$B(x,\, t) = \mathrm{v}\, M\big[f'(x - ct)\big].$$

Formally, Equation (38) leads to the convection equation $Lw = B$ (cf. (26) with $s = 0$ and $\mathrm{v} \in (0, +\infty)$) for $h \to 0$ and $k \to 0$. We seek for solutions $w : G \to \mathbb{R}$ to (38) together with the initial condition

$$w(x, 0) = f(x). \tag{39}$$

If we set $\lambda = k/h$, then (38) becomes

$$w(x,\, t+k) = \big\{1 + \lambda(c+\mathrm{v})\big\}\, w(x,\, t) + \tag{40}$$

$$-\lambda\,(c+\mathrm{v})\, w(x+h,\, t) + k\, B(x,\, t).$$

The advantage of the form (40) of the finite-difference equation is to express the values of w at the moment $t+k$ in terms of the values of w at the moment t. To solve (40), we need the shift operator $(Ew)(x,\, t) = w(x+h,\, t)$ acting (in the x-variable) on functions $w(x,\, t)$. Equation (40) becomes

$$w(x,\, t+k) = (Tw)(x,\, t) + k\, B(x,\, t), \tag{41}$$

$$T \equiv \big\{1 + \lambda\,(c+\mathrm{v})\big\}\, I - \lambda\,(c+\mathrm{v})\, E,$$

where $(Iw)(x,\, t) = w(x,\, t)$. If $t = nk$, then

$$w(x,\, t) = w(x,\, nk) = w\big(x,\, (n-1)k + k\big) \Longrightarrow$$

(by (40)) for $t = (n-1)k$)

$$w(x,\, nk) = (Tw)\big(x,\, (n-1)k\big) + k\, B\big(x,\, (n-1)k\big)\,. \tag{42}$$

We need

Lemma 2. *For every* $0 \le m \le n-1$

$$\big(T^m\, w\big)\big(x,\ (n-m)k\big) = \big(T^{m+1}\, w\big)\big(x,\ (n-m-1)k\big) + \tag{43}$$

$$+k\,\big(T^m\, B\big)\big(x,\ (n-m-1)k\big)$$

where $T^m = T \circ \cdots \circ T$ *(m terms).*

Proof. Let us denote the predicate (42) by $P(x,\, n)$. Let us set $\mu := (c+\mathrm{v})\,\lambda$ for simplicity so that $T = (1+\mu)\, I - \mu\, E$. Then,

$$\big(T^2\, w\big)\big(x,\ (n-2)k\big) = \tag{44}$$

$$= (1+\mu)(Tw)\big(x,\, (n-2)k\big) - \mu\,(Tw)\big(x+h,\, (n-2)k\big).$$

On the other hand (by $P(x,\, n-1)$ and $P(x+h,\, n-1)$)

$$(Tw)\big(x,\, (n-2)k\big) = w\big(x,\, (n-1)k\big) - k\, B\big(x,\, (n-2)k\big), \tag{45}$$

$$(Tw)\big(x+h,\, (n-2)k\big) = w\big(x+h,\, (n-1)k\big) - k\, B\big(x+h,\, (n-2)k\big). \tag{46}$$

Substitution from (45) to (46) into (44) yields

$$\big(T^2 w\big)\big(x,\, (n-2)k\big) =$$

$$= (1+\mu)\Big\{w\big(x,\, (n-1)k\big) - k\, B\big(x,\, (n-2)k\big)\Big\} +$$

$$-\mu\,\Big\{w\big(x+h,\, (n-1)k\big) - k\, B\big(x+h,\, (n-2)k\big)\Big\} =$$

$$= (Tw)\big(x,\, (n-1)k\big) - (1+\mu)\, k\, B\big(x,\, (n-2)k\big) + \mu\, k\, B\big(x+h,\, (n-2)k\big)$$

or

$$(Tw)\big(x,\, (n-1)k\big) = \big(T^2 w\big)\big(x,\, (n-2)k\big) + k\,(TB)\big(x,\, (n-2)k\big) \tag{47}$$

which is (43) for $m = 1$. The proof of Lemma 2 may now be completed by induction over m. Iteration of (43) for $0 \le m \le n-1$ gives

$$w(x,\, nk) = (T^n w)(x,\, 0) + k \sum_{m=0}^{n-1} \big(T^m B\big)\big(x,\, (n-m-1)k\big) \tag{48}$$

and

$$\big(T^n w\big)(x,\, 0) = \sum_{j=0}^{n} \binom{n}{j} (1+\mu)^j (-\mu)^{n-j} \big(E^{n-j} w\big)(x,\, 0)$$

or

$$\big(T^n w\big)(x,\, 0) = \sum_{j=0}^{n} \binom{n}{j} (1+\mu)^j (-\mu)^{n-j} f\big(x + (n-j)h\big). \tag{49}$$

Similarly

$$\big(T^m B\big)\big(x,\, (n-m-1)k\big) = \tag{50}$$

$$= \sum_{j=0}^{m} \binom{m}{j} (1+\mu)^j\, (-\mu)^{m-j}\, B\big(x + (m-j)h,\; (n-m-1)\,k\big).$$

□

The solution to the initial value problem (39) for the difference Equation (38) is

$$w(x,\, nk) = \sum_{j=0}^{n} \binom{n}{j} (1+\mu)^j (-\mu)^{n-j} f\big(x + (n-j)h\big) + \tag{51}$$

$$+ k\mathrm{v} \sum_{j=0}^{m} \binom{m}{j} (1+\mu)^j\, (-\mu)^{m-j}\, M\Big[f'\big(x + (m-j)h - (n-m-1)\,k\,c\big)\Big].$$

Hence, the domain of dependence for $w(x,\, t) = w(x,\, nk)$ consists of the set of points on the x-axis

$$\Big\{x,\; x+h,\; x+2h,\; \cdots\; x+nh\Big\} \cup \bigcup_{m=0}^{n-1} A_m\,,$$

$$A_m := \Big\{x - (n-m-1)\,k\,c + j\,h \;:\; 0 \le j \le m\Big\},$$

and

$$\Big\{x + jh \;:\; 0 \le j \le n\Big\} \subset \Big[x,\; x+nh\Big] \subset \mathbb{R}\,,$$

$$A_m \subset I_m\,, \quad 0 \le m \le n-1,$$

$$I_m := \Big[x - (n-m-1)\,c\,k\,,\; x - (n-m-1)\,c\,k + m\,h\Big] \subset \mathbb{R}.$$

As v is a constant $H = 0$, the exact solution to the initial value problem for (26) is (by (37) with $s = 0$ and $\mathrm{v} \in (0, +\infty)$)

$$u(x,\, t) = f\big(x - (c+\mathrm{v})t\big) + \mathrm{v} \int_0^t M\Big[f'\big(x - (c+\mathrm{v})t + \mathrm{v}\tau\big)\Big]\, d\,\tau. \tag{52}$$

Thus, the domain of dependence of $u(x,\, t)$ on the initial values f and on the expectation of their first derivatives f' consists, respectively, of the single point $\xi = x - (c+\mathrm{v})\,t$, and of the interval $\big[x - (c+\mathrm{v})t,\; x - ct\big] \subset \mathbb{R}$. Note that

$$\big[x,\; x+nh\big] \cap \big[x - (c+\mathrm{v})t,\; x - ct\big] = \varnothing\,,$$

$$I_m \cap \big[x - (c+\mathrm{v})t,\; x - ct\big] = \varnothing\,, \quad 0 \le m \le n-1.$$

Hence, one cannot expect w to converge for $h,\, k \to 0$ to the exact solution. The high degree of instability of the scheme (38) may be ascertained as follows. Let $\epsilon \in L^2(\Omega)$ be a random variable such that $\epsilon(\omega) > 0$ for a.e. $\omega \in \Omega$, and let $f_\epsilon \in C^1\big(\mathbb{R},\, L^2(\Omega)\big)$ such that

$$f_\epsilon(x + jh) = f(x + jh) + (-1)^j \epsilon\,, \quad x \in \mathbb{R}, \quad 0 \le j \le n.$$

