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Minimal Terracini Loci in a Plane and Their Generalizations

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Abstract: We study properties of the minimal Terracini loci, i.e., families of certain zero-dimensional schemes, in a projective plane. Among the new results here are: a maximality theorem and the existence of arbitrarily large gaps or non-gaps for the integers x for which the minimal Terracini locus in degree d is non-empty. We study similar theorems for the critical schemes of the minimal Terracini sets. This part is framed in a more general framework.

Keywords: projective plane; Terracini locus; double points; Veronese embedding; zero-dimensional scheme; Hilbert function

MSC: 14N05; 14N07; 15A69

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1. Introduction

Terracini loci came to life from the so-called Terracini Lemma ([1], Cor. 1.11), which helped to prove a huge number of important theorems on the dimensions of secant varieties, even in cases important for applications, e.g., low-rank approximation of tensors [2,3], the additive decompositions of forms [4–6], or cases of partially symmetric tensors [7–11]. They are an active topic of research [12–18].

For all positive integers x and any variety X, let $S(X,x)$ denote the set of all $A \subset X$ such that $\#A = x$. For any smooth point p of X, let $(2p,X)$ (or just $2p$ if $X = \mathbb{P}^n$) be the closed subscheme of X, with $(\mathcal{I}_p)^2$ as its ideal sheaf. Hence, $(2p,X)$ is a zero-dimensional scheme of degree $\dim X + 1$ with $\{p\}$ as its reduction. For any finite subset S of X contained in the smooth locus of X, set $(2S,X) := \cup_{p\in S} 2p$. If $X = \mathbb{P}^n$, set $2S := (2S,\mathbb{P}^n)$. For any set $A \subset \mathbb{P}^n$, let $\langle A\rangle$ denote its linear span. Fix positive integers n, d and x. The Terracini locus $\mathbb{T}(n,d;x)$ is the set of all $S \in S(\mathbb{P}^n,x)$ such that $\langle S\rangle = \mathbb{P}^n$, $h^0(\mathcal{I}_{2S}(d)) > 0$ and $h^1(\mathcal{I}_{2S}(d)) > 0$ [12–14]. More important is the minimal Terracini locus $\mathbb{T}(n,d;x)'$, which is the set of all $S \in \mathbb{T}(n,d;x)$ such that $h^1(\mathcal{I}_{2A}(d)) = 0$ for all $A \subsetneq S$.

To the best of our knowledge, the notion of minimality for Terracini sets was explicitly defined for Veronese embeddings in [13] and for arbitrary varieties in [12]. Since it is a very natural notion, it occurs "in nature" even if it is not explicitly defined. For instance, in the list in [19] of cardinality 3 Terracini sets for the Segre embeddings, the non-minimal ones are [19], Examples 4.1 and 4.2.

The minimal Terracini locus is usually very different from the non-minimal one [12–14]. In the setup of the Veronese embeddings on $\mathbb{P}^n$, the minimal one and the non-minimal one were considered in [13]. In that paper, many differences were pointed out. For instance, for almost all pairs (n,d), we have $\mathbb{T}(n,d;x) \neq \emptyset$ for all $x \gg 0$ ([13], Th. 1.1(iii)), while $\mathbb{T}(n,d;x)' = \emptyset$ for all $x > \lceil \binom{n+d}{n}/(n+1)\rceil$ ([13], Prop. 3.1). In this paper, we only consider the case $n = 2$ (as in [13]), and our tools (mainly the Hilbert function of the critical schemes of the elements of $\mathbb{T}(2,d;x)'$) are strong enough only for $n = 2$. For the case $n > 2$, we raise several questions.

We prove the following results.

Theorem 1. *Fix an integer $d \geq 6$, and set $\rho := \lceil (d+2)(d+1)/6\rceil$. Then $\mathbb{T}(2,d;\rho)' \neq \emptyset$ and $\mathbb{T}(2,d;x)' = \emptyset$ for all $x > \rho$.*

Theorem 2. *Fix a positive integer e. Then there is an integer $d(e) \geq 3$ such that for all integers $d \geq d(e)$, we have $\mathbb{T}(2,d;x)' \neq \emptyset$ for e consecutive integers x.*

Theorem 3. *Fix an integer $e > 0$. Then there is an integer $d_1(e)$ such that for all integers $d \geq d_1(e)$, there are integers $x > 0$, $y < x - e$ such that $\mathbb{T}(2,d;x)' \neq \emptyset$, $\mathbb{T}(2,d;y)' \neq \emptyset$ and $\mathbb{T}(2,d;c)' = \emptyset$ for all $x - e \leq c < x$.*

Question 1. *Are Theorems 1, 2 and/or 3 true in $\mathbb{P}^n$, $n \geq 3$ with $\rho := \lceil \binom{n+d}{m}/(n+1) \rceil$ and large integers $d(n,e)$ and $d_1(n,e)$ depending on n and e?*

Theorem 3 shows that for $d \gg 0$, there are arbitrarily large consecutive gaps and arbitrarily large consecutive non-gaps.

Question 2. *Is it possible (taking a larger $d_1(e)$ or a larger $d(e)$) to get that there are exactly c consecutive gaps or non-gaps?*

Our tools for making large consecutive gaps or large consecutive non-gaps seems not to be able to address Question 2.

As in [14], a key tool is the **numerical character** of any **critical scheme** of any $S \in \mathbb{T}(2,d;x)'$ (see Section 2 on the preliminaries).

In Section 3, we prove Theorems 1–3.

In Section 4, we prove the results on the possible degrees of the critical schemes of $S \in \mathbb{T}(2,d;x)'$ (Theorem 6). In particular, we prove that $Z \neq S$ (Proposition 1). Then, we prove the following theorem.

Theorem 4. *Fix an integer $c \geq 3$, and set $d_0(c) := 6c + 3$. Then for all $d \geq d_0(c)$, there are integers x_i, $1 \leq i \leq c$ with the following properties:*

1. *$x_i \geq x_{i-1} + 2$ for all $i = 2, \ldots, c$;*
2. *There is $S \in \mathbb{T}(2,d;x_i)'$ with a critical scheme Z with $\deg(Z) = 2x_i$;*
3. *There is no positive integer y such that there is $A \in \mathbb{T}(2,d;y)'$ with a critical scheme Z' with $2x_i - 2 \leq \deg(Z') \leq 2x_i - 1$.*

Theorem 4 is analogous to [14], Th. 1.3 for the degrees of the critical schemes of minimally Terracini sets.

In Section 5, we classify the pairs (d,x) such that $\mathbb{T}(2,d;x)' \neq \emptyset$ and $d \leq 8$.

In Section 6, we consider several related definitions of Terracini sets. One of the main results (Theorem 7) applies also to the degrees of the critical schemes of elements of $\mathbb{T}(2,d;x)'$. It says that for $d \gg 0$, there are arbitrarily large gaps in the degrees of critical schemes.

In the last section, we discuss some questions related to the maximal integer x such that $\mathbb{T}(n,d;x)' \neq \emptyset$.

It would be very interesting to extend [20,21] to some or all toric surfaces. Even an extension to only $\mathbb{P}^1 \times \mathbb{P}^1$ would be nice.

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2. Preliminaries

We work over an algebraically closed field with characteristic 0.

Each set $\mathbb{T}(n,d;x)$ and $\mathbb{T}(n,d;x)'$ is constructible ([22], Ex. II.3.18, Ex. II.3.19), and hence, we may speak about the dimensions and the irreducible components of the Terracini loci and the minimal Terracini loci.

For any zero-dimensional scheme $Z \subset \mathbb{P}^2$, $Z \neq \emptyset$, let $\tau(Z)$ denote the maximal integer ≥ -1 such that $h^1\mathcal{I}_Z(d)) > 0$. Let $s(Z)$ be the first integer s such that $h^0(\mathcal{I}_Z(s)) > 0$. The numerical character $n_0, \ldots, n_{s-1}$, $s := s(Z)$ is a string of s integers $n_0 \geq n_1 \geq \cdots \geq n_{s-1}$ that uniquely determines the Hilbert function of Z [14,20,21]. We have $n_0 = \tau(Z) + 2$ and

$n_{s-1} \ge s$. The numerical character $n_0, \dots, n_{s-1}$ is said to be **connected** if $n_i \le n_{i+1} + 1$ for $i = 0, \dots, s-2$. Fix any $S \in \mathbb{T}(n, d; x)$. A **critical scheme** of S is a subscheme $Z \subset 2S$ such that each connected component of Z has a degree of at most 2. If $S \in \mathbb{T}(n, d; x)'$, then $Z_{\mathrm{red}} = S$ for all critical schemes Z of S ([13], Lemma 2.11). The numerical character of any critical scheme of any element of $\mathbb{T}(2, d; x)'$ is connected ([14], Th. 2.10).

Remark 1. *Fix integers $d > t > 0$. Let $T \subset \mathbb{P}^2$ be any integral degree t curve. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(d - t)) = 0$, the long cohomology exact sequence associated with the inclusion $D \subset \mathbb{P}^2$ gives $h^0(T, \mathcal{O}_T(d)) = \binom{d+2}{2} - \binom{d+2-t}{2} = t(2d + 3 - t)/2$, and the restriction map $H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(T, \mathcal{O}_T(d))$ is surjective.*

Remark 2. *Take any $S \in \mathbb{T}(2, d; x)'$ and any critical scheme Z of S. By ([13], Lemma 2.11), we have $Z_{\mathrm{red}} = S$, and hence, $\deg(Z) \ge x$. Easy examples show that the latter inequality is not true (for many d and x) for the critical schemes of elements of $\mathbb{T}(2, d; x)$ that are not minimal.*

We use the following result ([14], Lemma 2.9).

