

Article

Analysis of Stability and Bifurcation in Nonlinear Mechanics with Dissipation

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Abstract: The analysis of stability and bifurcation is studied in nonlinear mechanics with dissipative mechanisms: plasticity, damage, fracture. The description is based on introduction of a set of internal variables. This framework allows a systematic description of the material behaviour via two potentials: the free energy and the potential of dissipation. In the framework of standard generalized materials the internal state evolution is governed by a variational inequality which depends on the mechanism of dissipation. This inequality is obtained through energetic considerations in an unified description based upon energy and driving forces associated to the dissipative process. This formulation provides criterion for existence and uniqueness of the system evolution. Examples are presented for plasticity, fracture and for damaged materials.

Keywords: stability; bifurcation; plasticity; damage; fracture; normality law

1. Introduction

Thermodynamics with an energetic analysis is providing a large framework for the description of irreversibility of anelastic structures. Various approaches has been developed. As an example, functional analysis for dissipative systems have been successfully applied in the case of viscoelasticity.

The introduction of the internal variables allows a systematic description of the material behaviour via two potentials: the free energy and the potential of dissipation.

The development of such description is due to the works of several authors: [1–4]. They propose a general framework for the usual constitutive laws in the modelling of the anelasticity. The purpose of

this article is to study the quasistatic evolution of anelastic structure. The system evolution is analysed using the definition of functionals presented here in the case of nonlinear dynamics, firstly for internal variables associated to volume dissipation in nonlinear mechanics (plasticity and damage), secondly for dissipation due to singularities and discontinuities propagation (fracture, phase transformation). After a short account for the description of the system motion and of the mechanical interactions, the Lagrangian and the Hamiltonian of the system is introduced to describe the evolution of the system [5,6]. After that, the quasistatic evolution is studied for dissipative materials. The criteria of stability and uniqueness of the response of the system are given. Finally illustrations in case of plasticity, of fracture and of damage mechanics are proposed.

The discussion of stability and bifurcation is well known in elasticity [7] and discussion in elastoplasticity have been studied in the same spirit [8–10], extensions for fracture mechanics or for damage are more recent [11–13]. The discussion is founded essentially on the analysis of the properties of the rate boundary value problem [8,14,15].

2. Some General Features

In order to describe the motion and the equilibrium of bodies or structures subjected to various physical interactions, a kinematical description of the motion is performed. In the case of a continuous medium, this description must ensure the continuity of the medium during its motion.

Usually one looks for the motion of a material point \underline{X} from a reference configuration by describing its displacement $\underline{u}(\underline{X}, t)$.

After the kinematical description of the body, one has to describe the mechanical interactions. Many choices for the description of these interactions are available. For example, the virtual power principle can be used.

This principle describes the mechanical interaction between each material point of the body with respect to a given loading distribution. For sake of simplicity and conciseness of this presentation, a thermodynamic description of interaction is then chosen.

3. Description of the Motion

Let a body Ω submitted to external forces described by vector fields \underline{f} over Ω and vector fields \underline{T} along the boundary $\partial\Omega$. The external forces are generally functions of time.

Under these loadings the body is deformed. The actual position \underline{x} of a material point is a function $\underline{\Phi}$ of its initial position \underline{X} and of the time. The displacement \underline{u} is then defined by :

$$\underline{x}(\underline{X}, t) = \underline{\Phi}(\underline{X}, t) = \underline{X} + \underline{u}(\underline{X}, t) \quad (1)$$

Consider now, two material points \underline{X} and $\underline{X} + d\underline{X}$, then we have:

$$\underline{x}(\underline{X}, t) + d\underline{x} = \underline{\Phi}(\underline{X} + d\underline{X}, t) = \underline{\Phi}(\underline{X}, t) + \frac{\partial \underline{\Phi}}{\partial \underline{X}} \cdot d\underline{X} + o(d\underline{X}) \quad (2)$$

Hence, a material element $d\underline{X}$ is transported by the motion to the material element $d\underline{x}$. The corresponding transformation is the linear application associated with the gradient of transformation \mathbf{F} :

$$d\underline{x} = \frac{\partial \underline{x}}{\partial \underline{X}} \cdot d\underline{X} = \mathbf{F} \cdot d\underline{X} \quad (3)$$

The present length of the material element is given by:

$$d\underline{x}.d\underline{x} = d\underline{X}.\mathbf{F}^T.\mathbf{F}.d\underline{X} = d\underline{X}.\mathbf{C}.d\underline{X} \tag{4}$$

The changes of the local geometry, the stretching and the shearing of material fibers are determined by the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T.\mathbf{F}$. In small perturbations the gradient of the displacement is small and the deformation is reduced to its linearized contribution $\underline{\varepsilon}(\underline{u})$:

$$2 \underline{\varepsilon}(\underline{u}) = \nabla \underline{u} + \nabla^T \underline{u} \tag{5}$$

or in terms of components in a Cartesian orthonormal frame

$$2\varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \tag{6}$$

All subsequent studies will be made in small perturbations for sake of simplicity.

4. The Mobility and the Interactions

The body Ω is considered as a continuous set of elements, positions of which are denoted by \underline{x} . This material element of volume $d\Omega$ posses an elementary mass $dm = \rho d\Omega$, where ρ is the mass density.

The mobility of the body is defined by its set of virtual motions associated with any vector field $\underline{\tilde{v}}^*$, interpreted as a virtual velocity field.

The notation \tilde{f} defines the field of f . The field is the set of the values $f(\underline{X})$ for $\underline{X} \in \Omega$. The local value is denoted simply by f .

The interactions are defined by linear forms $\underline{\tilde{v}}^* \rightarrow \mathcal{P}(\underline{\tilde{v}}^*)$ where \mathcal{P} is a real number. \mathcal{P} is named the virtual power of the forces developed by the virtual motion $\underline{\tilde{v}}^*$.

In classical continuum mechanics, the external forces applied on the body are given by vector field \tilde{f} defined over the volume Ω and by vector fields \tilde{T} defined over the boundary $\partial\Omega$. In this case the virtual power of external forces is expressed as:

$$\mathcal{P}_e(\underline{\tilde{v}}^*) = \int_{\Omega} \underline{f}.\underline{v}^* d\Omega + \int_{\partial\Omega} \underline{T}.\underline{v}^* da \tag{7}$$

The power of internal interactions is given by a field of second order tensors $\tilde{\sigma}(\underline{x})$ such that:

$$\mathcal{P}_i(\underline{\tilde{v}}^*) = - \int_{\Omega} \underline{\sigma} : \nabla \underline{v}^* d\Omega \tag{8}$$

The tensor $\underline{\sigma}$ is the Cauchy stress tensor. This expression is the simplest form for a local description of internal interactions compatible with external loadings defined only in terms of vector fields.

Taking account of the axiom of objectivity, the power of local interactions \mathcal{P}_i is equal to zero for any rigid body motion:

$$\mathcal{P}_i(\underline{\tilde{v}}^*) = 0, \quad \forall \underline{\tilde{v}}^* \in \mathcal{R}.\mathcal{B}.\mathcal{M} \tag{9}$$

The set $\mathcal{R}.\mathcal{B}.\mathcal{M}$ is the set of rigid body motion:

$$\mathcal{R}.\mathcal{B}.\mathcal{M} = \{ \underline{\tilde{v}}^*/\underline{v}^*(\underline{X}) = \underline{v}_o + \underline{\omega}_o.\underline{X} \} \tag{10}$$

with \underline{v}_o uniform and $\underline{\omega}_o^T = -\underline{\omega}_o$ a rigid body rotation. By application of (9) for any domain Ω , the Cauchy stress $\underline{\sigma}$ is a second order symmetric tensor.

If we assume that there is no jump for the velocity, the virtual power of acceleration quantities is:

$$\mathcal{A}(\underline{\tilde{v}}^*) = \int_{\Omega} \rho \underline{\gamma} \cdot \underline{v}^* \, d\Omega, \quad \underline{\gamma} = \dot{\underline{v}} = \frac{d}{dt} \underline{v} \tag{11}$$

The virtual power principle. The fundamental principle of dynamics is written in terms of virtual power, *i.e.*, the sum of the virtual power of internal interactions and of the virtual power of external forces is equal to the virtual power of acceleration quantities:

$$\mathcal{P}_i(\underline{\tilde{v}}^*) + \mathcal{P}_e(\underline{\tilde{v}}^*) = \mathcal{A}(\underline{\tilde{v}}^*) \tag{12}$$

for any virtual motion $\underline{\tilde{v}}^*$.

Using this equality for any virtual motion and any subdomain Ω^* , the momentum conservation is obtained in a local form:

$$\text{div } \underline{\sigma} + \rho \underline{f} = \rho \underline{\gamma}, \quad \underline{\sigma}^T = \underline{\sigma}, \quad \underline{\sigma} \cdot \underline{n} = \underline{T} \tag{13}$$

\underline{n} is the outward normal vector to domain Ω . The divergence operator is defined classically as

$$\text{div } \underline{\sigma} = \sum_i \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \underline{e}_i \tag{14}$$

By the same reasoning, the principle of action and reaction is recovered. On an elementary internal surface da , with normal $\underline{\nu}$, the stress vector is continuous, if there is no jump of displacement velocity:

$$\underline{\sigma}^+ \cdot \underline{\nu} = \underline{\sigma}^- \cdot \underline{\nu} \tag{15}$$

These equations of conservation are not sufficient to determine the internal state, and some complementary information are needed.

A thermodynamical point of view is chosen. The body is a thermodynamical system formed by a collection of small elements defined as material points. Each small element has a mass density ρ and the local state is characterized by a set of state variables.

5. Conservation of Energy

To describe effectively the behaviour of the material, we must measure a great number of mechanical quantities. In order to be more efficient, the concept of internal parameters is adopted. The internal state is described by the present value of a set of state variables.

The choice of these parameters is governed by the observation and the ability of the modelization to describe the studied phenomenon with accuracy.

The state variables are the strains $\underline{\varepsilon}$, the temperature θ and a set of internal variables α . Attached with these parameters, an internal energy density e and a entropy density s are determined. Then the internal energy and the entropy of the body Ω are given by integration over the whole body:

$$E = \int_{\Omega} \rho e \, d\Omega, \quad S = \int_{\Omega} \rho s \, d\Omega \tag{16}$$

Conservation of energy. The conservation of energy is written as:

$$\dot{\mathcal{K}} + \dot{E} = \mathcal{P}_{cal} + \mathcal{P}_e \quad (17)$$

We assume that the caloric power is due to conduction:

$$\mathcal{P}_{cal} = - \int_{\partial\Omega} \underline{q} \cdot \underline{n} \, da = - \int_{\Omega} \operatorname{div} \underline{q} \, d\Omega \quad (18)$$

The local expression of the energy conservation is deduced from (12,17,18) applied to any volume Ω :

$$\dot{e} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \operatorname{div} \underline{q} \quad (19)$$

Entropy production. The second law for the whole system is written as:

$$\dot{S} + \int_{\partial\Omega} \frac{\underline{q} \cdot \underline{n}}{\theta} \, da \geq 0 \quad (20)$$

We assume ρ constant in time. After integration by parts, we obtain:

$$\int_{\Omega} \left(\rho \dot{s} + \frac{1}{\theta} \operatorname{div} \underline{q} \right) \, d\Omega - \int_{\Omega} \frac{\underline{q} \cdot \operatorname{grad} \theta}{\theta^2} \, d\Omega \geq 0 \quad (21)$$

The two terms are of different nature. The first one is due to internal mechanical irreversibility, and the second one is due to conduction. In nonlinear mechanics, the internal state is generally associated with irreversibility. Then the fundamental inequality of thermodynamics implies that the internal production of entropy must be non-negative. In the total dissipation, we distinguish the part due to the conduction and the part due to internal forces. We assume that the choice of state parameters is a normal set of variables. In this case, a variation of temperature does not induce variation on kinetic energy, so the two dissipations are individually positive:

$$D_m = \rho \dot{s} + \frac{1}{\theta} \operatorname{div} \underline{q} \geq 0, \quad (22)$$

$$D_{th} = - \frac{\underline{q} \cdot \operatorname{grad} \theta}{\theta^2} \geq 0 \quad (23)$$

By introducing the energy conservation in the first equation, we can use the free energy ψ instead of internal energy $e = \psi + s\theta$. Then, the intrinsic dissipation D_m is rewritten as:

$$D_m = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\underline{v}) - \rho(\dot{\psi} + s\dot{\theta}) \geq 0 \quad (24)$$

This inequality must be satisfied by any real evolution from the current state defined by the values of the state variables $\boldsymbol{\varepsilon}, \alpha, \theta$.