Let $w_\epsilon(x,\, t)$ be the solution to the initial value problem $w(x,\, 0) = f_\epsilon(x)$ for the difference Equation (38), i.e.,

$$w_\epsilon(x,\, t) = w_\epsilon(x,\, nk) = w(x,\, t) + \big(1 + 2\,(c + \mathrm{v})\,\lambda\big)^n\, \epsilon$$

hence,

$$M\big[w_\epsilon(x,\, t)\big] - M\big[w(x,\, t)\big] = \big(1 + 2\,(c + \mathrm{v})\,\lambda\big)^n\, M[\epsilon]$$

i.e., for a fixed mesh ratio λ, the possible error in the expectation of w grows exponentially with the number n of steps in the t-direction. Alternatively, let us consider the difference equation

$$\begin{aligned} &\frac{1}{k}\,\big\{w(x,\, t + k) - w(x,\, t)\big\} + \\ &+ (c + \mathrm{v})\,\frac{1}{h}\,\big\{w(x,\, t) - w(x - h,\, t)\big\} = B(x,\, t). \end{aligned} \tag{53}$$

The shift operator is invertible and $(E^{-1}w)(x,\, t) = w(x - h,\, t)$. Equation (53) may then be recast as

$$\begin{aligned} w(x,\, t + k) &= (Tw)(x,\, t) + k\, B(x,\, t)\,, \\ T &\equiv \big\{1 - (c + \mathrm{v})\,\lambda\big\}\, I + (c + \mathrm{v})\lambda\, E^{-1}\,, \end{aligned} \tag{54}$$

The solution to the initial value problem (39) for (54) is

$$\begin{aligned} w(x,\, nk) &= \\ &= \sum_{j=0}^{n} \binom{n}{j}\, \big[1 - (c + \mathrm{v})\,\lambda\big]^j\, \big[\lambda(c + \mathrm{v})\big]^{n-j}\, f\big(x - (n - j)\,h\big) + \\ &+ k\,\mathrm{v} \sum_{m=0}^{n-1} \sum_{j=0}^{m} \binom{m}{j}\, \big[1 - (c + \mathrm{v})\,\lambda\big]^j\, \big[\lambda(c + \mathrm{v})\big]^{m-j} \times \\ &\times M\Big[f'\big(x - (m - j)\,h - (n - m - 1)\,c\,k\Big] \end{aligned} \tag{55}$$

whose dependence domain is

$$\Big\{x,\; x - h,\; x - 2h,\; \cdots\;,\; x - nh = x - \frac{t}{\lambda}\Big\} \cup \Big\{\bigcup_{m=0}^{n-1} A_m\Big\}\,,$$

$$A_m := \Big\{x - c(n - m - 1)\,k - j\,h \;:\; 0 \le j \le m\Big\},$$

$$\Big\{x,\; x - h,\; x - 2h,\; \cdots\;,\; x - nh\Big\} \subset \Big[x - \frac{t}{\lambda},\, x\Big] \subset \mathbb{R},$$

$$A_m \subset I_m := \big[x - (n - m - 1)\,c\,k - m\,h,\; x - (n - m - 1)\,c\,k\big] \subset \mathbb{R}.$$

The point $\xi = x - (c + \mathrm{v})t$ lies in the interval $\Big[x - \dfrac{t}{\lambda},\, x\Big]$ if and only if

$$(c + \mathrm{v})\,\lambda \le 1. \tag{56}$$

On the other hand, as $h \to 0$ and $t/n = k \to 0$

$$\bigcup_{m=0}^{n-1} I_m = \bigcup_{j=1}^{n} \left[x - ct + (n-j+1)\,c\,k - \left(1 - \frac{j}{n}\right)\frac{t}{\lambda}\,,\, x - ct + (n-j+1)\,c\,k\right] \to$$

$$\to \left[x - ct - \frac{t}{\lambda}\,,\, x - ct\right]$$

and $\xi = x - (c + \mathrm{v})t \in \left[x - ct - t/\lambda\,,\, x - ct\right] \iff \lambda \mathrm{v} \le 1$, which follows from (56). Therefore, the Courant–Friedrichs–Lewy test (requiring that the limit of the domain of dependence for the difference Equation (53) contains the domain of dependence for (37) with $s = 0$ and $\mathrm{v} \in (0, +\infty)$) is satisfied when the mesh ratio λ satisfies (56). Let us look at the stability of the scheme (53). To this end, let $\epsilon \in L^2(\Omega)$, and let $f_\epsilon(x) = f(x) + \epsilon$. Let $w_\epsilon(x,\, t)$ be the solution to Equation (53) with the initial condition $w(x, 0) = f_\epsilon(x)$, i.e.,

$$w_\epsilon(x,\, nk) =$$

$$= \sum_{j=0}^{n} \binom{n}{j} \left[1 - (c + \mathrm{v})\,\lambda\right]^j \left[\lambda(c + \mathrm{v})\right]^{n-j} f_\epsilon(x - (n-j)\,h) +$$

$$+ k\,\mathrm{v} \sum_{m=0}^{n-1} \sum_{j=0}^{m} \binom{m}{j} \left[1 - (c + \mathrm{v})\,\lambda\right]^j \left[\lambda(c + \mathrm{v})\right]^{m-j} \times$$

$$\times M\Big[f'_\epsilon(x - (m-j)\,h - (n-m-1)\,c\,k\Big] = w(x,\, nk) + \epsilon,$$

yielding

$$M\big[w_\epsilon(x,\, n\,k)\big] = M\big[w(x,\, n\,k)\big] + M[\epsilon]$$

that is to say an error of size $M[\epsilon]$ in the mean value of the initial random signal f results in a maximum possible error of size $M[\epsilon]$ in the expectation of $w(x,\, t) = w(x,\, nk)$.

To parallel the exact solution (52) to the solution $w(x,\, t)$ to the initial value problem (39) for (54), we start by recasting (52) as

$$u(x,\, t) = f\big(x - (c + \mathrm{v})\,t\big) + M\big[f(x - c\,t) - f\big(x - (c + \mathrm{v})\,t\big)\big]. \tag{57}$$

We need the following

Lemma 3.

(i) *$M : L^2(\Omega) \to \mathbb{R}$ is continuous.*

(ii) *For every continuous function $\varphi : \mathscr{T} \to L^2(\Omega)$ and every $t \in \mathscr{T}$,*

$$\int_0^t M\big[\varphi(\tau)\big]\, d\,\tau = M\Big[\int_0^t \varphi(\tau)\, d\,\tau\Big]. \tag{58}$$

Proof. (i) By the Cauchy–Schwartz inequality (and $P(\Omega) = 1$)

$$\big|M[g]\big| \le \int_\Omega |g|\, d\,P \le \|g\|_{L^2(\Omega)}$$

for every $g \in L^2(\Omega)$.

(ii) One starts with the observation that (58) holds for simple functions $\varphi : [0,\, t] \to L^2(\Omega)$ (by the mere linearity of M). Next, as $\varphi : [0,\, t] \to L^2(\Omega)$ is a Bochner integrable, there is a sequence of simple functions $\varphi_\nu : [0,\, t] \to L^2(\Omega)$ such that $\varphi_\nu \to \varphi$ pointwise a.e. in $[0,\, t]$ and $\lim_{\nu\to\infty} \int_0^t \|\varphi(\tau) - \varphi_\nu(\tau)\|_{L^2(\Omega)}\, d\,\tau = 0$, and

$$\int_0^t \varphi(\tau)\,d\tau = \lim_{\nu\to\infty}\int_0^t \varphi_\nu(\tau)\,d\tau$$

in $L^2(\Omega)$. Then,

$$M\Big[\int_0^t \varphi(\tau)\,d\tau\Big] = \quad \text{(by the continuity of } M)$$

$$= \lim_{\nu\to\infty} M\Big[\int_0^t \varphi_\nu(\tau)\,d\tau\Big] = \quad \text{(by (58) on simple functions)}$$

$$= \lim_{\nu\to\infty}\int_0^t M\big[\varphi_\nu(\tau)\big]\,d\tau\,.$$

Note that $M\circ g : [0,\,t] \to \mathbb{R}$ is simple for every simple function $g : [0,\,t] \to L^2(\Omega)$ (by the linearity of M). Hence, $M\circ\varphi_\nu$ are simple. Also,

$$\big|M[\varphi_\nu(\tau)] - M[\varphi(\tau)]\big| \le \|\varphi_\nu(\tau) - \varphi(\tau)\|_{L^2(\Omega)}$$

implies both that $M\circ\varphi_\nu \to M\circ\varphi$ pointwise a.e. in $[0,\,t]$, and that $\lim_{\nu\to\infty}\int_0^t \Big|M[\varphi(\tau)] - M[\varphi_\nu(\tau)]\Big|\,d\tau = 0$. Therefore, $M\circ\varphi$ is a (scalar-valued) Bochner integrable function and

$$\int_0^t M[\varphi(\tau)]\,d\tau = \lim_{\nu\to\infty}\int_0^t M[\varphi_\nu(\tau)]\,d\tau.$$