Lemma 1. *Let $Z \subset \mathbb{P}^2$, $Z \neq \emptyset$ be a zero-dimensional scheme. Set $z := \deg(Z)$, $s := s(Z)$ and $d := \tau(Z)$. Assume that the numerical character $n_0, \dots, n_{s-1}$ is connected, $s \le (d+3)/2$, and there exists an integer t such that $t^2 \le z$ and $\frac{z}{t} + t - 3 \le d$. Then $t = s$, $z = s(d + 3 - s)$ and Z is the complete intersection of a curve of degree z/t and a curve of degree t.*

Remark 3. *Let $Z \subset \mathbb{P}^2$ be any zero-dimensional scheme that is the complete intersection of a curve of degree a and a curve of degree b. We have $\deg(Z) = ab$, $h^1(\mathcal{I}_Z(a + b - 3)) = 1$, $h^1(\mathcal{I}_Z(t)) = 0$ for all $t \ge a + b - 2$ and $h^1(\mathcal{I}_W(a + b - 3)) = 0$ for all $W \subsetneq Z$.*

Remark 4. *Take any $S \in \mathbb{T}(2, d; x)'$ and any critical scheme Z of S. Set $z := \deg(Z)$. Obviously $z \le 2x$. Since $Z_{\mathrm{red}} = S$ ([13], Lemma 2.11), we have $z \ge x$. Later we will prove that $z > x$ (Proposition 1 for $d \ge 6$, Remark 6 for $d = 4$ and Proposition 2 for $d = 5$). Let $\mu = n_0, \dots, n_{s-1}$ be the numerical character of Z. Since $\tau(Z) = d$ and $h^1(\mathcal{I}_Z(d)) = 1$ ([13], Lemma 2.10 and Th. 3.1). $d = \tau(S) = n_0 - 2$, $n_1 < n_0$, and μ is connected, i.e., $n_i \le n_{i+1} + 1$ for all $i = 0, \dots, s-2$ ([14], Th. 2.10). By [14], Equation (2), we have*

$$\sum_{i=0}^{s-1} n_i = z + \binom{s}{2} \tag{1}$$

Remark 5. *Fix $S \in \mathbb{T}(2, d; x)'$ and take any critical scheme Z of S. We have $h^1(\mathcal{I}_Z(d)) = 1$ ([13], Lemma 2.10 or Th. 3.1). We have $Z_{\mathrm{red}} = S$ ([13], Lemma 2.11).*

Remark 6. *Assume $\mathbb{T}(2, d; x)' \neq \emptyset$. By [13], Proposition 3.5 and Theorem 1, we have $x < \lceil (d+2)(d+1)/6 \rceil$. By [13], Proposition 5.2, we have $x \ge d + 1$, and if $x = d + 1$, then any element of $\mathbb{T}(2, d; x)'$ is contained in a unique reduced conic, C. If $x = d + 1$ is odd, then C is smooth. If $x = d + 1$ is even, C may be singular with $x/2$ points on each of its irreducible components. Thus, $\mathbb{T}(2, 4; x)' \neq \emptyset$ if and only $x = 5$, and $\mathbb{T}(2, 4; 5)'$ is the set of all $S \in S(\mathbb{P}^2, 5)$ such that no 3 of the points of S are collinear (or, equivalently, the set of all S contained in a smooth conic). Hence, $\mathbb{T}(2, 4; 5)'$ is irreducible of dimension 10.*

3. Proofs of Theorems 1–3

Remark 7. *Fix integers $c \ge 0$ and t such that $t \ge (c-1)(c-2)/2$. There is an integral nodal curve $D \subset \mathbb{P}^2$ with exactly a nodes. Moreover, if $3a < \binom{t+2}{2}$ and $t \ge 6$, we may take as $\mathrm{Sing}(D)$ a general subset of $\mathbb{P}^2$ with cardinality a [23–25].*

Proof of Theorem 1: By [13], Proposition 3.5, we have $\mathbb{T}(2, d; x)' = \emptyset$ for all $x > \rho$. By [12], Th. 2, we have $\mathbb{T}(2, d; \rho)' \neq \emptyset$ if $d \equiv 1, 2 \pmod 3$, i.e., if $\rho = (d+2)(d+1)/6$.

Now, assume $d \equiv 0 \pmod 3$.

For all integers x such that $0 \le x \le (d-1)(d-2)/2$, let $W(d,x)$ denote the Severi variety of all integral and nodal curves with exactly x nodes. The set $W(d,x)$ is an irreducible variety of dimension $\binom{d+2}{2} - 1 - x$ [23,24,26,27]. Take a general $C \in W(d,\rho)$ and set $S := \mathrm{Sing}(C)$. Since S is contained in the singular locus of a degree d curve, $h^0(\mathcal{I}_{2S}(d)) > 0$. Since $\deg(2S) = 3\rho = \binom{d+2}{2} + 2$, $h^1(\mathcal{I}_{2S}(d)) > 0$. Thus, to conclude the proof, it is sufficient to prove that S is minimal. Let E be the set of all subsets of S with cardinality $\rho - 1$. The semicontinuity theorem for cohomology gives that, restricting $W(d,\rho)$ to an open dense subset W, we may assume that all $\mathrm{Sing}(D)$, $D \in W$ have subsets of cardinality $\rho - 1$, B with the same $h^1(\mathcal{I}_{2B}(d))$ (so either all $\mathrm{Sing}(D)$ are minimal or none is minimal). We may assume $C \in W$. Fix $A \in E$ and assume $h^1(\mathcal{I}_{2A}(d)) > 0$. Thus, $h^0(\mathcal{I}_{2A}(d)) \ge 2$. Hence, there is a 1-dimensional family of curves with A contained in their singular locus. Since C is irreducible, the general element of this 1-dimensional family is irreducible. Varying D in W, we get a family $\mathcal{W}$ of integral degree d curves with at least $\rho - 1$ nodes, and $\dim \mathcal{W} = \dim V(d,\rho) + 1 = \dim V(d,\rho-1)$ and $h^1(\mathcal{I}_{2B}(d)) > 0$ for all $B \in S(\mathbb{P}^2, \rho - 1)$ arising from some $D \in W$. The Severi conjecture proved in [23] also proves that each integral plane curve with at least $\rho - 1$ singular points is in the closure $\overline{W(d,\rho-1)}$ of $W(d,\rho-1)$ (see the beginning of the Introduction of [26] or see [27] (in Italian) for a full proof). A general $D \in W(d,\rho-1)$ has as its singular locus a general element of $S(\mathbb{P}^2, \rho-1)$, and hence, $h^1(\mathcal{I}_{2\mathrm{Sing}(D)}(d)) = 0$. Hence, $\overline{V(d,\rho-1)} \ne \overline{\mathcal{W}}$. Thus, $\dim \mathcal{W} < \dim W(d,\rho-1)$, which is a contradiction. $\square$

Remark 8. *As in [12], Th. 2, the proof of Theorem 1 gives the existence of an irreducible family of dimension $2\rho - 3$ of the family of all $S \in S(\mathbb{P}^2, \rho)$ formed by minimal Terracini sets.*

Proof of Theorem 2: Set $t := 4e + 4$ and $d(e) := 8t$. Fix an integer $d \ge d(e)$. Note that $d \ge 8t$. Since $t \equiv 0 \mod 4$, the integer $\binom{d+2}{2} - \binom{d+2-t}{2} = t(2d+3-t)/2$ is even. Fix a general $E \subset \mathbb{P}^2$ such that $\#E = 2e - 1$. Remark 7 and the assumption on t give $h^1(\mathcal{I}_{2E}(d)) = 0$ and the existence of an integral and nodal degree t curve D such that $\mathrm{Sing}(D) = E$. Take an odd integer a such that $1 \le a \le 2e - 1$. Since a is odd, the integer $\binom{d+2}{2} - \binom{d+2-t}{2} - 3a$ is odd. Fix $A_a \subseteq E$ such that $\#A_a$, and set $f_a := (\binom{d+2}{2} - \binom{d-t+2}{2} + 1 - 3a)/2$. Note that $3a + 2f_a = h^0(D_a, \mathcal{O}_{D_a}(d)) + 1$. Fix a general $B_a \subset D$ such that $\#B_a = f_a$ and set $S_a := A_a \cup B_a$. Since B_a is general in D, $B_a \cap E = \emptyset$, and hence, $\deg(2B_a \cap D) = 2f_a$. The set A_a is a general subset of $\mathbb{P}^2$ with cardinality a because E is a general subset with $\#E = 2e - 1$. Note that $2A_a \subset D$. Since $d \ge 5$ and $d \ge t > 3a$, $h^1(\mathcal{I}_{2A_a}(d)) = 0$ [28]. Thus, $h^1(D, \mathcal{I}_{2A_a,D}(d)) = 0$. Thus, $h^0(D, \mathcal{I}_{2A_a,D}(d)) = \binom{d+2}{2} - \binom{d-t+2}{2} - 3a$. Since B_a is general in D, $(2B_a, D)$ gives the maximal possible number of independent conditions to the vector space $H^0(D, \mathcal{I}_{2A_a,D}(d))$. Thus, $h^0(D, \mathcal{I}_{2A_a \cup (2B_a,D),D}(d)) = 0$. Hence, $h^1(D, \mathcal{I}_{2A+a \cup (2B_a,D),D}(d)) = 1$.

Claim 1: We have $h^1(\mathcal{I}_{B_a}(d-t)) = 0$ and $h^0(\mathcal{I}_{B_a}(d-t)) > 0$.

Proof of Claim 1: Remember that $d > 3t$. Since $d \ge 2t$ and B_a is contained in the degree t curve D, $h^0(\mathcal{I}_{B_a}(d-t)) > 0$. We have $3a + 2f_a = h^0(D, \mathcal{O}_D(d)) + 1$. Since B_a is general in D, $d > t$ and $g = h^1(\mathcal{O}_{\mathbb{P}^2}(d-2t)) = 0$, $h^1(\mathcal{I}_{B_a}(d-t)) = 0$ if and only if $h^1(D, \mathcal{I}_{B_a,D}(d-t)) = 0$. Hence, to prove that $h^1(\mathcal{I}_{B_a}(d-t)) = 0$ for all a, it is sufficient to prove that $f_1 \le h^0(D, \mathcal{O}_D(d-t))$. Since $d \ge 3t$, we have $h^0(D, \mathcal{O}_D(d-t)) = t(2d+3-3t)/2$ and $3 + 2f_1 = t(2d+3-t)/2$. Since $3 \ge 0$, it is sufficient to prove that $2t(2d+3-3t) \ge t(2d+3-t)$, i.e., $2d + 3 - 6t \ge -t$. The last inequality is true because $d > 3t$. $\square$

Claim 2: $S_a \in \mathbb{T}(2, d; a + f_a)'$ and $h^1(\mathcal{I}_{2S_a}(d)) = 1$.