6. The Linear Thermoelasticity

When the constitutive behaviour is elastic, all thermodynamical quantities are functions only of the current state $\boldsymbol{\varepsilon}, \theta$.

In linear thermoelasticity and for small perturbations around a natural state at the temperature θ_o , the free energy has the following form ($\tau = \theta - \theta_o$):

$$\psi = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} + \mathbf{k} \cdot \boldsymbol{\varepsilon} \tau - \frac{1}{2} C \tau^2 \tag{25}$$

The positivity of entropy production is satisfied by any real variations of the state variables around a thermodynamical equilibrium (*i.e.*, mechanical equilibrium under uniform temperature), hence we deduce the equations of state:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_r = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \quad s = -\frac{\partial \psi}{\partial \theta} \tag{26}$$

In this case, the reversible stresses $\boldsymbol{\sigma}$ satisfy the conservation of the momentum. Therefore the elastic behaviour is essentially reversible.

7. More General Cases

In general, the intrinsic dissipation D_m has a form derived from the free energy ψ , which depends on the strain $\boldsymbol{\varepsilon}$, of internal parameters α and of temperature θ . The entropy production is rewritten as:

$$D_m = (\boldsymbol{\sigma} - \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}) : \dot{\boldsymbol{\varepsilon}} - \rho (\frac{\partial \psi}{\partial \theta} + s) \dot{\theta} - \rho \frac{\partial \psi}{\partial \alpha} \cdot \dot{\alpha} \geq 0 \tag{27}$$

The thermodynamical driving forces associated with the state variables are defined by the state equations:

$$\boldsymbol{\sigma}_r = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}}, \quad A = -\frac{\partial \psi}{\partial \alpha}, \quad s = -\frac{\partial \psi}{\partial \theta} \tag{28}$$

Then, the dissipation takes the form:

$$D_m = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_r) : \dot{\boldsymbol{\varepsilon}} + A \cdot \dot{\alpha} = \boldsymbol{\sigma}_{ir} : \dot{\boldsymbol{\varepsilon}} + A \cdot \dot{\alpha} \geq 0 \tag{29}$$

There exist two sources of entropy production, one due to the variations of internal parameters and the other due to the strain rates. The equations of state do not provide the full constitutive equations, and some complementary laws are needed to describe the evolution of the irreversibility. Such laws are determined by observations and experimentations. The domain of reversibility must be determined. The influence of the strain rates must be analysed and finally constitutive relations should be given between the rates $\dot{\boldsymbol{\varepsilon}}, \dot{\alpha}$ and the thermodynamical driving forces $(\boldsymbol{\sigma}_{ir}, A)$.

Generalized standard materials. A powerful method is to consider the existence of potential for the dissipation. Let us assume that the behavior belongs to the class of the generalized standard materials [4]. This ensures the existence of a potential of dissipation. The evolution of the internal state satisfies the normality rule:

$$\dot{\alpha} = \frac{\partial d^*}{\partial A}, \quad \dot{\boldsymbol{\varepsilon}} = \frac{\partial d^*}{\partial \boldsymbol{\sigma}_{ir}} \quad \text{or} \quad A = \frac{\partial d}{\partial \dot{\alpha}}, \quad \boldsymbol{\sigma}_{ir} = \frac{\partial d}{\partial \dot{\boldsymbol{\varepsilon}}} \tag{30}$$

The potentials d^* and d are convex functions of the variables, having a minimum value at the origin.

Case of linear viscoelasticity. For example, the potential of dissipation is:

$$d(\dot{\epsilon}) = \frac{1}{2} \dot{\epsilon} : \eta : \dot{\epsilon} \tag{31}$$

with η a positive definite operator. The complementary law gives:

$$\sigma_{ir} = \frac{\partial d}{\partial \dot{\epsilon}} = \eta : \dot{\epsilon} \tag{32}$$

The stresses σ used in the balance of momentum is decomposed in two terms $\sigma = \sigma_r + \sigma_{ir}$.

- For the model of Kelvin-Voigt in linear viscoelasticity, the two potentials take the form:

$$\psi(\epsilon) = \frac{1}{2} \epsilon : \mathbb{C} : \epsilon ; \quad d(\dot{\epsilon}) = \frac{1}{2} \dot{\epsilon} : \eta : \dot{\epsilon} \tag{33}$$

and then the constitutive behaviour implies that:

$$\sigma = \mathbb{C} : \epsilon + \eta : \dot{\epsilon} \tag{34}$$

- The Maxwell's description is obtained by choosing the thermodynamical potential in the following form:

$$\psi(\epsilon, \alpha) = \frac{1}{2} (\epsilon - \alpha) : \mathbb{C} : (\epsilon - \alpha) \tag{35}$$

and a pseudo potential of dissipation in a quadratic manner:

$$d = \frac{1}{2} \dot{\alpha} : \eta : \dot{\alpha} \tag{36}$$

Then $\sigma_{ir} = 0, A = \mathbb{C} : (\epsilon - \alpha)$ and the complementary law gives the relation:

$$A = \eta : \dot{\alpha} \tag{37}$$

Then the constitutive behaviour is given by:

$$\sigma = \mathbb{C} : (\epsilon - \alpha), \quad A = \mathbb{C} : (\epsilon - \alpha) = \eta : \dot{\alpha} \tag{38}$$

Normality rule. In the case of a regular and differentiable function, the convexity of the potential of dissipation gives us the characterization of internal state evolution by the equalities:

$$\sigma_{ir} = \frac{\partial d}{\partial \dot{\epsilon}}, \quad A = \frac{\partial d}{\partial \dot{\alpha}} \tag{39}$$

More generally the definition of the gradient is replaced by the subgradient.

Normality rule and subgradient of a convex function. The internal state satisfies the evolution law given by the normality rule

$$(\sigma_{ir}, A) \in \partial d(\dot{\epsilon}, \dot{\alpha}) \tag{40}$$

that is by definition the set of thermodynamical forces (σ_{ir}, A) whose satisfy the inequality:

$$d(\dot{\epsilon}, \dot{\alpha}) + \sigma_{ir} : (\epsilon^* - \dot{\epsilon}) + A : (\alpha^* - \dot{\alpha}) \leq d(\epsilon^*, \alpha^*) \tag{41}$$

for all admissible values (ϵ^*, α^*) .

We can notice that the existence of a convex potential for the dissipation ensures the positivity of the entropy production:

$$\sigma_{ir} : \dot{\epsilon} + A : \dot{\alpha} \geq d(\dot{\epsilon}, \dot{\alpha}) - d(0, 0) \geq 0 \tag{42}$$

8. The Quasistatic Evolution

Consider a body Ω submitted to imposed boundary conditions. The boundary is decomposed into $\partial\Omega_u$ on which the displacement is imposed and $\partial\Omega_T$ along the stress vector is prescribed, ($\partial\Omega = \partial\Omega_u \cup \partial\Omega_T$ and $\emptyset = \partial\Omega_u \cap \partial\Omega_T$). A solution of the problem of quasistatic evolution is defined by the displacement field \underline{u} and the internal state which satisfy:

- the compatibility equations for strains and displacements: the strain field ε is associated to the displacement \underline{u} by $\varepsilon(\underline{u}) = \nabla\underline{u} + \nabla^T\underline{u}$. The displacement satisfies the boundary conditions $\underline{u} = \underline{u}^d$ over $\partial\Omega_u$,

- the state equations:

$$\sigma_r = \frac{\partial\psi}{\partial\varepsilon}, \quad A = -\frac{\partial\psi}{\partial\alpha}, \quad s = -\frac{\partial\psi}{\partial\theta} \tag{43}$$

- the equations of evolution for the state variables:

$$(\sigma_{ir}, A) \in \partial d(\dot{\varepsilon}, \dot{\alpha}) \tag{44}$$

- the constitutive law:

$$\sigma = \sigma_r + \sigma_{ir} \tag{45}$$

- the momentum conservation and the boundary conditions:

$$\text{div } \sigma = 0 \text{ on } \Omega, \quad \sigma \cdot \underline{n} = \underline{T}^d \text{ over } \partial\Omega_T \tag{46}$$

For the overall system, the rule of the local free energy is replaced by the global free energy:

$$W(\tilde{\varepsilon}, \tilde{\alpha}, \tilde{\theta}) = \int_{\Omega} \rho \psi(\varepsilon, \alpha, \theta) \, d\Omega \tag{47}$$

We recall the definition of the Gâteaux differential:

$$\frac{\partial F}{\partial \tilde{q}} \cdot \tilde{q}^* = \lim_{\eta \rightarrow 0} \frac{F(\tilde{q} + \eta \tilde{q}^*) - F(\tilde{q})}{\eta} \tag{48}$$

Then we get for our particular case:

$$\frac{\partial W}{\partial \tilde{q}} \cdot \tilde{q}^* = \int_{\Omega} \frac{\partial \psi}{\partial q} \cdot q^* \, d\Omega \tag{49}$$

For the whole system, the equations of state are relations between fields:

$$\tilde{\sigma}_r = \frac{\partial W}{\partial \tilde{\varepsilon}}, \quad \tilde{A} = -\frac{\partial W}{\partial \tilde{\alpha}}, \quad \tilde{s} = -\frac{\partial W}{\partial \tilde{\theta}} \tag{50}$$

In a global description, the equations of state have the same form as in the local description. The state of the system is defined by fields of state variables. Connection can be made with the theory of homogenization [16], the state of the whole system is defined by the field $\tilde{\alpha}$ of the internal parameters α .

Dissipative function. By integration of the potential of dissipation and of evolution law (41), we define the dissipative function:

$$D(\tilde{\epsilon}, \tilde{\alpha}) = \int_{\Omega} \rho d(\dot{\epsilon}, \dot{\alpha}) \, d\Omega \tag{51}$$

The evolution of internal state satisfies the normality rule:

$$(\tilde{\sigma}_{ir}, \tilde{A}) \in \partial D(\tilde{\epsilon}, \tilde{\alpha}) \tag{52}$$

That is by definition:

$$D(\tilde{\epsilon}, \tilde{\alpha}) - D(\tilde{\epsilon}^*, \tilde{\alpha}^*) + \tilde{\sigma}_{ir} \cdot (\tilde{\epsilon}^* - \tilde{\epsilon}) + \tilde{A} \cdot (\tilde{\alpha}^* - \tilde{\alpha}) \leq 0 \tag{53}$$

for all admissible fields $(\tilde{\epsilon}^*, \tilde{\alpha}^*)$.

For example, in the case of regular function we have:

$$\begin{aligned} \frac{\partial D}{\partial \tilde{\epsilon}} \cdot \tilde{\epsilon}(\delta u) &= \int_{\Omega} \sigma_{ir} : \epsilon(\delta u) \, d\Omega \\ \frac{\partial D}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} &= \int_{\Omega} A \cdot \delta \alpha \, d\Omega \end{aligned}$$

These equations can be rewritten as:

$$\tilde{\sigma}_{ir} = \frac{\partial D}{\partial \tilde{\epsilon}}, \quad \tilde{A} = \frac{\partial D}{\partial \tilde{\alpha}} \tag{54}$$

The isothermal boundary value problem. We consider now for sake of simplicity only isothermal processes. Let us consider that the external loading derives from a potential given in terms of traction \underline{T}^d applied on the external surface $\partial\Omega_T$ of the body. Then, the global free energy W is replaced by the potential energy \mathcal{E} of the system:

$$\mathcal{E}(\tilde{u}, \tilde{\alpha}, \tilde{T}^d) = \int_{\Omega} \rho \psi(\epsilon(u), \alpha) \, d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} \, da \tag{55}$$

By combining all the equations in terms of fields of state variables, the quasistatic evolution is then given in a global manner by the variational system:

$$0 = \frac{\partial \mathcal{E}}{\partial \tilde{u}} \cdot \delta \tilde{u} + \frac{\partial D}{\partial \tilde{\epsilon}} \cdot \tilde{\epsilon}(\delta u) \tag{56}$$

$$0 = \frac{\partial \mathcal{E}}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} + \frac{\partial D}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} \tag{57}$$

These equations are defined on a set of admissible fields. The displacement satisfies the boundary conditions $\underline{u} = \underline{u}^d$ over $\partial\Omega_u$. Then the fields $\delta \tilde{u}$ must satisfy $\delta \underline{u} = 0$ over $\partial\Omega_u$. The admissible variations $\delta \tilde{\alpha}$ are submitted to constraints depending on the nature of irreversibility.

