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# On the Role of Entropy Generation in Processes Involving Fatigue

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**Abstract:** In this paper we describe the potential of employing the concept of thermodynamic entropy generation to assess degradation in processes involving metal fatigue. It is shown that empirical fatigue models such as Miner's rule, Coffin-Manson equation, and Paris law can be deduced from thermodynamic consideration.

**Keywords:** entropy generation; irreversible thermodynamics; fatigue failure; degradation

## 1. Introduction

Fatigue due to cyclic loading is one of the most predominant modes of failure in a diverse array of man-made components and natural systems. Given that a fatigue process is always accompanied by transformation of energy, it is logical to attempt at developing a thermodynamic framework for studying its characteristics. Naturally, energy dissipation represents an irreversible phenomenon, making the concept of thermodynamic entropy production an ideal tool for probing into its behavior [1]. In this paper we show that fatigue degradation and entropy generation are intimately related and that their relationship can be used for prediction of failure and making fundamental advances in the study of fatigue without having to resort to traditional approaches that depend on empirical models.

## 2. Thermodynamics of Fatigue

### *2.1. Entropy Balance Equation*

The statement of the second law of thermodynamics for deformation of a body as described by Clausius-Duhem inequality [2–5] reads:

$$\dot{\gamma} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p / T - A_k \dot{V}_k / T - \boldsymbol{J_q} \cdot \nabla T / T^2 \geq 0 \tag{1}$$

where $\dot{\gamma}$ denotes the entropy production per unit volume per unit time, $\boldsymbol{J}_q$ is the heat flux, $T$ is the absolute temperature, $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}_p$ is the plastic part of strain tensor, $V_k$ represents the internal variables associated with microstructure, $A_k$ are the thermodynamic forces associated with the internal variables.

Entropy generation presented in Equation (1) consists of three dissipation terms: plastic dissipation $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p$, dissipation associated with evolution of internal variables $A_k \dot{V}_k$, and thermal dissipation due to the conduction of heat $\boldsymbol{J_q} \cdot \nabla T / T$. However, research shows that in metals the dissipation associated with evolution of internal variables $A_k \dot{V}_k$ represents only 5–10% of the entropy generation due to plastic dissipation $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p$, and is often negligible [3,4,6–9]. Therefore, assuming $A_k \dot{V}_k / T \approx 0$, the Clausius-Duhem inequality reduces to:

$$\dot{\gamma} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p / T - \boldsymbol{J_q} \cdot \nabla T / T^2 \geq 0 \tag{2}$$

### *2.2. Entropy Generation Approach to Fatigue Failure*

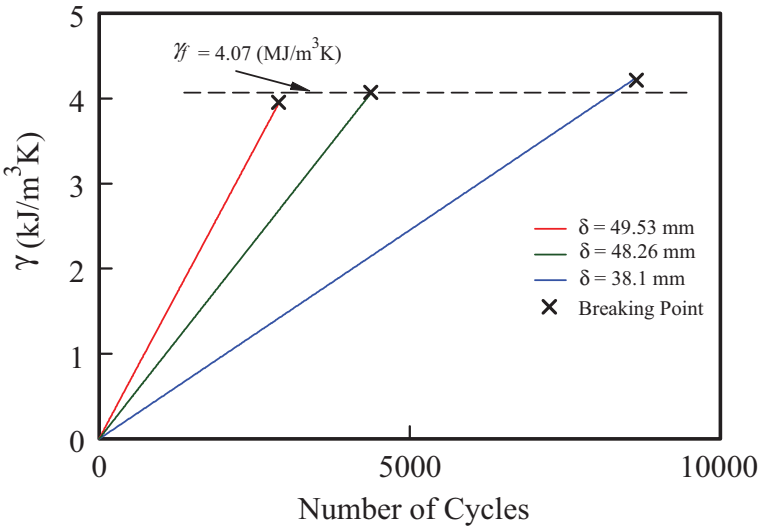
According to [7], the total accumulated entropy of metals, $\gamma_f$, undergoing repeated cyclic load as it reaches the point of fracture is a constant value, independent of load amplitude, geometry, size of specimen, frequency and stress state; see also [8–11]. The total entropy gain, or the so-called fatigue fracture entropy, can be evaluated by integrating Equation (2) from time $t = 0$ to $t = t_f$, when fracture occurs:

$$\gamma_f = \int_0^{t_f} \left( \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p / T - \boldsymbol{J_q} \cdot \nabla T / T^2 \right) dt \tag{3}$$

Naderi *et al.* [7] report an extensive series of experiments carried out to determine the fatigue fracture entropy for two different metals. Specifically, they show that the maximum value of entropy accumulation for Aluminum 6061-T6 is about 4 MJ/m$^3$K and about 60 MJ/m$^3$K for Stainless Steel 304L regardless of the load amplitude, geometry, size of specimen, frequency and stress state. In what follows typical result of accumulation of entropy generation is given to illustrate the concept.