Proof of Claim 2: Note that $2S_a \cap D = 2A_a \cup (2B_a, D)$. Since $h^1(D, \mathcal{I}_{2S_a \cup D,D}(d)) = 1$ and $h^1(\mathcal{O}_{\mathbb{P}^2}(d-t)) = 0$, we get $h^1(\mathcal{I}_{2S_a \cap D}(d)) = 1$. Since $2A_a \subset D$ and $B_a \cap E = \emptyset$, the residual exact sequence of D is the following exact sequence:

$$0 \to \mathcal{I}_{B_a}(d-t) \to \mathcal{I}_{2S_a}(d) \to \mathcal{I}_{2S_a \cap D,D}(d) \to 0 \qquad (2)$$

By Claim 1, we have $h^0(\mathcal{I}_{B_a}(d-t)) > 0$ and $h^1(\mathcal{I}_{B_a}(d-t)) = 0$. Thus, the long cohomology exact sequence of (2) gives $h^0(\mathcal{I}_{2S_a}(d)) > 0$ and $h^1(\mathcal{I}_{2S_a}(d)) = 1$. Hence, to conclude the proof of Claim 2, it is sufficient to prove that $h^1(\mathcal{I}_{2S'}(d)) = 0$ for all $S' \subset S_a$ such that $\#S' = a + f_a - 1$. First, assume $A_a \subset S'$, and hence, $S' = A_a \cup B'$ with $B' \subset B_a$ and $\#B' = f_a - 1$. In this case, we have the following residual exact sequence:

$$0 \to \mathcal{I}_{B'}(d-t) \to \mathcal{I}_{2S'}(d) \to \mathcal{I}_{2A_a \cup (2B',D),D}(d) \to 0 \tag{3}$$

Since $B' \subset B_a$ and $h^1(\mathcal{I}_{B_a}(d-t)) = 0$, we have $h^1(\mathcal{I}_{B'}(d-t)) = 0$. Recall that $h^1(D, \mathcal{I}_{3A_a}(d)) = 0$ and that $h^0(D, \mathcal{I}_{2A_a,D}(d))) = 2f_a - 1$. Since B' is a general subset of D with $\#B' = 2f_a - 1$, ref. [29] gives $h^1(D, \mathcal{I}_{2A_a \cup (2B',D),D}(d)) = 0$. Thus, the long cohomology exact sequence of (3) gives $h^1(\mathcal{I}_{2S'}(d)) = 0$. Now assume $A_a \nsubseteq S'$, and hence, $S' = B_a \cup A'$ with $A' \subset A_a$ and $\#A' = a - 1$. Since $2A' \subset 2A \subset D$, the residual exact sequence of D gives the following exact sequence:

$$0 \to \mathcal{I}_{B_a}(d-t) \to \mathcal{I}_{2S'}(d) \to \mathcal{I}_{2A' \cup (2B_a,D),D}(d) \to 0 \tag{4}$$

Since $\#A' = \#A_a - 1$ and $h^1(D, \mathcal{I}_{2A_a,D}(d)) = 0$, we have $h^1(D, \mathcal{I}_{2A',D}(d)) = 0$, and hence, $h^0(D, \mathcal{I}_{2A',D}(d)) = h^0(D, \mathcal{I}_{2A_a,D}(d)) + 3 = 2f_a + 2$. Since B_a is general in D and $2f_a \le h^0(D, \mathcal{I}_{2A_a}(d))$, ref. [29] gives $h^1(D, \mathcal{I}_{2A_a \cup (2B_a,D),D}(d)) = 0$. The long cohomology exact sequence of (4) gives $h^1(\mathcal{I}_{2S'}(d)) = 0$, concluding the proof of Claim 2. □

Take an odd integer a such that $1 \le a \le 2e - 3$. Thus, $h^0(D, \mathcal{O}_D(d)) - 3(a+2) \equiv h^0(D, \mathcal{O}_D(d)) - 3a \pmod 2)$, and A_{a+2}, f_{a+2} and B_{a+2} are well-defined. Since $3a + 2f_a = h^0(\mathcal{O}_D(d)) - 1 = 3(a+2) + 2f_{a+2}$, we have $f_{a+2} = f_a - 3$, and hence, $\#S_{a+2} = \#S_a - 1$. Thus, taking all odd integers a between 1 and $2e - 1$, we see that Claim 2 proves that $\mathbb{T}(2, d; x)' \neq \emptyset$ for e consecutive integers. □

Proof of Theorem 3: Set $t := 2e + 4$ and $d_1(e) := 8t$. Note that t is even. Fix an integer $d \ge d_1(e)$. We have $d \ge 8t$. Set $x := t(d + 3 - t)/2$ and $y := (t-2)(d + 5 - t)/2$. By [14], Proof of Prop. 3.1, a general complete intersection of a curve of degree $t/2$ and a curve of degree $d + 3 - t$ is an element of $\mathbb{T}(2, d; x)'$, while a general complete intersection of a curve of degree $(t-2)/2$ and a curve of degree $d + 5 - t$ is an element of $\mathbb{T}(2, d; y)'$. Since $d \ge 2t + e + 2$, we have $y < x - e$. Fix an integer c such that $1 \le c \le x$. Assume, by contradiction, the existence of $S \in \mathbb{T}(2, d; x - c)'$, and let Z be a critical scheme of S. Set $z := \deg(Z)$. Since $Z_{\mathrm{red}} = S$ ([13], Lemma 2.11) and each connected component of Z has a degree of at most 2, $x - c \le z \le 2x - 2c$. Since $2x = t(d + 3 - t)$, we have $d \ge t - 3 + z/t$. Since $x = t(d + 3 - t)/2$, $z \ge x$ and $d + 3 - t \ge 2t$, we have $t^2 \le z$. Let $n_0, \dots, n_{s-1}$, $s := s(Z)$ be the numerical of Z.)

Claim 1: We have $s \le (d+3)/2$.

Proof of Claim 1: Assume $s \ge (d+4)/2$. Since $n_{s-1} \ge s$, (1) and Lemma give $z \ge \binom{s+1}{2} \ge (d+5)(d+3)/8$. Since $z \le 2x - 2c$ with $t(d + 3 - t)/2$ and $t \le d/8$, we get a contradiction. □

Since the numerical character of Z is connected ([14], Th. 2.10), Claim 1 and Lemma 1 give $c = 0$, which is a contradiction. □

4. Gaps for the Critical Schemes

In this section, we prove Theorem 4 and give several results on the degrees of critical schemes.

Proposition 1. *Take any $S \in \mathbb{T}(2, d; x)'$, $d \ge 6$ and any critical scheme Z of S. Then $Z \neq S$.*

Proof. Since $S = Z_{\mathrm{red}}$ (Remark 5), we have $S \subseteq Z$. Assume $S = Z$. Set $s := s(Z)$ and let $\mu = n_0, \dots, n_{s-1}$ be the numerical character of S. Since $S = Z$, $d = \tau(S) = n_0 - 2$, $n_1 < n_0$ and μ is connected, i.e., $n_i \le n_{i+1} + 1$ for all $i = 0, \dots, s-2$. By (1), we have

$$\sum_{i=0}^{s-1} n_i = x + \binom{s}{2} \tag{5}$$

Since $n_0 = d+2$ and μ is connected, $n_i \ge d + 2 - i$ for all i, and hence, $\sum_{i=0}^{s-1} n_i \ge s(d+2) - \binom{s}{2}$. Thus, (5) gives $x \ge s(d+3-s)$. Fix any $T \in |\mathcal{I}_S(s)|$ and any $S' \subset S$ such that $\#S' = x - 1$. Since $d \ge 6$, Lemma 2 gives $h^0(T, \mathcal{O}_T(d)) = sd + 1 - (s-1)(s-2)/2$. Since S is minimal, $h^1(\mathcal{I}_{2S'}(d)) = 0$. Hence, $h^1(T, \mathcal{I}_{((2S) \cap T, T}(d)) = 0$. Note that $\deg(T \cap 2S') \ge 2x - 2$. Hence, $2x - 2 \le sd + 1 - (s-1)(s-2)/2$. Recall that $x \ge s(d+3-s)$. Thus, $2x - 2 \ge s(2d + 6 - s) - 2$. Hence, $sd + 1 - (s-1)(s-2)/2 \ge s(2d+6-s) - 2$, i.e., $3 - (s-1)(s-2)/2 \ge sd + 6s - s^2$, i.e., $3 + s^2/2 - 6s + (3/2)s \ge sd$. Since $d \ge s - 2$ (Lemma 2), we get $3 - s^2/2 - 4s + (3/2)s \ge 0$, which is a contradiction. □

Lemma 2. *Take $S \in \mathbb{T}(2, d; x)'$, and set $s := s(Z)$. We have $d \ge s - 2$ if $d \ge 6$.*

Proof. Assume $s \ge d - 1$. Since $n_{s-1} \ge s$ and $n_i \ge n_{i+1}$ for all $i \le s-2$, we get $z \ge s^2 \ge (d-1)^2$. Hence, $x \ge (d-1)^2/2$. Recall that $x \le (d+2)(d+1)/6$ if $d \equiv 1, 2 \mod 3$ and $x \le (d^2 + 3d + 6)/6$ if $d \equiv 0 \mod 3$ (Remark 6), contradicting the assumption $d \ge 6$ and the inequality $z \le 2x$. □

Theorem 5. *Fix an integer $t \ge 4$ and an integer $d \ge 3t$ such that $d + 3 - t$ is even. Set $x := t(d+3-t)/2$. Then there is $S \in \mathbb{T}(2, d; x)'$ with a critical scheme of degree $2x$.*

Moreover, for all integers w such that

$$2x - td + 3t^2/2 + t/2 + 3 < w < 2x \tag{6}$$

there is no pair (y, A) such that $A \in \mathbb{T}(2, d; y)'$ and A has a critical scheme of degree w.