The preceding equations are general. They contain the essential structure of a problem of quasistatic evolution. The first equation of this system explains the momentum conservation taking account of the constitutive law:

$$\text{div } \sigma = 0, \quad \sigma = \sigma_r + \sigma_{ir}, \quad \sigma \cdot \underline{n} = \underline{T}^d \text{ over } \partial\Omega_T \tag{58}$$

the second one explains the complementary law as a relation between the forces A and the internal parameters:

$$A \cdot \delta \tilde{\alpha} = -\frac{\partial \mathcal{E}}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} = \frac{\partial D}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha}. \tag{59}$$

9. The Lagrangian and the Dynamical Case

By definition the Lagrangian is the difference between the kinetic energy and the interaction potential of the system. For all kinematically admissible fields, the interaction potential is the potential energy of the whole body:

$$\mathcal{E}(\underline{\tilde{u}}, \tilde{\alpha}, \tilde{\theta}, \underline{\tilde{T}}^d) = \int_{\Omega} \psi(\varepsilon, \alpha, \theta) \rho \, d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} \, da \tag{60}$$

The kinetic energy is defined as:

$$\mathcal{K}(\underline{\tilde{v}}) = \int_{\Omega} \frac{1}{2} \rho \underline{v}^2 \, d\Omega \tag{61}$$

Therefore the Lagrangian has the expression:

$$\mathcal{L}(\underline{\tilde{u}}, \underline{\tilde{v}}, \tilde{\alpha}, \tilde{\theta}, \underline{\tilde{T}}^d) = \mathcal{K}(\underline{\tilde{v}}) - \mathcal{E}(\underline{\tilde{u}}, \tilde{\alpha}, \tilde{\theta}, \underline{\tilde{T}}^d) \tag{62}$$

The acceleration is denoted by $\underline{\gamma}$, $\underline{\gamma} = \dot{\underline{v}}$. The variations of the Lagrangian are useful, in particular we have:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \underline{\tilde{u}}} \cdot \delta \underline{\tilde{u}} &= - \int_{\Omega} \underline{\sigma}_r : \varepsilon(\delta \underline{u}) \, d\Omega + \int_{\partial\Omega_T} \underline{T}^d \cdot \delta \underline{u} \, da \\ \frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} \cdot \delta \underline{\tilde{v}} &= \int_{\Omega} \rho \underline{v} \cdot \delta \underline{v} \, d\Omega, \\ \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} \cdot \delta \underline{\tilde{v}} \right) &= \frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} \cdot \delta \underline{\tilde{v}} + \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} \right) \cdot \delta \underline{\tilde{v}} \\ &= \int_{\Omega} (\rho \underline{v} \cdot \delta \underline{v} + \rho \underline{\gamma} \cdot \delta \underline{u}) \, d\Omega + \int_{\Gamma} m[\underline{v}]_{\Gamma} \cdot \delta \underline{u} \, da \end{aligned}$$

The notation m represents the mass flux through the moving surface Γ along which the displacement velocity v has discontinuities $[v]_{\Gamma}$. The equations of the motion are related to the momentum conservation:

$$\text{div } \underline{\sigma} = \rho \underline{\gamma} \text{ over } \Omega \tag{63}$$

$$[\underline{\sigma}]_{\Gamma} \cdot \underline{n} = m[\underline{v}]_{\Gamma} \text{ along } \Gamma \tag{64}$$

Hence, a variational form for the momentum conservation is easily deduced:

$$\int_{\Omega} \underline{\sigma} : \varepsilon(\delta \underline{u}) \, d\Omega = \int_{\partial\Omega} \underline{n} \cdot \underline{\sigma} \cdot \delta \underline{u} \, da + \int_{\Gamma} \underline{n} \cdot [\underline{\sigma}]_{\Gamma} \cdot \delta \underline{u} \, da + \int_{\Omega} \rho \underline{\gamma} \cdot \delta \underline{u} \, d\Omega \tag{65}$$

The stresses $\underline{\sigma}$ are decomposed as previously as $\underline{\sigma} = \underline{\sigma}_r + \underline{\sigma}_{ir}$ taking account of the constitutive law. The surface Γ is a moving surface, where the displacement velocities undergo discontinuities. Taking all these relations, the evolution of the system is governed by the generalized Lagrange's equations:

$$- \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} - \frac{\partial \mathcal{L}}{\partial \underline{\tilde{u}}} \right) \cdot \delta \underline{\tilde{u}} = \frac{\partial D}{\partial \underline{\tilde{\varepsilon}}} \cdot \underline{\tilde{\varepsilon}}(\delta \underline{u}) \tag{66}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha} = \frac{\partial D}{\partial \tilde{\alpha}} \cdot \delta \tilde{\alpha}, \tag{67}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{\theta}} \cdot \delta \tilde{\theta} = \int_{\Omega} \rho s \, \delta \theta \, d\Omega \tag{68}$$

These equations are a generalization to non linear dynamics [5] of the classical Lagrange’s formulation, they have the same form as the expression given in viscoelasticity [1]. In this formulation, we have defined as previously the dissipative function as:

$$D(\tilde{\underline{\epsilon}}, \tilde{\underline{\alpha}}) = \int_{\Omega} \rho d(\dot{\underline{\epsilon}}, \dot{\underline{\alpha}}) d\Omega \tag{69}$$

The first term is identical to the equations of motion, the second corresponds to the evolution of the internal variables, the last one defines the local entropy. Adding to these equations, a conduction law for determining the temperature is needed.

10. The Hamiltonian

The Hamiltonian is a Legendre transformation of the Lagrangian, with respect to the velocity and to the temperature [5,6]:

$$\mathcal{H}(\tilde{\underline{u}}, \tilde{\underline{p}}, \tilde{\underline{\alpha}}, \tilde{s}, \tilde{\underline{T}}^d) = \int_{\Omega} (\underline{p} \cdot \underline{u} + \theta \rho s) d\Omega - \mathcal{L}(\tilde{\underline{u}}, \tilde{\underline{v}}, \tilde{\underline{\alpha}}, \tilde{\theta}, \tilde{\underline{T}}^d) \tag{70}$$

After transformation, we have:

$$\mathcal{H}(\tilde{\underline{u}}, \tilde{\underline{p}}, \tilde{\underline{\alpha}}, \tilde{s}, \tilde{\underline{T}}^d) = \int_{\Omega} \frac{1}{2} \frac{\underline{p}^2}{\rho} d\Omega + \int_{\Omega} \rho e(\underline{\epsilon}(\underline{u}), \alpha, \theta) d\Omega - \int_{\partial\Omega} \underline{T}^d \cdot \underline{u} da \tag{71}$$

Here \underline{p} is the quantity of momentum $\rho \underline{v}$. In this expression appears the density of internal energy: $e = \psi + \theta s$. In a global formulation, we obtain successively:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \tilde{\underline{p}}} \cdot \tilde{\underline{p}}^* &= \tilde{\underline{v}} \cdot \tilde{\underline{p}}^* = \frac{d}{dt} \tilde{\underline{u}} \cdot \tilde{\underline{p}}^*, \\ \frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta \tilde{\underline{u}} &= \int_{\Omega} \rho \frac{\partial \psi}{\partial \underline{\epsilon}} : \underline{\epsilon}(\delta \underline{u}) d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \delta \underline{u} da \\ &= \int_{\Omega} \underline{\sigma}_r : \underline{\epsilon}(\delta \underline{u}) d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \delta \underline{u} da \end{aligned}$$

Taking account of the momentum conservation, of the decomposition of the stresses into reversible and irreversible parts, of the boundary conditions and of jump conditions, the expressions are then modified:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta \tilde{\underline{u}} = - \int_{\Omega} \underline{\sigma}_{ir} : \underline{\epsilon}(\delta \underline{u}) d\Omega - \int_{\Omega} \rho \underline{\gamma} \cdot \delta \underline{u} d\Omega + \int_{\Gamma} \underline{n} \cdot [\underline{\sigma}]_{\Gamma} \cdot \delta \underline{u} da \tag{72}$$

Recall that $\underline{\sigma}_{ir} = \partial D / \partial \tilde{\underline{\epsilon}}$ and considering the relation:

$$\frac{d}{dt} \int_{\Omega} \underline{p} \cdot \delta \underline{u} d\Omega = \int_{\Omega} \rho \underline{\gamma} \cdot \delta \underline{u} d\Omega + \int_{\Gamma} m[v]_{\Gamma} \cdot \delta \underline{u} da, \tag{73}$$

we obtain the momentum conservation in the Hamiltonian’s form:

$$\frac{\partial \mathcal{H}}{\partial \tilde{\underline{u}}} \cdot \delta \tilde{\underline{u}} = - \frac{\partial D}{\partial \tilde{\underline{\epsilon}}} \cdot \tilde{\underline{\epsilon}}(\delta \underline{u}) - \frac{d}{dt} \int_{\Omega} \underline{p} \cdot \delta \underline{u} d\Omega \tag{74}$$

Finally, the Hamiltonian formulation of the evolution problem is obtained:

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \underline{\tilde{p}}} \cdot \underline{\tilde{p}}^* &= \underline{\tilde{v}} \cdot \underline{\tilde{p}}^* = \frac{d}{dt} \underline{\tilde{u}} \cdot \underline{\tilde{p}}^* \\ \frac{\partial \mathcal{H}}{\partial \underline{\tilde{u}}} \cdot \underline{\tilde{u}}^* &= -\frac{\partial D}{\partial \underline{\tilde{\epsilon}}(\underline{v})} \cdot \underline{\tilde{\epsilon}}(\underline{u}^*) - \frac{d}{dt} (\underline{\tilde{p}} \cdot \underline{\tilde{u}}^*) \\ \frac{\partial \mathcal{H}}{\partial \underline{\tilde{\alpha}}} \cdot \underline{\tilde{\alpha}}^* &= -\frac{\partial D}{\partial \underline{\tilde{\alpha}}} \cdot \underline{\tilde{\alpha}}^* \\ \frac{\partial \mathcal{H}}{\partial \underline{\tilde{s}}} \cdot \underline{\tilde{s}}^* &= \underline{\tilde{\theta}} \cdot \underline{\tilde{\rho s}}^* \end{aligned}$$

As previously a conduction law must be taken into account and the positivity of the entropy production must be satisfied to determine the evolution of the system.

Generalization. The definition of the Lagrangian and of the Hamiltonian can be extended to generalized media as beams or plates. The proposed description is performed when the behavior of the system is described by two potentials: a global free energy and a dissipative function. If some particular internal constraints exist, the preceding description must be revisited.

Expression of the conservation of energy. For the real motion, the value of the Hamiltonian is the sum of the kinetic energy, of the internal energy and of the potential energy of the external loading, then the conservation of the energy of the system can be easily rewritten as:

$$\frac{d}{dt} \mathcal{H} - \frac{\partial \mathcal{H}}{\partial \underline{\tilde{T}}^d} \cdot \frac{d}{dt} \underline{\tilde{T}}^d = \mathcal{P}_{cal} \tag{75}$$

When the external loading is time independent $\frac{d}{dt} \underline{\tilde{T}}^d = 0$, the exchange of energy is only due to the heat rate supply \mathcal{P}_{cal} . Generally this quantity has the form:

$$\mathcal{P}_{cal} = - \int_{\partial \Omega} \underline{q} \cdot \underline{n} \, da, \tag{76}$$

where \underline{q} is the heat flux. This result is useful in fracture mechanics to discuss the heat generated by the propagation of the crack as presented in the following section.

Conservation law. In the case of conservative system, in an adiabatic evolution ($\mathcal{P}_{cal} = 0$), the Hamiltonian is constant:

$$\mathcal{H}(t) = \mathcal{H}(0) \tag{77}$$

This property can be rewritten in terms of the Lagrangian:

$$\mathcal{L} - \underline{\tilde{v}} \cdot \frac{\partial \mathcal{L}}{\partial \underline{\tilde{v}}} = \mathcal{H}(0) \tag{78}$$

Property of stationarity. The Lagrangian has properties of stationarity in elasticity or viscoelasticity: let us consider a variation of the Lagrangian in isothermal evolution:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \tilde{\underline{v}}} \cdot \delta \tilde{\underline{v}} + \frac{\partial \mathcal{L}}{\partial \tilde{\underline{u}}} \cdot \delta \tilde{\underline{u}} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \tilde{\underline{v}}} \cdot \delta \tilde{\underline{u}} \right) + \frac{\partial D}{\partial \tilde{\underline{\epsilon}}} \cdot \tilde{\underline{\epsilon}}(\delta \underline{u}) \tag{79}$$

then

$$\delta \int_{t_1}^{t_2} \mathcal{L} dt = \delta D \tag{80}$$

where δD is the total viscous dissipation during the variation.

Finally let us note that the above results may be adapted in the case of other type of boundary conditions, taking care they give rise to a well posed problem.

11. Quasistatic Evolution and Time Independent Processes

If the potential of dissipation is convex and the normality rule is adopted, the evolution is well defined by the notion of subgradient as we have seen previously. We are interested now in particular cases of time-independent processes, hence there is no viscosity. This framework permits description of dry friction, of plasticity, damage and fracture.