Figure 1 shows the evolution of entropy generation for bending fatigue tests of Aluminum 6061-T6 samples clamped at one end and the other end cyclically bends with frequency of 10 Hz. Tests are carried out at three different displacement amplitudes of $\delta = 49.53$ mm, $\delta = 48.26$ mm and $\delta = 38.1$ mm. Note that values obtained for the accumulated entropy generation is nearly constant, averaging to $\gamma_f = 4.07$ MJ/m$^3$K, regardless of the displacement amplitude. At the beginning of the test, the accumulation of entropy is nil and it linearly increases until it reaches roughly 4.07 MJ/m$^3$K, at which point fracture occurs.

**Figure 1.** Evolution of entropy accumulation during fatigue tests pertaining to bending load of Aluminum 6061-T6.




According to [7], the entropy generation due to heat conduction inside the solid—the second term on the right hand side of Equation (3)—is negligibly small. That is, Equation (3) reduces to:

$$\gamma_f = \int_0^{t_f} \left( f \lhd w_p / T \right) dt \tag{4}$$

where $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_p = f \, \Delta \boldsymbol{w}_p$ is the plastic energy dissipation with $f$ as testing frequency.

*2.3. Application to Fatigue Life Prediction (Coffin-Manson Equation)*

The plastic energy generation per cycle, $\Delta w_p$, can be estimated using the following formula presented in the pioneering work of Morrow [12] on the assessment of energy generation during fatigue as:

$$\Delta w_p = \frac{4\sigma_f' \left( \frac{1-n'}{1+n'} \right)}{\left( \varepsilon_f' \right)^{n'}} \left( \frac{\Delta \varepsilon_p}{2} \right)^{1+n'} \tag{5}$$

where $\Delta\varepsilon_p$ is the plastic strain range, $n'$ is the cyclic strain hardening exponent, $\varepsilon_f'$ and $\sigma_f'$ are fatigue ductiliy and strength coefficients of the material. Morrow [12] experimentally demonstrates that in fully reveresed fatigue tests, the amount of energy generation per cycle is aproximately constant, but varies with the strain level, $\Delta\varepsilon$, and the cyclic properties of the material. Considering this assumption, Equation (4) yields to the following:

$$\gamma_f = \left( \frac{\Delta w_p}{T} \right) N_f \tag{6}$$

Substituting the plastic strain energy per cycle, $\Delta w_p$, from Equation (5), into Equation (6) and rearranging the resulting equation we obtain:

$$N_f = C \left( \Delta \varepsilon_p / 2 \right)^{\beta} \tag{7a}$$

This is the well-known Coffin-Manson relationship with constants $C$ and $\beta$ defined as following:

$$C = \frac{\gamma_f T\left(\varepsilon'_f\right)^{n'}}{4\sigma'_f\left(\frac{1-n'}{1+n'}\right)} \tag{7b}$$

$$\beta = \frac{-1}{1-n'} \tag{7c}$$

Equation (7a) is a direct consequence of the thermodynamic definition of entropy production as presented by Equation (4). It implies that empirical correlations such as Coffin-Manson equation can be subsumed into a more general thermodynamic analysis of the system taking into account the entropy generation.

It is to be mentioned that in derivation of Equation (6) it is, also, assumed that the temperature during fatigue process is constant. It is discussed by Amiri and Khonsari [13] that under environmentally undisturbed testing condition most of the fatigue life is spent in thermally steady-state condition wherein temperature remains almost constant. Discussion on the temperature variation of the samples under fatigue loading is beyond the scope of the present paper. Readers interested in further detail can refer to [13,14].

### *2.4. Application to Variable Load Amplitude (Miner's Rule)*

Let us assume that a specimen undergoes a series of stress levels $\sigma_i$, $i$=1, 2, …, $n$. Let $D$ represent the material degradation defined as the ratio of the accumulation of entropy generation divided by the fracture fatigue entropy, *viz.*:

$$D = \frac{\gamma_1 + \gamma_2 + \gamma_3 + \cdots}{\gamma_f} \tag{8}$$

where $\gamma_1$, $\gamma_2$, $\gamma_3$, …, are the entropy generations at stress levels $\sigma_1$, $\sigma_2$, $\sigma_3$, …, respectively. Employing Equation (6), $\gamma_i$ can be written as:

$$\gamma_i = \left(\frac{\Delta w_p}{T}\right)_i N_i \tag{9}$$

where the subscript $i$ = 1, 2, … corresponds to the stress level $\sigma_i$, and $N_i$ denotes the number of cycles elapsed at the corresponding stress level. Given that the fracture fatigue entropy, $\gamma_f$, is a material property and that it is independent of the stress level [7], the following relationship can be obtained from Equation (6):

$$\gamma_f = \left(\frac{\Delta w_p}{T}\right)_1 N_{f,1} = \left(\frac{\Delta w_p}{T}\right)_2 N_{f,2} = \left(\frac{\Delta w_p}{T}\right)_3 N_{f,3} = \cdots \tag{10}$$

where $N_{f,1}$, $N_{f,2}$, $N_{f,3}$, …, are the fatigue lives from constant stress amplitude at stresses $\sigma_1$, $\sigma_2$, $\sigma_3$, …, respectively. Substituting Equations (9) and (10) into Equation (8), yields:

$$D = \frac{\left(\Delta w_p/T\right)_1 N_1}{\left(\Delta w_p/T\right)_1 N_{f,1}} + \frac{\left(\Delta w_p/T\right)_2 N_2}{\left(\Delta w_p/T\right)_2 N_{f,2}} + \cdots = \frac{N_1}{N_{f,1}} + \frac{N_2}{N_{f,2}} + \cdots = \sum_i \frac{N_i}{N_{f,i}} \tag{11}$$

Now, failure occurs when the accumulation of the entropy generation reaches its maximum, *i.e.*, $\gamma_f$. This condition corresponds to $D = 1$. Therefore, from Equation (11) it follows:

$$\sum_i \frac{N_i}{N_{f,i}} = 1 \tag{12}$$

Equation (12) represents the linear fatigue damage hypothesis known as the Miner's rule.

### *2.5. Degradation Coefficient (DEG Theorem)*

In this section, we take advantage of the notion of thermodynamic forces, $X$, and thermodynamic flows, $J$, to explicitly express the rate of entropy production, $\mathrm{d}_i S$, in terms of experimentally measureable quantities. Following the notation of Bryant *et al.* [15], suppose that a system is divided into $j = 1, 2, \ldots, n$ subsystems with dissipative processes $p_j$, where each $p_j = p_j(\zeta_j^k)$ depends on a set of time-dependent phenomenological variables $\zeta_j^k = \zeta_j^k(t)$, $k = 1, 2, \ldots, m_j$. The entropy production of the entire system is the summation of the entropy production in each subsystem as follows:

$$\frac{d_i S}{dt} = \sum_j \sum_k \left( \frac{\partial_i S}{\partial p_j} \frac{\partial p_j}{\partial \zeta_j^k} \right) \frac{\partial \zeta_j^k}{\partial t} = \sum_j \sum_k X_j^k J_j^k \tag{13}$$

where $X_j^k$ are the thermodynamic forces and $J_j^k$ are the conjugate thermodynamic flows. It is to be noted that $\gamma$, explained in Equation (1) is the volumetric representation of entropy generation $\mathrm{d}_i S$. The entropy generation presented in Equation (1) consists of a group of thermodynamic forces $X = \left\{ \boldsymbol{\sigma}/T,\ A_k/T, -\nabla T/T^2 \right\}$ and thermodynamic rates or flows $J = \left\{ \dot{\boldsymbol{\varepsilon}}_{\boldsymbol{p}}, -\dot{V}_k, \boldsymbol{J}_{\boldsymbol{q}} \right\}$.

In conjunction with thermodynamic forces, Bryant *et al*. [15] introduce the concept of degradation forces to define degradation parameter $w = w\left\{ p_j(\zeta_j^k) \right\}$ as follows:

$$\frac{dw}{dt} = \sum_j \sum_k \left( \frac{\partial w}{\partial p_j} \frac{\partial p_j}{\partial \zeta_j^k} \right) \frac{\partial \zeta_j^k}{\partial t} = \sum_j \sum_k Y_j^k J_j^k \tag{14}$$

where $Y_j^k$ are the degradation forces. It is to be noted that the degradation of the system depends on the same dissipative processes $p_j$, as does the entropy generation. Considering the fact that thermodynamic flow $J_j^k$ is the common parameter in Equations (13) and (14), a degradation coefficient can be defined as [15]:

$$B_j = \frac{Y_j^k}{X_j^k} = \frac{\left(\partial w/\partial p_j\right)\left(\partial p_j/\partial \zeta_j^k\right)}{\left(\partial_i S/\partial p_j\right)\left(\partial p_j/\partial \zeta_j^k\right)} = \left. \frac{\partial w}{\partial_i S} \right|_{p_j} \tag{15}$$

Equation (15) suggests that $B_j$ measures how entropy generation and degradation interact on the level of dissipative processes $p_j$. Bryant *et al*. [15] refer to this model as Degradation-Entropy Generation (DEG) theorem.