Proof. Let $S \subset \mathbb{P}^2$ be a finite set that is the complete intersection of a smooth curve C of degree t and a curve of degree $(d+3-t)/2$. Set $Z := C \cap 2S = (2S, C)$. Since Z is the complete intersection of C and a curve of degree $d + 3 - t$, $h^1(\mathcal{I}_Z(d)) = 1$ and $h^1(\mathcal{I}_{Z'}(d)) = 0$ for all $Z' \subsetneq Z$ (Remark 3). Thus, $h^1(C, \mathcal{I}_{Z,C}(d)) = 1$ and $h^1(C, \mathcal{I}_{Z',C}(d)) = 0$ for all $Z' \subsetneq Z$. Since $d + 3 - t \ge t \ge 2$, $\langle S \rangle = \mathbb{P}^2$. Since C is smooth, for any $A \subseteq S$, the residual exact sequence of C gives the following exact sequence:

$$0 \to \mathcal{I}_A(d-t) \to \mathcal{I}_{2A}(d) \to \mathcal{I}_{((2A \cap C), C}(d) \to 0 \tag{7}$$

Recall that $h^1(C, \mathcal{I}_{Z,C}(d)) = 1$ and $h^1(C, \mathcal{I}_{Z',C}(d)) = 0$ for all $Z' \subsetneq Z$. Thus, the long cohomology exact sequence of (7) shows that to prove that $h^1(\mathcal{I}_{2S}(d)) = 1$ and that $h^1(\mathcal{I}_{2A}(d)) = 0$ for all $A \subsetneq S$ (and hence, to prove that S is minimal), it is sufficient to prove that $h^1(\mathcal{I}_S(d-t)) = 0$. This is true by [14], Proof of Prop. 3.1 because S is the complete intersection of a curve of degree t and a curve of degree $(d+3-t)/2$ and $d - t \ge t + (d+3-t)/2 - 2$. Now take $t \ge 4$ such that $d + 3 - t \equiv 0 \pmod 2$, and fix $w < 2x$. If $w < t^2$, then we are done, and hence, we may assume $w \ge t^2$. Assume the existence of y, $E \in \mathbb{T}(2, d; y)'$ and a critical scheme W for E such that $\deg(W) = w$. Since $w < 2x$, we have $d > t - 3 + w/t$. Recall that the numerical character of W is connected ([14], Th. 2.10). By [21], Cor. 2 and the inequality $w \ge t^2$, there is an integer $m \in \{1, \dots, t-1\}$ and a degree m curve $D \subset \mathbb{P}^2$ such that $W \subset D$ and $m(d+3-m) \le w \le m(d + (3-m)/2)$. Thus, $E \subset D$. Since $w < 2x$ and $m < t$, $2x - w \ge td - t^2/2 + t/2 + 3$. Thus, we get the theorem. □

Remark 9. *Fix integers $d \ge 3t \ge 12$. Then $td \ge 3t^2/2 + t/2 + 5$.*

Proof of Theorem 4: Note that if $t \geq 4$, we have $3t^2 \geq 5 + 3t^2/2 + t/2$. Thus, if $t \geq 4$, $x = t(d+3-t)/2$, $d \geq 3t$ and $d+3-t$ is even, then $td \geq 5 + 3t^2/2 + t/2$. Hence, the range of values of w in the equality (6) contains the integers $2x-2$ and $2x-1$.

Consider the function $f(t) := t(d+3-t)/2$, which is strictly increasing in the interval $(0, (d+3)/2)$. For any $i = 1, \dots, c$, we define:

$$x_i = \begin{cases} f(2i) = i(d+3-2i) & \text{if } d \text{ is odd,} \\ f(2i+1) = (2i+1)(d+2-2i)/2 & \text{if } d \text{ is even} \end{cases}$$

Set $t := 2i$ if d is odd and $t := 2i+1$ if d is even. To conclude the proof of the proposition, it is sufficient to prove that the assumptions of Proposition 5 are satisfied. Use Remark 9. □

Theorem 6. *Fix positive integer $d \geq 3$ and x such that $\mathbb{T}(2,d;x)' \neq \emptyset$, and take any $S \in \mathbb{T}(2,d;x)'$ and any critical scheme Z of S. Then $\deg(Z) \leq 2x$, and*

$$2x - \deg(Z) \leq s(d+3-s) - 3 + (s-1)(s-2)/2. \tag{8}$$

Proof. Set $z := \deg(Z)$. Since every connected component of Z has degree 1 or degree 2, $\deg(Z) \leq 2x$. Set $s := s(Z)$ and $\tau := \tau(Z)$. Let $n_0, \dots, n_{s-1}$ denote the numerical character of Z. Thus, $n_{s-1} \geq s$, $n_0 = d+2$ (Remark 4). Since $n_0 = d+2$ and μ is connected, $n_i \geq d+2-i$ for all i, and hence, $\sum_{i=0}^{s-1} n_i \geq s(d+2) - \binom{s}{2}$. Thus, (1) gives

$$z \geq s(d+3-s). \tag{9}$$

Now assume $z \neq 2x$. Take $T \in |\mathcal{I}_Z(s)|$. Thus, there is a union W of $x-1$ connected components of Z such that $\deg(W) = z-1$. Since Z is a critical scheme, $h^1(\mathcal{I}_W(d)) = 0$. Since $W \subset Z$, $W \subset T$. The restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(T, \mathcal{O}_T(d))$ gives $h^1(T, \mathcal{I}_{W,T}(d)) = 0$. If $d \leq 5$, we conclude by Remark 1. Now assume $d \geq 6$. Lemma 2 gives $d \geq s-2$. Since T is a degree s plane curve, we get $h^1(T, \mathcal{O}_T(d)) = 0$. Thus, Riemann–Roch gives $h^0(T, \mathcal{O}_T(d)) = sd + 1 - (s-1)(s-2)/2$. Fix any $S' \subset S$ such that $\#S' = x-1$. Since $d \geq 6$, Lemma 2 gives $h^0(T, \mathcal{O}_T(d)) = sd+1-(s-1)(s-2)/2$. Since S is minimal, $h^1(\mathcal{I}_{2S'}(d)) = 0$. The restriction map $\rho : H^0(\mathcal{O}_{\mathbb{P}^2}(d)) \to H^0(T, \mathcal{O}_T(d))$ gives $h^1(T, \mathcal{I}_{((2S')\cap T, T}(d)) = 0$. Note that $\deg(T \cap 2S') \geq 2x-2$. Hence,

$$2x - 2 \leq sd + 1 - (s-1)(s-2)/2. \tag{10}$$

From (9) and (10) we get (8). □

Example 1. *Fix integers $d \geq r+2 \geq 5$. There is a line $L \subset \mathbb{P}^r$ and a smooth degree d non-degenerate rational curve $X \subset \mathbb{P}^r$ such that X contains exactly 3 points of X and L is not a tangent line of X. Set $S := L \cap X$. Obviously $S \in \mathbb{T}(X;3)$. Since L is not one of the tangent lines of X, S is minimal. Obviously, S is the unique critical scheme of itself.*

The following result is the equivalent of Theorem 3 for the degrees of the critical schemes:

Theorem 7. *Fix a positive integer e. Then there is a positive integer $d_0(e)$ such that for all $d \geq d_0(e)$, there are integers $0 < x_1 < x_2$ such that $\mathbb{T}(2,d;x_i)' \neq \emptyset$, $i = 1,2$, and there are $S_i \in \mathbb{T}(2,d;x_i)'$ and critical schemes Z_i of S_i with $\deg(Z_i) = 2x_i$, while there is no (y, A, Z) with y a positive integer. $A \in \mathbb{T}(2,d;y)'$, and Z is a critical scheme of A; hence, $2x_2 - e \leq deg(Z) < 2x_2$.*

Proof. We take $x_1 = d+1$ (for any $d \geq 5$). By Remark 6, there is $S_1 \in \mathbb{T}(2,d;d+1)$ with critical scheme Z_1 of degree $2d+2$ and contained in a smooth conic.

Set $t := 4e+4$ and $d(e) := 8t$. Fix an integer $d \geq d(e)$. Mimic the proof of Theorem 3 with S_2 as a complete intersection or apply Theorem 6. □

5. Classification for $d \le 8$

In this section, we consider pairs (d, x) such that $\mathbb{T}(2, d; x)' \neq \emptyset$ for $d \le 8$. The cases with $d \le 4$ are well-known [13], Remark 2.3 and Lemmas 3.6, 3.7 or Remark 6).

Remark 6 gives the following result.

Proposition 2. *We have $\mathbb{T}(2, 5; x)' \neq \emptyset$ if and only if $x \in \{6, 7\}$. Every element of $\mathbb{T}(2, 5; 6)'$ is contained in a reduced conic.*

Proposition 3. *We have $\mathbb{T}(2, 6; x)' \neq \emptyset$ if and only if $x \in \{7, 9, 10\}$.*

Proof. By Remark 6, we have $7 \le x \le 10$, $\mathbb{T}(2, 6; 7)' \neq \emptyset$ and $\mathbb{T}(2, 6; 10)' \neq \emptyset$. Remark 6 also gives a description of $\mathbb{T}(2, 6; 7)'$. The case $x = 10$ is described in [12], Prop. 13.