For these particular behaviour, the potential of dissipation $d(\dot{q})$ is a positive homogeneous function of degree one:

$$d(m\dot{q}) = m d(\dot{q}), \forall m > 0 \tag{81}$$

The potential is not differentiable for $\dot{q} = 0$. In this case the sub-gradient at this point is a convex set in the space of the driving force A . This point of view is nothing else the description associated to Hill's principle of maximal dissipation. The value of d is given by

$$d(\dot{\alpha}) = \max_{A \in \mathcal{C}} A \cdot \dot{\alpha} \tag{82}$$

This explains the normality rule and its consequences.

11.1. The Problem of Evolution

Assuming now that the convex \mathcal{C} is defined by a regular convex function f of the driving force A

$$A \in \mathcal{C} = \{B / f(B) \leq 0\} \tag{83}$$

In this case, the normality rule implies that

$$\dot{\alpha} = \lambda \frac{\partial f}{\partial A} = \lambda \mathcal{N}, \quad \lambda \geq 0, \quad f(A)\lambda = 0 \tag{84}$$

The internal parameter α evolves if the driving force reaches the critical value given by $f(A) = 0$, otherwise the internal parameter cannot evolve. The rate of α is normal to the equipotential surface $f = 0$, the notation $\mathcal{N} = \frac{\partial f}{\partial A}$ is then adopted.

As previously, we introduce the free energy $\psi(\underline{\epsilon}, \alpha)$ and the potential energy

$$\mathcal{E}(\tilde{\underline{u}}, \tilde{\alpha}, \underline{T}^d) = \int_{\Omega} \psi(\underline{\epsilon}(\underline{u}), \alpha) d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} da. \tag{85}$$

The potential energy is defined on the set of admissible fields \underline{u} kinematically admissible with the given displacement \underline{u}^d prescribed on $\partial\Omega_u$.

At each time t , the driving force $A(x, t)$ satisfies the inequality $f(A(x, t)) \leq 0$ and the state equations:

$$A = -\frac{\partial\psi}{\partial\alpha} \tag{86}$$

The system is in equilibrium during the time, hence at the current state, the potential energy is stationary among the set of admissible displacements $\delta\underline{u}$ which satisfy $\delta\underline{u} = 0$ over $\partial\Omega_u$:

$$\frac{\partial\mathcal{E}}{\partial\underline{u}} \cdot \delta\underline{u} = 0 \tag{87}$$

The variations of the potential energy are equivalent to

$$\text{div } \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}, \quad \boldsymbol{\sigma} \cdot \underline{n} = \underline{T}^d \text{ over } \partial\Omega_T \tag{88}$$

These equations are true during the time, so the evolution of equilibrium is given by

$$\text{div } \dot{\boldsymbol{\sigma}} = 0 \text{ and } \dot{\boldsymbol{\sigma}} = \frac{\partial^2\psi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial^2\psi}{\partial\boldsymbol{\varepsilon}\partial\alpha} : \dot{\alpha} \text{ in } \Omega \text{ and } \dot{\boldsymbol{\sigma}} \cdot \underline{n} = \dot{\underline{T}}^d \text{ over } \partial\Omega_T \tag{89}$$

which is equivalent to

$$0 = \frac{d}{dt} \left(\frac{\partial\mathcal{E}}{\partial\underline{u}} \right) \cdot \delta\underline{u} \tag{90}$$

or

$$0 = \int_{\Omega} \boldsymbol{\varepsilon}(\delta\underline{u}) : \left(\frac{\partial^2\psi}{\partial\boldsymbol{\varepsilon}\partial\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial^2\psi}{\partial\boldsymbol{\varepsilon}\partial\alpha} : \dot{\alpha} \right) d\Omega - \int_{\partial\Omega_T} \dot{\underline{T}}^d \cdot \delta\underline{u} da \tag{91}$$

The current state is determined by the evolution of the internal state $\dot{\alpha}$.

Evolution of α . Considering the normality rule, we can conclude that

$$\lambda \geq 0, f \leq 0, \quad \lambda f = 0 \tag{92}$$

For an internal state such that $f = 0$, the evolution of f satisfies $\dot{f} \leq 0$, and simultaneously the time derivative of the condition $\lambda f = 0$ implies that

$$\overline{\lambda\dot{f}} = \dot{\lambda}f + \lambda\dot{f} = 0 \tag{93}$$

then

$$\lambda\dot{f} = 0, \text{ if } f = 0 \tag{94}$$

Hence $\lambda > 0$ if and only if $\dot{f} = 0$, which is the classical consistency condition. This provides the definition of the set \mathcal{P} of the admissible fields $\lambda(x)$.

At each instant τ , the domain Ω is decomposed into two complementary subdomains Ω_r and I^τ such that:

$$x \in \Omega_r, \quad f(A(x, t)) < 0 \tag{95}$$

$$x \in I^\tau, \quad f(A(x, t)) = 0 \tag{96}$$

Then \mathcal{P} is defined as:

$$\mathcal{P} = \{\beta(x)/\beta(x) = 0 \quad \forall x \in \Omega_r \text{ and } \beta(x) \geq 0 \quad \forall x \in I^\tau\}. \tag{97}$$

It is obvious that the field $\lambda(x)$ is an element of \mathcal{P} .

Considering now a point $x \in I^\tau$ then $\lambda(x) \geq 0$ and $\dot{f}\lambda = 0$. For $\beta(x) \geq 0$, because $\dot{f} \leq 0$ we deduce:

$$(\lambda(x) - \beta(x))\dot{f} \geq 0 \tag{98}$$

so we have

$$\int_{\Omega} (\lambda(x) - \beta(x))\dot{f} \, d\Omega \geq 0 \tag{99}$$

among the set \mathcal{P} of admissible fields β . This is a variational inequality.

By using now the definition of f , considering the equations of state for A and the normality rule for $\dot{\alpha}$, the inequality Equation 99 is rewritten as ($\mathcal{N} = \frac{\partial f}{\partial A}$):

$$\int_{\Omega} (\lambda(x) - \beta(x)) \left(\mathcal{N} : \frac{\partial^2 \psi}{\partial \alpha \partial \epsilon} : \dot{\epsilon} + \frac{\partial^2 \psi}{\partial \alpha \partial \alpha} : \mathcal{N} \lambda \right) \, d\Omega \leq 0. \tag{100}$$

11.2. The Rate Boundary Value Problem

We define the functional \mathcal{F} based on the velocities:

$$\mathcal{F}(\underline{\tilde{v}}, \tilde{\lambda}, \underline{\tilde{T}}^d) = \int_{\Omega} \left(\frac{1}{2} \epsilon(\underline{v}) : \mathbb{C} : \epsilon(\underline{v}) + \epsilon(\underline{v}) : M \lambda + \frac{1}{2} \lambda H \lambda \right) \, d\Omega - \int_{\partial\Omega_T} \underline{\tilde{T}}^d \cdot \underline{v} \, da \tag{101}$$

where we have adopted the notations: $\mathbb{C} = \frac{\partial^2 \psi}{\partial \epsilon \partial \epsilon}$, $M = \frac{\partial^2 \psi}{\partial \epsilon \partial \alpha} : \mathcal{N}$, $H = \mathcal{N} : \frac{\partial^2 \psi}{\partial \alpha \partial \alpha} : \mathcal{N}$.

Variational inequality. The solution of the rate boundary value problem satisfies the variational inequality

$$\frac{\partial \mathcal{F}}{\partial \underline{\tilde{v}}} \cdot (\underline{\tilde{v}} - \underline{\tilde{v}}^*) + \frac{\partial \mathcal{F}}{\partial \tilde{\lambda}} \cdot (\tilde{\lambda} - \tilde{\lambda}^*) \leq 0 \tag{102}$$

among the set of admissible fields $\underline{\tilde{v}}^* \in \mathcal{K}, \tilde{\lambda}^* \in \mathcal{P}$.

Generally the modulus of elasticity \mathbb{C} is a quadratic positive-definite operator, then the value of the field \underline{v} is unique, if the $\dot{\alpha}$ is a given data. So the velocity \underline{v} can be eliminated: $\underline{v}(\tilde{\lambda}, \underline{\tilde{T}}^d, \tilde{v}^d)$. Then, we introduce a new functional which is defined only on the internal variables:

$$\mathcal{F}^*(\tilde{\lambda}) = \mathcal{F}(\underline{\tilde{v}}(\tilde{\lambda}, \underline{\tilde{T}}^d, \tilde{v}^d), \tilde{\lambda}) \tag{103}$$

$$= \int_{\Omega} \int_{\Omega} \frac{1}{2} \lambda(x) Q(x, y) \lambda(y) \, d\Omega_x \, d\Omega_y - \int_{\partial\Omega} \int_{\partial\Omega} \lambda(x) T(\underline{\tilde{T}}^d, \underline{v}^d)(y, x) \, da_x \, da_y \tag{104}$$

The solution of the boundary value problem satisfies the variational inequality

$$\frac{\partial \mathcal{F}^*}{\partial \tilde{\lambda}} \cdot (\tilde{\lambda} - \tilde{\lambda}^*) \leq 0 \tag{105}$$

among the set of admissible fields $\tilde{\lambda}^* \in \mathcal{P}$.

Stability condition. It is known that a solution exist if

$$\forall \tilde{\beta} \in \mathcal{P}, \int_{\Omega} \int_{\Omega} \beta(x)Q(x,y)\beta(y) d\Omega_x d\Omega_y \geq 0 \tag{106}$$

where \mathcal{P} is

$$\mathcal{P} = \{ \tilde{\beta} / \beta(x) = 0 \ \forall x \in \Omega_r, \text{ and } \beta(x) \geq 0 \ \forall x \in I^r \} \tag{107}$$

this ensures that the current state is stable.

Uniqueness and no-bifurcation. The solution of the boundary value problem is also unique if

$$\forall \tilde{\beta} \in \mathcal{P}^* \int_{\Omega} \int_{\Omega} \beta(x)Q(x,y)\beta(y) d\Omega_x d\Omega_y \geq 0, \tag{108}$$

where \mathcal{P}^* is the set

$$\mathcal{P}^* = \{ \tilde{\beta} / \beta(x) = 0 \ \forall x \in \Omega_r \} \tag{109}$$

This conditions ensures that there is no bifurcation.

11.3. Property of the Functional

The functional \mathcal{F} is decomposed in a quadratic term and a linear contribution

$$\mathcal{F}(\tilde{v}, \tilde{\lambda}) = \frac{1}{2}Q((\tilde{v}, \tilde{\lambda}), (\tilde{v}, \tilde{\lambda})) + F(\tilde{T}^d, \tilde{v}) \tag{110}$$

Consider now that a solution is determined, the domain I^r is decomposed in three different domains depending on $\lambda > 0$ or not:

- the loading zone

$$I_+^r = \{ x \in \Omega / x \in I^r, \mu^r(x) > 0, \dot{f}^r(x) = 0 \}$$

- the unloading zone

$$I_-^r = \{ x \in \Omega / x \in I^r, \mu^r(x) = 0, \dot{f}^r(x) < 0 \}$$

- the neutral zone

$$I_o^r = \{ x \in \Omega / x \in I^r, \mu^r(x) = 0, \dot{f}^r(x) = 0 \}$$

Introducing asymptotic expansion to define a loading path

$$\underline{T}^d = \underline{T}_0^d + \tau \underline{T}_1^d + \tau^2 \underline{T}_2^d + \dots \tag{111}$$

$$\underline{u}^d = \underline{u}_0^d + \tau \underline{u}_1^d + \tau^2 \underline{u}_2^d + \dots \tag{112}$$

We may find a local response in terms of displacement field and internal variable fields. It is assumed that the response is developed as an asymptotic expansion in the form of:

$$\alpha = \alpha_0 + \tau \alpha_1 + \tau^2 \alpha_2 + \dots \tag{113}$$

$$\underline{u} = \underline{u}_0 + \tau \underline{u}_1 + \tau^2 \underline{u}_2 + \dots \tag{114}$$

The term of order one corresponds to the solution of the boundary value problem in velocities. As similar asymptotic expansions can be introduced for the function f to express the normality rule, we can derive constraints on the successive orders of the internal state. Hence the characterization of order two shows that

$$\lambda_2 f_1 + \lambda_1 f_2 = 0 \tag{115}$$

The properties of λ_2 are given related to the decomposition of I^τ and the field λ_2 is an element of the set \mathcal{P}_2

$$\begin{aligned} \mathcal{P}_2 = \{ \tilde{\mu}/\mu(x) &= 0, \text{ if } x \notin I^\tau \cup I_+^\tau \cup I_o^\tau, \\ \mu(x) &\geq 0, \text{ if } x \in I_o, \\ \mu(x) &\in \mathfrak{R}, \text{ if } x \in I_+^\tau \} \end{aligned} \tag{116}$$

The boundary value problem for the order two has the same form for that of order one, except that the linear term contains terms due to order one [17].

$$\mathcal{F}_2(\tilde{u}_2, \tilde{\lambda}_2) = \frac{1}{2}Q((\tilde{u}_2, \tilde{\lambda}_2), (\tilde{u}_2, \tilde{\lambda}_2)) + F_2((\tilde{u}_1, \tilde{\lambda}_1), (\tilde{u}_2, \tilde{\lambda}_2)) \tag{117}$$

and the solution of the rate boundary value problem of order 2 satisfies

$$\frac{\partial \mathcal{F}_2}{\partial \tilde{u}_2}(\tilde{u}_2^* - \tilde{u}_2) + \frac{\partial \mathcal{F}_2}{\partial \tilde{\lambda}_2}(\tilde{\lambda}_2^* - \tilde{\lambda}_2) \geq 0 \tag{118}$$

among the set of admissible field \tilde{u}_2^* which may satisfy the boundary conditions at order two and $\tilde{\lambda}_2^*$ is an element of \mathcal{P}_2 .