□

At this point, we may check (57). Indeed, the last term in (52) is

$$\mathrm{v}\int_0^t M[f'(x-(c+\mathrm{v})\,t+\mathrm{v}\,\tau)]\,d\tau =$$

$$= \int_0^t M\Big[\frac{d}{d\tau}\Big\{f\big(x-(c+\mathrm{v})\,t+\mathrm{v}\,\tau\big)\Big\}\Big]\,d\tau = \quad \text{(by Lemma 3)}$$

$$= M\Big[\int_0^t \frac{d}{d\tau}\Big\{f\big(x-(c+\mathrm{v})\,t+\mathrm{v}\,\tau\big)\Big\}\,d\tau\Big] = M\big[f(x-c\,t) - f\big(x-(c+\mathrm{v})\,t\big].$$

Let $u(x,\,t)$ be the solution to the problem

$$u_t + (c+\mathrm{v})\,u_x = \mathrm{v}\,M\big[f'(x-c\,t)\big], \quad u(x,\,0) = f(x).$$

Let $w(x,\,t)$ be the solution to the problem

$$\frac{1}{k}\Big\{w(x,\,t+k) - w(x,\,t)\Big\} + \frac{c+\mathrm{v}}{h}\Big\{w(x,\,t) - w(x-h,\,t)\Big\} =$$

$$= \mathrm{v}\,M\big[f'(x-c\,t)\big], \quad w(x,\,0) = f(x).$$

Let us set (for the sake of brevity)

$$(\mathscr{L}\,\phi)(x,\,t) \equiv \phi(x,\,t+k) - \big\{1-(c+\mathrm{v})\lambda\big\}\,\phi(x,\,t) - (c+\mathrm{v})\,\lambda\,\phi(x-h,\,t)$$

for any function $\phi(x,\,t)$. By Taylor's formula with Bochner integral reminder, and base point $\xi = x-(c+\mathrm{v})\,t$

$$f(X) = \sum_{p=0}^{1}\frac{1}{p!}\,(X-\xi)^p\,f^{(p)}(\xi) + r_1(X\,;\,f,\,\xi),$$

$$r_1(X\,;\,f,\,\xi) = 2\int_0^{|X-\xi|} r^2\,f''\Big(X - \frac{X-\xi}{|X-\xi|}\,r\Big)\,dr\,,$$

$$\left\|r_1(X\,;\,f,\,\xi)\right\|_{L^2(\Omega)} \leq |X-\xi|^2 \sup_{0\leq\theta\leq 1} \left\|f''\big((1-\theta)\,X+\theta\,\xi\big)\right\|_{L^2(\Omega)}.$$

In particular for $X \in \big\{x-(c+\mathrm{v})\,(t+k),\ x-h-(c+\mathrm{v})\,t\big\}$

$$f\big(x-(c+\mathrm{v})\,(t+k)\big) = f\big(x-(c+\mathrm{v})\,t\big)-(c+\mathrm{v})\,k\,f'\big(x-(c+\mathrm{v})\,t\big)+ \\ +r_1\big(x-(c+\mathrm{v})\,(t+k)\,;\,f,\ x-(c+\mathrm{v})\,t\big), \tag{59}$$

$$f\big(x-h-(c+\mathrm{v})\,t\big) = f\big(x-(c+\mathrm{v})\,t\big)-h\,f'\big(x-(c+\mathrm{v})\,t\big)+ \\ +r_1\big(x-h-(c+\mathrm{v})\,t\,;\,f,\ x-(c+\mathrm{v})\,t\big). \tag{60}$$

Similarly, if $\xi = x-c\,t$ and $X \in \big\{x-c(t+k),\ x-h-ct\big\}$, then

$$f\big(x-c\,(t+k)\big) = f(x-c\,t)-c\,k\,f'(x-c\,t)+ \\ +r_1\big(x-c\,(t+k)\,;\,f,\ x-c\,t\big), \tag{61}$$

$$f(x-h-c\,t) = f(x-c\,t)-h\,f'(x-c\,t)+ \\ +r_1(x-h-c\,t\,;\,f,\ x-c\,t). \tag{62}$$

Next (by (57)),

$$\begin{aligned}(\mathscr{L}u)(x,t) &= f\big(x-(c+\mathrm{v})(t+k)\big)-f\big(x-(c+\mathrm{v})t\big)+ \\ &-(c+\mathrm{v})\lambda\Big\{f\big(x-h-(c+\mathrm{v})t\big)-f\big(x-(c+\mathrm{v})t\big)\Big\}+ \\ &+M\big[f\big(x-c(t+k)\big)-f(x-ct)\big]+ \\ &-M\Big[f\big(x-(x+\mathrm{v})(t+k)\big)-f\big(x-(c+\mathrm{v})t\big)\Big]+ \\ &-(c+\mathrm{v})\lambda\Big\{M\big[f(x-h-ct)-f(x-ct)\big]+ \\ &-M\Big[f\big(x-h-(c+\mathrm{v})t\big)-f\big(x-(c+\mathrm{v})t\big)\Big]\Big\}=\end{aligned}$$

(by (59)–(62))

$$\begin{aligned}&= r_1\big(x-(c+\mathrm{v})(t+k)\,;\,f,\ x-(c+\mathrm{v})t\big)+ \\ &-(c+\mathrm{v})\lambda\,r_1\big(x-h-(c+\mathrm{v})t\,;\,f,\ x-(c+\mathrm{v})t\big)+ \\ &+M\Big[r_1\big(x-c(t+k)\,;\,f,\ x-ct\big)\Big]+ \\ &-M\Big[r_1\big(x-(c+\mathrm{v})(t+k)\,;\,f,\ x-(c+\mathrm{v})t\big)\Big]+ \\ &-(c+\mathrm{v})\lambda\,\Big\{M\Big[r_1\big(x-h-ct\,;\,f,\ x-ct\big)\Big]+ \\ &-M\Big[r_1\big(x-h-(c+\mathrm{v})t\,;\,f,\ x-(c+\mathrm{v})t\big)\Big]\Big\}.\end{aligned}$$

We adopt the structural assumption

(x) $f'' : \mathbb{R} \to L^2(\Omega)$ is bounded.