(a) Assume $x = 8$. Assume, by contradiction, the existence of $S \in \mathbb{T}(2, 6; 8)'$ and take a critical scheme Z of S. We have $8 \le z := \deg(Z) \le 16$. Take $Y \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ such that $S \subset Y$, and among the cubic containing S, one with $w := \deg(Z \cap Y)$ maximal. Assume for the moment $Z \nsubseteq Y$. Consider the residual exact sequence of Y:

$$0 \to \mathcal{I}_{\mathrm{Res}_Y(Z)}(3) \to \mathcal{I}_Z(6) \to \mathcal{I}_{Z \cap Y, Y}(6) \to 0 \tag{11}$$

Since $Z \cap Y \nsubseteq Z$ and Z is critical, $h^1(\mathcal{I}_{Z \cap Y}(6)) = 0$. The restriction map $H^0(\mathcal{O}_{\mathbb{P}^2}(6)) \to H^0(\mathcal{O}_Y(6))$ gives $h^1(Y, \mathcal{I}_{Z \cap Y, Y}(6)) = 0$. Thus, the long cohomology exact sequence of (11) gives $h^1(\mathcal{I}_{\mathrm{Res}_Y(Z)}(3)) > 0$. We have $\deg(\mathrm{Res}_Y(Z)) = z - w \le 16 - 9 = 7$. By [30], Lemma 34 there is a line L such that $\deg(\mathrm{Res}_Y(Z)) \ge 5$. Since $S \subset Y$, $\mathrm{Res}_Y(Z) \subseteq S \subset Y$. Thus, the theorem of Bezout gives that L is an irreducible component of Y. Note that $\mathrm{Res}_L(Z) \supseteq \mathrm{Res}_Y(Z)$. Since each connected component of Z has degree ≤ 2, we get $\#(S \cap L) \ge 5$, contradicting the minimality of S.

Now assume $Z \subset Y$. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(3)) = 0$, the long cohomology exact sequence of (11) gives $h^1(Y, \mathcal{I}_{Z,Y}(6)) > 0$. This inequality is false if Y is irreducible because $\mathcal{I}_{Z,Y}(6)$ is a positive degree rank 1 torsion free sheaf on Y and Y has arithmetic genus 1. Now assume that Y is reducible. Since S is minimal, $\#(S \cap R) \le 3$ for all lines R and $\#(S \cap D) \le 6$ for each conic D.

First, assume $Y = M \cup D$ with D a reduced conic, M a line and $\#(S \cap D) = 6$. Thus, $\#(S \cap (M \setminus M \cap D)) = 2$. The long cohomology exact sequence of the residual exact sequence of D gives $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(4)) > 0$. Since $\#(S \cap M) \le 3$, we have $\#(S \cap M \cap D) \le 1$, and hence, $\deg(\mathrm{Res}_D(Z)) \le 5$, contradicting [30, Lemma 34]. Now assume the non-existence of such a reduced conic. We get $Y = R \cup T$ with R a line, T a reduced conic, $\#(S \cap R) = 3$ and $S \cap R \cap T = \emptyset$. The long cohomology exact sequence of R gives $h^1(\mathcal{I}_{\mathrm{Res}_R(Z)}(5)) > 0$. Since $Z \subset Y$ and $S \cap R \cap T = \emptyset$, $\deg(\mathrm{Res}_R(Z)) \le 2(\#(S \cap T)) = 10$. By [30], Lemma 34, there is a line L such that $\deg(L \cap \mathrm{Res}_R(Z)) \ge 7$. Thus, $\#(S \cap L) \ge 4$, which is a contradiction.

(b) Assume $x = 9$. Take the complete intersection $S = C \cap C'$ of 2 smooth cubics. Set $Z := C \cap 2C'$. Remark 3 gives $h^1(\mathcal{I}_Z(6)) = 1$ and hence, $S \in \mathbb{T}(2, 6; 9)$. Fix $A \subsetneq S$. Since C has genus 1, any degree 8 zero-dimensional subscheme W of C (respectively, C') satisfies $h^1\mathcal{I}_W(3)) = 0$; the long cohomology exact sequence of C' gives that S is minimal. □

Proposition 4. *We have $\mathbb{T}(2, 7; x)' \neq \emptyset$ if and only if $x \in \{8, 11, 12\}$.*

(i) An element $S \in S(\mathbb{P}^2, 8)$ is contained in $\mathbb{T}(2, 7; 8)'$ if and only if S is contained in a reduced conic D, with the restriction that if D is reducible, each irreducible component of D contains exactly 4 points of S.

(ii) No element of $\mathbb{T}(2, 7; 12)'$ is contained in a plane cubic.

Proof. Fix $S \in \mathbb{T}(2, 7; x)'$, and call Z a critical scheme of S. Thus, $z := \deg(Z) \le 2x$. By Remark 6, we have $8 \le x \le 12$. Remark 6 also gives part (i). Thus, from now on, we assume $9 \le x \le 12$. Since S is minimal, $\#(S \cap L) \le 4$ for all lines L and $\#(S \cap D) \le 6$ for any reduced conic D. Set $s := s(Z)$. Recall that the numerical character $n_0, \ldots, n_{s-1} \ge s$

of Z is connected, $n_1 < n_0$ and $n_0 = d+2$. For any plane cubic C, we have the following residual exact sequence:

$$0 \to \mathcal{I}_{\mathrm{Res}_C(Z)}(4) \to \mathcal{I}_Z(7) \to \mathcal{I}_{C\cap Z,C}(7) \to 0 \tag{12}$$

For any plane conic D, we have the following residual exact sequence:

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)}(5) \to \mathcal{I}_Z(7) \to \mathcal{I}_{D\cap Z,D}(7) \to 0 \tag{13}$$

Consider the restriction maps $\rho_C : H^0(\mathcal{O}_{\mathbb{P}^2}(7)) \to H^0(C,\mathcal{O}_C(7))$ of C and $\rho_D : H^0(\mathcal{O}_{\mathbb{P}^2}(7)) \to H^0(D,\mathcal{O}_D(7))$ of D. Since S is minimal, no line contains 5 points of S and no conic contains 8 points of S. Since S is minimal, $Z_{\mathrm{red}} = S$ [13], Lemma 2.11, and if $S \nsubseteq C$ (respectively, $S \nsubseteq D$), then the restriction map ρ_C (respectively, ρ_D) gives $h^1(C,\mathcal{I}_{Z\cap C,C}(7)) = 0$ (respectively, $h^1(D,\mathcal{I}_{Z\cap D,D}(7)) = 0$). Thus, the long cohomology exact sequence of (12) (respectively, (13)) gives $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(4)) > 0$ (respectively, $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(5)) > 0$). Since $h^1(\mathcal{O}_{\mathbb{P}^2}(4)) = 0$ (respectively, $h^1(\mathcal{O}_{\mathbb{P}^4}(5)) = 0$), ρ_C (respectively, ρ_D) is surjective. We have $h^0(\mathcal{O}_C(7)) = h^0(\mathcal{O}_{\mathbb{P}^2}(7)) - h^0(\mathcal{O}_{\mathbb{P}^2}(3)) = \binom{9}{2} - \binom{6}{2} = 21$.

(a) Assume $x = 9$. We take $C \in |\mathcal{I}_S(3)|$. First, assume $Z \nsubseteq C$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(4)) > 0$. Since $S \subset C$ and every connected component of Z has degree ≤ 2, we have $\mathrm{Res}_C(Z) \subseteq S$. Hence, $h^1(\mathcal{I}_S(4)) > 0$. By [30], Lemma 34, there is a line L such that $\#(L\cap S) \ge 6$, contradicting the minimality of S.

Now assume $Z \subset C$. Since $h^1(\mathcal{O}_{\mathbb{P}^2}(4)) = 0$, the long cohomology exact sequence of C gives $h^1(C,\mathcal{I}_{Z,C}(7)) > 0$. First assume that C irreducible. Since C has arithmetic genus 1, $h^1(C,\mathcal{F}) = 0$ for each rank 1 torsion free sheaf $\mathcal{F}$ of degree > 0. Since $\deg(\mathcal{I}_{Z,C}(4)) \ge 3$, we get a contradiction. Now assume that C is reducible. Since $\#(S\cap D) \le 6$ for any reduced conic D, C has no multiple component. Write $Y = L\cup D$, with D a reduced conic. Since $\#(S\cap L_1) \le 4$ for all lines L_1 and $\#(S\cap D_1) \le 6$ for any reduced conic D_1, we have $\#(S\cap D\cap L) \le 1$. First, assume $S\cap D\cap L = \emptyset$. We get $3 \le \#(S\cap L) \le 4$, $\#(S\cap D) = 9 - \#(S\cap L)$ and $\deg(\mathrm{Res}_L(Z)) \le 2\#(S\cap D)$. The residual exact sequence of D gives $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(6)) > 0$. Since $\deg(\mathrm{Res}_L(Z)) \le 12$, ref. [30], Lemma 34 gives the existence of a line R such that $\deg(R\cap \mathrm{Res}_L(Z)) \ge 8$. Since $S\cap D\cap L = \emptyset$ and $Z \subset C$, we get that R is a component of D (the theorem of Bezout), $\#(S\cap R) = 4$ and all connected components of Z with reductions contained in R are contained in Z. Thus, $\deg(\mathrm{Res}_R(Z)) \le 10$. The residual exact sequence of R gives $h^1(\mathcal{I}_{\mathrm{Res}_R(Z)}(6)) > 0$, and hence, there is a line M such that $\deg(M\cap \mathrm{Res}_R(Z)) \ge 8$. We get $\#(S\cap M) = 4$, and hence, the conic $R\cup M$ contains at least 7 points of S, which is a contradiction.

Now assume $\#(S\cap D\cap L) = 1$. We get $\#(S\cap L) = 4$ and $\#(S\cap D) = 6$. The residual exact sequence of L gives $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(6)) > 0$ with $\deg(\mathrm{Res}_L(Z)) \le 11$. Thus, there is a line J such that $\deg(J\cap \mathrm{Res}_L(Z)) \ge 8$. Hence, $\#(S\cap J) \ge 4$ and $\#(S\cap L\cap J) \le 1$. The reduced conic $J\cup R$ contains at least 7 points of S, which is a contradiction.

(b) Assume $x = 10$. Take a cubic curve C such that $\#(S\cap C) \ge 9$.