The condition of stability on order two is different than the condition of order one, due to the presence of unloading zone. The condition of no bifurcation is also changed taking account of $\lambda = 0$ on I_o^τ . The lost of positivity of Q on these new spaces changes the critical value $\underline{T}_c^d, \underline{u}_c^d$.

12. Some Typical Examples

The column is simulated here by a model equivalent to that used previously by many authors [10]. We study the plastic buckling of a simple column. This system is discussed in detail in [9].

The rigid rod model has two degrees of freedom : the downward vertical displacement u and the rotation θ .

12.1. The Discrete Shanley Column

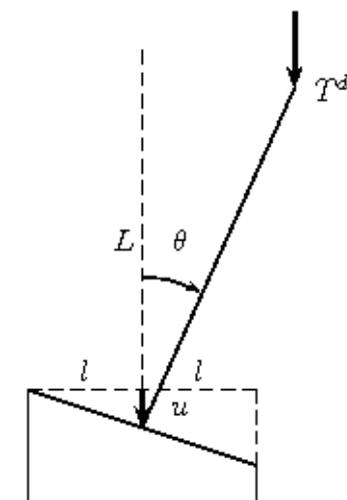
The increment in the compressive force is balanced by the deformation of the two support springs as shown in Figure 1. The springs have an elastoplastic behaviour with linear hardening with free energy:

$$\psi(\varepsilon, \alpha) = \frac{1}{2}(\varepsilon - \alpha)E(\varepsilon - \alpha) + \frac{1}{2}\alpha H\alpha \tag{119}$$

where

$$\varepsilon = u - z\theta \tag{120}$$

Figure 1. The discrete Shanley column.



The springs are located at points $x = -l$ and $x = l$. The internal variables are denoted by $\alpha_i, i = 1, 2$. The potential energy of the system is then:

$$\mathcal{E}(u, \theta, T^d) = \sum_i \psi_i + T^d(u + L(1 - \theta^2/2)) \tag{121}$$

The local tension of the spring are

$$\sigma_i = E(\varepsilon - \alpha_i) \tag{122}$$

The domain of reversibility is defined by the convex function f on the driving forces associated to α :

$$A_i = E(\varepsilon - \alpha_i) - H\alpha_i, \quad f(A_i) = A_i^2 - Y_c^2 \leq 0 \tag{123}$$

where Y_c^2 is the plastic limit and we adopt the normality rule for governing the evolution of α :

$$\dot{\alpha}_i = \lambda A_i, \quad \lambda \geq 0, \quad \lambda f = 0 \tag{124}$$

During loading, the local behaviour is purely elastic or elastoplastic. The local behaviour can be rewritten as:

$$\dot{\sigma}_i = E_i \dot{\varepsilon} \tag{125}$$

where $E_i = E$ for elastic response or $E_i = E_T = \frac{EH}{E + H}$ for loading path in elastoplasticity.

The potential energy stationarity gives the conditions of equilibrium:

$$-T^d = \sigma_1 + \sigma_2; \quad -LT^d\theta = l(\sigma_2 - \sigma_1) \tag{126}$$

These equations are preserved during the loading path:

$$-\dot{T}^d = \dot{\sigma}_1 + \dot{\sigma}_2; \quad -L(\dot{T}^d\theta + T^d\dot{\theta}) = l(\dot{\sigma}_2 - \dot{\sigma}_1) \tag{127}$$

In these relations, the local behaviour must be taken into account. Assume now that the local behaviour is conserved during the equilibrium path, the equations of equilibrium can be integrated since the initial position T_o^d, θ_o to the current state:

$$-L(T^d\theta - T_o^d\theta_o) = A(T^d - T_o^d) + B(\theta - \theta_o) \tag{128}$$

where

$$A = \frac{l}{L} \frac{E_1 - E_2}{E_1 + E_2}, \quad B = 4 \frac{l^2}{L} \frac{E_1 E_2}{E_1 + E_2}. \tag{129}$$

Three cases are then studied depending on whether plastic loading or elastic unloading occurs:

- The two springs are elastic,

$$E_1 = E_2 = E$$

hence $A=0$, the solution $\theta \neq 0$ is given by the condition $T^d = T_o^d = B/L = T_E = E \frac{2l^2}{L}$. That is the critical value of Euler. This can be found if the plastic limit Y_c is not reached, then $T_E \leq 2Y_c$.

- The two springs are in plastic loading.

$$E_1 = E_2 = E_T$$

The value $T^d = T_T = E_T \frac{2l^2}{L}$ is a critical value, but in this case due to the vertical equilibrium, the only possibility is to have $\varepsilon_i = 0$ in these case $\dot{\theta} = 0$. This state is reduced to the curve $\theta = 0$. That is the fundamental path, the condition of stability is fulfilled.

- One is elastic and two is in plastic regime:

$$E_1 = E, \quad E_2 = E_T$$

Introducing the notation

$$T^d = T_R = E_R \frac{2l^2}{L}, \quad E_R = \frac{2EE_T}{E + E_T}$$

T_o^d satisfies $T_R \geq T_o^d \geq T_T$, that is due to the signs of $\dot{\varepsilon}_i$ at $\theta = 0$. The (θ, T^d) curves are branches of hyperboles with (θ, T_R) as asymptotes. Along this new path the condition of stability is fulfilled.

This simple example shows the influence of the domain of unloading.

12.2. A Simple Continuum : The Shanley Column

Now the springs are uniformly distributed along the segment $[-l, l]$. The domain I_+^T is assumed to be $[d, l]$. The value of d is determined by the condition of neutral loading

$$\dot{\varepsilon}(d) = 0 = \dot{u} - d \dot{\theta}, \quad [\dot{\alpha}](d, t) = 0 \tag{130}$$

The equations of equilibrium are deduced from the potential energy

$$\mathcal{E}(u(x), \alpha(x), T^d) = \int_{-l}^l \psi(\varepsilon, \alpha) dx + T^d(u + L(1 - \theta^2/2)) \tag{131}$$

as previously, $\varepsilon(x) = u - x\theta$, then the state of equilibrium obeys to

$$0 = T^d + \int_{-l}^{+l} E(x)(\varepsilon - \alpha) dx \tag{132}$$

$$0 = -T^d L\theta + \int_{-l}^{+l} E(x)x(\varepsilon - \alpha) dx \tag{133}$$

These equations are valid during the loading process taking into account of the condition Equation 130. Then, we obtain

$$0 = \dot{T}^d + \int_{-l}^d E \dot{\epsilon} dx + \int_d^l E_T \dot{\epsilon} dx \tag{134}$$

$$0 = -L \overline{T}^d \dot{\theta} + \int_{-l}^{+d} E x \dot{\epsilon} dx + \int_d^l E_T x \dot{\epsilon} dx \tag{135}$$

A non-trivial solution in θ is obtained by introducing the time-scale τ such that the velocity $x_1 = \dot{d}$ of propagation of the unloading domain is finite. The domain $I_+^\tau = [x_\tau, l]$ is parametrized with

$$d = x_\tau = \sum_i x_i \tau^i. \tag{136}$$

At point x_τ , the condition $[\dot{\alpha}(x_\tau, \tau)] = 0$ where $\alpha(x, t) = \sum_i \alpha_i(x) \tau_i$ gives conditions on the asymptotic expansion.

$$\begin{aligned} 0 &= [\alpha_1(x_o)] \\ 0 &= [\alpha_2(x_o)] + x_1 [\alpha_1'(x_o)] \\ 0 &= [\alpha_3(x_o)] + 2x_1 [\alpha_2'(x_o)] + (2x_1^2 + x_2) [\alpha_1'(x_o)] \end{aligned} \tag{137}$$

A non-trivial solution is then obtained as

$$\alpha^*(\tau) = \alpha(\tau) + m(\tau)x \tag{138}$$

We can take the time-derivative of the equilibrium equations taking account of the position of x_τ and of discontinuities (137) of the mechanical quantities on this boundary. It is obvious that we have:

$$\frac{d}{dt} \int_{-l}^l f(x, \tau) dx = \int_{-l}^l \dot{f}(x, \tau) dx + [f(x, \tau)]_{x_\tau^-}^{x_\tau^+} \dot{x}_\tau \tag{139}$$

We find $T_c = T_T = \frac{2l^3 E}{3L} = \frac{H}{E + H} T_E$, $m_1 = 0$ and $m_2 = -T_2/2Hl^2$,

$$\begin{aligned} x_1^2 &= \frac{4l^2}{3} \frac{T^2}{T_E - T_T} \\ T &= T_T + T_2 \frac{\tau^2}{2} \\ \theta &= \frac{E l T_2}{3LH(T_E - T_T)} \frac{\tau^2}{2} + \dots \end{aligned}$$

This is a bifurcated path. The condition of stability of the fundamental path ($\theta = 0, T$) is preserved for loading near $T = T_c$ but for $T \geq T_c$ another path exists which is also a stable path.

Applications. More applications can be found in many papers for elastoplasticity [17,18] with implications on the constitutive laws [14,19]. Influence of prebifurcation conditions have been also analysed [20,21].

13. Stability of Crack Propagation in Fracture Mechanics

Consider an elastic solid Ω containing linear cracks of length $l_i, i = 1, \dots, N$, subjected to an external loading T^d on $\partial\Omega_T$ and prescribed displacement on $\partial\Omega_u$. Clearly the potential energy depends of the crack lengths l_i . For the whole structure, these parameters play the role of internal parameters. The displacement \underline{u} belongs to the set of admissible field:

$$\tilde{u} \in K.A = \{ \tilde{u}/u(x) = \underline{u}^d(x) \text{ over } \partial\Omega_u, [\underline{u}] \cdot \underline{n} \geq 0 \text{ along } S_i \} \tag{140}$$

The surface S_i defines the geometry of the crack i , and the prescribed condition is that the crack is opened. As previously stated, a state of equilibrium for a given distribution of crack of length l_i is related to the potential energy stationarity:

$$\mathcal{E}(\tilde{u}, l_i, T^d) = \int_{\Omega} \frac{1}{2} \varepsilon(\underline{u}) : \mathbb{C} : \varepsilon(\underline{u}) \, d\Omega - \int_{\partial\Omega_T} T^d \cdot \underline{u} \, da \tag{141}$$

The state of equilibrium is defined by the stress field $\sigma = \mathbb{C} : \varepsilon(\underline{u})$ such that: $\sigma \cdot \underline{n} = T^d$ over $\partial\Omega_T$ and the local equation $\text{div } \sigma = 0$. The strain $\varepsilon(\underline{u}) = \frac{1}{2}(\nabla^T \underline{u} + \nabla \underline{u})$ and the displacement \underline{u} satisfies the boundary condition $\underline{u} = \underline{u}^d$ over $\partial\Omega_u$.

Along the open cracks, there is no stress $\sigma \cdot \underline{n} = 0$.

Using now these equations for any closed loop C inside Ω we have:

$$\int_C \left(\sigma : \varepsilon(\underline{u}) n_x - \underline{n} \cdot \nabla_x \sigma \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \nabla_x \underline{u} \right) da = 0 \tag{142}$$

where we have adopted the notation $n_x = \underline{n} \cdot \underline{e}_x, \nabla_x \sigma = \nabla \sigma \cdot \underline{e}_x, \nabla_x \underline{u} = \nabla \underline{u} \cdot \underline{e}_x$.