Finally, the estimates on the Taylor rest

$$\left\|r_1\big(x-(c+\mathrm{v})(t+k)\,;\,f,\ x-(c+\mathrm{v})t\big)\right\| \leq (c+v)^2k^2 \sup_{X\in\mathbb{R}} \left\|f''(X)\right\|,$$

$$\left\|r_1\big(x-h-(c+\mathrm{v})t\,;\,f,\ x-(c+\mathrm{v})t\right\| \leq h^2 \sup_{X\in\mathbb{R}} \left\|f''(X)\right\|,$$

$$\left\| r_1\left(x - c(t+k)\,;\, f,\; x - ct\right) \right\| \leq c^2 k^2 \sup_{X \in \mathbb{R}} \left\| f''(X) \right\| ,$$

$$\left\| r_1\left(x - h - ct\,;\, f,\; x - ct\right) \right\| \leq h^2 \sup_{X \in \mathbb{R}} \left\| f''(X) \right\| ,$$

yield

$$\left\| (\mathscr{L}u)(x,\, t) \right\| \leq K\, h^2 \sup_{X \in \mathbb{R}} \left\| f''(X) \right\| \tag{63}$$

where $K = \left\{ 2(c + \mathrm{v})^2 + c^2 \right\} \lambda^2 + 3\,(c + \mathrm{v})\,\lambda$. Let us set $\phi(x,\, t) := u(x,\, t) - w(x,\, t)$ so that $\phi(x,\, 0) = 0$. Then,

$$\left\| (\mathscr{L}\phi)(x,\, t) \right\| \leq \left\| (\mathscr{L}u)(x,\, t) \right\| + \left\| (\mathscr{L}w)(x,\, t) \right\| \leq$$

$$\leq K\, h^2 \sup_{X \in \mathbb{R}} \left\| f''(x) \right\| + k \left| B(x,\, t) \right|$$

so that

$$\left\| \phi(x,\, t+k) \right\| \leq \left\| (\mathscr{L}\phi)(x,\, t) \right\| +$$

$$+ \left| 1 - (c + v)\lambda \right| \left\| \phi(x,\, t) \right\| + (c + v)\lambda \left\| \phi(x - h,\, t) \right\| \leq$$

(by (56))

$$\leq K\, h^2 \sup_{x \in \mathbb{R}} \left\| f''(x) \right\| + \sup_{x \in \mathbb{R}} \left\| \phi(x,\, t) \right\|,$$

yielding

$$\sup_{x \in \mathbb{R}} \left\| \phi(x,\, t+k) \right\| \leq \sup_{x \in \mathbb{R}} \left\| \phi(x,\, t) \right\| + \tag{64}$$

$$+ K\, h^2 \sup_{x \in \mathbb{R}} \left\| f''(x) \right\| + k \left| B(x,\, t) \right| .$$

Let us set $t = nk$ with $n \in \mathbb{Z}_+$ and use (64) to iterate. We obtain (as $nh^2 = (ht)/\lambda$)

$$\sup_{x \in \mathbb{R}} \left\| \phi(x,\, nk) \right\| \leq \frac{h\,t\,K}{\lambda} \sup_{x \in \mathbb{R}} \left\| f''(x) \right\| + k \sum_{j=0}^{n-1} \left| B(x,\, jk) \right|. \tag{65}$$

We adopt the structural assumption

(xi) $M\left[f'(x) \right] \geq 0$ for any $x \in \mathbb{R}$.

Note that

$$B(x,\, t) = \mathrm{v}\, M\left[f'(x - ct) \right] = -\frac{\mathrm{v}}{c}\, M\left[\frac{d}{dt} \left\{ f(x - c\,t) \right\} \right] \approx B_k(x,\, t),$$

where

$$B_k(x,\, t) := -\frac{\mathrm{v}}{c\,k}\, M\left[f\left(x - c\,(t + k)\right) - f\left(x - c\,t\right) \right].$$

From now on, let $w(x,\, t)$ be the solution to (53) with $B(x,\ ,t)$ replaced by $B_k(x,\, t)$. Then,

$$\sum_{j=0}^{n-1} \left| B(x,\, jk) \right| = -\frac{\mathrm{v}}{c\,k}\, M\left[\sum_{j=0}^{n-1} \left\{ f\left(x - c\,(j+1)\,k\right) - f\left(x - c\,j\,k\right) \right] = \right.$$

$$= -\frac{\mathrm{v}}{c\,k}\, M\left[f(x - c\,t) - f(x) \right]$$

and (65) reads

$$\sup_{x \in \mathbb{R}} \left\| \phi(x,\, t) \right\| \leq \frac{h\,t\,K}{\lambda} \sup_{x \in \mathbb{R}} \left\| f''(x) \right\| + \frac{\mathrm{v}}{c}\, M\left[f(x) - f(x - c\,t) \right]$$

yielding (11).

6. Taylor's Formula with Bochner Integral Reminder for Vector-Valued Functions

Let $\mathfrak{X}$ be a topological vector space on which $\mathfrak{X}^*$ separates points, and let $\mathfrak{m}$ be a Borel probability measure on a compact Hausdorff space Q. If $f : Q \to \mathfrak{X}$ is continuous and $\overline{\text{co}}\big[f(Q)\big]$ is a compact subset of $\mathfrak{X}$, then (cf. e.g., Theorem 3.27 in [7] (p. 78)) there is a unique $y \in \overline{\text{co}}\big[f(Q)\big]$ such that

$$\Lambda(y) = \int_Q \Lambda(f)\, d\,\mathfrak{m}, \quad \Lambda \in \mathfrak{X}^* .$$

A *partition* of Q is a finite family of mutually disjoint Borel subsets of Q whose union is Q. When $\mathfrak{X}$ is a Fréchet space, the (Bochner) integral $\int_Q f\, d\,\mathfrak{m} := y$ may be exhibited as a strong limit of "Riemann sums", i.e., for every neighborhood $V \subset \mathfrak{X}$ of the origin, there is a partition $\{E_j \,:\, 1 \le j \le N\}$ of Q such that

$$\int_Q f\, d\,\mathfrak{m} - \sum_{j=1}^{N} \mathfrak{m}(E_j)\, f(x_j) \in V$$

for any $x_j \in E_j$, $1 \le j \le N$, cf. [7] (p. 89). Approximations of Lebesgue integrals by Riemann sums were considered by S. Gy. Révész and I. Z. Ruzsa [11] and R. Nair [12] for scalar-valued functions. Recovering their results to the case of vector-valued functions is an open problem.

Let $\mathfrak{X}$ be a locally convex space over $\mathbb{R}$, on which $\mathfrak{X}^*$ separates points, and let $f \in C^{n+2}(U, \mathfrak{X})$, where $U \subset \mathbb{R}$ is an open set. For our needs (as to the applications to flows of quantities of random density), $\mathfrak{X} = L^2(\Omega)$ (with a given probability field $(\Omega\,,\, \mathscr{K}\,,\, P)$). Let $t_0 \in U$, and let us set

$$P_n(t\,;\, f,\, t_0) = \sum_{k=0}^{n} \frac{1}{k!}\, (t - t_0)^k\, f^{(k)}(t_0), \tag{66}$$

$$R_n(t\,;\, f,\, t_0) = f(t) - P_n(t\,;\, f,\, t_0). \tag{67}$$

Lemma 4.

(i) *The reminder is of order $o(|t - t_0|^n)$, i.e.,*

$$\lim_{t \to t_0} \frac{1}{(t - t_0)^n}\, R_n(t\,;\, f,\, t_0) = 0.$$

(ii) *The reminder admits the integral representation formula*

$$R_n(t\,;\, f,\, t_0) = \frac{1}{n!} \int_{t_0}^{t} (t - s)^n f^{(n+1)}(s)\, ds.$$

Proof. (i) Linear and continuous functionals $\Lambda \in \mathfrak{X}^*$ commute with derivatives of any order. Hence,

$$\Lambda[P_n(t\,;\, f,\, t_0)] = P_n(t\,;\, \Lambda(f),\, t_0) \tag{68}$$

where $\Lambda(f) := \Lambda \circ f$. As $\Lambda(f)$ is a scalar-valued function of class C^{n+1}, for any $t \in U$, there is $\alpha = \alpha(t,\, t_0\,,\, \Lambda(f)) \in [0, 1]$ such that

$$R_n(t\,;\, \Lambda(f),\, t_0) = \frac{(t - t_0)^{n+1}}{(n+1)!}\, \frac{d^{n+1}\Lambda(f)}{dt^{n+1}}(\xi) \tag{69}$$

where $\xi = (1 - \alpha)t_0 + \alpha\, t$. Consequently, as $\Lambda(f)$ is of class C^{n+2}, the representation formula (69) implies

$$\lim_{t\to t_0} \frac{R_n(t\,;\,\Lambda(f),\,t_0)}{(t-t_0)^{n+1}} = \frac{1}{(n+1)!}\,\frac{d^{n+1}\Lambda(f)}{dt^{n+1}}(t_0). \tag{70}$$