(b1) Assume $Z \nsubseteq C$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(4)) > 0$ with $\deg(\mathrm{Res}_C(Z)) \le 11$. Either there is a line L such that $\deg(L\cap \mathrm{Res}_C(Z)) \ge 6$ or there is a reduced conic D such that $\deg(D\cap \mathrm{Res}_D(Z)) \ge 10$.

(b1.1) Assume the existence of the line L. We get $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(6)) > 0$. Since $\deg(\mathrm{Res}_L(Z)) \le 14$, either there is a line R such that $\deg(R\cap \mathrm{Res}_L(Z)) \ge 8$ or there is a conic T such that $\deg(T\cap \mathrm{Res}_L(Z)) = 14$. The conic T does not exist because it would contain at least 7 points of S. The line R does not exist because the reducible conic $L\cup R$ would contain at least 7 points of S.

(b1.2) Now assume the existence of the conic D. We have $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(5)) > 0$ with $\deg(\mathrm{Res}_D(Z)) \le 10$. Thus, there is a line J such that $\deg(J\cap \mathrm{Res}_D(Z)) \ge 7$, and hence, $\#(J\cap S) \ge 4$. The theorem of Bezout gives $J \subset C$. Since $\deg(J\cap \mathrm{Res}_D(Z)) \ge 7$, we get $C = J\cup D$. We use the proof of step (b1.1) with J instead of L.

(b2) Now assume $Z \subset C$. First, assume that C is irreducible. Since C has arithmetic genus 1, $h^1(C, \mathcal{F}) = 0$ for every rank 1 torsion free sheaf $\mathcal{F}$ on C. Since $\mathcal{I}_{Z,C}(7)$ is a rank 1 torsion free sheaf on C with positive degree, we get a contradiction.

Now assume that C is reduced. Since $\#(S \cap L) \le 4$ for all lines L and $\#(S \cap D) \le 6$ for any reduced conic D, we have $C = L \cup D$, with L a line and D a reduced conic; $\#(L \cap S) = 4$, $\#(L \cap D) = 6$ and $S \cap L \cap D) = \emptyset$. Since $S \cap L \cap D) = \emptyset$ and $Z \subset C$, $\deg(\mathrm{Res}_L(Z)) \le 12$. We conclude as in step (b1).

(c) The case $x = 11$ is described in [12], Prop. 8.

(d) Assume $x = 12$. We have $\mathbb{T}(2,7;12)' \neq \emptyset$ ([12], Th. 2). Since $h^0(C, \mathcal{O}_C(7)) = 21$, no minimal S is contained in a plane cubic. □

Proposition 5. *We have $\mathbb{T}(2,8;x)' \neq \emptyset$ if and only if $x \in \{9,12,13,15\}$.*

Proof. By Remark 6, we have $9 \le x \le 15$. We have $\mathbb{T}(2,8;15)' \neq \emptyset$ by [12], Cor. 1. The case $x = 12$ is described in [12], Prop. 7. The case $x = 13$ is described in [12], Prop. 13. Thus, to conclude, we only need to prove that $\mathbb{T}(2,8;x)' = \emptyset$ for all $x \in \{10,11,14\}$. Fix $x \in \{10,11,14\}$. Assume, by contradiction, $\mathbb{T}(2,8;x)' \neq \emptyset$. Fix $S \in \mathbb{T}(2,8;x)'$ and let Z be a critical scheme of S. Set $z := \deg(Z)$. We have $x \le z \le 2z$. Since S is minimal, $\#(S \cap L) \le 4$ for all lines L and $\#(S \cap D) \le 8$ for all reduced conics D. Recall that $\dim |\mathcal{O}_{\mathbb{P}^2}(3)| = 9$ and $\dim |\mathcal{O}_{\mathbb{P}^2}(4)| = 14$. Fix $A \subset S$ such that $\#A = 9$. Since $\dim |\mathcal{O}_{\mathbb{P}^2}(3)| = 9$, there is $|\mathcal{I}_A(3)| \neq \emptyset$ containing A. Among the plane cubics containing A, we take one, C, such that $w := \deg(C \cap Z)$ is maximal. Consider the residual exact sequence of C:

$$0 \to \mathcal{I}_{\mathrm{Res}_C(Z)}(5) \to \mathcal{I}_Z(8) \to \mathcal{I}_{Z\cap C,C}(8) \to 0 \tag{14}$$

Since $h^1(\mathcal{O}_{\mathbb{P}^2}(5)) = 0$, the restriction map $H^0(\mathcal{I}_{Z\cap C}(8)) \to H^0(C, \mathcal{I}_{Z\cap C,C}(8))$ is surjective (Remark 1). Thus, $h^1(\mathcal{I}_{Z\cap C}(8)) = 0$ if and only if $h^1(C.\mathcal{I}_{Z\cap C,C}(8)) = 0$. Since Z is critical, the long cohomology exact sequence of (14) gives $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(5)) > 0$ if $Z \nsubseteq C$. We have $\deg(\mathrm{Res}_C(Z)) = z - w \le 2x - 9$.

Observation 1. *Assume C is integral. Since C has arithmetic genus 1, we have $h^1(C, \mathcal{F}) = 0$ for every positive degree rank 1 torsion free sheaf. If $Z \subset C$, we have $\deg(\mathcal{I}_{Z,C}(8)) = 24 - z > 0$ for $x \in \{10,11\}$. Thus, if $Z \subset C$ and $x \in \{10,11\}$, C is not integral. Since any reduced conic contains at most 6 points of S, C has no multiple component.*

(a) Assume $x = 10$.

(a1) Assume $Z \nsubseteq C$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(5)) > 0$ with $\deg(\mathrm{Res}_C(Z)) \le 11$. By [30], Lemma 34 there is a line L such that $\#(Z \cap L) \ge 7$. Thus, $\#(L \cap S) \ge 4$. The minimality of S gives $\#(S \cap L) = 4$. Consider the residual exact sequence of L:

$$0 \to \mathcal{I}_{\mathrm{Res}_L(Z)}(7) \to \mathcal{I}_Z(8) \to \mathcal{I}_{Z\cap L,L}(8) \to 0 \tag{15}$$

Since $S \nsubseteq L$, $Z \nsubseteq L$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(7)) > 0$. We have $\deg(\mathrm{Res}_L(Z)) \le 11 - 7 = 4$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_L(Z)}(7)) = 0$ ([30], Lemma 34), which is a contradiction.

(a2) Assume $Z \subset C$. Hence, $S \subset C$ ([13], Lemma 2.11). By Observation 1 C is reducible and without multiple components. Thus, $C = D \cup L$, with L a line and D a reduced conic. Since $\#(S \cap L) \le 4$ and $\#(S \cap D) \le 6$, we get $\#(S \cap L) = 4$, $\#(S \cap D) = 6$ and $S \cap D \cap L = \emptyset$. Since this is true for any decomposition of C as the union of a line and a reduced conic, D is a smooth conic. Since $S \cap L \cap D = \emptyset$, $\mathrm{Res}_L(Z) = Z \cap D$. Thus, $\deg(\mathrm{Res}_L(Z)) \le 12$. By [30], Lemma 34 and the long cohomology exact sequence of (15) give the existence of a line $R \subset \mathbb{P}^2$ such that $\deg(R \cap \mathrm{Res}_L(Z)) \ge 9$. Since $\mathrm{Res}_L(Z) \subset D$, the theorem of Bezout gives that R is an irreducible component of D, which is a contradiction.

(b) Assume $x = 11$.

(b1) Assume $Z \nsubseteq C$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_C(Z)}(5)) > 0$ with $\deg(\mathrm{Res}_C(Z)) \le 13$. Since $5 \cdot 3 > 13$, ref. [21], Remarques at p. 116 gives that either there is a line L such that

$\deg(L \cap \mathrm{Res}_C(Z)) \geq 7$ or there is a reduced conic D such that $\deg(D \cap \mathrm{Res}_C(Z)) \geq 12$. The existence of the line L is excluded as in step (a1). Assume the existence of the reduced conic D. Consider the residual exact sequence of D:

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)}(6) \to \mathcal{I}_Z(8) \to \mathcal{I}_{Z\cap D,D}(8) \to 0 \tag{16}$$

Since $\#(S \cap D) \leq 6$, $\mathrm{Res}_D(Z) \neq \emptyset$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(6)) > 0$. We have $\deg(\mathrm{Res}_D(Z)) \leq 22 - 12 = 10$. By [30], Lemma 34, there is a line R such that $\deg(R \cap \mathrm{Res}_D(Z)) \geq 8$. We conclude as in step (a1).

(b2) Assume $Z \subset C$. Hence, $S \subset C$ ([13], Lemma 2.11). By Observation 1, C is reducible and without multiple components. Thus, $C = D \cup L$, with L a line and D a reduced conic. Since $\#(S \cap L) \leq 4$ and $\#(S \cap D) \leq 6$, we get $x \leq 10$, which is a contradiction.

(c) Assume $x = 14$, and hence, $14 \leq z \leq 28$.

(c1) Assume $z \geq 25$ and $s \leq 5$. Take $d := 8$ and $a = 5$. We have $s \leq (d+3)/2$ and $d > a - 3 + z/a$. Thus, [14], Lemma 2.9 gives a contradiction.

(c2) Assume $z \geq 25$ and $s = 6$. Since $n_5 \geq 6$, $n_0 = 10$ and $n_0, \dots, n_5$ is connected, we have $\sum_{i=0}^{5} n_i \geq 10 + 9 + 8 + 7 + 6 + 6 = 46$. Thus, (1) gives $z \geq 46 - 15$, which is a contradiction.

(c3) Assume $z \geq 25$ and $s > 6$. Since $\dim |\mathcal{O}_{\mathbb{P}^2}(7)| = 28$, we get $s = 7$ and $z = 28$. Since $n_6 \geq 7$, $n_0 = 10$ and $n_0, \dots, n_6$ is connected, we get $\sum_{i=0}^{6} n_i \geq 10 + 9 + 8 + 7 + 7 + 7 + 7 = 55$, and hence, (1) gives $z \geq 55 - 21$, which is a contradiction.