With this property, the integral J is rewritten as:

$$J = \frac{1}{2} \int_{\Gamma} \left(\underline{n} \cdot \nabla_x \sigma \cdot \underline{u} - \underline{n} \cdot \sigma \cdot \nabla_x \underline{u} \right) da \tag{143}$$

The local behaviour is reversible, the dissipation is only due to the propagation of cracks. The characterization of the dissipation for the case of a linear crack have been studied, and the dissipation is defined by

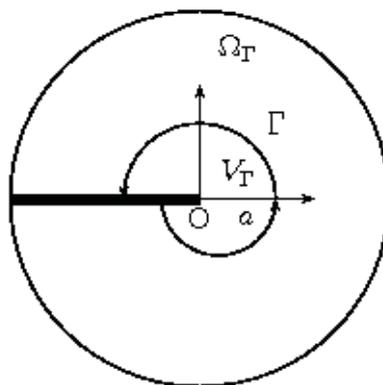
$$D_m = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \left(\psi \underline{n} \cdot \underline{e}_x - \underline{n} \cdot \sigma \cdot \nabla \underline{u} \cdot \underline{e}_x \right) da \quad \dot{l} = J \dot{l} \tag{144}$$

The integral $J_{\Gamma} = \int_{\Gamma} \left(\psi \underline{n} \cdot \underline{e}_x - \underline{n} \cdot \sigma \cdot \nabla \underline{u} \cdot \underline{e}_x \right) da$ in linear elasticity is independent of the curve Γ surrounding the crack tip. The propagation of the crack obeys the Griffith's law

$$J \leq G_c, \quad \dot{l} = 0, \quad (G_c - J) \dot{l} = 0 \tag{145}$$

In classical linear elasticity, the strain ε near the crack tip is singular as $\frac{1}{\sqrt{r}}$. To follow the evolution of J in the motion of the crack, we must take the singularity into account. The singularity is preserved for a frame in motion with the crack tip. The crack tip is surrounded by a curve Γ delimiting a domain V_{Γ} , see Figure 2. This domain moves with the position of the crack tip which is given by the function $l(t)$. All mechanical quantities are expressed in terms of the classical fixed coordinates X, y outside V_{Γ} and in terms of the moving frame inside V_{Γ} : the change of coordinates is $x = X - l(t), y = Y$.

Figure 2. Decomposition of Ω in $\Omega_\Gamma \cup V_\Gamma$.



The time derivative of any mechanical quantities F is given by $\overset{\circ}{f}$ in the moving frame

$$F(X, Y, t) = f(x, y, t), \quad \dot{f} = \frac{\partial F}{\partial t} = -a \frac{\partial f}{\partial x} + \overset{\circ}{f}$$

Following this process, it is easy to evaluate the evolution of J

$$\dot{J} = \int_{\Gamma} \left(\underline{n} \cdot \nabla_x \underline{\sigma} \cdot \underline{\dot{u}} - \underline{n} \cdot \overset{\circ}{\underline{\sigma}} \cdot \nabla_x \underline{u} \right) da \tag{146}$$

The direction \underline{e}_x is the direction of the crack. The notations $f_x = f \cdot \underline{e}_x$ and $f, x = \nabla f \cdot \underline{e}_x = \nabla_x f$ are adopted.

For a set of cracks, the boundary value problem is defined on the field $\hat{u} = \underline{u}$ over Ω/V_{Γ_i} and $\hat{u} = \underline{\dot{u}}$ inside V_{Γ_i} with the velocity $a^i = \dot{l}_i$. The direction of the crack is \underline{e}_i and the notations $f_i = f \cdot \underline{e}_i$ and $f, i = \nabla f \cdot \underline{e}_i = \nabla_i f$ are adopted.

The \hat{u} has a jump along Γ_i and the stress $\hat{\sigma}$ satisfies some discontinuity relations along each Γ_i

$$[\hat{u}]_{\Gamma} + a^i [\nabla_i]_{\Gamma} u = 0, \quad \underline{n} \cdot [\hat{\sigma}]_{\Gamma} + a^i \underline{n} \cdot \nabla_i \underline{\sigma} = 0 \tag{147}$$

The solution for the rate boundary value problem is now written by introducing the functional \mathcal{F}

$$\begin{aligned} \mathcal{F}(\hat{u}, a) &= \int_{\Omega/V_{\Gamma_i}} U(\hat{\epsilon}) \, d\Omega + \int_{\Omega} U(\hat{\epsilon} - a^i \nabla_i \underline{\epsilon}) \, d\Omega \\ &+ \int_{V_{\Gamma_i}} (a^i \nabla_i \underline{\sigma} : \hat{\epsilon} - \frac{1}{2} (a^i)^2 \nabla_i \underline{\sigma} : \nabla_i \underline{\epsilon}) \, d\omega \\ &+ \int_{\Gamma} \left(-a^i \underline{n} \cdot \nabla_i \underline{\sigma} \cdot \hat{u}^- + \frac{1}{2} a^{i2} \underline{n} \cdot \nabla \underline{\sigma} \cdot \nabla_i \underline{u} \right) da - \int_{\partial\Omega_T} \underline{\dot{T}}^d \cdot \hat{u} \, da \end{aligned} \tag{148}$$

This is the general form for the propagation of the crack, when the local behaviour is hypoelastic in the sense of Hill, due to the existence of a potential $U(\hat{\epsilon})$ such that

$$\overset{\circ}{\underline{\sigma}} = \frac{\partial U}{\partial \hat{\epsilon}}$$

More details are given in many papers for elastic media [11,12,22] and extension to other cases in [23–25].

Property of a solution. A solution of the boundary value problem (\hat{u}, a) satisfies the variational inequalities

$$\frac{\partial \mathcal{F}}{\partial \hat{u}} \cdot (\hat{u}^* - \hat{u}) + \frac{\partial \mathcal{F}}{\partial a} \cdot (a^* - a) \geq 0 \tag{149}$$

among the set of admissible fields

$$\mathcal{K} = \{(\hat{u}^*, a^*) / [\hat{u}^*]_{\Gamma} + a^* \nabla_x \underline{u} = 0, \quad a^* \geq 0, \quad \hat{u}^* = \underline{v}^d \text{ over } \partial\Omega_u\} \tag{150}$$

Some example are given in [12].

13.1. A Simple Model

Consider a straight beam in flexure fixed at the extremities l_1, l_2 , where the beam is clamped. The strain is defined by the vertical displacement $v : \varepsilon = v''y$. We applied a load at the origin, and we study the possibility of decohesion at point l_1, l_2 . We study two cases, first the applied load is a vertical displacement $v(o) = V$ and second we control the load at the origin $T(o) = F$

For the first case the potential energy at the equilibrium is

$$W(l_1, l_2, V) = \frac{3}{2} EIV^2 \frac{(l_1 + l_2)^3}{l_1^3 l_2^3} = \frac{1}{2} kV^2 \tag{151}$$

As previously $J_i = -\frac{\partial W}{\partial l_i}$ and $Q_{ij} = -J_{ij} = \frac{\partial^2 W}{\partial l_i \partial l_j}$ governs the evolution of the system.

For $l_1 = l_2$ the matrix Q is always positive definite, the position is always stable and we have no bifurcation.

For the case when the force is controlled

$$W(l_1, l_2, F) = -\frac{F^2}{k}$$

The associated Q matrix is always negative definite. The symmetric equilibrium is always a unstable state with a possible bifurcation.

Analysis of a system of two colinear cracks. In a recent paper, the case of two colinear cracks have been studied [26]. The stress intensity factors are expressed and the variation of these quantities relative to the length of the cracks are also given. The interaction matrix of the crack is explicitly given and then the condition of stability and bifurcation can be explicitly given.

14. Moving Surfaces

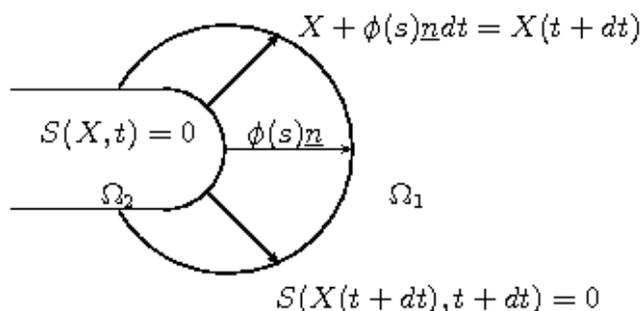
The propagation of moving surface inside a body is analysed. Here, the moving surface is associated with a change of mechanical properties. This framework is used to describe damage or phase transformation. Variational formulations were performed to describe the evolution of the surface between the sound and the damaged material [13,27,28]. Connection can be made with the notion of configurational forces [29].

14.1. Some Features

The domain Ω is composed of two distinct volumes Ω_1, Ω_2 of two materials with different mechanical characteristics. The bounding between the two phases is perfect and the interface is denoted by Γ , ($\Gamma = \partial\Omega_1 \cap \partial\Omega_2$). The external surface $\partial\Omega$ is decomposed in two parts $\partial\Omega_u$ and $\partial\Omega_T$ on which the displacement \underline{u}^d and the loading \underline{T}^d are prescribed respectively.

The material 1 changes into material 2 along the interface Γ by an irreversible process. Hence Γ moves with the normal velocity $\underline{c} = \phi \underline{\nu}$ in the reference state, $\underline{\nu}$ is the outward Ω_2 normal, then ϕ is positive; see Figure 3.

Figure 3. Propagation of the interface.



When the surface Γ is moving, all the mechanical quantities f can have a jump denoted by $[f]_\Gamma = f_1 - f_2$, and any volume average has a rate defined by

$$\frac{d}{dt} \int_{\Omega(\Gamma)} f \, d\Omega = \int_{\Omega(\Gamma)} \dot{f} \, d\Omega - \int_\Gamma [f]_\Gamma \, c \cdot \underline{\nu} \, da \tag{152}$$

The state of the system is characterized by the displacement field \underline{u} , from which the strain field ϵ is derived. The other parameters are the temperature θ and the spatial distribution of the two phases given by the position of the boundary Γ . We analyse quasistatic evolution of Γ under given loading prescribed on the boundary $\partial\Omega$.

The behaviour of the phase i is defined by the free energy density ψ_i , function of the strain ϵ and of the temperature θ . The mass density ρ of the two phases is the same. The state equations of each phase are

$$\sigma = \rho \frac{\partial \psi_i}{\partial \epsilon}, \quad s = - \frac{\partial \psi_i}{\partial \theta}, \tag{153}$$

where σ is the reversible stress and s the entropy. If the materials have no viscosity then σ is the stress satisfying the momentum equation.

To simplify the analysis, we assume that the two phases are linear elastic materials.

The two phases are linear elastic. The potential energy \mathcal{E} of the structure Ω ($\Omega_1 \cup \Omega_2$) has the following form

$$\mathcal{E}(\tilde{\underline{u}}, \Gamma, \tilde{\underline{T}}^d) = \sum_{i=1,2} \int_{\Omega_i} \rho \psi_i(\epsilon(u)) \, d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \underline{u} \, da.$$

The potential energy plays the role of the global free energy in a thermodynamical description. We can notice that the position of the interface Γ becomes an internal parameter for the global system. The characterization of an equilibrium state is given by the stationarity of the potential energy

$$\frac{\partial \mathcal{E}}{\partial \underline{u}} \cdot \delta \underline{u} = \sum_{i=1,2} \int_{\Omega_i} \rho \frac{\partial \psi_i}{\partial \underline{\varepsilon}} : \underline{\varepsilon}(\delta \underline{u}) \, d\Omega - \int_{\partial\Omega_T} \underline{T}^d \cdot \delta \underline{u} \, da = 0, \tag{154}$$

for all $\delta \underline{u}$ kinematically admissible field satisfying $\delta \underline{u} = 0$ over $\partial\Omega_u$. This formulation is equivalent to the set of local equations:

- local constitutive relations:

$$\underline{\sigma} = \rho \frac{\partial \psi_i}{\partial \underline{\varepsilon}} = \mathbb{C}_i : \underline{\varepsilon}, \text{ on } \Omega_i, \tag{155}$$

- momentum equations

$$\text{div } \underline{\sigma} = 0 \text{ on } \Omega, [\underline{\sigma}]_{\Gamma} \cdot \underline{\nu} = 0 \text{ over } \Gamma, \underline{\sigma} \cdot \underline{n} = \underline{T}^d \text{ over } \partial\Omega_T, \tag{156}$$

- compatibility relations

$$2\underline{\varepsilon} = \nabla \underline{u} + \nabla^t \underline{u}, [\underline{u}]_{\Gamma} = 0 \text{ over } \Gamma, \underline{u} = \underline{u}^d \text{ over } \partial\Omega_u. \tag{157}$$

They are equations of a problem of heterogeneous elasticity. The solution is denoted by \underline{u}^{sol} , this field depends upon the quantities $(\underline{u}^d, \underline{T}^d, \Gamma)$. For an equilibrium state

$$\mathcal{E}(\underline{u}_{sol}, \underline{T}^d, \Gamma) = W(\underline{u}^d, \underline{T}^d, \Gamma). \tag{158}$$

This equation expresses the fact that the position of the interface Γ plays the role of internal parameters.