On the other hand (by (68)),

$$\frac{R_n(t\,;\,\Lambda(f),\,t_0)}{(t-t_0)^{n+1}} = \frac{1}{(t-t_0)^{n+1}}[\Lambda(f(t)) - P_n(t\,;\,\Lambda(f),\,t_0)] =$$

$$= \Lambda\left[\frac{1}{(t-t_0)^{n+1}}\, R_n(t\,;\,f,\,t_0)\right]$$

hence (by (70)), for any sequence $\{t_\nu\}_{\nu\ge1} \subset U$ with $\lim_{\nu\to\infty} t_\nu = t_0$ the limit

$$\lim_{\nu\to\infty} \Lambda\left[\frac{1}{(t_\nu-t_0)^{n+1}}\, R_n(t_\nu\,;\,f,\,t_0)\right]$$

exists and is finite. Consequently,

$$\left\{\Lambda\left[\frac{1}{(t_\nu-t_0)^{n+1}}\, R_n(t_\nu\,;\,f,\,t_0)\right] : \nu\ge1\right\} \subset \mathbb{R}$$

is a bounded set for every $\Lambda \in \mathfrak{X}'$, i.e., the set

$$E = \left\{\frac{1}{(t_\nu-t_0)^{n+1}}\, R_n(t_\nu\,;\,f,\,t_0) : \nu\ge1\right\} \subset \mathfrak{X}$$

is weakly bounded. From now on, we assume that $\mathfrak{X}$ is a locally convex space. By Theorem 3.18 in [7] (p. 70), every weakly bounded set in $\mathfrak{X}$ is also strongly bounded. Hence, for any neighborhood V of $0 \in \mathfrak{X}$, there is $s_V > 0$ such that $E \subset sV$ for any $s > s_V$, i.e.,

$$\frac{1}{(t_\nu-t_0)^{n}}\, R_n(t_\nu\,;\,f,\,t_0) \in (t_\nu - t_0)\, s\, V$$

for any $\nu \ge 1$. Consequently, the sequence

$$\left\{\frac{1}{(t_\nu-t_0)^{n}}\, R_n(t_\nu\,;\,f,\,t_0)\right\}_{\nu\ge1}$$

is strongly convergent to 0 as $\nu\to\infty$.

(ii) Let $\mathfrak{X}$ be a topological vector space on which $\mathfrak{X}^*$ separates points. By a classical representation formula

$$R_n(t\,;\,\Lambda(f),\,t_0) = \frac{1}{n!}\int_{t_0}^{t} (t-s)^n \frac{d^{n+1}(\Lambda\circ f)}{dt^{n+1}}(s)\,ds$$

for any $t \in U$. Then,

$$\Lambda\left[R_n(t\,;\,f,\,t_0) - \frac{1}{n!}\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\,ds\right] = 0$$

for any $\Lambda \in \mathfrak{X}^*$, where $\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\,ds$ is a Bochner integral. As $\mathfrak{X}^*$ separates points, we may conclude that

$$R_n(t\,;\,f,\,t_0) = \frac{1}{n!}\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\,ds \tag{71}$$

for any $t \in U$. $\square$

Lemma 5. *Let $\mathfrak{X}$ be a locally convex space on which $\mathfrak{X}^*$ separates points. Let $U \subset \mathbb{R}$ be an open neighborhood of $t_0 \in \mathbb{R}$, and let $F \in C^{n+1}(U, \mathfrak{X})$ such that $F(t_0) = F'(t_0) = \cdots = F^{(n)}(t_0) = F^{(n+1)}(t_0) = 0$. Then,*

$$\lim_{t \to t_0} \left[\frac{1}{(t-t_0)^n} F(t) \right] = 0.$$

Proof. $\Lambda(F) \in C^{n+1}(U, \mathbb{R})$ for every $\Lambda \in \mathfrak{X}'$ and (by applying repeatedly the classical l'Hôspital theorem)

$$\lim_{t \to t_0} \frac{\Lambda(F(t))}{(t-t_0)^{n+1}} = \lim_{t \to t_0} \frac{\frac{d}{dt}[\Lambda(F(t))]}{(n+1)(t-t_0)^n} = \cdots = \lim_{t \to t_0} \frac{\frac{d^n}{dt^n}[\Lambda(F(t))]}{(n+1)!\,(t-t_0)} =$$

$$= \lim_{t \to t_0} \frac{\frac{d^{n+1}}{dt^{n+1}}[\Lambda(F(t))]}{(n+1)!} = \frac{1}{(n+1)!} \Lambda\left[F^{(n+1)}(t_0)\right] = 0$$

hence,

$$\lim_{\nu \to \infty} \frac{\Lambda(F(t_\nu))}{(t_\nu - t_0)^{n+1}} = 0$$

for any sequence $\{t_\nu\}_{\nu \ge 1} \subset U$ such that $\lim_{\nu \to \infty} t_\nu = t_0$. Consequently, the set

$$\left\{ \frac{1}{(t_\nu - t_0)^{n+1}} F(t_\nu) : \nu \ge 1 \right\} \subset \mathfrak{X}$$

is weakly bounded, and then strongly bounded, in $\mathfrak{X}$. Then, for any neighborhood $V \subset \mathfrak{X}$ of 0, there is $s > 0$ such that

$$\frac{1}{(t_\nu - t_0)^n} F(t_\nu) \in (t - t_0)\, s\, V, \quad \nu \ge 1,$$

yielding $\lim_{\nu \to \infty} \left[\frac{1}{(t_\nu - t_0)^n} F(t_\nu) \right] = 0$ strongly in $\mathfrak{X}$. $\square$

Let $\mathfrak{X}$ be a Fréchet space. Let $A \subset \mathbb{R}^n$ be an open set, and let $f \in C^k(A, \mathfrak{X})$ and $x_0 \in A$. There is $R > 0$ such that $\overline{B}_R(x_0) \subset A$. Let $w \in \mathbb{R}^n$ such that $\|w\| = 1$, and let us consider the function

$$F : (-R, R) \to \mathfrak{X}, \quad F(t) = f(x_0 + tw), \quad |t| < R.$$

Then, $F \in C^k((-R, R), \mathfrak{X})$ and then by Taylor's formula with a reminder for $\mathfrak{X}$-valued functions of one real variable, cf. (66) and (67)

$$F(t) = \sum_{j=0}^{k} \frac{t^j}{j!} F^{(j)}(0) + R_k(t; F, 0). \tag{72}$$

On the other hand,

$$F^{(h)}(t) = h! \sum_{|\alpha| = h} \frac{w^\alpha}{\alpha!} (D^\alpha f)(x_0 + tw) \tag{73}$$

and choosing

$$w = \frac{1}{\|x - x_0\|}(x - x_0), \quad t = \|x - x_0\|, \quad x \in B_R(x_0),$$

the Formula (72)

becomes

$$f(x) = F(\|x - x_0\|) = \sum_{j=0}^{k} \frac{\|x - x_0\|^j}{j!} F^{(j)}(0) + R_k(\|x - x_0\|;\, F,\, 0) =$$

(for (73) with $t = 0$)

$$= \sum_{j=0}^{k} \|x - x_0\|^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \frac{(x - x_0)^\alpha}{\|x - x_0\|^{|\alpha|}} (D^\alpha f)(x_0) + r_k(x; f,\, x_0)$$

where we have set

$$r_k(x; f,\, x_0) = R_k(\|x - x_0\|;\, F,\, 0),$$
$$F(t) = f\left(x_0 + \frac{t}{\|x - x_0\|}(x - x_0)\right), \quad |t| < R.$$

The Taylor formula we seek for is

$$f(x) = \sum_{|\alpha| \le k} \frac{(x - x_0)^\alpha}{\alpha!} (D^\alpha f)(x_0) + r_k(x;\, f,\, x_0). \tag{74}$$