(c4) Assume $z \leq 24$. Take $T \in |\mathcal{O}_{\mathbb{P}^2}(5)|$ such that $w := \deg(T \cap Z)$ is maximal.

Assume for the moment $Z \nsubseteq T$, and hence, $h^1(\mathcal{I}_{\mathrm{Res}_T(Z)}(3)) > 0$. Since $\dim |\mathcal{O}_{\mathbb{P}^2}(5)| = 20$, we have $w \geq \min\{20, z\}$, and hence, $\deg(\mathrm{Res}_T(Z)) \leq 4$, contradicting [30], Lemma 34.

Thus, $Z \subset T$. The restriction map $H^0(\mathcal{O}_{\mathbb{P}^2}(8)) \to H^0(T, \mathcal{O}_T(8))$ gives the inequality $h^1(T, \mathcal{I}_{Z,T}(8)) > 0$. First, assume that T is integral. By the adjunction formula, the curve T has arithmetic genus 6, and hence, $h^1(T, \mathcal{F}) = 0$ for every rank 1 torsion free sheaf $\mathcal{F}$ on T such that $\deg(\mathcal{F}) > 10$. We have $\deg(\mathcal{I}_{Z,T}(8)) = 40 - z > 10$, which is a contradiction. Hence, T is reducible. Since S is minimal, $\#(S \cap L) \leq 4$ for all lines L, $\#(S \cap D) \leq 6$ for any reduced conic D; $\#(S \cap C) \leq 12$ for every cubic curve. Thus, $T = L \cup C$, with L a line and C an integral curve. Set $\alpha := \#(S \cap L \cap C)$, $\beta := \#(S \cap L)$ and $\gamma := \#(S \cap C)$. We have $14 = \beta + \gamma - \alpha$, $\deg(\mathrm{Res}_L(Z)) \leq 2\gamma - \alpha$ and $\deg(\mathrm{Res}_C(Z)) \leq \beta - \alpha$. If $\beta = 4$, we use the residual exact sequence of L. If $\beta \leq 3$, we use the residual exact sequence of C. $\square$

Question 3. *Is $\mathbb{T}(2, 9; 18)' = \emptyset$? Is $\mathbb{T}(2, 9; \rho - 1)' = \emptyset$ for all large d?*

We proved that $\mathbb{T}(2, d; \rho - 1)' \neq \emptyset$ for $d = 5, 6, 7$.

6. Generalized Terracini Loci

Definition 1. *Fix a positive integer d and a zero-dimensional scheme $W \subset \mathbb{P}^2$ such that $h^1(\mathcal{I}_W(d)) > 0$. A zero-dimensional scheme $Z \subset \mathbb{P}^2$ is said to be a* **critical scheme** *of W in degree d if $Z \subseteq W$, $h^1(\mathcal{I}_Z(d)) > 0$ and $h^1(\mathcal{I}_{Z'}(d)) = 0$ for all $Z' \subsetneq Z$.*

Definition 1 is a key definition because if Z is as in Definition 1 and A is any zero-dimensional scheme containing Z, then $h^1(\mathcal{I}_A(d)) > 0$, and hence the zero-dimensional schemes W such that $h^1(\mathcal{I}_W(d)) > 0$ are, roughly speaking, built from its critical schemes. The next result, Theorem 8, says that each W such that $h^1(\mathcal{I}_W(d)) > 0$ has a critical scheme. There are schemes W with several critical schemes (for instance the scheme $2S$ in [13], Th. 1.4 for odd values of d).

Theorem 8. *Fix a positive integer d and a zero-dimensional scheme $W \subset \mathbb{P}^2$ such that $h^1(\mathcal{I}_W(d)) > 0$.*

(a) W has at least one critical subscheme in degree d.

(b) Let Z be any critical subscheme of Z in degree d. Then $h^1(\mathcal{I}_Z(d)) = 1$, $\tau(Z) = d$ and the numerical character of Z is connected.

Proof. Let E be the set of all $A \subseteq W$ such that $h^1(\mathcal{I}_A(d)) \neq 0$. Since $W \in E$, $E \neq \emptyset$. Take $Z \in E$ with minimal degree such that $h^1(\mathcal{I}_Z(d)) > 0$. The assumption on the minimality of $\deg(Z)$ implies $h^1(\mathcal{I}_{Z'}(d)) = 0$ for all $Z' \subsetneq Z$. Thus, Z is critical for W in degree d.

Let $Z \subseteq W$ be any critical scheme of W in degree d. Since Z has subschemes of degree $\deg(Z) - 1$ and $h^1(\mathcal{I}_A(d)) - h^1(\mathcal{I}_B(d)) \leq \deg(B) - \deg(A)$ for all zero-dimensional schemes $A \subset B$, we have $h^1(\mathcal{I}_Z(d)) = 1$, and hence, $h^1(\mathcal{I}_Z(t)) = 0$ for all $t > d$. Set $s := s(Z)$ and $z := \deg(Z)$. Let $n_0, \dots, n_{s-1}$ be the numerical character of Z. Assume that $n_0, \dots, n_{s-1}$ is not connected and let t be the first integer $< s$ such that $n_t \leq n_{t-1} - 2$. By [21], Cor. 3.2 there is a degree t curve T such that the scheme $T \cap Z$ has numerical character $n_0, \dots, n_{t-1}$ (which is connected). Since $n_0 = d + 2$, $h^1(\mathcal{I}_{Z \cap T}(d)) > 0$. The minimality of Z gives $Z = T \cap Z$. By the definition of $s(Z)$, we get $s = t$, which is a contradiction. □

In the next example, we give a list of possible connected components of zero-dimensional schemes $A \subset \mathbb{P}^2$ that may be connected components of zero-dimensional schemes to which the easy Theorem 8 may be applied. It is important to notice that for interesting schemes W, the connected components may be completely different and with different degrees.

Example 2. *For any positive integer m and any $p \in \mathbb{P}^2$, let mp denote the closed subscheme of $\mathbb{P}^2$ with $(\mathcal{I}_p)^m$ as its ideal sheaf. We have $(mp)_{\mathrm{red}} = \{p\}$, $\deg(mp) = \binom{m+2}{2}$ and $mp \subset (m+1)p$. We have $1p = \{p\}$. By the Terracini Lemma ([1], Cor. 1.11), the double point $2p$ is the scheme used to define the Terracini loci. It is easy to check that $2p$ is a flat limit of sets of cardinality 3 and that $3p$ is flat limit of $2p$ and a family of sets of cardinality 3. The scheme $4p$ is a flat limit of a family of union of 5 disjoint double points ([31], part 1 of Prop. 22). Degree 5 subschemes of $3p$ containing $2p$ were used to compute secant varieties of tangential varieties of $\mathbb{P}^2$ [32]. General unions of schemes $4p$ (or its higher dimensional generalization) and double points are used to compute the dimension of the secant varieties of many varieties.*

Set $\mathcal{Z}(2;0) = \emptyset$. For each positive integer x and any $p \in \mathbb{P}^2$, let $\mathcal{Z}(x;p)$ denote the set of all *curvilinear schemes* $Z \subset \mathbb{P}^2$ such that $\deg(Z) = x$ and $Z_{\mathrm{red}} = \{p\}$. Note that we require that every $Z \in \mathcal{Z}(x;p)$ to be curvilinear. The curvilinearity assumption is automatic for $x = 1, 2$, but it is a restriction for $x > 2$. The set $\mathcal{Z}(x;p)$ has a natural structure of a smooth and connected quasi-projective variety of dimension $x - 1$ [33–36]. Since each $A \in \mathcal{Z}(x;p)$ is connected and curvilinear, for each integer $0 \leq y \leq x$, there is a unique $A \in \mathcal{Z}(y;p)$ such that $A \subseteq Z$. Set $\mathcal{Z}(x) := \cup_{p \in \mathbb{P}^2} \mathcal{Z}(x;p)$. The set $\mathcal{Z}(x)$ is a connected and smooth quasi-projective manifold of dimension $x + 1$. For any positive integer x and all $e_1, \dots, e_x \in \mathbb{N}$, let $\mathcal{Z}(x; e_1, \dots, e_x)$ denote the set of all $(A, Z_1, \dots, Z_x)$ such that $A = (p_1, \dots, p_x) \in S(\mathbb{P}^2, x)$ and $Z_i \in \mathcal{Z}(e_i; p_i)$. For any $(A, Z_1, \dots, Z_x) \in \mathcal{Z}(x; e_1, \dots, e_x)$, set $u(A, Z_1, \dots, Z_x) := Z_1 \cup \cdots \cup Z_x \subset \mathbb{P}^2$. The scheme $u(A, Z_1, \dots, Z_x)$ is a degree $e_1 + \cdots + e_x$ curvilinear scheme with exactly x connected components. Let $u(\mathcal{Z}; x; e_1, \dots, e_x)$ denote the set of all $u(A, Z_1, \dots, Z_x)$ for some $(A, Z_1, \dots, Z_x) \in \mathcal{Z}(x; e_1, \dots, e_x)$. For all positive integers d, x, $e_1, \dots, e_x$, let $\mathbb{T}(d; x; e_1, \dots, e_x)$ denote the set of all $Z \in u(A, Z_1, \dots, Z_x)$ such that $h^1(\mathcal{I}_Z(d)) > 0$ and $h^0(\mathcal{I}_Z(d)) > 0$. Take $Z \in \mathcal{T}(d; x; e_1, \dots, e_x)$. We say that Z is *minimal* if $h^1(\mathcal{I}_W(d)) = 0$ for all $W \subsetneq Z$. Let $\mathcal{T}(d; x; e_1, \dots, e_x)$ (respectively, $\mathcal{T}(d; x; e_1, \dots, e_x)'$) denote the set of all $Z \in \mathcal{T}(d; x; e_1, \dots, e_x)'$ (respectively, $Z \in \mathcal{T}(d; x; e_1, \dots, e_x)'$) such that $\langle Z \rangle = \mathbb{P}^2$.