### *2.6. Application to Paris-Erdogan Law*

Let us assume that the work of plastic deformation is the dominant dissipative process $p_j$. We define the crack length, $a$, as the degradation parameter, *i.e.*, $w = a$ in Equation (14). Therefore, degradation can be defined as $a = a\{W_p(N)\}$, where dissipative process is the plastic energy dissipation, $p = W_p$,

and the time-dependent phenomenological variable is the number of cycles, $\zeta = N$. Equations (13) and (14) yield:

$$\mathrm{d}_i S/\mathrm{d}t = XJ \text{ and } \quad \mathrm{d}a/\mathrm{d}t = YJ$$

where $J = \mathrm{d}N/\mathrm{d}t = fX = (\mathrm{d}_i S/\mathrm{d}W_p)(\mathrm{d}W_p/\mathrm{d}N)$, and $Y = (\mathrm{d}a/\mathrm{d}W_p)(\mathrm{d}W_p/\mathrm{d}N)$.

As mentioned before, $f$ denotes the frequency. The degradation coefficient, $B$, is expressed as $B = Y/X$. Assuming that the crack growth is occurring at steady rate and that all the plastic work is dissipated to increase entropy, *i.e.*, $\mathrm{d}W_p = T\mathrm{d}_i S$, the rate of irreversible entropy production due to plastic deformation can be obtained as follows:

$$\frac{\mathrm{d}_i S}{\mathrm{d}t} = \frac{\partial_i S}{\partial p}\frac{\partial p}{\partial N}\frac{\partial N}{\partial t} = \frac{f}{T}\frac{\mathrm{d}W_p}{\mathrm{d}N} \tag{16}$$

where $X = (1/T)(\mathrm{d}W_p/\mathrm{d}N)$. Applying Equation (14) yields:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = YJ = BXJ = B\frac{f}{T}\frac{\mathrm{d}W_p}{\mathrm{d}N} \tag{17}$$

It is interesting to note that right-hand-side of Equation (17) contains the degradation coefficient $B$ which shows how crack propagation and entropy generation interact on the level of dissipative plastic deformation process.

Methods are developed to assess the energy dissipation in the plastic zone ahead of the crack tip, $W_p$. Bodner *et al*. [16], for example, derive a correlation for the plastic energy dissipation at crack tip as follows:

$$\frac{\mathrm{d}W_p}{\mathrm{d}N} = At\frac{(\Delta K)^4}{\mu\sigma_y^2} \tag{18}$$

where $A$ is a dimensionless constant, $t$ is the specimen thickness, $\Delta K$ is the stress intensity factor, $\mu$ is the shear modulus and $\sigma_y$ is the yield stress. Substitution of Equation (18) into Equation (17) yields:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \frac{\mathrm{d}a}{f\mathrm{d}t} = B\frac{At}{T}\frac{(\Delta K)^4}{\mu\sigma_y^2} = C(\Delta K)^4 \tag{19a}$$

Equation (19a) is the well-known Paris-Erdogan law of fatigue crack propagation. Note that constant $C$ in the above equation includes degradation coefficient $B$, which is, in turn, related to entropy production via Equation (15). Having determined $B$, constant $C$ in Paris-Erdogan law can be evaluated as:

$$C = B\frac{At}{T\mu\,\sigma_y^2} \tag{19b}$$

The intensity of degradation coefficient $B$ determines how fast a crack propagates into the solid material. These relationships imply that entropy and crack propagation (as a measure of degradation) are intimately related via the degradation coefficient and that the empirical Paris-Erdogan law of crack propagation can be arrived at from consideration of the DEG theorem. It is to be noted that coefficient $C$ in Paris-Erdogan law and subsequently degradation coefficient $B$ is not necessarily constant and may vary depending, for example, on size of the specimen and/or grain size of material [17–20].

## 3. Conclusions

The concept of thermodynamic entropy generation offers a natural time base for developing the fundamental science for the study of dissipative processes. Processes involving fatigue are indeed governed by the principles of irreversible thermodynamics and useful insight can be gained by investigating their degradation behavior within this context. To illustrate the utility of the concepts, it is shown that many widely used empirical correlations for fatigue analysis can be arrived at by consideration of irreversible thermodynamics taking into account entropy generation as a degradation index.

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