Next, let us assume that $f \in C^{k+2}(A, \mathfrak{X})$ so that $F \in C^{k+2}((-R, R), \mathfrak{X})$ and hence

$$R_k(t; F, 0) = \frac{1}{k!} \int_0^t (t - s)^k F^{(k+1)}(s)\, ds$$

where from (by (73) for $h = k + 1$)

$$r_k(x; f,\, x_0) = \frac{1}{k!} \int_0^{\|x - x_0\|} (\|x - x_0\| - s)^k \times$$
$$\times (k+1)! \sum_{|\alpha|=k+1} \frac{(x - x_0)^\alpha}{\alpha!\, \|x - x_0\|^{|\alpha|}} (D^\alpha f)\left(x_0 + \frac{s}{\|x - x_0\|}(x - x_0)\right) ds =$$

or (by a change of variable $r = \|x - x_0\| - s$)

$$r_k(x; f,\, x_0) = \frac{k+1}{\|x - x_0\|^{k+1}} \times \tag{75}$$
$$\times \sum_{|\alpha|=k+1} (x - x_0)^\alpha \int_0^{\|x - x_0\|} r^k\, (D^\alpha f)\left(x - \frac{r}{\|x - x_0\|}(x - x_0)\right) dr.$$

Let $\mathfrak{X}$ be a complex Fréchet space. Let $\{\| \cdot \|_m : m \ge 0\}$ be a separating family of semi-norms defining the topology of $\mathfrak{X}$ as a local convex space.

Lemma 6. *Let $A \subset \mathbb{R}^n$ be an open set and let $f \in C^{k+2}(A, \mathfrak{X})$. Let $x_0 \in A$ and $R > 0$ such that $B_R(x_0) \subset A$. Then,*

$$\|r_k(x; f,\, x_0)\|_m \le M_{k+1} \|x - x_0\|^{k+1} \times \tag{76}$$
$$\times \max_{|\alpha|=k+1} \sup_{0 \le \tau \le 1} \|(D^\alpha f)((1 - \tau)x + \tau x_0)\|_m$$

for any $x \in B_R(x_0)$.

Here, M_ℓ is the cardinality of the set $\{\alpha \in \mathbb{Z}_+^n : |\alpha| = \ell\}$.

Proof of Lemma 6.

$$\|r_k(x; f,\, x_0)\|_m \le \frac{k+1}{\|x-x_0\|^{k+1}} \sum_{|\alpha|=k+1} \prod_{j=1}^{n} \left|x_j - x_j^0\right|^{\alpha_j} \times$$

$$\times \int_0^{\|x-x_0\|} r^k \left\| (D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}\,(x-x_0)\right)\right\|_m dr \le$$

(as $\prod_{j=1}^{n}\left|x_j - x_j^0\right|^{\alpha_j} \le \|x-x_0\|^{|\alpha|}$)

$$\le \left[(k+1) \int_0^{\|x-x_0\|} r^k\, dr \right] \times$$

$$\times \sum_{|\alpha|=k+1} \sup_{0\le r\le \|x-x_0\|} \left\| (D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}\,(x-x_0)\right)\right\|_m \le$$

$$\le M_{k+1} \|x-x_0\|^{k+1} \times$$

$$\times \max_{|\alpha|=k+1} \sup_{0\le r\le \|x-x_0\|} \left\| (D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}\,(x-x_0)\right)\right\|_m .$$

Setting $\tau = r/\|x-x_0\| \in [0,1]$, one may conclude that

$$\|r_k(x; f,\, x_0)\|_m \le M_{k+1} \|x-x_0\|^{k+1} \times$$

$$\times \max_{|\alpha|=k+1} \sup_{0\le\tau\le 1} \|(D^\alpha f)((1-\tau)x + \tau x_0)\|_m .$$

□

7. Examples of Initial Value Problems

Let $F_\xi \in L^1_{\mathrm{loc}}(\Omega)$ be the distribution function $F_\xi(a) = P\big(\xi \le a\big)$ of the random variable $\xi : \Omega \to \mathbb{R}$, and let $p_\xi(x) = F'_\xi(x)$ (distributional derivative) be the corresponding probability density. Let u be the solution to the initial value problem (23) for the Equation (26) with $s = 0$, expressed as in (57), i.e.,

$$u(x,\, t) = f\big(x - (c + \mathrm{v})\, t\big) + M\big[f(x - c\, t) - f\big(x - (c+\mathrm{v})\, t\big)\big].$$

The probability distribution of u at $\{(x_i\,,\, t_i)\}_{1\le i\le p}$ is the joint probability distribution of the random variables $\xi_i := u(x_i\,,\, t_i) \in L^2(\Omega)$, i.e.,

$$P_{(x_1\,, t_1), \cdots, (x_p\,, t_p)}\big(B\big) := P\big((\xi_1\,, \cdots, \xi_p) \in B\big)$$

for any Borelian set $B \subset \mathbb{R}^p$. For every $(x,\, t) \in \mathbb{R}^2$ and $B = (-\infty,\, a] \subset \mathbb{R}$

$$P_{(x,\, t)}\big(B\big) = P\big(u(x,\, t) \le a\big) = F_{u(x,\, t)}(a).$$

Through this section, we discuss the class of functions given by (57) for various choices of random signals $f : \mathscr{T} \to L^2(\Omega)$ (where $\mathscr{T}$ is $\mathbb{R}$ or $[0, +\infty)$), e.g., random sine signals, Poisson, and Wiener processes. Of course, Poisson and Wiener processes are not differentiable, yet the right-hand side of (57) is well defined for such choices of "initial values" f.

7.1. Deterministic Initial Values and Generalizations

Let $g \in C^1(\mathbb{R})$, and let $f : \mathbb{R} \to L^2(\Omega)$ be the corresponding deterministic signal, i.e., $f(x)(\omega) = g(x)$ for any $\omega \in \Omega$ and $x \in \mathbb{R}$. Then, (57) becomes $u(x,\, t) = f(x - c\, t)$ so the solution u is a deterministic signal as well. In particular, the probability density of u at

$(x,\, t)$ is $p_{u(x,t)} = \delta_{g(x-ct)}$. Here, $\delta_a \in \mathscr{D}'(\mathbb{R})$ is the Dirac distribution concentrated at $a \in \mathbb{R}$. As another example, let $\alpha_k \in L^2(\Omega)$ and $f_k \in C^1(\mathbb{R})$, $1 \le k \le p$, be, respectively, random variables and C^1 functions, and let us set

$$f : \mathbb{R} \to L^2(\Omega), \quad f(x) = \sum_{k=1}^{n} f_k(x)\, \alpha_k\,, \quad x \in \mathbb{R}. \tag{77}$$

Then, $f \in C^1\big(\mathbb{R},\, L^2(\Omega)\big)$, and the solution is

$$u(x,t) = \sum_{k=1}^{n} \Big\{ f_k\big(x-(c+\mathrm{v})t\big)\, \mathring{\alpha}_k + f_k(x-ct)\, M\big[\alpha_k\big] \Big\}.$$

The family of initial data (77) contains the random algebraic and trigonometric polynomials

$$\sum_{k=0}^{n} x^k\, a_k\,, \quad \frac{1}{2}\, a_0 + \sum_{k=0}^{n} \Big\{ (\cos\ kx)\, a_k + (\sin\ kx)\, b_k \Big\},$$

with $a_k\,,\, b_k \in L^2(\Omega)$.