At a given state of equilibrium for a given value of the prescribed loading $(\underline{u}^d, \underline{T}^d)$, the position of the interface Γ is known. At this time a variation of the loading is imposed, the mechanical quantities evolve and propagation of the interface can occur according to a given evolution law. For a prescribed history of the loading, we must determine the rate of all mechanical fields and the normal propagation ϕ to characterize the position of the interface Γ at each time. Along interface Γ perfect bounding is preserved at each time. Let us introduce the notion of convected derivative.

Convected Derivation. The convected derivative D_{ϕ} of any function $f(\underline{X}_{\Gamma}, t)$ is

$$D_{\phi} f = \lim_{\tau \rightarrow 0} \frac{f(\underline{X}_{\Gamma} + \phi \underline{\nu} \tau, t + \tau) - f(\underline{X}_{\Gamma}, t)}{\tau}. \tag{159}$$

With this definition, we can expressed the transport of the normal vector at point \underline{x}

$$D_{\phi} \underline{\nu} = -\nabla \phi \cdot \underline{e}_{\alpha} \underline{e}_{\alpha}, \tag{160}$$

where $\underline{e}_1, \underline{e}_2$ is a basis of the plane tangent to Γ at point \underline{x} . We can notice that the equation of the surface Γ , $S(\underline{X}, t) = 0$ satisfies immediately

$$D_{\phi} S = \frac{\partial S}{\partial \underline{X}} \cdot \dot{\underline{X}} + \frac{\partial S}{\partial t} = 0, \tag{161}$$

which defines the normal velocity \underline{c} of Γ ($||\underline{f}'||^2 = \underline{f} \cdot \underline{f}$):

$$\underline{c} = \phi \underline{\nu}, \quad \underline{\nu} = \frac{\partial S}{\partial \underline{X}} / \left\| \frac{\partial S}{\partial \underline{X}} \right\|, \tag{162}$$

and finally for any differentiable fields f the convected derivative is obvious

$$D_\phi f = \frac{\partial f}{\partial t} + \phi \nabla f \cdot \underline{\nu}. \tag{163}$$

Hadamard’s relations. The bounding being perfect between the phases, the displacement and the stress vector are continuous along Γ . Their rates have discontinuities according to the general compatibility conditions of Hadamard, rewritten with the convected derivative:

- continuity of displacement

$$[u]_\Gamma = 0 \Rightarrow D_\phi([u]_\Gamma) = [v]_\Gamma + \phi[\nabla u]_\Gamma \cdot \underline{\nu} = 0, \tag{164}$$

- continuity of the stress vector

$$[\sigma]_\Gamma \cdot \underline{\nu} = 0 \Rightarrow D_\phi([\sigma]_\Gamma \cdot \underline{\nu}) = [\dot{\sigma}]_\Gamma \cdot \underline{\nu} - \text{div}_\Gamma([\sigma]_\Gamma \phi) = 0. \tag{165}$$

The last equation is obtained taking the equilibrium equation into account. As we have

$$D_\phi([\sigma]_\Gamma \cdot \underline{\nu}) = D_\phi[\sigma]_\Gamma \cdot \underline{\nu} + [\sigma]_\Gamma \cdot D_\phi \underline{\nu}, \tag{166}$$

where

$$D_\phi[\sigma]_\Gamma = [\dot{\sigma}]_\Gamma + \phi \underline{\nu} \cdot [\nabla \sigma]_\Gamma \cdot \underline{\nu}, \tag{167}$$

and using the conservation of momentum

$$\underline{e}_\alpha \cdot \nabla \sigma \cdot \underline{e}_\alpha + \underline{\nu} \cdot \nabla \sigma \cdot \underline{\nu} = 0, \tag{168}$$

and the expression of the surface divergence given by

$$\text{div}_\Gamma F = \text{div} F - \underline{\nu} \cdot \nabla F \cdot \underline{\nu}, \tag{169}$$

the above result is obtained.

Orthogonality property for discontinuities. Since the displacement is continuous along the interface,

$$[u]_\Gamma = 0, \Rightarrow [\nabla u]_\Gamma \cdot \underline{e}_\alpha = 0, \tag{170}$$

the discontinuities of the gradient must satisfy

$$[\nabla u]_\Gamma = \underline{U}(x) \otimes \underline{\nu}. \tag{171}$$

Since the stress vector is continuous on Γ ,

$$[\sigma]_\Gamma \cdot \underline{\nu} = 0, \tag{172}$$

the discontinuities of σ and of ∇u have the property of orthogonality as pointed in [30]:

$$[\sigma]_\Gamma : [\nabla u]_\Gamma = 0. \tag{173}$$

14.2. Dissipation Analysis

We consider isothermal transformation and the dissipation is written as previously

$$D_m = \mathcal{P}_e - \frac{d}{dt}(W) \geq 0 \tag{174}$$

Taking into account of the discontinuities of the mechanical quantities along the moving surface Γ , we have

$$D_m = \int_{\partial\Omega} \underline{n} \cdot \underline{\sigma} \cdot \underline{v} \, da - \left(\int_{\Omega} \dot{\psi} \rho \, d\Omega - \int_{\Gamma} [\psi]_{\Gamma} \rho \phi \, da \right) \tag{175}$$

Taking account of equilibrium and Hadamard relations on the velocities the dissipation is rewritten as

$$D_m = \int_{\Gamma} \mathcal{G}(s) \phi(s) \, da \tag{176}$$

where

$$\mathcal{G}(s) = [\rho\psi]_{\Gamma} - \underline{\sigma} : [\underline{\varepsilon}]_{\Gamma} \tag{177}$$

This quantity has an analogous form to the driving traction force acting on a surface of strain discontinuity proposed in [31]. The criteria which guide the evolution of the interface may be written as function of this quantity.

In a thermomechanical coupling, two different release rates must be distinguished [6]. One defined in terms of variation of the total internal energy gives rise to the heat source associated with the moving surface; the second one gives rise to the production of entropy.

In the case of isothermal evolution the total dissipation is given in terms of the derivative of the potential energy relatively to the position of the interface

$$\frac{\partial \mathcal{E}}{\partial \Gamma} \cdot \dot{\Gamma} = - \int_{\Gamma} \mathcal{G}_s \phi \, da, \text{ or } \mathcal{G}_s(\underline{x}) = - \frac{\partial \mathcal{E}}{\partial \Gamma}(\underline{x}). \tag{178}$$

with $\mathcal{G}_s = \rho[\psi]_{\Gamma} - \underline{\sigma} : [\underline{\varepsilon}]_{\Gamma}$.

In this case, there is only one energy release rate to characterize the propagation, which gives the sources of entropy production and the dissipation.

These relations can be generalized in the dynamical case, by replacing the internal energy of the system by its Hamiltonian, and can be extended to the case of running cracks as well as more general behaviour and structures [6].

14.3. Quasistatic Evolution

In isothermal evolution we must give complementary relations to describe the irreversibility. An energy criterion is chosen as a generalized form of the well known theory of Griffith. Then, we assume

$$\phi \geq 0, \text{ if } \mathcal{G}_s = G_c \text{ on } \Gamma, \phi = 0, \text{ otherwise.} \tag{179}$$

This is a local energy criterion. At each equilibrium state, the interface Γ is decomposed into two subsets where the propagation is either possible or not. Let Γ^+ denote the subset of Γ where the critical value G_c

is reached. The evolution of the interface is governed by the consistency condition. If at the geometrical point $\underline{x}_\Gamma(t)$ the criterion is reached

$$\mathcal{G}_s(\underline{x}_\Gamma(t), t) = G_c, \tag{180}$$

then the derivative of \mathcal{G}_s following the moving surface vanishes $D_\phi \mathcal{G}_s = 0$. This leads to the consistency condition written for all point belonging to Γ^+

$$(\phi - \phi^*)D_\phi \mathcal{G}_s \geq 0, \forall \phi^* \geq 0, \text{ over } \Gamma^+, \tag{181}$$

otherwise $\phi = 0$.

Evaluation of $D_\phi \mathcal{G}_s$. Along the interface, the displacement is continuous, then the velocities satisfy the Hadamard relation:

$$\underline{v}_2 + \phi \nabla \underline{u}_2 \cdot \underline{\nu} = \underline{v}_1 + \phi \nabla \underline{u}_1 \cdot \underline{\nu}. \tag{182}$$

To take the convected derivative $D_\phi \mathcal{G}_s$, we consider each term of \mathcal{G} . The first term is the jump of free energy and the derivative is

$$D_\phi(\rho[\psi]_\Gamma) = -\sigma_2 : (\nabla \underline{v}_2 + \phi \nabla \nabla \underline{u}_2 \cdot \underline{\nu}) + \sigma_1 : (\nabla \underline{v}_1 + \phi \nabla \nabla \underline{u}_1 \cdot \underline{\nu}) \tag{183}$$

Hence, we obtain:

$$\begin{aligned} D_\phi \mathcal{G} &= D_\phi[\psi]_\Gamma - D_\phi \sigma_2 : [\nabla \underline{u}]_\Gamma - \sigma_2 : [D_\phi \nabla \underline{u}]_\Gamma \\ &= [\sigma]_\Gamma : (\nabla \underline{v}_1 + \phi \nabla \nabla \underline{u} \cdot \underline{\nu}) - (\dot{\sigma}_2 + \phi \nabla \sigma_2 \cdot \underline{\nu}) : [\nabla \underline{u}]_\Gamma \end{aligned} \tag{184}$$

Therefore after regrouping of terms

$$D_\phi \mathcal{G} = [\sigma]_\Gamma : \nabla \underline{v}_1 - \dot{\sigma}_2 : [\nabla \underline{u}]_\Gamma - \phi G_n \tag{185}$$

where the last term is given as

$$G_n = -[\sigma]_\Gamma : (\nabla \nabla \underline{u}_1 \cdot \underline{\nu}) + \nabla \sigma_2 \cdot \underline{\nu} : [\nabla \underline{u}]_\Gamma \tag{186}$$

15. The Rate Boundary Value Problem

The solution $(\tilde{v}, \tilde{\phi})$ must satisfy:

- the constitutive law: $\dot{\sigma} = \mathbb{C}_i : \dot{\epsilon}$, over Ω
- the compatibility for strain and displacement: $\dot{\epsilon} = \frac{1}{2}(\nabla \underline{v} + \nabla^T \underline{v})$ over Ω , and the boundary conditions $\underline{v} = \underline{v}^d$ along $\partial\Omega_u$,
- the conservation of the momentum: $\text{div } \dot{\sigma} = 0$ over Ω , and $\dot{\sigma} \cdot \underline{n} = \underline{\dot{T}}^d$ along $\partial\Omega_T$,
- the compatibility conditions along the moving perfect interface: $[D_\phi \underline{v}]_\Gamma = 0, [D_\phi(\sigma \cdot \underline{\nu})]_\Gamma = 0$,
- the propagation law: $\forall \beta \in \mathcal{K}, (\beta - \phi)D_\phi \mathcal{G} \geq 0$.

This system is written in a global formulation.