Remark 10. *Take $Z \in \mathcal{T}(d; x; e_1, \dots, e_x)'$. Since $h^1(\mathcal{I}_Z(d)) > 0$ and $h^1(\mathcal{I}_E(d)) = 0$ for all $E \subsetneq Z$, we have $h^1(\mathcal{I}_Z(d)) = 1$. Thus, $h^1(\mathcal{I}_Z(t)) = 0$ for all $t > d$, and hence, $\tau(Z) = d$.*

Remark 11. *Obviously, $\mathcal{T}(1; x; e_1, \dots, e_x) = \emptyset$ for all positive integers x, $e_1, \dots, e_x$, while $\mathcal{T}(1; x; e_1, \dots, e_x) \neq \emptyset$ if and only if $e_1 + \cdots + e_x \geq 3$.*

As a particular case of Theorem 8, we get the following result.

Corollary 1. *Take any $Z \in \mathcal{T}(d; x; e_1, \dots, e_x)'$. Then the numerical character of Z is connected.*

Proposition 6. *Fix an integer $d \geq 2$. If $\mathcal{T}(d;x;e_1,\dots.e_x)' \neq \emptyset$, then $x \geq 3$ and $e_1 + \cdots + e_x \leq \binom{d+2}{2}$.*

Proof. We saw that $x \geq 3$ (Remark 10). Fix $Z \in \mathcal{T}(d;x;e_1,\dots.e_x)$, assume $\deg(Z) > \binom{d+2}{2}$, and take $Z' \subset Z$ such that $\deg(Z') = \deg(Z) - 1$. Since $h^0(\mathcal{I}_Z(d)) > 0$, $h^0(\mathcal{I}_{Z'}(d)) > 0$. Since $\deg(Z') \geq \binom{d+2}{2}$, we get $h^1(\mathcal{I}_{Z'}(d)) > 0$. Thus, Z' is not minimal. □

Proposition 7. *Fix positive integers x, $e_1,\dots,e_x$ and $d \geq 2$.*

(i) We have $\mathcal{T}(d;x;e_1,\dots,e_x) \neq \emptyset$ if and only if $e_1 + \cdots + e_x \geq d+2$.

(ii) Assume $e_1 \geq \cdots \geq e_x > 0$. We have $\mathcal{T}(d;x;e_1,\dots,e_x) \neq \emptyset$ if and only if $x \geq 3$ and either $e_1 + \cdots + e_x \geq 2d+2$ or $e_1 + \cdots + e_{x-1} \geq d+2$.

Proof. For the existence part of (i), take a closed subscheme of a line with degree $e + 1 + \cdots + e_x \geq d+2$. Take $Z \in \mathbb{T}(d;x;e_1,\dots,e_x)$. By [30], Lemma 34, we have $\deg(Z) \geq d+2$ and $\deg(Z) > d+2$ if Z is not contained in a line, concluding the proof of part (ii).

Now we consider part (ii). Obviously, we need $x \geq 3$. Take a smooth conic C. If $e_1 + \cdots + e_x \geq 2d+2$ for the existence part, it is sufficient to take $Z \subset C$ with x connected components of degree $e_1,\dots,e_x$. Now assume $e_1 + \cdots + e_x \leq 2d+1$. By [30], Lemma 34, there is a line L such that $\deg(L \cap Z) \geq d+2$. Since $\langle Z_{\mathrm{reg}} \rangle = \mathbb{P}^2$ and $e_x \leq e_i$ for all i, we have $e_1 + \cdots + e_{x-1} \geq d+2$. For the existence part, we take $Z_1 \subset L$ with $x-1$ connected components of degree $e_1 + \cdots = e_{x-1}$ and add a degree e_x curvilinear scheme for which the reduction is a point of $\mathbb{P}^2 \setminus L$. □

Proposition 8. *Fix integers $d \geq t \geq 2$. Set $z := t(d+3-t)$.*

(i) We have $\mathcal{T}(d;z;1,\dots,1)' \neq \emptyset$.

(ii) Assume $d \geq t^2$ and $z \leq d^2/4$. All $Z \in \mathcal{T}(d;x;e_1\dots,e_x)'$ with $e_1 + \cdots + e_x = z$ are the complete intersection of a curve of degree t and a curve of degree $d+3-t$. We have $\mathcal{T}(d;x;e_1,\dots,e_x)' = \emptyset$ for all positive integers x and $e_1,\dots,e_x$ such that $e_1 + \cdots + e_x < z$.

Proof. To prove part (1), we take as Z a complete intersection of a general curve of degree t and a general curve of degree $d+3-t$ (Remark 3). We only use that $t \geq 2$ and $d+3-t \geq 2$, so that $\langle Z \rangle = \mathbb{P}^2$.

Now assume $d \geq t^2$ and $z \leq d^2/4$. By Theorem 1, the numerical character $n_0,\dots,n_{s-1}$ of any element of $\mathbb{T}(d;x;e_1,\dots,e_x)'$, $e_1 + \cdots + e_x \leq z$ is connected. Claim 1 of the proof of Theorem 2 gives $s \leq (d+3)/2$. Apply Lemma 1. □

7. Queries about the Maximal Non-Empty Terracini Loci

We fix an integer $n \geq 2$ and an integer $d \geq 5$. Set $\rho := \lceil \binom{n+d}{n}/(n+1) \rceil$. Recall that $\mathbb{T}(n,d;x)' = \emptyset$ for all $x > \rho$.

Question 4. *Is $\mathbb{T}(n,d;\rho)' \neq \emptyset$?*

From now on, we fix $n \geq 3$ and $d \geq 5$. For any positive integer x and any $A \in S(\mathbb{P}^n,x)$, let $\mathcal{B}_A$ denote the scheme-theoretic base locus of $\mathcal{I}_{2A}(d)$. Obviously, $\mathcal{B}_A \supseteq 2A$ Call $\mathcal{D}_A$ the schematic closure in $\mathbb{P}^n$ of the restriction of $\mathcal{B}_A$ to the open subset $\mathbb{P}^n \setminus A$ of $\mathbb{P}^n$. We always assume $x < \lfloor \binom{n+d}{n}/(n+1) \rfloor$ and that A is general in $S(\mathbb{P}^n,x)$. With these assumptions, $h^1(\mathcal{I}_{2A}(d)) = 0$ [4–6], and hence, $h^0(\mathcal{I}_A(d)) = \binom{n+d}{n} - (n+1)x \geq n+1$ with equality if and only if $\binom{n+d}{n}/(n+1) \in \mathbb{Z}$, i.e., if and only if $\rho = \binom{n+d}{n}/(n+1)$ and $x = \rho - 1$. By [37], a general $Y \in |\mathcal{I}_{2A}(d)|$ is smooth outside A and has ordinary double points at the points of A. This implies that $\mathcal{B}_A$ does not contain the scheme $2p$ for some $p \in \mathbb{P}^n \setminus A$, but unfortunately, as far as we know, it does not imply $\mathcal{D}_A = \emptyset$ (the base locus question is quite open even in dimension 1 for singular curves [38–40]), even when $h^0(\mathcal{I}_{2A}(d)) \geq n+2$.

Question 5. *Is $\mathcal{D}_A = \emptyset$ if $h^0(\mathcal{I}_{2A}(d)) \geq n+2$? For which values of x is $\mathcal{D}_A = \emptyset$?*

For low values of x, it easy to check that $\mathcal{D}_A = \emptyset$, but our methods are too crude to tackle large integers x.

Remark 12. *Take $x \geq n+1$ and a general $A \in S(\mathbb{P}^n, x)$. Assume $\mathcal{D}_A \neq \emptyset$ and take $p \in \mathbb{P}^n \setminus A$ in the support of A. We have $h^1(\mathcal{I}_{2A\cup 2p}(d)) > 0$. Hence, $h^0(\mathcal{I}_{2A\cup 2p}(d)) > 0$ if $h^0(\mathcal{I}_{2A}(d)) \geq n+1$, i.e., if $x \leq -1 + \binom{n+d}{n}/(n+1)$. Thus, $A \cup \{p\} \in \mathbb{T}(n,d;x+1)$. Now assume $h^0(\mathcal{I}_{2A}(d)) = n+1$, i.e., $\rho = \binom{n+d}{n}/(n+1)$ and $x = \rho - 1$. In this case, we expect that $\mathcal{D}_A \neq \emptyset$, that $\mathcal{D}_A$ is scheme-theoretically a finite set, and that $A \cup \{p\} \in \mathbb{T}(n,d;x+1)'$ for all $p \in \mathcal{D}_A$.*

8. Methods

There are no experimental data and no part of a proof is completed numerically. All results are given with full proofs.

9. Conclusions

We study properties of the minimal Terracini loci, i.e., families of certain zero-dimensional schemes, in the projective plane. Among the new results here are: a maximality theorem and the existence of arbitrarily large gaps or non-gaps for the integers x for which the minimal Terracini locus in degree d is non-empty. We study similar theorems for the critical schemes of the minimal Terracini sets.

We consider more general zero-dimensional schemes and give five open questions. Most of these question concern the extension of this paper to higher-dimensional projective space.

A different (and much more general) kind of extension would be to toric varieties. Even just for smooth toric surfaces, an extension should come with very nice examples and, for low cardinality sets, a full classification list. F. Galuppi, P. Santarsiero, D.A. Torrance and E. Teixeira Turatti studied in several (non-toric) cases the first non-empty Terracini locus [17]. In particular, they gave a full classification for all smooth Del Pezzo surfaces. All elements of the first non-empty Terracini set are minimal. In those cases (and in particular for Del Pezzo surfaces and for the Hirzebruch surfaces), two natural questions arise:

1. Are non-minimal Terracini loci non-empty for all numbers $x \gg 0$?
2. What is the computation of the cardinality of the second non-empty Terracini locus?

For (1), there should be finitely many classes of exceptional cases, i.e., of pairs (variety, embedding) in which all Terracini loci are empty and “almost all” the other pairs should have non-minimal Terracini sets for all $x \gg 0$. These statements are known in the case of Veronese embeddings [13].

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