7.2. Sine Random Initial Values

Let $A,\, \alpha,\, \varphi \,:\, \Omega \to \mathbb{R}$ be independent random variables such that (1) A is square integrable, (2) $A \ge 0$, $\alpha \ge 0$, (3) A and α have the same distribution function, and 4) φ is uniformly distributed over the interval $[0,\, 2\pi)$, i.e., its probability density is

$$p_\varphi(x) = \frac{1}{2\pi}\Big\{\sigma(x) - \sigma(x-2\pi)\Big\}$$

with σ the Heaviside unit step function. Let

$$f(x)(\omega) = A(\omega)\ \sin\big[x\,\alpha(\omega) + \varphi(\omega)\big] \tag{78}$$

be a sine random signal, with random amplitude, frequency, and phase. Then,

$$\int_\Omega \big|f(x)(\omega)\big|^2\, d\,P(\omega) \le \|A\|^2_{L^2(\Omega)} < \infty$$

i.e., $f(x) \in L^2(\Omega)$. The solution to (26) with the initial values (78) possesses the remarkable property that its distributions are translation invariant in the x-variable, i.e.,

$$P_{(x_1+a\,,\,t_1)\,,\,\cdots\,,\,(x_p+a\,,\,t_p)} = P_{(x_1\,,\,t_1),\,\cdots\,,\,(x_p\,,\,t_p)}\,, \quad (x_i\,,\, t_i) \in \mathbb{R}^2\,,\ a \in \mathbb{R}\,. \tag{79}$$

For every Borel set $B \subset \mathbb{R}^p$,

$$P_{(x_1+a\,,\,t_1)\,,\,\cdots\,,\,(x_p+a\,,\,t_p)}(B) = P\Big(\big(u(x_1+a\,,\, t_1)\,,\, \cdots\,,\, u(x_p+a\,,\, t_p)\big) \in B\Big).$$

Let us consider the Borel set $C \subset \mathbb{R}^3$ given by

$$C = \Big\{ (x,\, y,\, z) \in \mathbb{R}^3 \,:\, x \ge 0,\ \ y \ge 0,\ \ 0 \le z \le 2\pi,\ \sum_{i=1}^{p} \Big\{ x\, \sin\big((x_i - (c+\mathrm{v})t_i)\, y + z\big) +$$

$$+ M\Big[x\, \sin\big((x_i - ct_i)\, y + z\big) - x\, \sin\big((x_i - (c+\mathrm{v})t_i)\, y + z\big)\Big]\Big\}\, e_i \in B \Big\}$$

where $\{e_i \,:\, 1 \le i \le 3\} \subset \mathbb{R}^3$ is the canonical basis. For every $r \in \mathbb{R}$, let $\hat{r} \in \mathbb{R}$ denote the unique real number such that $0 \le \hat{r} \le 2\pi$ and $r - \hat{r} \in \big\{2m\pi \,:\, m \in \mathbb{Z}\big\}$. Then, (79) is equivalent to

$$P\Big(\big(A,\, \alpha,\, \widehat{\varphi + a\,\alpha}\big) \in C\Big) = P\big((A,\, \alpha\,,\, \varphi) \in C\big)$$

or (as A, α, φ are independent, i.e., $P_{A\alpha\varphi} = P_A\, P_\alpha\, P_\varphi$)

$$\int_0^\infty P_A(x)\, d\,x \int_0^\infty P_\alpha(y)\; P_\varphi\big(\{z\,:\,(x,\,y,\,\widehat{z+ay})\in C\}\big)\; d\,y =$$

$$= \int_0^\infty P_A(x)\, d\,x \int_0^\infty P_\alpha(y)\, P_\varphi\big(\{z\,:\,(x,\,y,\,z)\in C\}\big)\; d\,y.$$

The set $\{z\;:\;(x,\,y,\,\widehat{z+ay})\in C\}$ is obtained from the set $\{z\;:\;(x,\,y,\,z)\in C\}$ by translation with $a\,y$, followed by reduction modulo 2π. Finally, as the random variable φ is uniformly distributed, its distribution is translation invariant so that

$$P_\varphi\big(\{z\,:\,(x,\,y,\,\widehat{z+ay})\in C\}\big) = P_\varphi\big(\{z\,:\,(x,\,y,\,z)\in C\}\big).$$

7.3. Poisson Initial Values

Let $\lambda > 0$, and let $f : [0, +\infty) \to L^2(\Omega)$ be a function such that (1) $f(0) = 0$, (2) for any $n \in \mathbb{Z}_+$ and any $0 \le x_0 < x_1 < \cdots < x_n$ the random variables

$$f(x_1) - f(x_0), f(x_2) - f(x_1), \cdots, f(x_n) - f(x_{n-1}),$$

are independent, (3) for any $0 \le x < y$ the random variable $f(y) - f(x)$ is Poisson distributed with parameter $\lambda(y - x)$, i.e.,

$$f(y) - f(x) \;\sim\; \begin{pmatrix} 0 & 1 & \cdots & k & \cdots \\ & & & & \\ p_0 & p_1 & \cdots & p_k & \cdots \end{pmatrix},$$

$$p_k = P\big(f(y) - f(x) = k\big) = \frac{\lambda^k (y-x)^k}{k!}\, e^{-\lambda(y-x)}\,.$$

Hence, for every $t > 0$, the random variable $f(x - ct) - f(x - (c + v)t)$ is discrete and Poisson distributed with parameter $\lambda\, v\, t$; hence, its expectation is

$$M\big[f(x-ct) - f(x-(c+v)t)\big] = \sum_{n=0}^{\infty} n\, p_n = e^{-\lambda(x-ct)} \sum_{n=1}^{\infty} \frac{(\lambda\, v\, t)^n}{(n-1)!}$$

so that

$$u(x,\,t) = f\big(x - (c+\mathrm{v})t\big) + \lambda\,\mathrm{v}\,t\, e^{-\lambda\left(x-(c+\mathrm{v})t\right)}\,.$$

7.4. Wiener Initial Values

Let $f : [0, +\infty) \to L^2(\Omega)$ be a function such that (1) $f(0) = 0$, (2) for any $n \in \mathbb{Z}_+$ and any $0 \le x_0 < x_1 < \cdots < x_n$ the random variables $f(x_i) - f(x_{i-1})$, $1 \le i \le n$, are independent, and (3) for any $0 \le x < y$ the random variable $f(y) - f(x)$ is Gaussian with mean $m = 0$ and dispersion $\sigma^2 = y - x$, i.e., the probability distribution is

$$p_{f(y)-f(x)}(a) = \frac{1}{\sqrt{2\pi(y-x)}}\, \exp\Big\{ -\frac{a}{2(y-x)} \Big\}, \quad a \in \mathbb{R}.$$

Let $x \in \mathbb{R}$ and $t > 0$ such that $x - (c + \mathrm{v})t \ge 0$. Then, $f(x - ct) - f(x - (c + \mathrm{v})t)$ is a Gaussian random variable with mean 0 and dispersion $\mathrm{v}t$, i.e.,

$$p_{f(x-ct)-f(x-(c+\mathrm{v})t)}(z) = \frac{1}{\sqrt{2\pi \mathrm{v}t}}\, \exp\Big(-\frac{z}{2\mathrm{v}t} \Big), \quad z \in \mathbb{R},$$

hence

$$M\big[f(x-ct) - f(x-(c+\mathrm{v})t)\big] = \int_{-\infty}^{+\infty} z\, p_{f(x-ct)-f(x-(c+\mathrm{v})t)}(z)\; d\,z =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \zeta \, e^{-\zeta^2/2} \, d\,\zeta = 0$$

so that $u(x,t) = f\big(x - (c + \mathrm{v})t\big)$.