The rate boundary value problem. The evolution is determined by the functional

$$\begin{aligned} \mathcal{F}(\underline{v}, \tilde{\phi}, \dot{\underline{T}}^d) &= \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}(\underline{v}) : \mathbb{C} : \boldsymbol{\varepsilon}(\underline{v}) \, d\Omega - \int_{\partial\Omega_T} \dot{\underline{T}}^d \cdot \underline{v} \, da \\ &- \int_{\Gamma} \phi[\boldsymbol{\sigma}]_{\Gamma} : \nabla \underline{v}_1 \, da + \int_{\Gamma} \frac{1}{2} \phi^2 G_n \, da. \end{aligned}$$

The solution satisfies the inequality

$$0 \leq \frac{\partial \mathcal{F}}{\partial \underline{v}}(\underline{v} - \underline{v}^*) + \frac{\partial \mathcal{F}}{\partial \phi}(\beta - \phi), \tag{187}$$

among the set $\mathcal{K.A}$ of admissible fields $(\underline{v}^*, \phi^*)$:

$$\mathcal{K.A} = \left\{ (\underline{v}, \phi / \underline{v} = \underline{v}^d \text{ over } \partial\Omega_u, [\underline{v}]_{\Gamma} + \phi[\nabla \underline{u}]_{\Gamma} = 0, \phi \in \mathcal{K} \right\}, \tag{188}$$

$$\mathcal{K} = \{ \beta / \beta \geq 0 \text{ on } \Gamma^+, \beta = 0 \text{ otherwise} \}. \tag{189}$$

Proof The variations of the functional is given by

$$\begin{aligned} \delta \mathcal{F} &= \int_{\Omega} \boldsymbol{\varepsilon}(\underline{v}) : \mathbb{C} : \boldsymbol{\varepsilon}(\delta \underline{v}) \, d\Omega - \int_{\partial\Omega_T} \dot{\underline{T}}^d \cdot \delta \underline{v} \, da \\ &- \int_{\Gamma} \delta \phi[\boldsymbol{\sigma}]_{\Gamma} : \nabla \underline{v}_1 \, da + \int_{\Gamma} \phi \delta \phi G_n \, da - \int_{\Gamma} \phi[\boldsymbol{\sigma}]_{\Gamma} : \nabla \delta \underline{v}_1 \, da, \end{aligned}$$

after integration by part we obtain:

$$\begin{aligned} \delta \mathcal{F} &= \int_{\Gamma} \underline{n} \cdot [\dot{\boldsymbol{\sigma}} \cdot \delta \underline{v}]_{\Gamma} \, da + \int_{\partial\Omega_T} (\dot{\boldsymbol{\sigma}} \cdot \underline{n} - \dot{\underline{T}}^d) \cdot \delta \underline{v} \, da - \int_{\Gamma} \phi[\boldsymbol{\sigma}]_{\Gamma} : \nabla \delta \underline{v}_1 \, da \\ &- \int_{\Gamma} \delta \phi[\boldsymbol{\sigma}]_{\Gamma} : \nabla \underline{v}_1 \, da + \int_{\Gamma} \phi \delta \phi G_n \, da. \end{aligned}$$

Using now the compatibility conditions for the variation:

$$\delta \underline{v} = 0, \text{ over } \partial\Omega_u, \quad \delta \underline{v}_2 + \delta \phi \nabla \underline{u}_2 \cdot \underline{\nu} = \delta \underline{v}_1 + \delta \phi \nabla \underline{u}_1 \cdot \underline{\nu}, \text{ over } \Gamma, \tag{190}$$

we obtain finally:

$$\begin{aligned} \delta \mathcal{F} &= \int_{\Gamma} (\underline{\nu} \cdot [\dot{\boldsymbol{\sigma}}]_{\Gamma} - \text{div}_{\Gamma}(\phi[\boldsymbol{\sigma}]_{\Gamma})) \cdot \delta \underline{v}_1 \, da + \int_{\partial\Omega_T} (\underline{n} \cdot \dot{\boldsymbol{\sigma}} - \dot{\underline{T}}^d) \cdot \delta \underline{v} \, da \\ &- \int_{\Gamma} \delta \phi([\boldsymbol{\sigma}]_{\Gamma} : \nabla \underline{v}_1 - \dot{\boldsymbol{\sigma}}_2 \cdot [\nabla \underline{u}]_{\Gamma} - \phi G_n) \, da. \end{aligned}$$

Hence, we recover the conservation of the momentum and the propagation law.

Stability and Bifurcation The discussion of the stability and bifurcation along an evolution process can be investigated as presented in [13].

Consider the velocity \underline{v} solution of the rate boundary value problem for any given velocity ϕ . The field \underline{v} satisfies:

$$\begin{aligned} \operatorname{div} \dot{\boldsymbol{\sigma}} &= 0, \quad \dot{\boldsymbol{\sigma}} = \rho \frac{\partial^2 \psi}{\partial \boldsymbol{\varepsilon} \partial \boldsymbol{\varepsilon}} : \boldsymbol{\varepsilon}(\underline{v}), \text{ over } \Omega, \\ \underline{v} &= \underline{v}^d \text{ along } \partial\Omega_u, \quad \dot{\boldsymbol{\sigma}} \cdot \underline{n} = \underline{\dot{T}}^d \text{ along } \partial\Omega_T, \end{aligned}$$

and non-classical boundary conditions on Γ :

$$D_\phi([\boldsymbol{\sigma}]_\Gamma \cdot \underline{\nu}) = 0, \quad D_\phi[\underline{u}]_\Gamma = 0.$$

Consider the value W of \mathcal{F} for this solution $\underline{v}(\phi, \underline{v}^d, \underline{\dot{T}}^d)$

$$W(\phi, \underline{v}^d, \underline{\dot{T}}^d) = \mathcal{F}(\underline{v}(\phi, \underline{v}^d, \underline{\dot{T}}^d), \phi, \underline{\dot{T}}^d). \tag{191}$$

The stability of the actual state is determined by the condition of the existence of a solution

$$\delta\phi \frac{\partial^2 W}{\partial \phi \partial \phi} \delta\phi \geq 0, \quad \delta\phi \geq 0 \text{ on } \Gamma^+, \delta\phi \neq 0, \tag{192}$$

and the uniqueness and no-bifurcation is characterized by

$$\delta\phi \frac{\partial^2 W}{\partial \phi \partial \phi} \delta\phi \geq 0, \quad \delta\phi \neq 0 \text{ on } \Gamma^+. \tag{193}$$

The functional W is complex and has the general form

$$W = \int_\Gamma \int_\Gamma \frac{1}{2} \phi(s) \cdot \mathbb{B}(s, s') \phi(s') \, da \, da' - \int_\Gamma \bar{Q} \cdot \phi(s) \, da, \tag{194}$$

where $\mathbb{B}(s, s')$ is an integral operator.

Applications The main idea of this framework is that the constitutive law is composed of a potential ψ to describe the reversibility and a potential of dissipation and the normality rule for describing the evolution of the internal parameters. So this approach can be applied in generalized continuum media for study the delamination of laminates or thin film decohesion [32].

Moving surfaces and moving layers are used in a thermodynamical approach of wear and friction [33].

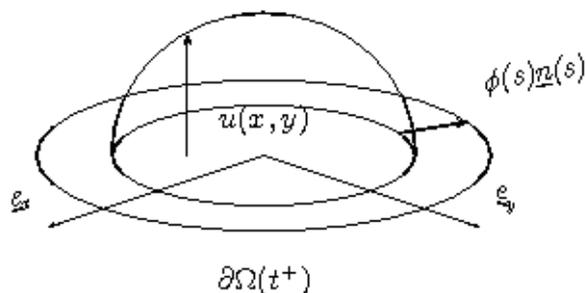
Using of stability and bifurcation analysis, this framework is also used to determine criterion of initiation of defects [34,35].

Many papers are concerned on damaged structure constituted by elastic-brittle material [36] with discussion of conditions of stability [27,37].

15.1. Delamination of a Thin Membrane under Pressure

In this case the volume $\Omega = Sxe$ with e small. The strain energy is given as $\psi = \frac{1}{2}K(\nabla u)^2$ where u is the transverse displacement as depicted in Figure 4.

Figure 4. Delamination of a thin membrane.



The potential energy of the whole system is:

$$\mathcal{E}(u(x, y), p) = \int_{\Omega} \frac{1}{2} K (\nabla u)^2 \, da - \int_{\Omega} pu \, da \tag{195}$$

The displacement $u = 0$ over $\partial\Omega$. When the boundary $\partial\Omega$ is moving the variation of energy gives the dissipation

$$D_m = \int_{\Omega} \frac{\partial(\psi - pu)}{\partial u} \delta u \, d\Omega - \int_{\partial\Omega} (\psi - pu) \delta\phi(s) \, da \tag{196}$$

The displacement δu is related to the boundary $\partial\Omega$ which moves with the velocity $\delta\phi$. Along this front $u = 0$ at each instant, then we have a constrain over the variations:

$$\delta u + \nabla u \cdot \underline{n} \, \delta\phi = 0$$

In the domain, the variations of the solution satisfies

$$K \Delta \delta u = 0 \quad \underline{x} \in \Omega$$

The dissipation is then given by

$$D_m = \int_{\partial\Omega} \psi \delta\phi \, da = \int_{\partial\Omega} \mathcal{G}(s) \delta\phi(s) \, da$$

The normality rule is used

$$\phi(s) \geq 0, \quad \mathcal{G}(s, t) - G_c \leq 0, \quad \phi(s)(\mathcal{G}(s, t) - G_c) = 0$$

This ensures the fact that if $\phi(s) > 0$ then $\mathcal{G}(s, t) = G_c$, this equation defines the motion of the boundary $\partial\Omega$. The variational inequality takes the form

$$\int_{\partial\Omega} \frac{d}{dt} (\mathcal{G} - G_c) (\delta\phi - \phi) \, da \geq 0. \tag{197}$$

for all $\delta\phi \geq 0$.

The boundary value problem is given by the functional

$$\mathcal{F}(\tilde{v}, \tilde{\phi}) = \int_{\Omega} \frac{1}{2} K (\nabla v)^2 \, d\Omega - \sqrt{2KG_c} \int_{\partial\Omega} \underline{n} \cdot \nabla \nabla u \cdot \underline{n} \phi^2 \, da \tag{198}$$

v and ϕ are linked by the constrain $v + \phi \nabla u \cdot \underline{n} = 0$ over $\partial\Omega$.

The evolution of \mathcal{G} is given by

$$\delta\mathcal{G} = K \nabla u \cdot \nabla \delta u + (K \nabla u \cdot \nabla \nabla u \cdot \underline{n}) \delta\phi$$

For a circular delamination, $\delta\phi$ is decomposed in Fourier series of θ . The second variations of energy is

$$\delta\phi \cdot \mathcal{E}'' \cdot \phi = \int_{\partial\Omega} (\dot{\mathcal{G}} + \phi \frac{\mathcal{G}}{R}) \delta\phi \, da$$

and the variation of the dissipation is given in the same time by

$$\delta D_m = \int_{\partial\Omega} G_c \delta\phi \frac{\phi}{R} \, da$$

The primal functional takes the form

$$\mathcal{F} = \int_{\Omega} \frac{1}{2} K \nabla v \cdot \nabla v \, d\Omega - \int_{\Omega} \dot{p} v \, d\Omega - \phi^2 K \nabla u \cdot \nabla \nabla u \cdot \underline{n} \, da \tag{199}$$

The set of the admissible velocities v is

$$\mathcal{K} = \{(\tilde{v}, \tilde{\phi})/v(s) + \phi(s) \nabla u \cdot \underline{n} = 0, \quad \phi \geq 0, \quad \mathcal{G} \leq G_c, \quad \phi(\mathcal{G} - G_c) = 0\} \tag{200}$$

By combining the equations, the solution is $u = \frac{p}{4K}(R^2 - r^2)$. The propagation is possible when $\mathcal{G} = G_c$ that defines the pressure $p = p_c = \frac{2}{R} \sqrt{2KG_c}$. For a change of shape of

$$\delta\phi = a_o + \sum_i a_i \cos(i\theta) + b_i \sin(i\theta) \tag{201}$$

the displacement is obtained as

$$\delta u = \frac{pR}{2K} \left(a_o + \sum_i (a_i \cos(i\theta) + b_i \sin(i\theta)) \left(\frac{r}{R}\right)^i \right) \tag{202}$$

The condition of stability is given by

$$2\pi G_c (-2a_o^2 + \sum_i (i-1)(a_i^2 + b_i^2)) \geq 0 \tag{203}$$

hence the circular shape is unstable for pressure controlled system.

Conversely, if the volume is controlled, the pressure is the lagrange multiplier associated to the condition

$$\int_{\Omega} u \, d\Omega = V^d \tag{204}$$

The condition of stability is governed by

$$2\pi G_c \left(6a_o^2 + \sum_i (i-1)(a_i^2 + b_i^2) \right) \geq 0 \tag{205}$$

The condition for stability is fulfilled but the condition of uniqueness is not, a_1 and b_1 can be defined such that $\delta\phi = a_o + a_1 \cos \theta + b_1 \sin \theta \geq 0$.

Many other examples are founded in literature for more complicated situations.

16. Conclusions

We have presented an introduction to the analysis of bifurcation and stability during the evolution of nonlinear system governed by potential energy, potential of dissipation and normality rule. The frame work is used in elastoplasticity, in fracture and for moving interfaces.

The rate boundary value problem has a formal identical structure and leads to variational inequalities that the evolution must satisfy. These inequalities are based on the second derivative of the energy of the system, and are quadratic operators. The properties of these operators give the condition of existence and uniqueness of the system evolution.

Some applications has been also presented. Many other situations can be investigated as in phase transformation [13]. This last example show how the analysis of stability bifurcation has strong implications in homogeneization of the existence of an homogeneized constitutive behaviour.

The conditions of stability and no-bifurcation can also be used to determine new criterion of initiation of defect as pointed out in [34,35].

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