8. Conclusions and Open Problems

The equation (6)

$$u_t + (c + \mathrm{v})\, u_x - \mathrm{v}\, M\big[u_x\big] + \mathrm{v}_x\, \mathring{u} = s$$

was derived from the conservation law (2) under the structural assumptions (iii) $M\phi = c\,Mu$ and (iv) $\big|r_{u(x,t)\,\phi(x,t)}\big| = 1$ for any $(x,\, t) \in [0,\, \ell] \times \mathscr{T}$. The correlation coefficient $r_{\xi\eta}$ of two random variables $\xi,\, \eta \in L^2(\Omega)$ is known to satisfy $\big|r_{\xi\eta}\big| \le 1$, and $\xi,\, \eta$ are uncorrelated if $r_{\xi\,\eta} = 0$. Therefore, the meaning of the assumption (iv) is that the random variables $u(x,\, t)$ and $\phi(x,\, t)$ are as far from being uncorrelated as possible. Note that (iv) is a necessary condition for $\phi = c\,u$ to hold, which is the deterministic model for transport. In the probabilistic setting, we refuted the state Equation (16) and instead modeled convection by the assumption (iii) that the mean values of the flux ϕ and density u at any point of a x-section, and at any time moment t, be proportional. An elementary, yet crucial, consequence of (iii) and (iv) was that the (non-vanishing, in general) flux $\phi - c\,u$ is proportional to $\mathring{u}$ (for every random variable $\xi \in L^2(\Omega)$ the L^2 norm of $\mathring{\xi}$ is the mean square deviation of ξ), i.e., (5) holds for some function $\mathrm{v} : [0,\, \ell] \times \mathscr{T} \to \mathbb{R}$ with the dimensions of a velocity. When $\mathrm{v} = 0$ Equation (6) becomes $u_t + c\,u_x = s$; hence, (6) is a probabilistic version of the ordinary convection equation (with sinks/sources). Through the present paper, we discussed the initial value problem $u(x,\, 0) = f(x)$ for the Equation (6), where $f : \mathbb{R} \to L^2(\Omega)$ is a given random signal. Our structural assumption (vii) $M\,s = 0$ is the probabilistic version of the simpler "no sinks or sources" case that we wish to analyze. An undesirable feature of (6) is that it contains both the spatial derivative of the density and its expectation. A remedy we adopt is to take the expectations of both members of (6) and of the initial condition (23) so that to obtain the initial value problem (24) governing the evolution of the mean value $M\,u$ of the density, and uniquely determining it (cf. (25)). The initial value problem for (6) is then recast as

$$u_t + (c + \mathrm{v})\, u_x + \mathrm{v}_x\, u = s + \frac{\partial}{\partial x}\Big\{\mathrm{v}\, M\big[f(x - ct)\big]\Big\}, \tag{80}$$

$$u(x,\, 0) = f(x). \tag{81}$$

When diffusion, rather than convection, is manifest, the appropriate probabilistic version of Fick's law is perhaps

$$M\,\phi = -D\, M\big[u_x\big], \tag{82}$$

for some constant $D > 0$ with $\big[D\big]_{\mathrm{SI}} = L^2 \cdot T^{-1}$. It is an open problem to derive a probabilistic diffusion equation from the conservation law (2) together with the constitutive Equation (82). Under the simplifying assumption that the velocity v is a positive constant, i.e., (ix) $\mathrm{v} \in (0,\, +\infty)$, a difference equation scheme, approximating (80) over a grid $G \subset \mathbb{R}^2$ of mesh ratio λ, is proposed and recognized to be both stable and satisfying the Courant–Friedrichs–Lewy test, provided that the mesh ratio and the velocities c and v obey to (10), i.e., $(c + \mathrm{v})\,\lambda \le 1$. The more general problem with arbitrary $\mathrm{v} \in C^1\big([0,\, \ell] \times \mathscr{T}\big)$ is so far open. To estimate (in the L^2 norm) the difference between the analytic solution and the discrete solution, we recover the integral representation of the Taylor rest in the truncated Taylor development for functions $f : U \subset \mathbb{R} \to L^2(\Omega)$. We do this in the more general setting of C^{k+1} functions $f : A \subset \mathbb{R}^n \to \mathfrak{X}$, where $\mathfrak{X}$ is an arbitrary Fréchet space, and integrals there are Bochner integrals, i.e.,

$$r_k(x; f,\, x_0) = \frac{k+1}{\|x - x_0\|^{k+1}} \times \tag{83}$$

$$\times \sum_{|\alpha|=k+1} (x-x_0)^\alpha \int_0^{\|x-x_0\|} r^k \, (D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}\,(x-x_0)\right) dr$$

yielding the needed estimate on the Taylor rest (cf. (76))

$$\|r_k(x; f, x_0)\|_m \le M_{k+1} \|x-x_0\|^{k+1} \times$$
$$\times \max_{|\alpha|=k+1} \sup_{0\le\tau\le 1} \|(D^\alpha f)((1-\tau)x + \tau x_0)\|_m \,,$$

with respect to a given family $\{\|\cdot\|_m : m \in \mathbb{Z}_+\}$ of semi-norms on $\mathfrak{X}$ (defining its topology as a locally convex space). The proof of (83) relies on standard techniques of functional analysis, and in particular on the fact that any weakly bounded subset of a locally convex space is also strongly bounded (cf. [7]). Convection and convection + diffusion models are useful in understanding ion translocation (cf. C-Y. Kong and M. Muthukumar [13]) across cellular membranes (cf. A. Parsegian [14], L. Movileanu, S. Howorka, O. Braha, and H. Bayley [15], A.G. Cherstvy [16]), which in turn is known to control a large number of biological processes. Cf. also R.I. Stefan-van Staden [17], for a discussion of classical sensing *versus* stochastic sensing. If small free ions in water are treated as a continuous medium with charge density $C_i(\mathbf{r}, t)$ located at $\mathbf{r}$ at time t of the, say, i-th ion species, the ionic flux $\mathbf{J}_i$ and the local electrostatic potential $V(\mathbf{r})$ are related (cf. Equations (2.6)–(2.8) in [13] (p. 18254)) by

$$\frac{\partial C_i(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{J}_i \,, \tag{84}$$

$$\mathbf{J}_i = -D_i \left\{ \nabla C_i(\mathbf{r}, t) + \frac{Z_i\, C_i}{k_B\, T} \nabla V \right\}, \tag{85}$$

$$\epsilon_0 \nabla \cdot \left\{ \epsilon(\mathbf{r}) \nabla V(\mathbf{r}) \right\} = -\rho_{\text{ex}} - \sum_i Z_i\, C_i(\mathbf{r}), \tag{86}$$

where Z_i, D_i are, respectively, the charge and diffusion coefficients of the i-th species contributing to the ionic current. Equation (84) is a conservation law, while (85) exhibits the flux $\mathbf{J}_i$ as having both a diffusive and a convective term (and (86) is the Poisson equation governing the electrostatic potential V (with ϵ_0 = permittivity of vacuum, $\epsilon(\mathbf{r})$ = inhomogeneous dielectric constant)). The model (84)–(86) may be further refined by assuming (as in [10] (p. 134706-2)) a *position-dependent* diffusion coefficient $D_i(\mathbf{r})$. Probabilistic versions of (84)–(86) may be derived in the spirit of the model proposed in the current paper, as suggested by the reviewer (the problem will be addressed in future work). A cell's membrane is thought of (cf. e.g., H. Davson and J. Danielli [18]) as a region of low polarizability acting as a barrier to the passage of solute material between two aqueous solutions. In a low-dielectric, hardly polarizable environment of lipid membranes, the ion electrostatic self-energy (the *self-energy*, or *energy of charging*, is the leading term in the energy of a charge in a given medium) $E_c = e_0^2/(2\,\epsilon_c\, a)$ (cf. M. Born [19], with a = ion's radius, e_0 = elementary charge) is known to increase with several factors of $k_B\, T$ (k_B = Boltzmann constant, T = absolute temperature) resulting into strong barriers that would stop ionic current across a membrane that were void of further intimate structure, such as ion-transport proteins incorporated into cell's membrane, that actually allow for ion permeation. The possibility of including (as once again suggested by the reviewer) low-dielectric membranes into the model developed in the present work, is left as an open problem.

Author Contributions: E.B., S.D. and F.E. contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data sharing not applicable.

Acknowledgments: Francesco Esposito has been supported by NOP Research and Innovation 2014–2020 (Action IV.4 and Actions IV.6)—FSE-REACT—"Reproducing Kernel Hilbert Spaces and Applications: Signal Theory, Machine Learning, Robotics, and AI" (CUP C39J21042630006). The paper was written while the authors were members of G.N.S.A.G.A. of INdAM, Rome, Italy. Elisabetta Barletta and Sorin Dragomir drew inspiration from the beautiful monograph by D. Stanomir and O. Stănăşilă [20]. The authors are grateful to the anonymous reviewer for drawing Authors' attention to the papers [10,13,14,16] and for remarks that improved the original manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

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